Problem 1
Consider the eigenproblem
\[ y'' + \epsilon f(x)y + \lambda y = 0, \tag{1} \]
posed on \(0 < x < \pi\), with Dirichlet boundary conditions \(y(0) = y(\pi) = 0\). If \(\epsilon = 0\), the gravest mode is \(y_0 = \sin x\) with eigenvalue \(\lambda_0 = \pi^2\). Calculate the first-order shift in the eigenvalue \(\lambda_0\) induced by the perturbation \(\epsilon f(x)\). (An unevaluated integral involving \(f(x)\) is acceptable.)

Problem 2
Consider the boundary value problem
\[ \epsilon y'' + 2\sigma xy' + 4x^2 y = 0, \tag{2} \]
posed on the interval \(0 < x < 1\) with boundary conditions \(y(0) = 0\) and \(y(1) = 1\). (i) First, with \(\sigma = +1\), \(\tag{3} \)
construct a uniformly valid leading-order \(\epsilon \to 0\) solution. (ii) Next, do the same with \(\sigma = -1\). \(\tag{4} \)

Problem 3
Find \(\lim_{t\to\infty} x(t)\), where \(x(t)\) is the solution of
\[ \frac{dx}{dt} = e^x - 1.02 - x, \quad \text{with initial condition} \quad x(0) = 0. \tag{5} \]
Give a numerical answer with one significant figure.

Problem 4
Consider the eigenvalue problem
\[ \epsilon y'' + 2xy' + \lambda x \int_0^1 y(t) \, dt = 0, \tag{6} \]
posed on \(0 < x < 1\) with boundary conditions \(y(0) = y(1) = 0\). Notice that \(y(x) = 0\) is a trivial solution for every value of \(\lambda\). The problem has a nontrivial solution only if \(\lambda(\epsilon)\) (the eigenvalue) has some special value. (a) Show that
\[ 2\epsilon [y'(1) - y'(0)] + (\lambda - 4) \int_0^1 y(t) \, dt = 0. \tag{7} \]
(b) In the limit \(\epsilon \to 0\), construct a nontrivial solution with boundary-layer theory and determine \(\lambda(0)\). (c) Use the exact result (7) to deduce the next correction to the \(\epsilon \to 0\) expansion of \(\lambda(\epsilon)\).
Problem 5

Figure 1 shows the solution of one of the following differential equations

\begin{align*}
\epsilon^2 y_1'' + (1 + x)^2 y_1 &= 0, \\
\epsilon^2 y_2' - (1 + x)^2 y_2 &= 0, \\
\epsilon^2 y_3'' - (1 + x)^2 y_3 &= 0, \\
\epsilon^2 y_4'' + (1 + x)^2 y_4 &= 0.
\end{align*}

(8) \quad (9) \quad (10) \quad (11)

Which \( y_n(x) \) is shown in the figure? Estimate the value of \( \epsilon \) used in this calculation.