SIO203C/MAE294C: Perturbation theory and asymptotics

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Lecture 1

Algebraic perturbation theory

1.1 An introductory example

The quadratic equation
\[ x^2 - \pi x + 2 = 0, \]  
has the exact solutions
\[ x = \frac{\pi}{2} \pm \sqrt{\left(\frac{\pi}{2}\right)^2 - 2} = 2.254464 \quad \text{and} \quad 0.887129. \]  

To introduce the main idea of perturbation theory, let’s pretend that calculating a square root is a big deal. We notice that if we replace \( \pi \) by 3 in (1.1) then the resulting quadratic equation nicely factors and the roots are just \( x = 1 \) and \( x = 2 \). Because \( \pi \) is close to 3, our hope is that the roots of (1.1) are close to 1 and 2. Perturbation theory makes this intuition precise and systematically improves our initial approximations \( x \approx 1 \) and \( x \approx 2 \).

A regular perturbation series

We use perturbation theory by writing
\[ \pi = 3 + \epsilon, \]  
and assuming that the solutions of
\[ x^2 - (3 + \epsilon)x + 2 = 0, \]  
are given by a regular perturbation series (RPS):
\[ x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots \]  

We are assuming that the \( x_n \)’s above do not depend on \( \epsilon \). Putting (1.5) into (1.4) we have
\[ x_0^2 + \epsilon 2x_0x_1 + \epsilon^2 (2x_0x_2 + x_1^2) + \epsilon^3 (2x_0x_3 + 2x_1x_2) - (3 + \epsilon)(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3) + 2 = \text{ord} (\epsilon^4). \]  
The notation “\( \text{ord}(\epsilon^4) \)” means that we are suppressing some unnecessary information, but we are indicating that the largest unwritten terms are all proportional to \( \epsilon^4 \). We’re assuming that the expansion (1.5) is unique so we can match up powers of \( \epsilon \) in (1.7) and obtain a hierarchy of equations for the unknown coefficients \( x_n \) in (1.5).
The leading-order terms from (1.7) are
\[ \epsilon^0 : \quad x_0^2 - 3x_0 + 2 = 0, \quad \Rightarrow \quad x_0 = 1 \text{ or } 2. \] (1.8)

Quadratic equations have two roots and the important point is that indeed the leading order approximation above delivers two roots: this is a regular perturbation problem. We'll soon see examples in which the leading approximation provides only one root: these problems are singular perturbation problems.

Let's take \( x_0 = 1 \). The next three orders are then
\[ \epsilon^1 : \quad (2x_0 - 3)x_1 - x_0 = 0, \quad \Rightarrow \quad x_1 = \frac{x_0}{2x_0 - 3} = -1, \] (1.9)
\[ \epsilon^2 : \quad (2x_0 - 3)x_2 + x_1^2 - x_1 = 0, \quad \Rightarrow \quad x_2 = \frac{x_1 - x_1^2}{2x_0 - 3} = 2, \] (1.10)
\[ \epsilon^3 : \quad (2x_0 - 3)x_3 + 2x_1x_2 - x_2 = 0, \quad \Rightarrow \quad x_3 = \frac{x_2(1 - 2x_1)}{2x_0 - 3} = -6, \] (1.11)

The resulting perturbation expansion is
\[ x = 1 - \epsilon + 2\epsilon^2 - 6\epsilon^3 + \text{ord}(\epsilon^4). \] (1.12)

With \( \epsilon = \pi - 3 \) we find \( x = 0.881472 + \text{ord}(\epsilon^4) \).

Exercise: Determine a few terms in the RPS of the root \( x_0 = 2 \).

This example illustrates the main features of perturbation theory. When faced with a difficult problem one should:

1. Find an easy problem that’s close to the difficult problem. It helps if the easier problem has a simple analytic solution.
2. Quantify the difference between the two problems by introducing a small parameter \( \epsilon \).
3. Assume that the answer is in the form of a perturbation expansion, such as (1.5).
4. Compute terms in the perturbation expansion.
5. Solve the difficult problem by summing the series with the appropriate value of \( \epsilon \).

Step 4 above also deserves some discussion. In this step we are sequentially solving a hierarchy of linear equations:
\[ (2x_0 - 3)x_{n+1} = R(x_0, x_1, \ldots x_n). \] (1.13)

We can determine \( x_{n+1} \) because we can divide by \( 2x_0 - 3 \): fortunately \( x_0 \neq 3/2 \). This structure occurs in other simple perturbation problems: we confront the same linear problem at every level of the hierarchy and we have to solve this problem repeatedly.

**Iteration**

Now let’s consider the method of iteration — this is an alternative to the RPS. Iteration requires a bit of initial ingenuity. But in cases where the form of the expansion is not obvious, iteration is essential. (One of the strengths of H is that it emphasizes the utility of iteration.)

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1Mathematicians would say that we are inverting the linear operator — this language seems pretentious in this simple example. But we’ll soon see examples in which it is appropriate.
Solution of a quadratic equation by iteration

We can rewrite the quadratic equation (1.4) as
\[(x - 1)(x - 2) = \epsilon x.\] (1.14)

If we interested in the effect of \(\epsilon\) on the root \(x = 1\) then we rearrange this further as
\[x = 1 + \frac{\epsilon}{x - 2}.\] (1.15)

We iterate by first dropping the \(\epsilon\)-term on the right — this provides the first guess \(x^{(0)} = 1\). At the next iteration we keep the \(\epsilon\)-term with \(f\) evaluated at \(x^{(0)}\):
\[x^{(1)} = 1 + \epsilon f\left(x^{(0)}\right) = 1 - \epsilon.\] (1.16)

We continue to improve the approximation with more and more iterates:
\[x^{(n+1)} = 1 + f\left(x^{(n)}\right).\] (1.17)

So the second iteration is
\[x^{(2)} = 1 + \epsilon f\left(x^{(1)}\right) = 1 - \frac{1 - \epsilon}{1 + \epsilon}.\] (1.18)

This is not an RPS — if we want an answer ordered in powers of \(\epsilon\) then we must simplify (1.18) further as
\[x^{(2)} = 1 - \epsilon \left(1 - \epsilon\right) \left(1 - \epsilon + \text{ord}(\epsilon^2)\right) = 1 - \epsilon + 2\epsilon^2 + \text{ord}(\epsilon^3).\] (1.19)

I suspect there is no point in keeping the \(\epsilon^3\) in (1.19) because it is probably not correct — I am guessing that we have to iterate one more time to get the correct \(\epsilon^3\) term.

Exercise: use iteration to locate the root near \(x = 2\).

1.2 Singular perturbation of polynomial equations

Consider the example
\[\epsilon x^2 + x - 1 = 0.\] (1.20)

If we simply set \(\epsilon = 0\) we find \(x = 1\) and we can proceed to nail down this root with an RPS:
\[x = 1 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\] (1.21)

Exercise: Find \(x_1\) and \(x_2\) by both RPS and iteration.

But a quadratic equation has two roots: we’re missing a root. If we peek at the answer we find that
\[x = \begin{cases} 1 - \epsilon + 2\epsilon^2 - 5\epsilon^3 + \cdots \\ -\epsilon^{-1} - 1 + \epsilon - 2\epsilon^2 + 5\epsilon^3 + \cdots \end{cases}\] (1.22)

The missing root is going to infinity as \(\epsilon \to 0\). Notice that the term we blithely dropped, namely \(\epsilon x^2\), is therefore \(\text{ord}(\epsilon^{-1})\). Dropping a big term is a mistake.

We could have discovered the missing root by looking for two-term dominant balances in (1.20):
\[\epsilon x^2 + x - 1 = 0.\] (1.23)
The balance above implies that \( x = -\epsilon^{-1} \). The balance is consistent because the neglected term in (1.23) (the \(-1\)) is smaller than the two retained terms as \( \epsilon \to 0 \). Once we know that \( x \) is varying as \( \epsilon^{-1} \) we can *re-scale* by defining

\[
X \overset{\text{def}}{=} \epsilon x. \tag{1.24}
\]

The variable \( X \) remains finite as \( \epsilon \to 0 \), and substituting (1.24) into (1.20) we find that \( X \) satisfies the rescaled equation

\[
X^2 + X - \epsilon = 0. \tag{1.25}
\]

Now we can find the big root via an RPS

\[
X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots \tag{1.26}
\]

This procedure reproduces the expansion that begins with \(-\epsilon^{-1}\) in (1.22).

Notice that (1.24) is “only” a change in notation, and (1.25) is equivalent to (1.20). But notation matters: in terms of \( x \) the problem is singular while in terms of \( X \) the problem is regular.

**Example:** Find \( \epsilon \ll 1 \) expansions of the roots of

\[
\epsilon x^3 + x - 1 = 0. \tag{1.27}
\]

One root is obviously obtained via an RPS

\[
x = 1 - \epsilon + \text{ord}(\epsilon^2). \tag{1.28}
\]

But there are two missing roots. A dominant balance between the first two terms in (1.24),

\[
\epsilon x^3 + x \approx 0, \tag{1.29}
\]

implies that \( x \) varies as \( \epsilon^{-1/2} \). This balance is consistent, so rescale

\[
X \overset{\text{def}}{=} \epsilon^{1/2} x. \tag{1.30}
\]

This definition ensures that \( X \) is order unity as \( \epsilon \to 0 \). The rescaled equation is

\[
X^3 + X - \sqrt{\epsilon} = 0, \tag{1.31}
\]

and there is now a regular perturbation problem with small parameter \( \sqrt{\epsilon} \):

\[
X = X_0 + \sqrt{\epsilon} X_1 + (\sqrt{\epsilon})^2 X_2 + \cdots \tag{1.32}
\]

The leading order terms are

\[
X_0^3 + X_0 = 0 \quad \Rightarrow \quad X_0 = \pm i \quad \text{and} \quad X_0 = 0. \tag{1.33}
\]

The solution \( X_0 = 0 \) is reproducing the solution back in (1.28). Let’s focus on the other two roots, \( X_0 = \pm i \). At next order the problem is

\[
\sqrt{\epsilon} : \quad 3X_0^2 X_1 + X_1 - 1 = 0, \quad \Rightarrow \quad X_1 = \frac{1}{3X_0^2 + 1} = -\frac{1}{2}, \tag{1.34}
\]

For good value

\[
(\sqrt{\epsilon})^2 : \quad 3X_0^2 X_2 + X_2 + 3X_0 X_1^2 = 0, \quad \Rightarrow \quad X_2 = -\frac{3X_0 X_1^2}{3X_0^2 + 1} = \pm \frac{3}{8}. \tag{1.35}
\]

We write the expansion in terms of our original variable as

\[
x = \pm \frac{i}{\sqrt{\epsilon}} + \frac{1}{2} \pm \sqrt{\epsilon} \frac{3}{8} i + \text{ord}(\epsilon^1). \tag{1.36}
\]

**Example:** Find leading-order expressions for all six roots of

\[
\epsilon^2 x^6 - \epsilon x^4 - x^3 + 8 = 0, \quad \text{as} \quad \epsilon \to 0. \tag{1.37}
\]

This is from BO section 7.2.
1.3 Double roots

Now consider

\[ x^2 - 2x + 1 - \epsilon f(x) = 0. \]  \hspace{1cm} (1.38)

where \( f(x) \) is some function of \( x \). Section 1.3 of \( H \) discusses the case \( f(x) = x^2 \) — with a surfeit of testosterone we attack the general case.

We try the RPS:

\[ x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots \]  \hspace{1cm} (1.39)

We must expand \( f(x) \) with a Taylor series:

\[
 f \left( x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots \right) = f(x_0) + \epsilon x_1 f'(x_0) + \epsilon^2 \left( x_2 f''(x_0) + \frac{1}{2} x_1^2 f'''(x_0) \right) + \text{ord} \left( \epsilon^3 \right).
\]  \hspace{1cm} (1.40)

This is not as bad as it looks — we’ll only need the first term, \( f(x_0) \), though that may not be obvious at the outset.

The leading term in (1.38) is

\[ x_0^2 - 2x_0 + 1 = 0, \quad \Rightarrow \quad x_0 = 1, \ (\text{twice}). \]  \hspace{1cm} (1.41)

There is a double root. At next order there is a problem:

\[ \epsilon^1 : \quad 2x_1 - 2x_1 - f(1) = 0. \]  \hspace{1cm} (1.42)

Unless \( f(1) \) happens to vanish, we’re stuck. The problem is that we assumed that the solution has the form in (1.39), and it turns out that this assumption is wrong. The perturbation method kindly tells us this by producing the contradiction in (1.42).

To find the correct form of the expansion we use iteration: rewrite (1.38) as

\[ x = 1 \pm \sqrt{\epsilon f(x)}. \]  \hspace{1cm} (1.43)

and starting with \( x^{(0)} = 1 \), iterate with

\[ x^{(n+1)} = 1 \pm \sqrt{\epsilon f\left( x^{(n)} \right)}. \]  \hspace{1cm} (1.44)

At the first iteration we find

\[ x^{(1)} = 1 \pm \sqrt{\epsilon f\left( 1 \right)}. \]  \hspace{1cm} (1.45)

There is a \( \sqrt{\epsilon} \) which was not anticipated by the RPS back in (1.39).

Exercise: Go through another iteration cycle to find \( x^{(2)} \).

Iteration has shown us the way forward: we proceed assuming that the correct RPS is probably

\[ x = x_0 + \epsilon^{1/2} x_1 + \epsilon x_2 + \epsilon^{3/2} x_3 + \cdots \]  \hspace{1cm} (1.46)

At leading order we find \( x_0 = 1 \), and at next order

\[ \epsilon^{1/2} : \quad 2x_1 - 2x_1 = 0. \]  \hspace{1cm} (1.47)
This is surprising, but it is not a contradiction: $x_1$ is not determined at this order. We have to endure some suspense — we go to next order and find
\[
\epsilon^1 : \quad 2(x_0 - 1)x_2 + x_1^2 - f(x_0) = 0, \quad \Rightarrow \quad x_1 = \pm \sqrt{f(1)}.
\] (1.48)

The RPS has now managed to reproduce the first iterate $x^{(1)}$. Going to order $\epsilon^{3/2}$, we find that $x_3$ is undetermined and
\[
x_2 = \frac{1}{2} f'(1).
\] (1.49)

The solution we constructed is
\[
x = 1 \pm \sqrt{\epsilon f(1)} + \frac{\epsilon}{2} f''(1) + \text{ord} \left( \epsilon^{3/2} \right).
\] (1.50)

This example teaches us that a perturbation "splits" double roots. The splitting is rather large: adding the order $\epsilon$ perturbation in (1.38) moves the roots apart by order $\sqrt{\epsilon} \gg \epsilon$. This sensitivity to small perturbations is obvious geometrically — draw a parabola $P$ touching the $x$-axis at some point, and move $P$ downwards by small distance. The small movement produces two roots separated by a distance that is clearly much greater than the small vertical displacement of $P$. If $P$ moves upwards (corresponding to $f(1) < 0$ in the example above) then the roots split off along the imaginary axis.

### 1.4 An example with logarithms

I’ll discuss the example from H section 1.4:
\[
xe^{-x} = \epsilon.
\] (1.51)

It is easy to see that if $0 < \epsilon \ll 1$ there is a small solution and a big solution. It is straightforward to find the small solution in terms of $\epsilon$. Here we discuss the more difficult problem of finding the big solution.

**Exercise**: Show that the small solution is $x(\epsilon) = \epsilon + \epsilon^2 + \frac{3}{2} \epsilon^3 + \text{ord}(\epsilon^4)$.

To get a handle on (1.51), we take the logarithm and write the result as
\[
x = L_1 + \ln x,
\] (1.52)

where
\[
L_1 \overset{\text{def}}{=} \ln \frac{1}{\epsilon}.
\] (1.53)

Note if $0 < \epsilon < 1$ then $\ln \epsilon < 0$. To avoid confusion over signs it is best to work with the large positive quantity $L_1$.

Now observe that if $x \to \infty$ then there is a consistent two-term dominant balance in (1.52): $x \approx L_1$. This is consistent because the neglected term, namely $\ln x$, is much less then $x$ as $x \to \infty$. We can improve on this first approximation using the iterative scheme
\[
x^{(n+1)} = L_1 + \ln x^{(n)} \quad \text{with} \quad x^{(0)} = L_1.
\] (1.54)

The first iteration gives
\[
x^{(1)} = L_1 + L_2,
\] (1.55)

---

This example is related to the Lambert $W$-function, also known as the omega function and the product logarithm; try help ProductLog in Mathematica and lambertw in MATLAB.
Figure 1.1: Comparison of $\epsilon = xe^{-x}$ with increasingly accurate small-$\epsilon$ approximations to the inverse function $\epsilon(x)$.

where

$$L_2 \overset{\text{def}}{=} \ln L_1$$  \hfill (1.56)

is the iterated logarithm. The second iteration is

$$x^{(2)} = L_1 + \ln (L_1 + L_2) = L_1 + L_2 + \ln \left(1 + \frac{L_2}{L_1}\right),$$  \hfill (1.57)

$$= L_1 + L_2 + \frac{L_2}{L_1} - \frac{1}{2} \left(\frac{L_2}{L_1}\right)^2 + \cdots$$  \hfill (1.58)

We don’t need $L_3$.

At the third iteration a pattern starts to emerge

$$x^{(3)} = L_1 + \ln \left(\frac{L_1 + L_2 + \frac{L_2}{L_1} - \frac{1}{2} \left(\frac{L_2}{L_1}\right)^2 + \cdots}{L_1 + L_2 + \frac{L_2}{L_1} - \frac{1}{2} \left(\frac{L_2}{L_1}\right)^2 + \cdots}\right),$$

$$= L_1 + L_2 + \ln \left(1 + \frac{L_2}{L_1} + \frac{L_2^2}{L_1^2} - \frac{1}{2} \frac{L_2^3}{L_1^3} + \cdots\right),$$

$$= L_1 + L_2 + \left(\frac{L_2}{L_1} + \frac{L_2^2}{L_1^2} - \frac{1}{2} \frac{L_2^3}{L_1^3} + \cdots\right) - \frac{1}{2} \left(\frac{L_2}{L_1} + \frac{L_2^2}{L_1^2} + \cdots\right)^2 + \frac{1}{3} \left(\frac{L_2}{L_1} + \cdots\right)^3 + \cdots$$

$$= L_1 + L_2 + \frac{L_2}{L_1} + \frac{L_2 - \frac{1}{2} L_2^2}{L_1^2} + \frac{1}{3} \frac{L_2^3}{L_1^3} - \frac{3}{2} \frac{L_2^2}{L_1^2} + \cdots + \cdots$$  \hfill (1.60)

\footnote{We’re using the Taylor series

$$\ln(1 + \eta) = \eta - \frac{1}{2} \eta^2 + \frac{1}{3} \eta^3 + \frac{1}{4} \eta^4 + \cdots$$}
The final \dots above indicates a fraction with $L_4^1$ in the denominator.

The philosophy is that as one grinds out more terms the earlier terms in the developing expansion stop changing and a stable pattern emerges. In this example the expansion has the form

$$x = L_1 + L_2 + \sum_{n=1}^{\infty} \frac{P_n(L_2)}{L_1^n},$$

(1.61)

where $P_n$ is a polynomial of degree $n$. This was not guessable from (1.51).

1.5 Convergence

Usually we can’t prove that an RPS converges. The only way of proving convergence is to have a simple expression for the form of the $n$‘th term. In realistic problems this is not available. One just has to satisfied with consistency and hope for the best.

But with iteration there is a simple result. Suppose that $x = x_\ast$ is the solution of

$$x = f(x).$$

(1.62)

Start with a guess $x = x_0$ and proceed to iterate with $x_{n+1} = f(x_n)$. If an iterate $x_n$ is close to the solution $x_\ast$ then we have

$$x = x_\ast + \eta_n, \quad \text{with } \eta_n \ll 1.$$

(1.63)

The next iterate is:

$$x_\ast + \eta_{n+1} = f(x_\ast + \eta_n),$$

$$= x_\ast + \eta_n f'(x_\ast) + \text{ord}(\eta_n^2),$$

(1.64)

(1.65)

and therefore

$$\eta_{n+1} = f'(x_\ast) \eta_n.$$

(1.66)

The sequence $\eta_n$ will decrease exponentially if

$$|f'(x_\ast)| < 1.$$  

(1.67)

If the condition above is satisfied, and the first guess is good enough, then the iteration converges onto $x_\ast$. This is a loose version of the contraction mapping theorem.

1.6 Problems

**Problem 1.1.** Because 10 is close to 9 we suspect that $\sqrt{10}$ is close to $\sqrt{9} = 3$. (i) Define $x(\epsilon)$ by

$$x(\epsilon)^2 = 9 + \epsilon,$$

(1.68)

and assume that $x(\epsilon)$ has an RPS like (1.5). Calculate the first four terms, $x_0$ through $x_3$. (ii) Take $\epsilon = 1$ and compare your estimate of $\sqrt{10}$ with a six decimal place computation. (iii) Now solve (1.68) with the binomial expansion and verify that the resulting series is the same as the RPS from part (ii). What is the radius of convergence of the series?
Problem 1.2. Find two-term, $\epsilon \ll 1$ approximations to all roots of
\[ x^3 + x^2 + x + \epsilon = 0, \quad (1.69) \]
and
\[ y^3 - y^2 + \epsilon = 0, \quad (1.70) \]
and
\[ \epsilon z^4 - z + 1 = 0. \quad (1.71) \]

Problem 1.3. Assume that the Earth is a perfect sphere of radius $R = 6400\text{km}$ and that it is wrapped tightly at the equator with a rope. Suppose one cuts the rope and splices a length $\ell = 1\text{cm}$ into the rope. Then the rope is grabbed at a point and hoisted above the surface of the Earth as high as possible. How high is that?

Problem 1.4. Use perturbation theory to solve $(x + 1)^n = \epsilon x$, with $n$ an integer. How rapidly do the $n$ roots vary from $x = -1$ as a function of $\epsilon$? Give the first three terms in the expansion.

Problem 1.5. Here is a medley of algebraic perturbation problems, mostly from BO and H. Use perturbation theory to find two-term approximations ($\epsilon \to 0$) to all roots of:

(a) $x^2 + x + 6\epsilon = 0$,
(b) $x^3 - \epsilon x - 1 = 0$,
(c) $x^3 + \epsilon x^2 - x - \epsilon = 0$,
(d) $\epsilon^2 x^3 + x^2 + 2x + \epsilon = 0$,
(e) $\epsilon x^3 + x^2 - 2x + 1 = 0$,
(f) $\epsilon x^3 + x^2 + (2 + \epsilon)x + 1 = 0$,
(g) $\epsilon x^3 + x^2 + (2 - \epsilon)x + 1 = 0$,
(h) $\epsilon x^4 - x^2 - x + 2 = 0$,
(i) $\epsilon x^8 - \epsilon^2 x^6 + x - 2 = 0$,
(j) $\epsilon x^8 - \epsilon x^6 + x - 2 = 0$,
(k) $\epsilon^2 x^8 - \epsilon x^6 + x - 2 = 0$,
(l) $x^3 - x^2 + \epsilon = 0$,
(m) $x^2 + \epsilon = \frac{1}{x + 2\epsilon}$,
(n) $\epsilon e^{x^2} = 1 + \frac{\epsilon}{1 + x^2}$.

Problem 1.6. Consider $y(\epsilon, a)$ defined as the solution of
\[ \epsilon y^a = e^{-y}. \quad (1.72) \]
Note that $a = -1$ is the example [1.51]. Use the method of iteration to find a few terms in the $\epsilon \to 0$ asymptotic solution of (1.72) — “few” means about as many as in (1.60). Consider the case $a = +1$; use MATLAB to compare the exact solution with increasingly accurate asymptotic approximations (e.g., as in Figure 1.1).

Problem 1.7. Let us continue problem [1.6] by considering numerical convergence of iteration in the special case $a = 1$. Figure [1.2] shows numerical iteration of
\[ y_{n+1} = \ln \frac{1}{\epsilon} - \ln y_n. \quad (1.73) \]
Figure 1.2: Figure for problem 1.7. Numerical iteration of \( y_{n+1} = \ln \frac{1}{\epsilon} - \ln y_n \). At \( \epsilon = 0.45 \) the iteration diverges. In all three cases we start \( x_0 \) within 0.1% of the right answer.

With \( \epsilon = 0.25 \) everything is hunky-dory. At \( \epsilon = 0.35 \) the iteration is converging, but it is painfully slow. And at \( \epsilon = 0.45 \) it all goes horribly wrong. Explain this failure of iteration. To be convincing your explanation should include a calculation of the magic value of \( \epsilon \) at which numerical iteration fails. That is, if \( \epsilon > \epsilon^* \) then the iterates do not converge to the solution of \( \epsilon y = e^{-y} \). Find \( \epsilon^* \).

**Problem 1.8.** The relation

\[ xy = e^x - y \]  

implicitly defines \( y \) as a function of \( x \), or vice versa. View \( y \) as a function \( x \), and determine the large-\( x \) behavior of this function. Calculate enough terms to guess the form of the expansion.

**Problem 1.9.** Consider \( z(\epsilon) \) defined as the solution to

\[ z^\frac{1}{\epsilon} = e^z. \]  

(i) Use MATLAB to make a graphical analysis of this equation with \( \epsilon = 1/5 \) and \( \epsilon = 1/10 \). Convince yourself that as \( \epsilon \to 0 \) there is one root near \( z = 1 \), and second, large root that recedes to infinity as \( \epsilon \to 0 \). (ii) Use an iterative method to develop an \( \epsilon \to 0 \) approximation to the large solution. Calculate a few terms so that you understand the form of the expansion. (iii) Use MATLAB to compare the exact answer with approximations of various orders e.g., as in Figure 1.1. (iv) Find the dependance of the other root, near \( z = 1 \), on \( \epsilon \) as \( \epsilon \to 0 \).

**Problem 1.10.** Find the \( x \gg 1 \) solution of

\[ e^{e^x} = 10^{10}x^{10} \exp(10^{10}x^{10}) \]

with one significant figure of accuracy. (I think you can do this without a calculator if you use \( \ln 2 \approx 0.69 \) and \( \ln 10 \approx 2.30 \).)
Lecture 2

Regular perturbation of ordinary differential equations

2.1 The projectile problem

If one projects a particle vertically upwards from the surface of the Earth at \( z = 0 \) with speed \( u \) then the projectile reaches a maximum height \( h = \frac{u^2}{2g_0} \) and returns to the ground at \( t = \frac{2u}{g_0} \) (ignoring air resistance). At least that’s what happens if the gravitational acceleration \( g_0 \) is constant. But a better model is that the gravitational acceleration is

\[
g(z) = \frac{g_0}{(1 + z/R)^2},
\]

where \( g_0 = 9.8 \text{m s}^{-2}, R = 6,400 \text{kmeters} \) and \( z \) is the altitude. The particle stays aloft longer than \( 2u/g_0 \) because gravity is weaker up there.

Let’s use perturbation theory to calculate the correction to the time aloft due to the small decrease in the force of gravity. But first, before the perturbation expansion, we begin with a complete formulation of the problem. We must solve the second-order autonomous differential equation

\[
\frac{d^2 z}{dt^2} = -\frac{g_0}{(1 + z/R)^2},
\]

with the initial condition

\[
t = 0: \quad z = 0 \quad \text{and} \quad \frac{dz}{dt} = u.
\]

We require the time \( \tau \) at which \( z(\tau) = 0 \). Notice that if \( R = \infty \) we recover the elementary problem with uniform gravity.

An important part of this problem is non-dimensionalizing and identifying the small parameter used to organize a perturbation expansion. We use the elementary problem \((R = \infty)\) to motivate the following definition of non-dimensional variables

\[
z \overset{\text{def}}{=} \frac{g_0 z}{u^2}, \quad \text{and} \quad t \overset{\text{def}}{=} \frac{g_0 t}{u}.
\]

Notice that

\[
\frac{d}{dt} = \frac{g_0}{u} \frac{d}{dt}, \quad \text{and therefore} \quad \frac{d^2 z}{dt^2} = \left( \frac{g_0}{u} \right)^2 \frac{d^2 u^2}{dt^2} \frac{u^2}{g_0^2} \frac{dz}{dt^2} = g_0 \frac{d^2 z}{dt^2}
\]
Putting these expressions into (2.1) we obtain the non-dimensional problem

\[
\frac{d^2 \bar{z}}{dt^2} + \frac{1}{(1 + \epsilon \bar{z})^2} = 0, \quad (2.5)
\]

where

\[
\epsilon \overset{\text{def}}{=} \frac{u^2}{Rg_0}. \quad (2.6)
\]

We must also non-dimensionalize the initial conditions in (2.2):

\[
\bar{t} = 0 : \bar{z} = 0 \quad \text{and} \quad \frac{d \bar{z}}{d \bar{t}} = 1. \quad (2.7)
\]

At this point we have done nothing more than change notation. The original problem was specified by three parameters, \(g_0\), \(u\) and \(u\). The non-dimensional problem is specified by a single parameter \(\epsilon\), which might be large, small, or in between. If we’re interested in balls and bullets fired from the surface of the Earth then \(\epsilon \ll 1\).

OK, so assuming that \(\epsilon \ll 1\) we try a regular perturbation expansion on (2.5). We also drop all the bars that decorate the non-dimensional variables: we can restore the dimensions at the end of the calculation and it is just too onerous to keep writing all those little bars. The regular perturbation expansion is

\[
z(t) = z_0(t) + \epsilon z_1(t) + \epsilon^2 z_2(t) + \text{ord} \left( \epsilon^3 \right). \quad (2.8)
\]

We use the binomial theorem

\[
(1 + x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \text{ord} \left( x^3 \right), \quad (2.9)
\]

with \(n = -2\) to expand the nonlinear term:

\[
(1 + \epsilon z)^{-2} = 1 - 2\epsilon z + 3\epsilon^2 z^2 + \text{ord} \left( \epsilon^3 \right). \quad (2.10)
\]

Introducing (2.8) into the expansion above gives

\[
(1 + \epsilon z)^{-2} = 1 - 2\epsilon z_0 + \epsilon^2 (3z_0^2 - 2z_1) + \text{ord} \left( \epsilon^3 \right). \quad (2.11)
\]

So matching up equal powers of \(\epsilon\) in (2.5) (and denoting time derivatives by dots) we obtain the first three terms in perturbation hierarchy:

\[
\begin{align*}
\ddot{z}_0 &= -1, & \text{with } z_0(0) = 0, & \dot{z}_0(0) = 1, \\
\ddot{z}_1 &= 2z_0, & \text{with } z_1(0) = 0, & \dot{z}_1(0) = 0, \\
\ddot{z}_2 &= 2z_1 - 3z_0^2, & \text{with } z_2(0) = 0, & \dot{z}_2(0) = 0.
\end{align*}
\]

Above we have the first three terms in a hierarchy of linear equations of the form

\[
Lz_{n+1} = R(z_0, \cdots z_n), \quad (2.12)
\]

where the linear operator is

\[
L \overset{\text{def}}{=} \frac{d^2}{dt^2}. \quad (2.13)
\]

To solve each term in the hierarchy we must invert this linear operator, being careful to use the correct initial equations that \(z_{n+1}(0) = \dot{z}_{n+1}(0) = 0\).
The solution of the first two equations is

\[ z_0(t) = t - \frac{t^2}{2}, \quad \text{and} \quad z_1(t) = \frac{t^3}{3} - \frac{t^4}{12}. \]  \hfill (2.14)

To obtain \( z_2(t) \) we integrate

\[ \dot{z}_2 = -3t^2 + \frac{11t^3}{3} - \frac{11t^4}{12}, \]  \hfill (2.15)

to obtain

\[ z_2(t) = -\frac{t^4}{4} + \frac{11t^5}{60} - \frac{11t^6}{360}. \]  \hfill (2.16)

Thus the expanded solution is

\[ z(t) = t - \frac{t^2}{2} + 1\epsilon \left( \frac{t^3}{3} - \frac{t^4}{12} \right) + \epsilon^2 \left( -\frac{t^4}{4} + \frac{11t^5}{60} - \frac{11t^6}{360} \right) + \text{ord} (\epsilon^3). \]  \hfill (2.17)

We assume that the time aloft, \( \tau(\epsilon) \), also has a perturbation expansion

\[ \tau(\epsilon) = \tau_0 + \epsilon \tau_1 + \epsilon^2 \tau_2 + \text{ord} (\epsilon^3). \]  \hfill (2.18)

The terms in this expansion are determined by solving:

\[ z_0 \left( \tau_0 + \epsilon \tau_1 + \epsilon^2 \tau_2 \right) + \epsilon z_1 \left( \tau_0 + \epsilon \tau_1 \right) + \epsilon^2 z_2 \left( \tau_0 \right) = \text{ord} (\epsilon^3). \]  \hfill (2.19)

We have ruthlessly ditched all terms of order \( \epsilon^3 \) into the garbage heap on the right of (2.19). The left side is a polynomial of order \( \tau^6 \) so there are six roots. One of these roots is \( \tau = 0 \) and another root is close to \( \tau = 2 \). The other four roots are artificial creatures of the perturbation expansion and should be ignored — if we want the time aloft then we focus on the root near \( \tau = 2 \) by taking \( \tau_0 = 2 \) in (2.18). Expanding the \( z_n \)'s in a Taylor series about \( \tau_0 = 2 \), we have:

\[ z_0(2) + (\epsilon \tau_1 + \epsilon^2 \tau_2) \dot{z}_0(2) + \frac{1}{2} (\epsilon \tau_1)^2 z_0(2) + \epsilon z_1(2) + \epsilon^2 \tau_1 \dot{z}_1(2) + \epsilon^2 z_2(2) = \text{ord} (\epsilon^3). \]  \hfill (2.20)

Now we can match up powers of \( \epsilon \):

\[
\begin{align*}
z_0(2) &= 0, \\
\tau_1 \dot{z}_0(2) + z_1(2) &= 0, \\
\tau_2 \dot{z}_0(2) + \frac{1}{2} \tau_1^2 \dot{z}_0(2) + \tau_1 \dot{z}_1(2) + z_2(2) &= 0.
\end{align*}
\]

Solving these equations, one finds

\[ \tau = 2 + \frac{4}{3} \epsilon + \frac{4}{5} \epsilon^2 + O(\epsilon^3). \]

The Taylor series above is another procedure for generating the expansion of a regularly perturbed root of a polynomial.

---

1 Some intermediate results \( \dot{z}_0(2) = -1, z_1(2) = 4/3, \dot{z}_1(2) = 4/3 \) and \( z_2(2) = -4/45 \).
Attempted solution of the projectile problem by iteration

We’re considering the differential equation

$$\frac{d^2 z}{dt^2} + \frac{1}{(1 + \epsilon z)^2} = 0, \quad (2.21)$$

again. Our first iterate is

$$z^{(0)} = t - \frac{t^2}{2}, \quad (2.22)$$

which is the same as the first term in the earlier RPS. To obtain the next iterate, $z^{(1)}(t)$, we try to solve

$$\frac{d^2 z^{(1)}}{dt^2} + \frac{1}{(1 + \epsilon (t - \frac{t^2}{2}))^2} = 0, \quad (2.23)$$

with the initial condition

$$z^{(1)} = 0, \quad \dot{z}^{(1)}(0) = 1. \quad (2.24)$$

We could assault this problem with Mathematica or Maple:

```
DSolve[{z''[t] + 1/(1 + (t - t^2/2))^2 == 0, z[0] == 0, z'[0] == 1}, z[t], t]
```

However the answer is not presentable in polite company. In this example, the RPS back in (2.17) is definitely superior to iteration.

### 2.2 A boundary value problem: belligerent drunks

Imagine a continuum of drunks random-walking along a stretch of sidewalk, the $x$-axis, that lies between bars at $x = 0$ and $x = \ell$. When a drunk collides with another drunk they have a certain probability of mutual destruction: a fight breaks out that may result in the death of one or both participants. We desire the density of drunks on the stretch of sidewalk between $x = 0$ and $x = \ell$. The mathematical description of this problem is based on the density (drunks per meter) $u(x, t)$, which is governed by the partial differential equation

$$u_t = \kappa u_{xx} - \mu u^2, \quad (2.25)$$

with boundary conditions at the bars

$$u(0, t) = u(\ell, t) = U. \quad (2.26)$$

We’re modeling the bars using a Dirichlet boundary condition — there is constant density at each bar and drunks spill out onto the sidewalk. The parameter $\mu$ models the lethality of the interaction between pairs of drunks.

**Exercise:** How would the formulation change if the drunks are not belligerent? They peacefully ignore each other. But instead, drunks have a constant probability per unit time of dropping dead. How does the formulation of the continuum model change? (In this case you might prefer to think of $u(x, t)$ as the concentration of a radioactive element, rather than drunken walkers.)

If we integrate (2.25) from $x = 0$ to $\ell$ we obtain

$$\frac{d}{dt} \int_0^\ell u \, dx = [\kappa u_x]_0^\ell - \int_0^\ell \mu u^2 \, dx. \quad (2.27)$$
You should be able to interpret each term in this budget.

First order of business is to non-dimensionalize the problem. How many control parameters are there? With the definitions

\[ \bar{t} \overset{\text{def}}{=} \frac{\kappa t}{\ell^2}, \quad \bar{x} \overset{\text{def}}{=} \frac{x}{\ell} - \frac{1}{2}, \quad \text{and} \quad \bar{u} \overset{\text{def}}{=} \frac{u}{U}, \tag{2.28} \]

we quickly find that the scaled boundary value problem is

\[ u_t = u_{xx} - \beta u^2, \quad \text{with BCs} \quad u(\pm 1/2) = 1. \tag{2.29} \]

There is a single control parameter

\[ \beta \overset{\text{def}}{=} \frac{\ell^2 \mu U}{\kappa}. \tag{2.30} \]

We made an aesthetic decision to put the boundaries at \( x = \pm 1/2 \).

Looking for a steady solution \((u_t = 0)\) to the partial differential equation, we are lead to consider the nonlinear boundary value problem

\[ u_{xx} = \beta u^2, \quad \text{with BCs} \quad u(\pm 1) = 1. \tag{2.31} \]

The weakly interacting limit \( \beta \ll 1 \)

If \( \beta \ll 1 \) — the weakly interacting limit — we can use an RPS

\[ u = u_0(x) + \beta u_1(x) + \cdots. \tag{2.32} \]

The leading-order problem

\[ u_{0xx} = 0, \quad \text{with BCs} \quad u_0(\pm 1/2) = 1, \tag{2.33} \]

and solution

\[ u_0(x) = 1. \tag{2.34} \]

At subsequent orders, the BCs are homogeneous. For example, the first-order problem is

\[ u_{1xx} = \frac{u_0^2}{u_0^2} = 1, \quad u_1(\pm 1/2) = 0, \tag{2.35} \]
with solution
\[ u_1(x) = \frac{4x^2 - 1}{8}. \]  (2.36)

At second order, \( \beta^2 \),
\[ u_{2xx} = 2u_0u_1 = x^2 - \frac{1}{4}, \quad u_2(\pm1/2) = 0, \]  (2.37)
with solution
\[ u_2(x) = \frac{x^4}{12} - \frac{x^2}{8} + \frac{5}{192} = \frac{(4x^2 - 1)(4x^2 - 5)}{192}. \]  (2.38)

Figure 2.1 compares the perturbation solution with a numerical solution obtained using the MATLAB routine bvp4c. The three term expansion is not bad, even at \( \beta = 1 \). We resist the temptation to compute \( u_3 \).

At every step of the perturbation hierarchy we are inverting the linear operator \( \frac{d^2}{dx^2} \) with homogeneous boundary conditions. You should recognize that all the regular perturbation problems we’ve seen have this structure. There is a general result, called the implicit function theorem, which assures us that if we know how to solve these reduced linear problems, with invertible linear operators, then the original problem has a solution for some sufficiently small value of the expansion parameter (\( \beta \) in the problem above).

### 2.3 Failure of RPS — singular perturbation problems

Let's close by giving a few examples of differential equation which do not obligingly yield to regular perturbation methods.

#### Boundary layers

First, consider the boundary value problem (2.31) with \( \beta = \epsilon^{-1} \gg 1 \). In terms of \( \epsilon \), the problem is
\[ \epsilon u_{xx} = u^2, \quad \text{with BCs} \quad u(\pm1) = 1. \]  (2.39)
We try the RPS
\[ u = u_0(x) + \epsilon u_1(x) + \cdots \]  (2.40)
The leading order is
\[ 0 = u_0^2, \quad \text{with BCs} \quad u_0(\pm1) = 1. \]  (2.41)
Immediately we see that there is no solution to the leading-order problem.

What's gone wrong? Let's consider a linear problem with the same issues:
\[ \epsilon v_{xx} = v, \quad \text{with BCs} \quad v(\pm1) = 1. \]  (2.42)
Again the RPS fails because the leading-order problem,
\[ 0 = v_0, \quad \text{with BCs} \quad v_0(\pm1) = 1, \]  (2.43)
has no solution. The advantage of a linear example is that we can exhibit the exact solution:
\[ v = \frac{\cosh(x/\sqrt{\epsilon})}{\cosh(1/\sqrt{\epsilon})}, \]  (2.44)
see figure ???. The exact solution has boundary layers near \( x = -1 \) and \( x = +1 \). In these regions \( v \) varies rapidly so that the term \( \epsilon v_{xx} \) in (2.42) is not small relative to \( v \). Note that the
leading order interior solution, \( v_0 = 0 \) is a good approximation to the correct solution outside the boundary layers. In this interior region the exact solution is exponentially small e.g.,

\[
v(0, \epsilon) = \frac{1}{\cosh(1/\sqrt{\epsilon})} \sim 2e^{-1/\sqrt{\epsilon}} \quad \text{as} \quad \epsilon \to 0.
\]  

(2.45)

Our attempted RPS is using \( \epsilon^n \) as gauge functions and as \( \epsilon \to 0 \)

\[
2e^{-1/\sqrt{\epsilon}} = O(\epsilon^n), \quad \text{for all} \quad n \geq 0.
\]  

(2.46)

As far as the \( \epsilon^n \) gauge is concerned, \( e^{-1/\sqrt{\epsilon}} \) is indistinguishable from zero.

The problem in both the examples above is that the small parameter \( \epsilon \) multiplies the term with the most derivatives. Thus the leading-order problem in the RPS is of lower order than the exact problem. In fact, in the examples above, the leading-order problem is not even a differential equation.

Exercise: Considering the example above, does it indicate how to approximately solve the linear differential equation

\[
10^{-12}v_{xx} = e^x v, \quad \text{with BCs} \quad v(\pm 1) = 1?
\]  

If you can do this, you’ll be on your way to understanding boundary layer theory.

Rapid oscillations

Another linear problem that defeats a regular perturbation expansion is

\[
\epsilon w_{xx} = -w, \quad \text{with BCs} \quad w(\pm 1) = 1.
\]  

(2.48)

The exact solution, shown in figure ??, is

\[
w = \frac{\cos(x/\sqrt{\epsilon})}{\cos(1/\sqrt{\epsilon})}.
\]  

(2.49)

In this case the solution is rapidly varying throughout the domain. The term \( \epsilon w_{xx} \) is never smaller than \( w \).

Secular errors

Let’s consider a more subtle problem:

\[
\ddot{x} + (1 + \epsilon)x = 0, \quad \text{with ICs} \quad x(0) = 1, \quad \text{and} \quad \dot{x}(0) = 0.
\]  

(2.50)

The exact solution of this oscillator problem is

\[
x(t, \epsilon) = \cos \left( \sqrt{1 + \epsilon} t \right).
\]  

(2.51)

In this case it looks like the RPS

\[
x(t, \epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots
\]  

(2.52)

might work. The leading-order problem is

\[
\ddot{x}_0 + x_0 = 0, \quad \text{with ICs} \quad x_0(0) = 1, \quad \text{and} \quad \dot{x}_0(0) = 0,
\]  

(2.53)

with solution

\[
x_0 = \cos t.
\]  

(2.54)
In fact, this RPS does work for some time — see figure ?. But eventually the exact solution (2.51) and the leading-order approximation in (2.54) have different signs. That’s a bad error if \( x_0(t) \) is a clock.

Maybe we can improve the approximation by calculating the next term? The order \( \epsilon^1 \) problem is

\[
\ddot{x}_1 + x_1 = -\cos t, \quad \text{with homogeneous ICs} \quad x_1(0) = 0, \quad \text{and} \quad \dot{x}_1(0) = 0. \tag{2.55}
\]

I hope you recognize a resonantly forced oscillator when you see it: the solution of (2.55) is

\[
x_1 = -\frac{1}{2} t \sin t. \tag{2.56}
\]

Thus the perturbation solution is now

\[
x = \cos t - \epsilon \frac{1}{2} t \sin t + \text{ord}(\epsilon^2). \tag{2.57}
\]

This first-order “correction” makes matters worse — see figure ?. The RPS in (2.57) is “disordered” once \( \epsilon t = \text{ord}(1) \): we don’t expect an RPS to work if the higher order terms are larger than the earlier terms. Clearly there is a problem with this direct perturbative solution of an elementary problem.

In this example the term \( \epsilon x \) is small relative to the other two terms in differential equation at all time. Yet the small error slowly accumulates over long times \( \sim \epsilon^{-1} \). Astronomers call this a secular error\(^2\). We did not face secular errors in the projectile problem because we were solving the differential equation only for the time aloft, which was always much less than \( 1/\epsilon \).

\(^2\)From Latin saecula, meaning a long period of time. Saecula saeculorum is translated literally as “in a century of centuries”, or more poetically as “forever and ever”, or “world without end”.

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2.4 Problems

Problem 2.1. (i) Consider the projectile problem with linear drag:

\[
\frac{d^2 z}{dt^2} + \mu \frac{dz}{dt} = -g_0,
\]  

and the initial conditions \( z(0) = 0 \) and \( \frac{dz}{dt} = u \). Find the solution with no drag, \( \mu = 0 \), and calculate the time aloft, \( \tau \). (ii) Suppose that the drag is small — make this precise by non-dimensionalizing the equation of motion and exhibiting the relevant small parameter \( \epsilon \). (iii) Use a regular perturbation expansion to determine the first correction to \( \tau \) associated with non-zero drag. (iv) Integrate the non-dimensional differential equation exactly and obtain a transcendental equation for \( \tau(\epsilon) \). Asymptotically solve this transcendental equation approximately in the limits \( \epsilon \to 0 \) and \( \epsilon \to \infty \). Make sure the \( \epsilon \to 0 \) solution agrees with the earlier RPS.

Problem 2.2. Consider the projectile problem with quadratic drag:

\[
\frac{d^2 z}{dt^2} + \nu |\frac{dz}{dt}| \frac{dz}{dt} = -g_0,
\]  

and the initial conditions \( z(0) = 0 \) and \( \frac{dz}{dt} = u \). (i) Explain why the absolute value \( |\dot{z}| \) in (2.59) is necessary if this term is to model air resistance. (ii) What are the dimensions of the coefficient \( \nu \)? Nondimensionalize the problem so there is only one control parameter. (iii) Suppose that \( \nu \) is small. Use a regular perturbation expansion to determine the first correction to the time aloft. (iv) Solve the nonlinear problem exactly and obtain a transcendental equation for the time aloft. (This is complicated.)

Problem 2.3. (i) Solve the problem

\[
\ddot{x} + \left(1 + \epsilon e^{\alpha t}\right) x = 0,
\]

with IC

\[
x(0, \epsilon, \alpha) = 1, \quad \dot{x}(0, \epsilon, \alpha) = 0,
\]

with the RPS

\[
x(t, \epsilon, \alpha) = x_0(t, \alpha) + \epsilon x_1(t, \alpha) + \cdots
\]

Calculate \( x_0 \) and \( x_1 \). (ii) Bearing in mind that \( \alpha \) might be positive or negative, discuss the utility of the RPS when \( t \) is large.

Problem 2.4. Consider a partial differential equation analog to the boundary value problem in (2.31). The domain is the disc \( r = \sqrt{x^2 + y^2} < a \) in the \( (x, y) \)-plane and the problem is

\[
u_{xx} + u_{yy} = \alpha u^2,
\]

with BC: \( u(a, \theta) = U \).

Following the discussion in section 2.2, compute three terms in the RPS.

Problem 2.5. Let’s make a small change to the formulation of the belligerent-drunks example in (2.25) and (2.26). Suppose that we model the bars using a Neumann boundary condition. This means that the flux of drunks, rather than the concentration, is prescribed at \( x = 0 \) and \( \ell \): the boundary condition in (2.26) is changed to

\[
\kappa u_x(0, t) = -F, \quad \text{and} \quad \kappa u_x(\ell, t) = F,
\]

where \( F \), with dimensions drunks per second, is the flux entering the domain from the bars. Try to repeat all calculations in section 2.2 including the analog of the \( \beta \ll 1 \) perturbation expansion. You’ll find that it is not straightforward and that a certain amount of ingenuity is required to understand the weakly interacting limit with fixed-flux boundary conditions.
**Problem 2.6.** Consider the non-dimensional oscillator problem

\[ \ddot{x} + \beta \dot{x} + x = 0 \]  
(2.64)

with the initial conditions

\[ x(0) = 0, \quad \text{and} \quad \dot{x}(0) = 1. \]  
(2.65)

(i) Supposing that \( \beta > 2 \), solve the problem exactly. (ii) Show that if \( \beta \gg 1 \) then the long-time behaviour of your exact solution is

\[ x \propto e^{-t/\beta}, \]  
(2.66)

i.e., the displacement very slowly decays to zero. (iii) Motivated by this exact solution, “rescale” the problem (and the initial condition) by defining the slow time

\[ \tau \overset{\text{def}}{=} \frac{t}{\beta}, \]  
(2.67)

and \( X(\tau) = ? x(t) \). Show that with a suitable choice of \( ? \), the rescaled problem is

\[ \epsilon \ddot{X} + \dot{X} + X = 0 \quad \text{with the IC:} \quad X(0) = 0, \quad \dot{X}(0) = 1. \]  
(2.68)

Make sure you give the definition of \( X(\tau) \) and \( \epsilon \ll 1 \) in terms of the parameter \( \beta \gg 1 \) and the original variable \( x(t) \). (iv) Try to solve the rescaled problem (2.68) using an RPS

\[ X(\tau, \epsilon) = X_0(\tau) + \epsilon X_1(\tau) + \cdots \]  
(2.69)

Discuss the miserable failure of this approach by analyzing the dependence of the exact solution from part (i) on \( \beta \). That is, simplify the exact solution to deduce a useful \( \beta \to \infty \) approximation, and explain why the RPS (2.69) cannot provide this useful approximation.
Lecture 3

What is asymptotic?

3.1 Convergence and asymptoticity

We consider the error function

\[
erf(z) \overset{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \, dt.
\]  (3.1)

Figure 3.1 shows \( \text{erf} \) on the real line. The series on the right of

\[
e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!}
\]  (3.2)

has infinite radius of convergence i.e., \( e^{-t^2} \) is an entire function in the complex \( t \)-plane. Thus we can simply integrate term-by-term in (3.1) to obtain a series that converges in the entire complex plane:

\[
erf(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-z^2)^n}{(2n+1)n!}
\]  (3.3)

\[
= \frac{2}{\sqrt{\pi}} \left( z - \frac{1}{3}z^3 + \frac{1}{10}z^5 - \frac{1}{42}z^7 + \frac{1}{210}z^9 - \frac{1}{1320}z^{11} \right) + R_6,
\]  (3.4)

where \( \text{erf}_6(x) \) is the sum of the first six terms and \( R_6(z) \) is the remainder after 6 terms.

We denote the sum of the first \( n \) nonzero terms of the Taylor series as \( \text{erf}_n(x) \). Figure 3.1 shows \( \text{erf}_{10}(x) \). With matlab we find that

\[
\frac{\text{erf}(1) - \text{erf}_{10}(1)}{\text{erf}(1)} = 1.6217 \times 10^{-8}, \quad \text{and} \quad \frac{\text{erf}(2) - \text{erf}_{10}(2)}{\text{erf}(2)} = 0.0233.
\]  (3.5)

The Taylor series is useful if \(|z| < 1\), but as \(|z| \) increases convergence is slow. Moreover some of the intermediate terms are very large and there is a lot of destructive cancellation between terms of different signs. The lower panel of Figure 3.1 shows that this cancellation is already happening at \( z = 2 \), and it gets a lot worse as \(|z|\) increases. Thus, because of round-off error, a computer with limited precision cannot accurately sum the convergent Taylor series if \(|z|\) is too large. Convergence is not as useful as one might think.
Figure 3.1: Upper panel: the solid curve is \(\text{erf}(x)\) and the dashed curve is the sum of a Taylor series with 10 terms, \(\text{erf}_{10}(x)\). The truncated sum \(\text{erf}_{10}\) is wildly different from \(\text{erf}\) once \(x\) is greater than about 2.5. The lower panel shows the individual terms in \(\text{erf}_{10}(x)\) — note the cancellation that must occur to produce a sum close to one.

Now let’s consider an approximation to \(\text{erf}(x)\) that’s good for large \(x\). We work with the complementary error functions defined by

\[
\text{erfc}(x) \overset{\text{def}}{=} 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \, dt. \tag{3.6}
\]

We use integration by parts

\[
\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \left( -\frac{1}{2t} \right) \frac{d}{dt} e^{-t^2} \, dt, \tag{3.7}
\]

\[
= \frac{2}{\sqrt{\pi}} e^{-x^2} - \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{e^{-t^2}}{2t^2} \, dt. \tag{3.8}
\]

The above is an identity. But if we discard the final term in (3.8) we get a wonderful approximation for large \(x\):

\[
\text{erfc}(x) \sim \frac{2}{\sqrt{\pi}} e^{-x^2}. \tag{3.9}
\]

Figure 3.2 shows that this leading-order asymptotic approximation is reliable\(^2\) once \(x\) is greater than about 1.5.

**Exercise:** If we try integration by parts on \(\text{erf}\) (as opposed \(\text{erfc}\)) something bad happens: try it and see.

---

\(^1\)We restrict attention to the real line: \(z = x + iy\). The situation in the complex plane is tricky — we’ll return to this later.

\(^2\)We define **asymptotic approximation** later.
Figure 3.2: The complementary error function \( \text{erfc}(x) \), and the leading-order asymptotic approximation in (3.9).

Why does this approximation work? Notice that the final term in (3.8) can be bounded like this

\[
\frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \frac{1}{2t^2} \ dt = \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{1}{4t^3} \times 2te^{-t^2} \ dt, \tag{3.10}
\]

\[
\leq \frac{1}{4x^3} \int_x^\infty 2te^{-t^2} \ dt, \tag{3.11}
\]

\[
= \frac{2}{\sqrt{\pi}} e^{-x^2/4x^3}. \tag{3.12}
\]

The little trick we've used above in going from (3.10) to (3.11) is that

\[
t \geq x, \quad \Rightarrow \quad \frac{1}{4t^3} \leq \frac{1}{4x^3}. \tag{3.13}
\]

Pulling the \((4x)^{-3}\) outside, we're left with an elementary integral. Variants of this maneuver appear frequently in the asymptotics of integrals (try the exercise below).

Using the bound in (3.12) in (3.8) we have

\[
\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \frac{e^{-x^2/2x}}{2x} + \left[ \text{something which is much less than } \frac{2}{\sqrt{\pi}} \frac{e^{-x^2/2x}}{2x} \text{ as } x \to \infty. \right] \tag{3.14}
\]

Thus as \(x \to \infty\) there is a dominant balance in (3.14) between the left hand side and the first term on the right. The final term is smaller than the other two terms by a factor of at least \(x^{-2}\).

Exercise: Prove that

\[
\int_x^\infty \frac{e^{-t}}{t^N} \ dt < \frac{e^{-x}}{x^N}. \tag{3.15}
\]

More terms: the asymptotic series

We can develop an asymptotic series if we integrate by parts successively starting with (3.8). Thus, integrating by parts \(N\) times, we obtain an interesting exact expression for \(\text{erfc}(x)\):

\[
\text{erfc}(x) = \frac{e^{-x^2/\sqrt{\pi}x}}{2x} \sum_{n=0}^{N-1} (2n-1)!! \left( -\frac{1}{2x^2} \right)^n + \frac{2(2N-1)!!}{\sqrt{\pi}x} \int_x^\infty \frac{e^{-t^2}}{(2t^2)^N} \ dt. \tag{3.16}
\]
Above, $R_N(x)$ is the remainder after $N$ terms and the “double factorial” is $7!! = 7 \cdot 5 \cdot 3 \cdot 1$ etc. To bound the remainder we use our trick:

$$|R_N| = \frac{2(2N-1)!!}{\sqrt{\pi}} \int_x^\infty \frac{(e^{-t^2})_t}{2t \times (2t^2)^N} \, dt,$$

$$\leq \frac{(2N-1)!!}{\sqrt{\pi}2^N x^{2N+1}} e^{-x^2}. \quad (3.17)$$

Thus we have shown that

$$\frac{|R_N|}{N\text{th term of the series}} \leq \frac{2N - 1}{(2x)^2} \quad (3.19)$$

i.e., as $x \to \infty$ the remainder is going to zero faster than the $N$'th term: if we fix $N$ and then increase $x$ then the approximation to erfc$(x)$ obtained by dropping the remainder gets better and better.

**Numerical example**

We have obtained an asymptotic series

$$\text{erfc}(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}} \left(1 - \frac{1}{2x^2} + \frac{1 \times 3}{(2x^2)^2} - \frac{1 \times 3 \times 5}{(2x^2)^3} + \frac{1 \times 3 \times 5 \times 7}{(2x^2)^4} + O \left(x^{-10}\right)\right). \quad (3.20)$$

The numerators above are growing very quickly so at a fixed value of $x$ this series for erfc$(x)$ diverges as we add more terms. More specifically, if we sit at a fixed value of $x$ and add more and more terms then the answer gets better for awhile. But eventually successive terms get larger and larger and eventually it all goes horribly wrong. Nonetheless we can get spectacular accuracy if we’re not greedy. The guiding philosophy of asymptotics is that the first term — also known as the leading term — should be a faithful approximation in the limit $x \to \infty$. This indeed is the case with (3.9).

We illustrate the limited accuracy of an asymptotic series by calculating the fractional error in (3.20) at $x = 1.5$ as a function of the number of terms retained. With $x = 1.5$ the sum is

$$\text{erfc}(1.5) \sim 0.039643(1 - 0.222222 + 0.148148 - 0.164609 + 0.256059 - 0.512117 + \cdots) \quad (3.21)$$

After the first three terms the subsequent terms steadily increase in magnitude. If $e_n$ is the percentage error with $n$ terms, we find

$$e_1 = -17.0\%, \quad e_2 = 9.0\%, \quad e_3 = -8.3\%, \quad e_4 = 11.0\%, \quad e_5 = -19.0\%, \quad e_6 = 40.9\%. \quad (3.22)$$

The best result is obtained with three terms, although two terms is also pretty good.

In this example the answer lies between the sums with $n$ and $(n+1)$ terms. Moreover

$$\frac{\text{term } n + 1}{\text{term } n} = (-1)^n \frac{2n - 1}{2x^2} \quad (3.23)$$

If the ratio above is less than one then the terms are decreasing: this is the case if $n < x^2 + \frac{1}{2}$. So, if $x$ is large compared to $n$, then the terms decrease to a minimum — the smallest term — and then start to increase. If we stop at the term just before the smallest term then we know that the error is less than the smallest term. This is the **optimal stopping rule**.

### 3.2 The Landau symbols

Let’s explain the frequently used “Landau symbols”. In asymptotic calculations the Landau notation is used to suppress information while still maintaining some precision.
Big Oh

We frequently use “big Oh” — in fact I may have accidentally done this without defining $O$. One says $f(\epsilon) = O(\phi(\epsilon))$ as $\epsilon \to 0$ if we can find an $\epsilon_0$ and a number $A$ such that

$$|f(\epsilon)| < A|\phi(\epsilon)|, \quad \text{whenever } \epsilon < \epsilon_0.$$  

Both $\epsilon_0$ and $A$ have to be independent of $\epsilon$. Application of the big Oh notation is a lot easier than this definition suggests. Here are some $\epsilon \to 0$ examples

\[\sin 32\epsilon = O(\epsilon), \quad \sin 32\epsilon = O(\epsilon^{1/2}), \quad \epsilon^5 = O(\epsilon^2),\]

\[\cos \epsilon - 1 = O(\epsilon^{1/2}), \quad \epsilon + \epsilon^2 \sin \frac{1}{\epsilon} = O(\epsilon),\]

\[\sin \frac{1}{\epsilon} = O(1), \quad \epsilon^{-1/\epsilon} = O(\epsilon^n) \text{ for all } n.\]

The expression

\[\cos \epsilon = 1 - \frac{\epsilon^2}{2} + O(\epsilon^3)\]

means

\[\cos \epsilon - 1 + \frac{\epsilon^2}{2} = O(\epsilon^3).\]

In some of the cases above

\[\lim_{\epsilon \to 0} \frac{f(\epsilon)}{\phi(\epsilon)}\]

is zero, and that’s good enough for $O$. Also, according to our definition of $O$, the limit in (3.26) may not exist — all that’s required is that ratio $f(\epsilon)/\phi(\epsilon)$ is bounded by a constant independent of $\epsilon$ as $\epsilon \to 0$. One of the examples above illustrates this case.

The big Oh notation can be applied to other limits in obvious ways. For example, as $x \to \infty$

\[\sin x = O(1), \quad \sqrt{1 + x^2} = O(x^2), \quad \ln \cosh x = O(x).\]

As $x \to 1$

\[\ln \left(1 + x + x^2\right) - x = O(x^2).\]

Hinch’s ord

H uses the more precise notation $\text{ord}(\phi(\epsilon))$. We say

\[f(\epsilon) = \text{ord}(\phi(\epsilon)) \iff \lim_{\epsilon \to 0} \frac{f(\epsilon)}{\phi(\epsilon)} \text{ exists and is nonzero}.\]

For example, as $\epsilon \to 0$:

\[\sinh(37\epsilon + \epsilon^3) = \text{ord}(\epsilon), \quad \text{and} \quad \frac{\epsilon}{\ln(1 + \epsilon + \epsilon^2)} = \text{ord}(1).\]

Notice that $\sinh(37\epsilon + \epsilon^3)$ is not $\text{ord}(\epsilon^{1/2})$, but

\[\sinh(37\epsilon + \epsilon^3) = O(\epsilon^{1/2}), \quad \text{and} \quad \sin \left(\frac{1}{\epsilon}\right) \sinh(37\epsilon + \epsilon^3) = O(\epsilon^{1/2})\]

Big Oh tells one a lot less than ord.
Little Oh

Very occasionally — almost never — we need “little Oh”. We say \( f(\epsilon) = o(\phi(\epsilon)) \) if for every positive \( \delta \) there is an \( \epsilon_0 \) such that

\[
|f(\epsilon)| < \delta |\phi(\epsilon)|, \quad \text{whenever } \epsilon < \epsilon_0.
\]

Another way of saying this is that

\[
f(\epsilon) = o(\phi(\epsilon)) \iff \lim_{\epsilon \to 0} \frac{f(\epsilon)}{\phi(\epsilon)} = 0.
\] (3.32)

Obviously \( f(\epsilon) = o(\phi(\epsilon)) \) implies \( f(\epsilon) = O(\phi(\epsilon)) \), but not the reverse. Here are some examples

\[
\ln(1 + \epsilon) = o(\epsilon^{1/2}), \quad \cos \epsilon - 1 + \frac{\epsilon^2}{2} = o(\epsilon^3), \quad e^{o(\epsilon)} = 1 + o(\epsilon).
\] (3.33)

The trouble with little Oh is that it hides too much information: if something tends to zero we usually want to know how it tends to zero. For example

\[
\ln(1 + 2e^{-x} + 3e^{-2x}) = o\left(e^{-x/2}\right), \quad \text{as } x \to \infty,
\] (3.34)

is not as informative as

\[
\ln(1 + 2e^{-x} + 3e^{-2x}) = \text{ord} \left(e^{-x}\right), \quad \text{as } x \to \infty.
\] (3.35)

Asymptotic equivalence

Finally “asymptotic equivalence” \( \approx \) is useful. We say \( f(\epsilon) \approx \phi(\epsilon) \) as \( \epsilon \to 0 \) if

\[
\lim_{\epsilon \to 0} \frac{f(\epsilon)}{\phi(\epsilon)} = 1.
\] (3.36)

Notice that

\[
f(\epsilon) \approx \phi(\epsilon), \quad \iff \quad f(\epsilon) = \phi(\epsilon) \left[1 + o(\epsilon)\right].
\] (3.37)

Some \( \epsilon \to 0 \) examples are

\[
\epsilon + \frac{\sin \epsilon}{\ln(1/\epsilon)} \approx \epsilon, \quad \text{and} \quad \sqrt{1 + \epsilon} - 1 \approx \frac{\epsilon}{2}.
\] (3.38)

Some \( x \to \infty \) examples are

\[
\sinh x \approx \frac{e^x}{2}, \quad \text{and} \quad \frac{x^3}{1 + x^2} + \sin x \approx x, \quad \text{and} \quad x + \ln \left(1 + e^{2x}\right) \approx 3x.
\] (3.39)

Exercise: Show by counterexample that \( f(x) \approx g(x) \) as \( x \to \infty \) does not imply that \( \frac{df}{dx} \approx \frac{dg}{dx} \), and that \( f(x) \approx g(x) \) as \( x \to \infty \) does not imply that \( e^{f} \approx e^{g} \).

Gauge functions

The \( \phi(\epsilon) \)'s referred to above are gauge functions — simple functions that we use to compare a complicated \( f(\epsilon) \) with. A sequence of gauge functions \( \{\phi_0, \phi_1, \cdots\} \) is asymptotically ordered if

\[
\phi_{n+1}(\epsilon) = o[\phi_n(\epsilon)], \quad \text{as } \epsilon \to 0.
\] (3.40)

In practice the \( \phi \)'s are combinations of powers and logarithms:

\[
\epsilon^n, \quad \ln \epsilon, \quad e^n (\ln \epsilon)^p, \quad \ln \ln \epsilon \text{ etc.}
\] (3.41)
Exercise  Suppose $\epsilon \to 0$. Arrange the following gauge functions in order, from the largest to the smallest:

$$
\epsilon, \quad \ln \left( \frac{1}{\epsilon} \right), \quad e^{-\ln^2 \epsilon}, \quad e^{1/\sqrt{\epsilon}}, \quad \epsilon^0, \quad \ln \left( \frac{1}{\epsilon} \right) \quad (3.42)
$$

$$
e^{-1/\epsilon}, \quad \epsilon^{1/3}, \quad \epsilon^{1/\pi}, \quad \epsilon \ln \frac{1}{\epsilon}, \quad \ln \frac{1}{\epsilon}, \quad \epsilon^{\ln \epsilon} \quad (3.43)
$$

3.3 The definition of asymptoticity

Asymptotic power series

Consider a sum based on the simplest gauge functions $\epsilon^n$:

$$
\sum_{n=0}^{\infty} a_n \epsilon^n. \quad (3.44)
$$

This sum is an $\epsilon \to 0$ asymptotic approximation to a function $f(\epsilon)$ if

$$
\lim_{\epsilon \to 0} \frac{f(\epsilon) - \sum_{n=0}^{N} a_n \epsilon^n}{\epsilon^N} = 0. \quad (3.45)
$$

The numerator in the fraction above is the remainder after summing $N + 1$ terms — we call this $R_{N+1}(\epsilon)$. So the series in (3.44) is asymptotic to the function $f(\epsilon)$ if the remainder $R_{N+1}(\epsilon)$ goes to zero faster than the last retained gauge function $\epsilon^N$. We use the notation $\sim$ to denote an asymptotic approximation:

$$
f(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n, \quad \text{as} \quad \epsilon \to 0. \quad (3.46)
$$

The right hand side of (3.46) is called an asymptotic power series or a Poincaré series, or an asymptotic representation of $f(\epsilon)$.

Our erf-example satisfies this definition with $\epsilon = x^{-1}$. If we retain only one term in the series (3.20) then the remainder is

$$
R_1 = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} \frac{2t^2}{2t^2} dt. \quad (3.47)
$$

In (3.11) we showed that

$$
\frac{R_1}{e^{-x^2/\sqrt{\pi}}} \leq \frac{1}{4x^2}. \quad (3.48)
$$

Thus as $x \to \infty$ the remainder is much less than the last retained term. According to the definition above, this is the first step in justifying the asymptoticness of the series.

Exercise: Show from the definition of asymptoticity that

$$
e^{-1/\epsilon} = 0 + 0 \epsilon + 0 \epsilon^2 + 0 \epsilon^3 + \cdots \quad \text{as} \quad \epsilon \downarrow 0. \quad (3.49)
$$

A problem with applying the definition is that one has to be able to say something about the remainder in order to determine if a series is asymptotic. This is not the case with convergence. For example, one can establish the convergence of

$$
\sum_{n=0}^{\infty} \ln(n + 2) x^n, \quad (3.50)
$$

without knowing the function to which this mysterious series converges. Convergence is an intrinsic property of the coefficients $\ln(n + 2)$. The ratio test shows that the series in (3.50) converges if $|x| < 1$ and we don’t have to know what (3.50) is converging to. On the other hand, asymptoticity depends on both the function and the terms in the asymptotic series.
**Example** The famous *Stieltjes* series

\[ S(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} (-)^n n! x^n \]  

(3.51)

does not converge unless \( x = 0 \). In fact, as it stands, \( S(x) \) does not define a function of \( x \). \( S(x) \) is a *formal power series*. And we can’t say that \( S(x) \) is an asymptotic series because we have to ask asymptotic to what? But now observe that

\[ n! = \int_0^{\infty} t^n e^{-t} \, dt , \]  

(3.52)

and substitute this integral representation of \( n! \) into the sum (3.51). There is a moment of pleasure when we realize that if we exchange the order of integration and summation then we can evaluate the sum to obtain

\[ F(x) \stackrel{\text{def}}{=} \int_0^{\infty} \frac{e^{-t}}{1 + xt} \, dt . \]  

(3.53)

Because of the dubious steps between (3.51) and (3.53), I’ve simply defined \( F(x) \) by the integral above. But now that we have a well defined function \( F(x) \), we’re entitled to ask is the sum \( S(x) \) asymptotic to \( F(x) \) as \( x \to 0? \) The answer is yes.

The proof is integration by parts, which yields the identity

\[ F(x) = 1 - x + 2!x^2 - 3!x^3 + \cdots (-1)^{(N-1)}(N-1)!x^{N-1} + (-1)^N N!x^N \int_0^{R_N} \frac{e^{-t}}{(1 + xt)^{N+1}} \, dt . \]  

(3.54)

It is easy show that

\[ |R_N(x)| \leq N!x^N , \]  

(3.55)

and therefore

\[ \lim_{x \to 0} \frac{R_N(x)}{(N-1)!x^{N-1}} = 0 . \]  

(3.56)

Above we’re comparing the remainder to the last retained term in the truncated series. Because the ratio goes to zero in the limit the series is asymptotic.

**Exercise:** Find another function with the same \( x \to 0 \) asymptotic expansion as \( F(x) \) in (3.53).

**Other gauge functions**

Many — but not all— of the expansion expansions you’ll encounter have the form of an asymptotic power series. But in the previous lectures we also saw examples with fractional powers of \( \epsilon \) and \( \ln \epsilon \) and \( \ln[\ln(1/\epsilon)] \). These expansions have the form

\[ f(\epsilon) \sim \sum_{n=0}^{\infty} a_n \phi_n(\epsilon) , \]  

(3.57)

where \( \{\phi_n\} \) is an asymptotically ordered set of gauge functions. The definition of asymptoticity is generalized to say that the sum on the right of (3.57) is an asymptotic approximation to \( f(\epsilon) \) as \( \epsilon \to 0 \) if

\[ \lim_{\epsilon \to 0} \frac{f(\epsilon) - \sum_{n=0}^{N} a_n \phi_n(\epsilon)}{\phi_N(\epsilon)} = 0 . \]  

(3.58)

In other words, once \( \epsilon \) is sufficiently small the remainder is less than the last term. Mathematicians get excited about obtaining the *full* asymptotic series i.e., having an explicit formula for \( a_n \). But the main value of asymptotic analysis is that the first term in the expansion — the *leading order term* — reveals the structure of the function as \( \epsilon \to 0 \).
Manipulation of asymptotic expansions

Uniqueness

3.4 Stokes lines

3.5 Problems

Problem 3.1. Find a leading-order \( x \to \infty \) asymptotic approximation to

\[
A(x; p, q) \overset{\text{def}}{=} \int_{x}^{\infty} e^{-pt} \, dt.
\] (3.59)

Show that the remainder is asymptotically negligible as \( x \to \infty \). Above, \( p \) and \( q \) are both positive real numbers.

Problem 3.2. (i) Use integration by parts to find the leading-order term in the \( x \to \infty \) asymptotic expansion of the exponential integral:

\[
E_1(x) \overset{\text{def}}{=} \int_{x}^{\infty} \frac{e^{-v}}{v} \, dv.
\] (3.60)

Show that this approximation is asymptotic i.e., prove that the remainder is asymptotically less than the leading term as \( x \to \infty \). (ii) Find more terms in the series with further integration by parts. Find an expression for the \( n \)'th term, and the remainder after \( n \) terms.

Problem 3.3. Consider the first-order differential equation:

\[
y' - y = -\frac{1}{x}, \quad \text{with the condition } \lim_{x \to \infty} y(x) = 0.
\] (3.61)

(i) Find a valid two-term dominant balance in the differential equation and thus deduce the leading-order asymptotic approximation to \( y(x) \) for large positive \( x \). (ii) Use an iterative procedure to deduce the full asymptotic expansion of \( y(x) \). (iii) Is the expansion convergent? (iv) Use the integrating function method to solve the differential equation exactly in terms of the exponential integral in (3.60). Use MATLAB (help expint) to compare the exact solution of (3.61) with asymptotic expansions of different order. Summarize your study as in Figure 3.3.

Problem 3.4. The exponential integral of order \( n \) is

\[
E_n(x) \overset{\text{def}}{=} \int_{x}^{\infty} \frac{e^{-t}}{t^n} \, dt.
\] (3.62)

Show that

\[
E_{n+1}(x) = \frac{e^{-x}}{nx^n} - \frac{1}{n} E_n(x).
\] (3.63)

Find the leading-order asymptotic approximation to \( E_n(x) \) as \( x \to \infty \).

Problem 3.5. Find an example of a infinitely differentiable function satisfying the inequalities

\[
\max_{0<x<1} |f(x)| < 10^{-10}, \quad \text{and} \quad \max_{0<x<1} \left| \frac{df}{dx} \right| > 10^{10}.
\] (3.64)

This is why the differential operator \( d/dx \) is “unbounded”: \( d/dx \) can take a small function and turn it into a big function.
Figure 3.3: Solution of problem 3.3. Upper panel compares the exact solution with truncated asymptotic series. Lower panel shows the asymptotic approximation at $x = 5$ as a function of the truncation order $n$ i.e., $n = 1$ is the one-term approximation. The solid line is the exact answer.

**Problem 3.6.** Prove that

$$\int_0^\infty \frac{e^{-t}}{1 + xt^2} dt \sim \sum_{n=0}^{\infty} (-1)^n (2n)! x^n, \quad x \to 0.$$  \hspace{1cm} (3.65)

**Problem 3.7.** True or false as $x \to \infty$

$$\begin{align*}
(i) \quad x + \frac{1}{x} & \approx x, \\
(ii) \quad x + \sqrt{x} & \approx x, \\
(iii) \quad \exp \left( x + \frac{1}{x} \right) & \approx \exp(x), \\
(iv) \quad \exp \left( x + \sqrt{x} \right) & \approx \exp(x), \\
(v) \quad \cos \left( x + \frac{1}{x} \right) & \approx \cos x, \\
(v) \quad \frac{1}{x} & \approx 0?
\end{align*}$$  \hspace{1cm} (3.66)

**Problem 3.8.** Let’s investigate the Stieltjes series $S(x)$ in (3.51). (i) Compute the integral $F(0.1)$ numerically. (ii) With $x = 0.1$, compute partial sums of the divergent series (3.51) with $N = 2, 3, 4, \ldots 20$. Which $N$ gives the best approximation to $F(0.1)$? (iii) I think the best answer is obtained by truncating the series $S(0.1)$ just before the smallest term. Is that correct?
Lecture 4

Why integrals?

Integrals occur frequently as the solution of partial and ordinary differential equations, and as the definition of many “special functions”. The coefficients of a Fourier series are given as integrals involving the target function etc. Green’s function technology expresses the solution of a differential equation as a convolution integral etc. Integrals are also important because they provide the simplest and most accessible examples of concepts like asymptoticity and techniques such as asymptotic matching.

4.1 First-order linear differential equations

Linear first order differential equations, such as
\[ y' - xy = -1, \quad \text{with} \quad \lim_{x \to \infty} y(x) = 0, \]  
(4.1)
can be solved with the integrating-factor method. This delivers \( y(x) \) as an integral. In the case above we find
\[ y(x) = e^{x^2/2} \int_x^\infty e^{-t^2/2} \, dt. \]  
(4.2)
You might feel nervous because the factor \( e^{x^2/2} \) is growing very fast as \( x \to \infty \). Our hope is that the integral is decaying even faster so that the product on the right of (4.2) satisfies the requirement at \( x = \infty \) in (4.1). In fact, from our earlier asymptotic adventure with \( \text{erfc}(x) \), we know that everything in (4.2) is OK.

Exercise: write the solution in (4.2) in terms of \( \text{erfc} \) and use the asymptotic series in (3.20) to obtain the large-\( x \) behaviour of \( y(x) \).

Exercise: obtain the result in the previous exercise via asymptotic solution of the differential equation (4.1) i.e., without integrating the differential equation. (Hint: find an \( x \to \infty \) two-term dominant balance in (4.1) and use iteration to generate more terms.)

4.2 Second-order linear differential equations

Airy’s equation,
\[ y'' - xy = 0, \]  
(4.3)
is an important second-order differential equation. The two linearly independent solutions, \( \text{Ai}(x) \) and \( \text{Bi}(x) \), are shown in figure 4.1. The Airy function, \( \text{Ai}(x) \), is defined as the solution that decays as \( x \to \infty \), with the normalization
\[ \int_{-\infty}^\infty \text{Ai}(x) \, dx = 1. \]  
(4.4)
We obtain an integral representation of $\text{Ai}(x)$ by attacking (4.3) with the Fourier transform. Denote the Fourier transform of $\text{Ai}$ by

$$\tilde{\text{Ai}}(k) = \int_{-\infty}^{\infty} \text{Ai}(x)e^{-ikx} \, dx.$$  

(4.5)

Fourier transforming (4.3), we eventually find

$$\tilde{\text{Ai}}(k) = e^{ik^3/3}.$$  

(4.6)

Using the Fourier integral theorem

$$\text{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx+ik^3/3} \frac{dk}{2\pi},$$  

(4.7)

$$= \frac{1}{\pi} \int_{0}^{\infty} \cos \left( kx + \frac{k^3}{3} \right) dk.$$  

(4.8)

Notice that the integral converges at $k = \infty$ because of destructive interference or catastrophic cancellation.

We’ll develop several techniques for extracting information from integral representations such as (4.8). We’ll show that as $x \to -\infty$:

$$\text{Ai}(x) \sim \frac{1}{\sqrt{\pi|x|^{1/4}}} \cos \left( \frac{2|x|^{3/2}}{3} - \frac{\pi}{4} \right),$$  

(4.9)

and as $x \to +\infty$:

$$\text{Ai}(x) \sim e^{-\frac{2x^{3/2}}{3}} \frac{1}{2\sqrt{\pi x^{1/4}}}.$$  

(4.10)

**Exercise:** Fill in the details between (4.5) and (4.6).

### 4.3 Recursion relations: the example $n!$

The factorial function

$$a_n = n!$$  

(4.11)

satisfies the recursion relation

$$a_{n+1} = (n + 1)a_n, \quad a_0 = 1.$$  

(4.12)
The integral representation

\[ a_n = \int_0^\infty t^n e^{-t} \, dt \]  

(4.13)
is equivalent to both the initial condition and the recursion relation. The proof is integration by parts:

\[ \int_0^\infty t^{n+1} e^{-t} \, dt = - \int_0^\infty t^{n+1} \frac{d}{dt} e^{-t} \, dt \]

(4.14)

\[ = - \left[ t^{n+1} e^{-t} \right]_0^\infty + (n+1) \int_0^\infty t^n e^{-t} \, dt . \]

(4.15)

Exercise: Memorize

\[ n! = \int_0^\infty t^n e^{-t} \, dt . \]

(4.16)

Later we will use the integral representation (4.16) to obtain **Stirling’s approximation**:

\[ n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n , \text{ as } n \to \infty . \]

(4.17)

Exercise: Compare Stirling’s approximation to \( n! \) with \( n = 1, 2 \) and 3.

### 4.4 Special functions defined by integrals

We’ve already mentioned the error function and its complement:

\[ \text{erf}(z) \overset{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \, dt \quad \text{and} \quad \text{erfc}(z) \overset{\text{def}}{=} 1 - \text{erf}(z) . \]

(4.18)

Another special function defined by an integral is the exponential integral of order \( n \):

\[ E_n(z) \overset{\text{def}}{=} \int_z^\infty \frac{e^{-t}}{t^n} \, dt . \]

(4.19)

We refer to the case \( n = 1 \) simply as the “exponential integral”.

Example: Singularity subtraction — small \( z \) behavior of \( E_n(z) \).

**The Gamma function**: \( \Gamma(z) \overset{\text{def}}{=} \int_0^\infty t^{z-1} e^{-t} \, dt , \text{ for } \Re z > 0 . \)

There are many other examples of special functions defined by integrals. Probably the most important is the \( \Gamma \)-function, which is defined in the heading of this section — see Figure 4.2. If \( \Re z > 0 \) we can use integration by parts to show that \( \Gamma(z) \) satisfies the functional equation

\[ z\Gamma(z) = \Gamma(z+1) . \]

(4.20)

Using analytic continuation this result is valid for all \( z \neq 0, -1, -2 \cdots \) Thus the functional equation (4.20) is used to extend the definition of \( \Gamma \)-function throughout the complex plane. Notice that if \( z \) is an integer, \( n \), then

\[ \Gamma(n+1) = n! \]

(4.21)

The special value

\[ \Gamma \left( \frac{1}{2} \right) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}} \, dt = \int_{-\infty}^\infty e^{-u^2} \, du = \sqrt{\pi} \]

(4.22)
is important.

---

1If \( f(z) \) and \( g(z) \) are analytic in a domain \( D \), and if \( f = g \) in a smaller domain \( E \subset D \), then \( f = g \) throughout \( D \).
Exercise: Use the functional equation (4.20) to obtain \( \Gamma(3/2) \) and \( \Gamma(-1/2) \).

Exercise: Use the functional equation (4.20) to find the leading order behaviour of \( \Gamma(z) \) near \( z = 0 \) and \( z = -1 \), and other negative integers. Work backwards and show that

\[
\Gamma(x) \sim \frac{(-n)^n}{n! \, x + n}, \quad \text{as } x \to -n.
\]

Thus \( \Gamma(z) \) has poles at \( z = 0, -1, \ldots \) with residues \( (-n)^n/n! \).

## 4.5 Elementary methods for evaluating integrals

### Change of variables

How can we evaluate the integral

\[
\int_0^\infty e^{-t^3} \, dt
\]

Try a change of variable

\[
v = t^3 \quad \text{and therefore} \quad dv = 3t^2 \, dt = 3v^{2/3} \, dt.
\]

The integral is then

\[
\frac{1}{3} \int_0^\infty e^{-v} v^{-2/3} \, dv = \frac{1}{3} \Gamma \left( \frac{1}{3} \right) = \Gamma \left( \frac{4}{3} \right).
\]

Exercise: Evaluate in terms of the \( \Gamma \)-function

\[
U(\alpha, p, q) \overset{\text{def}}{=} \int_0^\infty t^\alpha e^{-qt} \, dt.
\]

Exercise: Show that

\[
L[t^p] = \int_0^\infty t^p e^{-st} \, dt = \frac{\Gamma(1 + p)}{s^{1+p}}.
\]
Differentiation with respect to a parameter

Given that
\[ \frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-x^2} \, dx, \]  
(4.27)
we can make the change of variables \( x = \sqrt{t}x' \) and find that
\[ \frac{1}{2\sqrt{t}} = \int_0^\infty e^{-tx^2} \, dx. \]  
(4.28)

We now have an integral containing the parameter \( t \).

To evaluate
\[ \int_0^\infty x^2 e^{-tx^2} \, dx, \]  
(4.29)
we differentiate (4.28) with respect to \( t \) to obtain
\[ \frac{1}{4\sqrt{t^3}} = \int_0^\infty x^2 e^{-tx^2} \, dx, \]  
(4.30)
and again
\[ \frac{3}{8\sqrt{t^5}} = \int_0^\infty x^4 e^{-tx^2} \, dx. \]  
(4.31)

Differentiation with respect to a parameter is a very effective trick. For some reason it is not taught to undergraduates.

How would you calculate \( \mathcal{L}[t^p \ln t] \)? No problem — just notice that
\[ \partial_p t^p = \partial_p e^{p \ln t} = t^p \ln t, \]  
(4.32)
and then take the derivative of (4.26) with respect to \( p \)
\[ \mathcal{L}[t^p \ln t] = \frac{\Gamma'(1+p)}{s^{1+p}} - \frac{\Gamma(1+p) \ln s}{s^{1+p}}, \]  
(4.33)
where the digamma function
\[ \psi(z) \overset{\text{def}}{=} \frac{\Gamma'(z)}{\Gamma(z)} \]  
(4.34)
is the derivative of \( \ln \Gamma \).

4.6 Complexification

Consider
\[ F(a, b) = \int_0^\infty e^{-at} \cos bt \, dt, \]  
(4.35)
where \( a > 0 \). Then
\[ F = \Re \int_0^\infty e^{-(a+ib)t} \, dt, \]  
(4.36)
\[ = \Re \frac{1}{a+ib} = \Re \frac{a-ib}{a^2+b^2}, \]  
(4.37)
\[ = \frac{a}{a^2+b^2}. \]  
(4.38)
As a bonus, the imaginary part gives us
\[ \frac{b}{a^2+b^2} = \int_0^\infty e^{-at} \sin bt \, dt. \]  
(4.39)

Derivatives with respect to the parameters \( a \) and \( b \) generate further integrals.
Figure 4.3: The pie contour $ABC$ in the complex plane ($z = x + iy = re^{iθ}$) used to evaluate $J(1)$ in (4.44). The ray $AC$ is a contour of constant phase: $z^3 = ir^3$ and $\exp(iz^3) = \exp(-r^3)$.

**Contour integration**

The theory of contour integration, covered in part B, is an example of complexification. As revision we’ll consider examples that illustrate important techniques.

**Example:** Consider the Fourier transform

$$f(k) = \int_{-\infty}^{\infty} e^{-ikx} \frac{1}{1 + x^2} \, dx.$$  

We evaluate this Fourier transform using contour integration to obtain

$$f(k) = \pi e^{-|k|}.$$  

Note particularly the $|k|$: if $k > 0$ we must close in the lower half of the $z = x + iy$ plane, and if $k < 0$ we close in the upper half plane.

**Example** Let’s evaluate

$$\text{Ai}(0) = \frac{1}{\pi} \int_{0}^{\infty} \cos \left( \frac{k^3}{3} \right) \, dk$$  

via contour integration. We consider a slightly more general integral

$$J(\alpha) = \int_{0}^{\infty} e^{i\alpha v^3} \, dv,$$

$$= |\alpha|^{-1/3} \int_{0}^{\infty} e^{i\text{sgn}(\alpha)x^3} \, dx.$$  

Thus if we can evaluate $J(1)$ we also have $J(\alpha)$, and in particular $\Re J(1/3)$, which is just what we need for $\text{Ai}(0)$. But at the moment it may not even be clear that these integrals converge — we’re relying on the destructive cancellation of increasingly wild oscillations as $x \to \infty$, rather than decay of the integrand, to ensure convergence.

To evaluate $J(1)$ we consider the entire analytic function

$$f(z) = e^{iz^3} = e^{ir^3e^{i\pi/6}} \quad \text{exp} \left[ y^3 - 3x^2 y + i(x^3 - 3xy^2) \right].$$  

Notice from Cauchy’s theorem that the integral of $f(z)$ over any closed path in the $z$-plane is zero. In particular, using the pie-shaped path $ABC$ in the figure,

$$0 = \int_{ABC} e^{iz^3} \, dz.$$  

The pie-path $ABC$ is cunningly chosen so that the segment $CA$ (where $z = re^{iπ/6}$) is a contour of constant phase, so called because

$$f(z) = e^{-r^3} \quad \text{on } AC.$$  

On $CA$ phase of $f(z)$ is a constant, namely zero.
Now write out (4.46) as

\[ 0 = \int_{-j(1)}^{R} e^{i x^3} \, dx + \int_{0}^{\pi/6} e^{i R^3 \cos \theta} e^{-R^3 \sin \theta} \, d\theta + \int_{R}^{0} e^{-r^3} \, dr. \] (4.48)

Note that on the arc BC, \( z = R e^{i \theta} \) and \( dz = i R e^{i \theta} \, d\theta \) — we’ve used this in \( M(R) \) above.

We consider the limit \( R \to \infty \). If we can show that the term in the middle, \( M(r) \), vanishes as \( R \to \infty \) then we will have

\[ J(1) = \int_{0}^{\infty} e^{-r^3} \, dr. \] (4.49)

The right of (4.49) is a splendidly convergent integral and is readily evaluated in terms of our friend the \( \Gamma \)-function.

So we now focus on the troublesome \( M(R) \):

\[
|M(R)| = R \left| \int_{0}^{\pi/6} e^{i R^3 \cos \theta} e^{-R^3 \sin \theta} \, d\theta \right| \leq R \int_{0}^{\pi/6} e^{-R^3 \sin \theta} \, d\theta \leq R \int_{0}^{\pi/6} e^{-R^3 \sin \theta} \, d\theta \leq R \int_{0}^{\pi/6} e^{-R^3} \, d\theta = \frac{\pi}{6} e^{-R^3} \to 0, \quad \text{as} \quad R \to \infty. \] (4.50)

At (4.50) we’ve obtained a simple upper bound using the inequality \( \sin 3\theta > \frac{6\theta}{\pi} \), for \( 0 < \theta < \frac{\pi}{6} \). (4.52)

An alternative is to change variables with \( v = \sin 3\theta \) so that

\[
\int_{0}^{\pi/6} e^{-R^3 \sin \theta} \, d\theta = \frac{1}{3} \int_{0}^{1} e^{-R^3 v} \frac{dv}{\sqrt{1-v^2}},
\] (4.53)

and then use Watson’s lemma (from the next lecture). This gives a sharper bound on the arc integral. The final answer is

\[ \text{Ai}(0) = \frac{3^{1/3}}{\pi} \int_{0}^{\infty} e^{-r^3} \, dr = \frac{\Gamma(1/3)}{3^{2/3} \sqrt{\pi}}. \] (4.54)

In the example above we used a constant-phase contour to evaluate an integral exactly. A constant-phase contour is also a contour of steepest descent. The function in the exponential is

\[ i z^3 = y^3 - 3 x^2 y + i (x^3 - 3 x y^2) \] (4.55)

On CA the phase is constant: \( \psi = 0 \). But from the Cauchy-Riemann equations

\[ \nabla \phi \cdot \nabla \psi = 0, \] (4.56)

and therefore as one moves along CA one is moving parallel to \( \nabla \phi \). One is therefore always ascending or descending along the steepest direction of the surface formed by \( \phi(x, y) \) above the \((x, y)\)-plane. Thus the main advantage to integrating along the constant-phase contour CA is that the integrand is decreasing as fast as possible without any oscillatory behavior.

Example: Let’s prove the important functional equation

\[ \Gamma(z) \Gamma(1 - z) = \int_{0}^{\infty} \frac{v^{z-1}}{1 + v} \, dv = \frac{\pi}{\sin \pi z}. \] (4.57)

\(^2\)This trick is a variant of Jordan’s lemma.
**Example:** Later, in our discussion of the method of averaging, we’ll need the integral
\[
A(\kappa) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{1 + \kappa \cos t}.
\] (4.58)

We introduce a complex variable
\[z = e^{it},\] so that \(dz = iz \, d\theta\), and \(\cos t = \frac{1}{2}z + \frac{1}{2}z^{-1}\). (4.59)

Thus
\[
A(\kappa) = -\frac{i}{\pi} \int_{C} \frac{dz}{\sqrt{\kappa z^2 + 2z + \kappa}},
\] (4.60)

where the path of integration, \(C\), is a unit circle centered on the origin. The integrand has simple poles at
\[z_\pm = \kappa^{-1} \pm \sqrt{\kappa^{-2} - 1}.
\] (4.62)

The pole at \(z_+\) is inside \(C\), and the other is outside. Therefore
\[
A(\kappa) = \frac{1}{\sqrt{1 - \kappa^2}}.
\] (4.64)

**Mathematica, Maple and Gradshteyn & Ryzhik**

*Tables of Integrals Series and Products* by I.S. Gradshteyn & I.M. Ryzhik is a good source for look-up evaluation of integrals. Get the seventh edition — it has fewer typos.

### 4.7 Problems

**Problem 4.1.** Use the elementary integral
\[
\frac{1}{n+1} = \int_0^1 x^n \, dx,
\] (4.65)

to evaluate
\[
S(n) \overset{\text{def}}{=} \int_0^1 x^n \ln \left( \frac{1}{x} \right) \, dx \quad \text{and} \quad R(n) \overset{\text{def}}{=} \int_0^1 x^n \ln^2 \left( \frac{1}{x} \right) \, dx.
\] (4.66)

**Problem 4.2.** Starting from
\[
\frac{a}{a^2 + \lambda^2} = \int_0^\infty e^{-ax} \cos \lambda x \, dx,
\] (4.67)
evaluate
\[
I(a, \lambda) = \int_0^\infty x \, e^{-ax} \cos \lambda x \, dx,
\] (4.68)
and for desert
\[
J(a) = \int_0^\infty \frac{e^{-ax} \sin x}{x} \, dx.
\] (4.69)

Notice that \(J(a)\) is an interesting Laplace transform.
Problem 4.3. Consider
\[ F(a,b) = \int_0^\infty e^{-a^2u^2-b^2u^2} \, du. \] (4.70)

(i) Using a change of variables show that \( F(a,b) = a^{-1}F(1,ab) \).

(ii) Show that \( \frac{\partial F(a,b)}{\partial b} = -2F(1,ab) \). (4.71)

(iii) Use the results above to show that \( f \) satisfies a simple first order differential equation; solve the equation and show that
\[ F(a,b) = \frac{\sqrt{2\pi}}{2a} e^{-2ab}. \] (4.72)

Problem 4.4. The harmonic sum is defined by
\[ H_N \equiv \sum_{n=1}^N \frac{1}{n}. \] (4.73)

In this problem you’re asked to show that
\[ \lim_{N \to \infty} (H_N - \ln N) = \gamma_E, \] where the Euler constant \( \gamma_E \) is defined in (4.82).

(i) Prove that \( H_N \) diverges by showing that
\[ \ln(1 + N) \leq H_N \leq 1 + \ln N. \] (4.75)

Hint: compare \( H_N \) with the area beneath the curve \( f(x) = x^{-1} \) — you’ll need to carefully select the limits of integration. Your answer should include a careful sketch.

(ii) Prove that \( H_N = \int_1^0 \frac{1-x^N}{1-x} \, dx. \) (4.76)

Hint: \( n^{-1} = \int_0^1 x^{n-1} \, dx. \) (iii) Use MATLAB to graph
\[ F_N(x) = \frac{1-x^N}{1-x}, \quad \text{for } 0 \leq x \leq 1, \] with \( N = 100 \). This indicates that \( F_N(x) \) achieves its maximum value at \( x = 1 \). Prove that \( F_N(1) = N \). These considerations should convince you that the integral in (4.76) is dominated by the peak at \( x = 1 \).

(iv) With a change of variables, rewrite (4.76) as
\[ H_N = \int_0^1 \left[ 1 - \left( 1 - \frac{y}{N} \right)^N \right] \frac{dy}{y}. \] (4.78)

(v) Deduce (4.74) by asymptotic evaluation, \( N \to \infty \), of the integral in (4.78).

Problem 4.5. Consider a harmonic oscillator that is kicked at \( t = 0 \) by singular forcing
\[ \ddot{x} + x = \frac{1}{t}. \] (4.79)

(i) Show that a particular solution of (4.79) is provided by the Stieltjes integral
\[ x(t) = \int_0^\infty \frac{e^{-st}}{1 + s^2} \, ds. \] (4.80)

(ii) Find the leading-order the behaviour of \( x(t) \) as \( t \to \infty \) from the integral representation (4.80).

(iii) Show that this asymptotic result corresponds to a two-term balance in (4.79).

(iv) Evaluate \( x(0) \). (v) Can you find \( x(0) \)? (vi) If your answer to (v) was "no", what can you say about the form of \( x(t) \) as \( t \to 0 \)? Do you get more information from the differential equation, or from the integral representation?
Problem 4.6. Evaluate the Fresnel integral

\[ F(\alpha) = \int_0^\infty e^{i\alpha x^2} \, dx. \] (4.81)

Problem 4.7. Euler's constant is defined by

\[ \gamma_E \overset{\text{def}}{=} -\Gamma'(1). \] (4.82)

(i) Show by direct differentiation of the definition of the \( \Gamma \)-function that:

\[ \gamma_E = -\int_0^\infty e^{-t} \ln t \, dt. \] (4.83)

(ii) Judiciously applying IP to the RHS, deduce that

\[ \gamma_E = \int_1^0 \frac{1 - e^{-t} - e^{-t-1}}{t} \, dt. \] (4.84)

Problem 4.8. This problem uses many\(^3\) of the elementary tricks you'll need for real integrals.

(i) Show that

\[ \ln t = \int_0^\infty \frac{e^{-x} - e^{-xt}}{x} \, dx. \] (4.85)

(ii) From the definition of the \( \Gamma \)-function,

\[ \Gamma(z) \overset{\text{def}}{=} \int_0^\infty e^{-t} t^{z-1} \, dt, \quad \Re z > 0, \] (4.86)

show that the digamma function is

\[ \psi(z) \overset{\text{def}}{=} \frac{d \ln \Gamma}{dz} = \frac{\Gamma'(z)}{\Gamma(z)} = \int_0^\infty \left[ e^{-x} - \frac{1}{(x+1)^z} \right] \frac{dx}{x}, \quad \Re z > 0. \] (4.87)

Hint: Differentiate the definition of \( \Gamma(z) \) in (4.86), and use the result from part (i). (iii) Noting that (4.87) implies

\[ \psi(z) = \lim_{\delta \to 0} \left[ \int_0^\infty \frac{e^{-x}}{x} \, dx - \int_0^\infty \frac{1}{(x+1)^z} \frac{dx}{x} \right], \quad \Re z > 0, \] (4.88)

change variables with \( x + 1 = e^u \) in the second integral and deduce that:

\[ \psi(z) = \int_0^\infty \left( \frac{e^{-u}}{u} - \frac{e^{-zu}}{1 - e^{-u}} \right) \, du, \quad \Re z > 0. \] (4.89)

Explain in ten or twenty words why it is necessary to introduce \( \delta \) in order to split the integral on the RHS of (4.87) into the two integrals on the RHS of (4.88). (iv) We define Euler's constant as

\[ \gamma_E \equiv -\psi(1) = -\Gamma'(1) = 0.57721 \ldots \] (4.90)

Show that

\[ \psi(z) = -\gamma_E + \int_0^\infty \frac{e^{-u} - e^{-ux}}{1 - e^{-u}} \, du, \]

\[ = -\gamma_E + \int_0^1 \frac{1 - v^{z-1}}{1 - v} \, dv. \]

\(^3\)But not all — there is no integration by parts.
(v) From the last integral representation, show that

\[ \psi(z) = -\gamma E + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+z} \right). \]

Notice we can now drop the restriction \( \Re z > 0 \) — the beautiful formula above provides an analytic extension of \( \psi(z) \) into the whole complex plane.

**Problem 4.9.** Use pie-shaped contours to evaluate the integrals

\[ A = \int_{0}^{\infty} \frac{dx}{1 + x^3}, \quad \text{and} \quad B = \int_{0}^{\infty} \cos x^2 \, dx. \] \hspace{1cm} (4.91)

**Problem 4.10.** Use the Fourier transform to solve the dispersive wave equation

\[ u_t = \nu u_{xxx}, \quad \text{with IC } u(x,0) = \delta(x). \] \hspace{1cm} (4.92)

Express the answer in terms of \( \text{Ai} \).

**Problem 4.11.** Solve the half-plane \( (y > 0) \) boundary value problem

\[ yu_{xx} + u_{yy} = 0 \] \hspace{1cm} (4.93)

with \( u(x,0) = \cos qx \) and \( \lim_{y \to \infty} u(x,y) = 0. \)
Lecture 5

Integration by parts (IP)

Our earlier example

\[
\text{erfc}(z) \sim \frac{e^{-z^2}}{z\sqrt{\pi}} \left[ 1 - \frac{1}{2z^2} + \frac{1 \times 3}{(2z^2)^2} - \frac{1 \times 3 \times 5}{(2z^2)^3} + O(z^{-8}) \right], \quad \text{as } z \to \infty, \quad (5.1)
\]

illustrated the use of integration by parts (IP) to obtain an asymptotic series. In this lecture we discuss other integrals that yield to IP.

5.1 Dawson’s integral

Dawson’s integral is

\[
D(x) = e^{-x^2} \int_0^x e^{t^2} \, dt. \quad (5.2)
\]

To estimate \(D(x)\) for \(x \gg 1\) we can try IP:

\[
\int_0^x e^{t^2} \, dt = \int_0^1 e^{t^2} \, dt + \int_1^x e^{t^2} \, dt, \quad (5.3)
\]

\[
= \left[ \frac{e^{t^2}}{2t} \right]_0^x + \int_0^x \frac{e^{t^2}}{2t^2} \, dt. \quad (5.4)
\]

The expression above is meaningless — we’ve taken a perfectly sensible integral and written it as the difference of two infinities.

The correct approach is to split the integral like this

\[
\int_0^x e^{t^2} \, dt = \int_0^1 e^{t^2} \, dt + \int_1^x \frac{1}{2t} \, dt, \quad (5.5)
\]

\[
= \int_0^1 e^{t^2} \, dt + \left[ \frac{e^{t^2}}{2t} \right]_1^x + \int_1^x \frac{e^{t^2}}{2t^2} \, dt, \quad (5.6)
\]

\[
\int_0^1 e^{t^2} \, dt - \frac{1}{2} e^{x^2} + \frac{e^{x^2}}{2x} + \int_1^x \frac{e^{t^2}}{2t^2} \, dt, \quad (5.7)
\]

\[
\sim \frac{e^{x^2}}{2x}, \quad \text{as } x \to \infty. \quad (5.8)
\]

Thus

\[
D(x) \sim \frac{1}{2x}, \quad \text{as } x \to \infty. \quad (5.9)
\]
Back in (5.5) we split the range at \( t = 1 \) — this was an arbitrary choice. We could split at another arbitrary value such as \( t = 32.2345465 \). The point is that as \( x \to \infty \) all the terms on the right of (5.5) are much less than the single dominant term \( e^{x^2}/2x \). If we want the next term in (5.9), then that comes from performing another IP on the next biggest term on the right of (5.6), namely

\[
R(x) = \int_1^x \frac{e^{t^2}}{2t^2} \, dt .
\]  

To show that (5.8) is a valid asymptotic approximation we should show that the term above is very much less than the leading term, or in other words that

\[
\lim_{x \to \infty} \frac{\int_1^x e^{t^2}/2t^2 \, dt}{e^{x^2}/2x} = 0 .
\]

**Exercise:** Use l’Hôpital’s rule to verify the result above.

**Example:** Find the \( x \to \infty \) behaviour of

\[
A(x) = \int_0^x \frac{e^{-v}}{v^{1/2}} \, dv .
\]

Direct integration by parts doesn’t work:

\[
A(x) = -\int_0^x \frac{1}{v^{1/2}} \frac{e^{-v}}{dv} \, dv = -\left[ \frac{e^{-v}}{v^{1/2}} \right]_0^x + \int_0^x \frac{e^{-v}}{v^{3/2}} \, dv .
\]

There are two infinities at \( v = 0 \) that must cancel to produce the finite \( A(x) \). The correct approach is to write

\[
A(x) = \int_0^\infty \frac{e^{-v}}{v^{1/2}} \, dv - \int_x^\infty \frac{e^{-v}}{v^{1/2}} \, dv
\]

\[
= \Gamma(1/2) = \sqrt{\pi} - \int_x^\infty \frac{e^{-v}}{\sqrt{v}} \, dv .
\]

Now we can use integration by parts to handle the final integral above

\[
A(x) = \sqrt{\pi} + \int_x^\infty \frac{1}{v^{1/2}} \frac{e^{-v}}{dv} \, dv ,
\]

\[
= \sqrt{\pi} - \frac{e^{-x}}{\sqrt{x}} + \frac{1}{2} \int_x^\infty \frac{e^{-v}}{v^{3/2}} \, dv .
\]

The final term above is asymptotically smaller than the other terms as \( x \to \infty \). Repeated integration by parts generates the full expansion.

**Example:** Find the first two terms in the \( x \to \infty \) asymptotic expansion of

\[
A(x) \overset{\text{def}}{=} \int_{-1}^1 e^{xt^3} .
\]

Explain why the remainder is asymptotically negligible relative to the second term. Do the same for \( x \to -\infty \).

### 5.2 Fourier Integrals

Recall that we can represent almost any function \( f(x) \) defined on the fundamental interval \(-\pi < x < \pi\) as a Fourier series

\[
f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots
+ b_1 \sin x + b_2 \sin 2x + \cdots
\]
Figure 5.1: Convergence of the Fourier series of \( \text{sqr}(t) \). The left panel shows the partial sum with 1, 4 and 16 terms. The right panel is an expanded view of the Gibbs oscillations round the discontinuity at \( x = 0 \). Notice that the overshoot near \( x = 0 \) does not get smaller if \( n \) is increased from 16 to 256.  

(see chapter 12 of RHB). Determining the coefficients in the series above devolves to evaluating the integrals:

\[
\begin{align*}
a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx, \\
a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos kx \, f(x) \, dx, \\
b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin kx \, f(x) \, dx.
\end{align*}
\]  

(Notice the irritating factors of 2 in \( a_0 \) versus \( a_k \).) We’re interested in how fast these Fourier coefficients decay as \( k \to \infty \): the series is most useful if the coefficients decay rapidly.

A classic example is the discontinuous square wave function

\[
\text{sqr}(x) \overset{\text{def}}{=} \text{sgn} \left[ \sin(x) \right].
\]  

(Applying the recipe above to \( \text{sqr}(x) \), we begin by observing that because \( \text{sqr}(x) \) is an odd function, all the \( a_k \)’s are zero. To evaluate \( b_k \) notice that the integrand of (5.21) is even so that we need only integrate from 0 to \( \pi \)

\[
b_k = \frac{2}{\pi} \int_{0}^{\pi} \sin kx \, dx = - \left[ \frac{2}{\pi k} \cos kx \right]_{0}^{\pi} = \left[ 1 - (-1)^k \right] \frac{2}{\pi k}.
\]  

The even \( b_k \)’s are also zero — this is clear from the anti-symmetry of the integrand above about \( x = \pi/2 \). A sensitive awareness of symmetry is often a great help in evaluating Fourier coefficients. Thus we have

\[
\text{sqr}(x) = \frac{4}{\pi} \left[ \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \frac{1}{7} \sin 7x + \cdots \right].
\]  

The wiggly convergence of (5.24) is illustrated in figure 5.1. (Perhaps we’ll have time to say more about the wiggles later.) The point of this square-wave example is that Fourier series is converging very slowly: the coefficients decrease only as \( k^{-1} \), and the series is certainly not absolutely convergent.

**Exercise:** Deduce the Gregory-Leibniz series

\[
\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots
\]  

from (5.24).
Now let’s go to the other extreme and consider very rapidly convergent Fourier series, such as
\[ \cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x . \]  
(5.26)

Another example of a rapidly convergent Fourier series is
\[ \frac{1 - r^2}{1 + r^2 - 2r \cos x} = 1 + 2r \cos x + 2r^2 \cos 2x + 2r^3 \cos 3x + \cdots \]  
(5.27)

If \(|r| < 1\) then the coefficients decrease as \(r^k = e^{k \ln r}\), which is faster than any power of \(k\). In examples like this we get a great approximation with only a few terms.

We can use IP to prove that if \(f(x)\) and its first \(p - 1\) derivatives is continuous and differentiable in the closed interval \(-\pi \leq x \leq \pi\), and the \(p\)’th derivative exists apart from jump discontinuities at some points, then the Fourier coefficients are \(O(n^{-p-1})\) as \(n \to \infty\). Functions such as \(\text{sqr}(x)\), with jump discontinuities, correspond to \(p = 0\). Very smooth functions such as (5.26) and (5.27) correspond to \(p = \infty\).

These results are obtained by evaluating the Fourier coefficient
\[ f_n = \int_{-\pi}^{\pi} f(x)e^{inx} \, dx , \]  
(5.28)

using integration by parts. Suppose we can break the fundamental interval up into sub-intervals so that \(f(x)\) is smooth (i.e., infinitely differentiable) in each subinterval. Non-smooth behavior, such a jump in some derivative, occurs only at the ends of the interval. Then the contribution of the sub-interval \((a, b)\) to \(f_n\) is
\[ I_n = \int_a^b f(x)e^{inx} \, dx , \]
\[ = \frac{1}{in} \int_a^b f(x) \frac{de^{inx}}{dx} \, dx , \]
\[ = \frac{1}{in} \left[ f(x)e^{inx} \right]_a^b - \frac{1}{in} \int_a^b f'(x)e^{inx} \, dx . \]  
(5.29)

Since \(f(x)\) is smooth, we can apply integration by parts to \(I_n\) to obtain
\[ I_n = \frac{1}{in} \left[ f(x)e^{inx} \right]_a^b - \frac{1}{n^2} \left[ f'(x)e^{inx} \right]_a^b + \frac{1}{n^2} \int_a^b f''(x)e^{inx} \, dx . \]  
(5.30)

Obviously we can keep going and develop a series in powers of \(n^{-1}\). Thus we can express \(I_n\) in terms of the values of \(f\) and its derivatives at the end-points.

It is sporting to show that we actually generate an asymptotic series with this approach. For instance, looking at (5.30), we should show that the ratio of the remainder, \(n^{-2}K_n\), to the previous term limits to zero as \(n\) increases. Assuming that \(f'\) is not zero at both end points, this requires that
\[ \lim_{n \to \infty} \int_a^b f''(x)e^{inx} \, dx = 0 . \]  
(5.31)

We can bound the integral easily
\[ \left| \int_a^b f''(x)e^{inx} \, dx \right| \leq \int_a^b |f''(x)||e^{inx}| \, dx \leq \int_a^b |f''(x)| \, dx . (??) \]  
(5.32)
But this doesn’t do the job.

Instead, we can invoke the Riemann-Lebesgue lemma\footnote{The statement above is not the most general and useful form of RL — see section 3.4 of Asymptotics and Special Functions by F.W.J. Olver — particularly for cases with \(a = -\infty\) or \(b = +\infty\).}: If \( \int_a^b |F(t)| \, dt \) exists then

\[
\lim_{\alpha \to \infty} \int_a^b e^{i\alpha t} F(t) \, dt = 0. \tag{5.33}
\]

Riemann-Lebesgue does not tell us how fast the integral vanishes. So, by itself, Riemann-Lebesgue is not an asymptotic estimate. But RL does assure us that the remainder in (5.30) is vanishing faster than the previous term as \( n \to 0 \) i.e., dropping the remainder we obtain an \( n \to \infty \) asymptotic approximation.

An alternative to Riemann-Lebesgue is to change our perspective and think of (5.30) like this:

\[
I_n = \frac{1}{in} [f(x)e^{inx}]_a^b b - \frac{1}{n^2} [f'(x)e^{inx}]_a^b + \frac{1}{n^2} \int_a^b f''(x)e^{inx} \, dx. \tag{5.34}
\]

The bound in (??) then shows that the new remainder is asymptotically less than the first term on the right as \( n \to \infty \). We can then continue to integrate by parts and prove asymptoticity by using the ultimate integral and the penultimate term as the remainder.

Some examples

Suppose, for example, we have a function such as those in (5.26) and (5.27). These examples are smooth throughout the fundamental interval. In this case we take \( a = -\pi \) and \( b = \pi \) and use the result above. Since \( f(x) \) and all its derivatives have no jumps, even at \( x = \pm \pi \), all the end-point terms vanish. Thus in this case \( f_n \) decreases faster than any power of \( n \) e.g., perhaps something like \( e^{-n} \), or \( e^{-\sqrt{n}} \). In this case integration-by-parts does not provide the asymptotic rate of decay of the Fourier coefficients — we must deploy a more potent method such as steepest descent.

Example: An interesting example of a Fourier series is provided by a square in the \((x, y)\)-plane defined by the four vertices \((1, 0)\), \((0, 1)\), \((-1, 0)\) and \((0, -1)\). The square can be represented in polar coordinates as \( r = R(\theta) \). In the first quadrant of the \((x, y)\)-plane, the edge of the square is line \( x + y = 1 \), or

\[
R(\theta) = \frac{1}{\cos \theta + \sin \theta}, \quad \text{if } 0 \leq \theta \leq \pi/2. \tag{5.35}
\]

With some huffing and puffing we could write down \( R(\theta) \) in the other three quadrants. But instead we simplify matters using the obvious symmetries of the square. Because \( R(\theta) = R(-\theta) \) we only need the cosines in the Fourier series. But we also have \( R(\theta) = R(\theta + \pi/2) \), and this symmetry implies that

\[
R(\theta) = a_0 + a_4 \cos 4\theta + a_8 \cos 8\theta + \cdots \tag{5.36}
\]

We can save some work by leaving out \( \cos \theta, \cos 2\theta, \cos 3\theta \) etc because these terms reverse sign if \( \theta \to \pi/2 + \theta \). Thus the \( a_k \)'s corresponding to these harmonics will turn out to be zero.

The first term in the Fourier series is therefore

\[
a_0 = \frac{1}{2\pi} \int_{\pi/2}^{\pi/2} R(\theta) \, d\theta, \tag{5.37}
\]

\[
= \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\cos \theta + \sin \theta}. \tag{5.38}
\]

We've used symmetry to reduce the integral from \(-\pi\) to \(\pi\) to four times the integral over the side in the first quadrant. The mathematica command
Figure 5.2: The first three terms in (5.36) make a rough approximation to the dotted square.

\[ \int \frac{1}{\sin(x) + \cos(x)} \, dx \quad \{x, 0, \pi/2\} \]
tells us that

\[ a_0 = \frac{2\sqrt{2}}{\pi} \tanh^{-1} \left( \frac{1}{\sqrt{2}} \right) \approx 0.7935. \]  
\[ (5.39) \]

The higher terms in the series are

\[ a_{4k} = 4 \frac{\pi}{\pi} \left[ \frac{4}{3} - \tau \right] \]
\[ = \frac{4}{\pi} \left[ 4364 - \tau \right] = 0.1106, \]
\[ a_{8} = -\frac{4}{\pi} \left[ \frac{128}{105} - \tau \right] \]
\[ = -4 \frac{\pi}{\pi} \left[ \frac{45045}{55808} - \tau \right] \]
\[ = -4 \pi \left[ \frac{55808}{45045} - \tau \right] = 0.0349, \]  
\[ a_{12} = \frac{4}{\pi} \left[ \frac{4364}{3465} - \tau \right] \]
\[ = 0.0166, \]
\[ a_{16} = -\frac{4}{\pi} \left[ \frac{55808}{45045} - \tau \right] \]
\[ = -\frac{4}{\pi} \left[ \frac{55808}{45045} - \tau \right] = 0.0096, \]  
\[ (5.40) \]

where

\[ \tau \) \]
\[ \left( \frac{1}{\sqrt{2}} \right) = 1.2464. \]  
\[ (5.43) \]

Figure 5.2 shows that the first three terms of the Fourier series can be used to draw a pretty good square. We might have anticipated this because the coefficients above decrease quickly. In fact, we now show that

\[ a_{4k} = O(k^{-2}) \text{ as } k \to \infty. \]

After this preamble, we consider the problem of estimating the Fourier integral

\[ S(N) = \int_{0}^{\pi/2} \frac{\cos N \theta}{\sin \theta + \cos \theta} \, d\theta, \]  
\[ (5.44) \]

as \( N \to \infty. \) (I've changed notation: \( N \) is a continuously varying quantity i.e., not necessarily integers 4, 8 etc.)

The Riemann-Lebesgue (RL) lemma assures us that

\[ \lim_{N \to \infty} S(N) = 0. \]  
\[ (5.45) \]

But in asymptotics we're not content with this — we want to know how \( S(N) \) approaches zero.

Let us try IP

\[ S(N) = \frac{1}{N} \int_{0}^{\pi/2} \frac{(\sin N \theta)_{\theta}}{\sin \theta + \cos \theta} \, d\theta, \]
\[ = \frac{1}{N} \int_{0}^{\pi/2} \sin N \theta \, d\theta + \frac{1}{N} \int_{0}^{\pi/2} \frac{\sin N \theta (\cos \theta - \sin \theta)}{(\sin \theta + \cos \theta)^2} \, d\theta. \]  
\[ (5.46) \]
We’ve invoked the Riemann-Lebesgue lemma above. Thus, provided that \( \sin(N\pi/2) \neq 0 \), the leading order term is

\[
S(N) \sim \frac{\sin(N\pi/2)}{N}.
\]  

(5.47)

If \( N \) is an even integer (and in the problem that originated this example, \( N = 4k \) is an even integer) then to find a non-zero result we have to integrate by parts again:

\[
S(N) = \sin\left(\frac{N\pi}{2}\right) - \frac{1}{N^2} \int_0^{\pi/2} \left(\frac{\sin \theta - \sin \sin \theta}{(\sin \theta + \cos \theta)^2} \cos N\theta \right) d\theta,
\]

\[
= \sin\left(\frac{N\pi}{2}\right) + \frac{1}{N^2} \left[ \frac{\cos \theta - \sin \theta}{(\sin \theta + \cos \theta)^2} \right]_0^{\pi/2} \cos N\theta d\theta,
\]

\[
\sim \sin\left(\frac{N\pi}{2}\right) N + \frac{1}{N^2} \left[ \frac{\cos \theta - \sin \theta}{(\sin \theta + \cos \theta)^2} \right]_0^{\pi/2} \cos N\theta d\theta + o\left(\frac{1}{N^2}\right).
\]  

(5.48)

We've used RL to justify the \( o(N^{-2}) \). We can keep integrating by parts and develop an asymptotic series in powers of \( N^{-1} \).

With \( N = 4 \) and \( 8 \) we find from (5.48)

\[
a_4 \approx \frac{1}{2\pi} = 0.1592, \quad a_8 \approx \frac{1}{8\pi} = 0.0398, \quad (5.49)
\]

\[
a_{12} \approx \frac{1}{18\pi} = 0.0177, \quad a_{16} = \frac{1}{32\pi} = 0.0099. \quad (5.50)
\]

Comparing these asymptotic estimates with (5.40), we see that the errors are 44%, 14%, 6% and 4% respectively.

**Example: partial failure of IP** Previously we evaluated the Fourier transform

\[
\pi e^{-|k|} = \int_{-\infty}^{\infty} e^{-ikx} \frac{1}{1 + x^2} dx.
\]  

(5.51)

Can we find a \( k \to \infty \) asymptotic expansion using IP? Let’s try:

\[
f(k) = \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \frac{e^{ikx}}{ik} \frac{d}{dk} \frac{d}{dk} \bigg|_{-\infty}^{\infty} dx,
\]

\[
= \left[ \frac{e^{ikx}}{1 + x^2} \right]_{-\infty}^{\infty} + \frac{1}{ik} \int_{-\infty}^{\infty} \frac{2x e^{ikx}}{(1 + x^2)^2} dx,
\]

\[
= O(k^{-1}), \quad \text{ (use RL).} \quad (5.53)
\]

We could IP again, but again the terms that fall outside the integral are zero. In retrospect, this can’t work — after \( n \) integrations we’ll find

\[
f(k) = O(k^{-n}).
\]

(5.54)

This is true: using the exact answer in (4.31)

\[
\lim_{k \to \infty} k^n e^{-k} = 0, \quad \text{ for all } n.
\]  

(5.55)

IP will never recover an exponentially small integral. I call this a partial failure, because at least integration by parts correctly tells us that the Fourier transform is smaller than any inverse power of \( k \). This is the case for any infinitely differentiable function: just keep integrating by parts.

### 5.3 The Taylor series, with remainder

We can very quickly use integration by parts to prove that a function \( f(x) \) with \( n \) derivatives can be represented exactly by \( n \) terms of a Taylor series, plus a remainder. The fundamental theorem of calculus is

\[
f(x) = f(a) + \int_a^x f'(\xi) d\xi.
\]  

(5.56)

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If we drop the final term, \( R_1(x) \), we have a one-term Taylor series for \( f(x) \) centered on \( x = a \). To generate one more term we integrate by parts like this

\[
f(x) = f(a) + \int_a^x f'(\xi) \frac{d}{d\xi} (\xi - x) \, d\xi,
\]

(5.57)

\[
= f(a) + (x - a) f'(a) - \int_a^x f''(\xi) (\xi - x) \, d\xi.
\]

(5.58)

And again

\[
f(x) = f(a) + (x - a) f'(a) - \int_a^x f''(\xi) \frac{d}{d\xi} \frac{1}{2} (\xi - x)^2 \, d\xi,
\]

(5.59)

\[
= f(a) + (x - a) f'(a) + \frac{1}{2} f''(a) (x - a)^2 + \frac{1}{2} \int_a^x f'''(\xi) (\xi - x)^2 \, d\xi.
\]

(5.60)

If \( f(x) \) has \( n \)-derivatives we can keep going till we get

\[
f(x) = f(a) + f'(a) (x - a) + \frac{f''(a)}{2} (x - a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!} (x - a)^{n-1} + R_n(x),
\]

(5.61)

where the remainder after \( n \)-terms is

\[
R_n(x) = \frac{1}{(n-1)!} \int_a^x f^{(n)}(\xi) (\xi - x)^{n-1} \, d\xi.
\]

(5.62)

Using the first mean value theorem, the remainder can be represented as

\[
R_n(x) = \frac{f^{(n)}(\bar{x})}{n!} (x - a)^n,
\]

(5.63)

where \( \bar{x} \) is some unknown point in the interval \([a, x]\). This is the form given in section 4.6 of RHB.

Some remarks about the result in (5.61) and (5.63) are:

1. \( f(x) \) need not have derivatives of all order at the point \( x \): the representation in (5.61) and (5.63) makes reference only to derivatives of order \( n \), and that is all that is required.

2. Using (5.63), we see that the ratio of \( R_n(x) \) to the last retained term in the series is proportional to \( x - a \) and therefore vanishes as \( x \to a \). Thus, according to our definition in (5.58), \( f_n(x) \) is an asymptotic expansion of \( f(x) \).

3. The convergence of the truncated series \( f_n(x) \) as \( n \to \infty \) is not assumed: (5.61) is exact. The remainder \( R_n(x) \) may decrease up to a certain point and then start increasing again.

4. Even if \( f_n(x) \) diverges with increasing \( n \), we may obtain a close approximation to \( f(x) \) — with a small remainder \( R_n \) — if we stop summing at a judicious value of \( n \).

5. The difference between the convergent case and the divergent case is that in the former instance the remainder can be made arbitrarily small by increasing \( n \), while in the latter case the remainder cannot be reduced below a certain minimum.
Figure 5.3: Sums obtained by truncating the $\epsilon = 16$ Taylor series in (5.67) keeping $n$ terms.

**Example:** Taylor series, even when they diverge, are still asymptotic series. Let’s investigate this by revisiting problem 1.1:

$$x(\epsilon)^2 = 9 + \epsilon.$$  \hspace{1cm} (5.64)

Notice that even before taking this class you could have solved this problem by arguing that

$$x(\epsilon) = 3 \left(1 + \frac{\epsilon}{9}\right)^{1/2},$$  \hspace{1cm} (5.65)

and then recollecting the standard Taylor series

$$(1 + z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha - 1)}{2!} z^2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!} z^3 + \cdots$$  \hspace{1cm} (5.66)

The perturbation expansion you worked out in problem 1.1 is laboriously reproducing the special case $\alpha = 1/2$ and $z = \epsilon/9$.

You should recall from part B that the radius of convergence of (5.66) is limited by the nearest singularity to the origin in the complex $z$-plane. With $\alpha = 1/2$ the nearest singularity is the branch point at $z = -1$.

So the series in problem 1.1 converges provided that $\epsilon < 9$. Let us ignore this red flag and use the Taylor series with $\epsilon = 16$ to estimate $x(16) = \sqrt{25} = 5$. We calculate a lot of terms with the mathematica command:

```
Series[Sqrt[9 + u], {u, 0, 8}].
```

This gives

$$x(\epsilon) = 3 + \frac{\epsilon}{6} - \frac{\epsilon^2}{216} + \frac{\epsilon^3}{3888} - \frac{5\epsilon^4}{279936} + \frac{7\epsilon^5}{5038848} - \frac{7\epsilon^6}{60466176} + \frac{11\epsilon^7}{1088391168} - \frac{143\epsilon^8}{156728328192} + \text{ord} (\epsilon^9).$$  \hspace{1cm} (5.67)

Figure 5.3 shows the truncated sums, $x_n(16)$, obtained by keeping $n$ terms in (5.67). The series (5.67) is asymptotic: $x_3 = 4.48$ and $x_4 = 5.53$ with errors of 10% and 11% respectively. Given that we're expanding around $x = 3$, I regard this accuracy as not bad at all.

### 5.4 Large-$s$ behaviour of Laplace transforms

The $s \to \infty$ behaviour of the Laplace transform

$$\hat{f}(s) \overset{\text{def}}{=} \int_0^\infty e^{-st} f(t) \, dt$$  \hspace{1cm} (5.68)

provides a typical and important example of IP. But before turning to IP, we argue that as $\Re s \to \infty$, the maximum of the integrand in (5.68) is determined by the rapidly decaying $e^{-st}$ and is therefore at $t = 0$. In fact, $e^{-st}$ is appreciably different from zero only in a peak at $t = 0,$
and the width of this peak is $s^{-1} \ll 1$. Within this peak $t = O(s^{-1})$ the function $f(t)$ is almost equal to $f(0)$ (assuming that $f(0)$ is non-zero) and thus

$$\tilde{f}(s) \approx f(0) \int_0^\infty e^{-st} \, dt = \frac{f(0)}{s}.$$  \hfill (5.69)

This argument suggests that the large $s$-behaviour of the Laplace transform of any function $f(t)$ with a Taylor series around $t = 0$ is given by

$$\int_0^\infty e^{-st} f(t) \, dt = \int_0^\infty e^{-st} \left[ f(0) + t f'(0) + \frac{t^2}{2!} f''(0) + \cdots \right] e^{-st} \, dt ,$$  \hfill (5.70)

$$\sim \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{f''(0)}{s^3} + \cdots$$  \hfill (5.71)

This heuristic answer is in fact a valid asymptotic series.

We obtain an improved version of (5.71) using successive integration by parts starting with (5.68):

$$\tilde{f}(s) = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{f''(0)}{s^3} + \cdots + \frac{f^{(n-1)}(0)}{s^n} \int_0^\infty e^{-st} (s) \, dt .$$  \hfill (5.72)

The improvement over (5.71) is that on the right of (5.72), IP has provided an explicit expression for the remainder $R_n(s)$.

**Example: A Laplace transform.** Find the large-$s$ behaviour of the Laplace transform

$$\mathcal{L} \left[ \frac{1}{\sqrt{1 + t^2}} \right] = \int_0^\infty \frac{e^{-st}}{\sqrt{1 + t^2}} \, dt.$$  \hfill (5.73)

When $s$ is large the function $e^{-st}$ is non-zero only in a peak located at $t = 0$. The width of this peak is $s^{-1} \ll 1$. In this region the function $(1 + t^2)^{-1/2}$ is almost equal to one. Hence heuristically

$$\int_0^\infty \frac{e^{-st}}{\sqrt{1 + t^2}} \, dt = \int_0^\infty e^{-st} \, dt = \frac{1}{s}.$$  \hfill (5.74)

This is the correct leading-order behaviour.

To make a more careful estimate we can use integration by parts:

$$\mathcal{L} \left[ \frac{1}{\sqrt{1 + t^2}} \right] = \frac{1}{s} \int_0^\infty \frac{e^{-st} \, dt}{\sqrt{1 + t^2}},$$  \hfill (5.75)

$$= \frac{1}{s} \left[ \frac{e^{-st}}{\sqrt{1 + t^2}} \right]_0^\infty - \frac{1}{s} \int_0^\infty \frac{te^{-st}}{(1 + t^2)^{3/2}} \, dt ,$$  \hfill (5.76)

$$= \frac{1}{s} - R_1(s) .$$  \hfill (5.77)

As $s \to \infty$ the remainder $R_1(s)$ is negligible with respect to $s^{-1}$ and the heuristic (5.74) is confirmed. Why is $R_1(s)$ much smaller than $s^{-1}$ in the limit? Notice that in the integrand of $R_1$

$$\frac{te^{-st}}{(1 + t^2)^{3/2}} \leq te^{-st} , \quad \text{and therefore} \quad R(s) < \frac{1}{s} \int_0^\infty e^{-st} \, dt = \frac{1}{s^2} .$$  \hfill (5.78)

The estimates between (5.75) and (5.78) are a recap of arguments we've been making in the previous lectures. The proof of Watson's lemma below is just a slightly more general version of these same estimates.

To get more terms in the asymptotic expansion we invoke Watson's lemma, so as $s \to \infty$:

$$\mathcal{L} \left[ \frac{1}{\sqrt{1 + t^2}} \right] = \int_0^\infty e^{-st} \left[ 1 - \frac{t^2}{2} + \frac{3t^4}{8} - \frac{5t^6}{16} + O(t^8) \right] \, dt ,$$  \hfill (5.79)

$$\sim \frac{1}{s} - \frac{2t^2}{2s^3} + \frac{3 \times 4!}{8s^5} - \frac{5 \times 6!}{16s^7} + O \left( s^{-9} \right) .$$  \hfill (5.80)

---

See the next section, *Watson's lemma*, for justification.
Because of the rapid growth of the numerators this is clearly an asymptotic series.

Notice that the Taylor series of \((1 + t^2)^{-1/2}\) does not converge beyond \(t = 1\). The limited radius of convergence doesn’t matter; Watson’s lemma assures us that we get the right asymptotic expansion even if we integrate into the region where the Taylor series diverges. In fact, the expansion of the integral is asymptotic, rather than convergent, because we’ve integrated a Taylor series beyond its radius of convergence.

**Example: Another Laplace transform.** Consider

\[
\mathcal{L} \left[ \frac{H(t)}{\sqrt{1 - t^2}} \right] = \int_0^1 \frac{e^{-st}}{\sqrt{1 - t^2}} \, dt ,
\]

\sim \frac{1}{s} + \frac{2!}{2s^3} + \frac{3 \times 4!}{8s^5} + \frac{5 \times 6!}{16s^7} + O \left( s^{-9} \right) .
\]

This is the same as (5.80), except that all the signs are positive. The integrable singularity at \(t = 1\) makes only an exponentially small contribution as \(s \to \infty\).

**Example: yet more Laplace transform.** Find the large-\(s\) behaviour of the Laplace transform

\[
\mathcal{L} \left[ \sqrt{1 + e^t} \right] = \int_0^\infty e^{-st} \frac{\sqrt{1 + e^t}}{f(t)} \, dt .
\]

In this case \(f(0) = \sqrt{2}\) and we expect that the leading order is

\[
\hat{f} \sim \frac{\sqrt{2}}{s} .
\]

Let’s confirm this using IP:

\[
\hat{f}(s) = \frac{\sqrt{2}}{s} - \frac{1}{s} \int_0^\infty e^{-st} \frac{e^t}{\sqrt{1 + e^t}} \, dt .
\]

Notice that in this example \(f'(t) \sim e^{t/2}\) as \(t \to \infty\), and thus we cannot bound the remainder using \(\max_{t>0} f'(t)\). Instead, we bound the reminder like this

\[
R_1 = \frac{1}{s} \int_0^\infty e^{-(s - 1/2)t} \frac{1}{2\sqrt{1 + e^{t/2}}} \, dt < \frac{1}{s} \frac{1}{2s - 1} .
\]

This maneuver works in examples with \(f(t) \sim e^{\alpha t}\) as \(t \to \infty\).

### 5.5 Watson’s Lemma

Consider a Laplace transform

\[
\hat{f}(s) = \int_0^\infty e^{-st} t^\xi g(t) \, dt ,
\]

where the factor \(t^\xi\) includes whatever singularity exists at \(t = 0\); the singularity must be integrable i.e., \(\xi > -1\). We assume that the function \(g(t)\) has a Taylor series with remainder

\[
g(t) = g_0 + g_1 t + \cdots + g_n t^n + R_{n+1}(t) .
\]

This is a \(t \to 0\) asymptotic expansion in the sense that there is some constant \(K\) such that

\[
|R_{n+1}| < K t^{n+1} .
\]

Notice we are not assuming that the Taylor series converges.

Of course, we do assume convergence of the Laplace transform (5.87) as \(t \to \infty\), which most simply requires that \(f(t) = t^\xi g(t)\) eventually grows no faster than \(e^{\gamma t}\) for some \(\gamma\). Notice that the possibility of a finite upper limit in (5.87) is encompassed if \(f(t)\) is zero once \(t > T\).
With these modest constraints on \( t^\xi g(t) \):

\[
\hat{f}(s) = \int_0^\infty e^{-st} t^\xi (g_0 + g_1 t + \cdots + g_n t^n) \, dt + \int_0^\infty e^{-st} t^\xi R_{n+1}(t) \, dt .
\]

(5.90)

The second integral in (5.90) is

\[
I_2 < K \int_0^\infty e^{-st} t^{n+1+\xi} \, dt = O\left( \frac{1}{s^{\xi+n+2}} \right).
\]

(5.91)

Using

\[
\int_0^\infty e^{-st} t^{\xi+n} \, dt = \frac{\Gamma(n+\xi+1)}{s^{\xi+n+1}},
\]

(5.92)

we integrate \( I_1 \) term-by-term and obtain Watson's lemma:

\[
\hat{f}(s) \sim g_0 \frac{\Gamma(\xi+1)}{s^{\xi+1}} + g_1 \frac{\Gamma(\xi+2)}{s^{\xi+2}} + \cdots + g_n \frac{\Gamma(\xi+n+1)}{s^{\xi+n+1}} + O\left( \frac{1}{s^{\xi+n+2}} \right).
\]

(5.93)

Watson's lemma justifies doing what comes naturally.

Example: Consider

\[
I(x, \nu) \overset{\text{def}}{=} \int_0^\infty t^\nu e^{-x \sinh t} \, dt.
\]

(5.94)

The minimum of \( \phi(t) = \sinh t \) is at \( t = 0 \), so

\[
I(x) \sim \int_0^\infty t^\nu e^{-xt} \, dt \sim \frac{\Gamma(v + 1)}{x^{v+1}}, \quad \text{as } x \to \infty.
\]

(5.95)

To get the next term in the asymptotic series, keep one more term in the expansion of \( \sinh t \):

\[
e^{-x \sinh t} \approx e^{-xt} e^{xt^3/6 - xt^7/120 + \cdots} \approx e^{-xt} \left( 1 - \frac{xt^3}{6} + O(xt^5) \right).
\]

(5.96)

Thus

\[
I(x) \sim \int_0^\infty t^\nu e^{-xt} \left( 1 - \frac{xt^3}{6} + O(xt^5) \right) \, dt,
\]

(5.97)

\[
\sim \frac{\Gamma(v + 1)}{x^{v+1}} - \frac{\Gamma(v + 4)}{6x^{v+3}} + O\left( x^{v-5} \right).
\]

(5.98)

Notice we have to keep the dominant term \( xt \) up in the exponential.

If we desire more terms, and are obliged to justify the heuristic above, we should change variables with \( u = \sinh t \) in (5.94), and use Watson's lemma. The transformed integral is a formidable Laplace transform:

\[
I(x, \nu) \overset{\text{def}}{=} \int_0^\infty e^{-xu} \ln^\nu \left( \sqrt{1 + u^2} + u \right) \frac{du}{\sqrt{1 + u^2}}.
\]

(5.99)

With mathematica

\[
\ln^\nu \left( \sqrt{1 + u^2} + u \right) = u^\nu \left[ 1 - \frac{3 + \nu}{6} u^2 + \frac{135 + 52\nu + 5\nu^2}{360} u^4 + O(u^6) \right].
\]

(5.100)

The coefficient of \( u^{2n} \) in this expansion is a polynomial — let's call it \((-)^n P_n(\nu)\) — of order \( n \). Substituting (5.100) into (5.99) and integrating term-by-term

\[
I(x, \nu) \sim \frac{1}{x^{v+1}} \left[ \Gamma(v + 1) - \frac{P_1(\nu)}{x^2} \Gamma(v + 3) + \frac{P_2(\nu)}{x^3} \Gamma(v + 5) + O(x^{-6}) \right].
\]

(5.101)
5.6 Problems

Problem 5.1. (i) Use IP to obtain the leading-order asymptotic approximation for the integral
\[ \int_{-1}^{1} e^{xt^5} \, dt, \quad \text{as } x \to \infty. \] (5.102)
(ii) Justify the asymptoticness of the expansion. (iii) Find the leading-order asymptotic approximation for \( x \to -\infty \).

Problem 5.2. (i) Use IP to obtain the leading-order asymptotic approximation for the integral
\[ \int_{x}^{\infty} \frac{e^{it}}{t} \, dt, \quad \text{as } x \to \infty. \] (5.103)
(ii) Justify the asymptoticness of the expansion.

Problem 5.3. In our earlier evaluation of \( \text{Ai}(0) \) we encountered a special case, namely \( n = 3 \), of the integral
\[ Z(n, x) \overset{\text{def}}{=} \int_{0}^{\pi/(2n)} e^{-x \sin n\theta} \, d\theta. \] (5.104)
Convert \( Z(n, x) \) to a Laplace transform and use Watson’s lemma to obtain the first few terms of the \( x \to \infty \) asymptotic expansion.

Problem 5.4. In lecture 3 we obtained the full asymptotic series for \( \text{erfc}(z) \) via IP:
\[ \text{erfc}(x) \sim \frac{e^{-x^2}}{\sqrt{\pi x}} \sum_{n=0}^{\infty} (2n-1)!! \left( -\frac{1}{2x^2} \right)^n. \] (5.105)
Obtain this result by making a change of variables that converts \( \text{erfc}(z) \) into a Laplace transform, and then use Watson’s lemma.

Problem 5.5. Using integration by parts to find \( x \to \infty \) asymptotic approximations of the integrals
\[ A(x) = \int_{0}^{x} e^{-t^4} \, dt, \] (5.106)
\[ B(x) = \int_{0}^{x} e^{+t^4} \, dt, \] (5.107)
\[ D(x) = \int_{1}^{2} \frac{\cos xt}{t} \, dt, \] (5.108)
\[ E(x) = \int_{0}^{\infty} e^{-xt} \ln(1 + t^2) \, dt, \] (5.109)
\[ F(x) = \int_{0}^{\infty} \frac{e^{-xt}}{ta(1 + t)} \, dt, \quad \text{with } a > 1; \] (5.110)
\[ H(x) = \int_{1}^{\infty} e^{-xt^p} \, dt, \quad \text{with } p > 0; \] (5.111)
\[ I(x) = \int_{0}^{1} \cos t^2 e^{ixt} \, dt. \] (5.112)
In each case obtain a two-term asymptotic approximation and exhibit the remainder as an integral. Explain why the remainder is smaller than the second term as \( x \to \infty \).
Problem 5.6. Find $x \to 0$ approximations to the integrals in problem [5.5] (Some examples may be difficult.)

Problem 5.7. Using repeated IP, find the full $x \to \infty$ asymptotic expansion of Dawson's integral (5.2). Is this series convergent?

Problem 5.8. Consider

$$f(x) = \int_{\pi/4}^{\infty} \cos(xt^2) \tan^2 t \, dt, \quad \text{as } x \to \infty. \tag{5.113}$$

Show that IP can be used to compute the leading-order term, but not the second term. Compute the second term using stationary phase.

Problem 5.9. Find two terms in the $x \to 0$ and $x \to \infty$ expansion of the Fresnel integrals

$$C(x) = \int_{x}^{\infty} \cos t^2 \, dt, \quad \text{and} \quad S(x) = \int_{x}^{\infty} \sin t^2 \, dt. \tag{5.114}$$

Problem 5.10. Consider $f(x) = (1 + x)^{5/2}$, and the corresponding Taylor series $f_n(x)$ centered on $x = 0$. (i) Show that for $n \geq 3$ and $x > 0$:

$$R_n < \frac{f^{(n)}(0)}{n!} x^n,$$

i.e., the remainder is smaller than the first neglected term for all positive $x$. (ii) The Taylor series converges only up to $x = 1$. But suppose we desire $f(2) = 3^{5/2}$. How many terms of the series should be summed for best accuracy? Sum this optimally truncated series and compare with the exact answer. (iii) Argue from the remainder in (5.63) that the error can be reduced by adding half the first neglected term. Compare this corrected series with the exact answer.
Lecture 6

Laplace’s method

Laplace’s method applies to integrals in which the integrand is concentrated in the neighbourhood of a few (or one) isolated points. The value of the integral is determined by the dominant contribution from those points. This happens most often for integrals of the form

\[ I(x) = \int_a^b f(t) e^{-x\phi(t)} \, dt, \quad \text{as } x \to \infty. \] (6.1)

If \( \phi(t) \geq 0 \) for all \( t \) in the interval \( a \leq t \leq b \) then as \( x \to +\infty \) the integrand will be maximal where \( \phi(t) \) is smallest. This largest contribution becomes more and more dominant as \( x \) increases.

Look what happens if we apply IP to (6.1):

\[ I(x) = -\int_a^b \frac{f(t)}{x\phi'(t)} e^{-x\phi(t)} \, dt, \quad \text{as } x \to \infty. \] (6.2)

\[ = -\left[ \frac{f}{x\phi'} e^{-x\phi} \right]_a^b + \int_a^b e^{-x\phi} \frac{d}{dt} \left( \frac{f}{x\phi'} \right) \, dt. \] (6.3)

There is a problem if \( \phi' \) has a zero anywhere in the closed interval \([a,b]\). However if \( \phi' \) is non-zero throughout \([a,b]\) then IP delivers the goods. For example, suppose

\[ \phi' > 0 \quad \text{for } a \leq t \leq b, \] (6.4)

then from (6.3)

\[ I(x) \sim \frac{f(a)}{x\phi'(a)} e^{-x\phi(a)}, \quad \text{as } x \to \infty. \] (6.5)

Our earlier Laplace transform example (5.69) is a special case with \( a = 0, b = \infty \) and \( \phi = t \).

In the case of (6.5), the integrand is concentrated near \( x = a \) and the asymptotic approximation in (6.3) depends only on \( f(a), \phi(a) \) and \( \phi'(a) \). We can quickly obtain (6.3) with the following approximations in (6.3):

\[ I(x) \sim \int_a^\infty f(a) e^{-x\phi(a) - x\phi'(a)t} \, dt. \] (6.6)

**Exercise:** Find the leading order asymptotic approximation to \( I(x) \) if \( \phi' < 0 \) for \( a \leq t \leq b \). Show that

\[ A(x) \overset{\text{def}}{=} \int_0^\pi e^{x \cosh t} \, dt \sim \frac{e^{x \cosh \pi}}{x \sinh \pi}, \quad \text{as } x \to \infty. \] (6.7)
To summarize, if \( \phi' \) is non-zero throughout \([a, b]\) then the integrand is concentrated at one of the end points, and IP quickly delivers the leading-order term. And, if necessary, one can change variables with
\[
v = \phi(t)
\]
in (6.1) and then use Watson’s lemma to get the full asymptotic expansion. We turn now to discussion of the case in which \( \phi'(t) \) has a zero somewhere in \([a, b]\).

**Example:** Let’s use Watson’s lemma to obtain the full asymptotic expansion of
\[
\text{erfc}(x) \overset{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} \, dt.
\]

### 6.1 An example — the Gaussian approximation

As an example of Laplace’s method with a zero of \( \phi' \) we study the function defined by
\[
U(x, y) \overset{\text{def}}{=} \int_{0}^{y} e^{-x \cosh t} \, dt,
\]
and ask for an asymptotic approximations as \( x \to +\infty \) with \( y \) fixed. In this example \( \phi' = \sinh t \) is zero at \( t = 0 \) and IP fails.

**Exercise:** Find a two-term approximation of \( U(x, 1) \) when \( |x| \ll 1 \).

With \( x \to \infty \), the main contribution to \( U(x, y) \) in (6.10) is from \( t \approx 0 \). Thus, according to Laplace, the leading-order behaviour is
\[
U(x, y) \sim \int_{0}^{\infty} e^{-x(1+\frac{1}{2}t^2)} \, dt,
\]
\[
= e^{-x} \sqrt{\frac{\pi}{2x}}, \quad \text{as} \; x \to +\infty.
\]

The peak of the integrand is centered on \( t = 0 \) and has width \( x^{-1/2} \ll 1 \). All the approximations we’ve made above are good in the peak region. They’re lousy approximations outside the peak e.g., near \( t = 1/2 \). But both the integrand and our approximation to the integrand are tiny near \( t = 1/2 \) and thus those errors do not seriously disturb our estimate of the integral.

Notice that in (6.11) the range of integration is extended to \( t = \infty \) — we can then do the integral without getting tangled up in error functions. The point is that the leading-order behaviour of \( U(x, y) \) as \( x \to \infty \) is independent of the fixed upper limit \( y \). If you’ve understood the argument above regarding the peak width, then you’ll appreciate that if \( y = 1/10 \) then \( x \) will have to be roughly as big as 100 in order for (6.12) to be accurate.

Let’s bash out the second term in the \( x \to \infty \) asymptotic expansion. According to MATHEMATICA, the integrand is
\[
e^{-x \cosh t} = e^{-x-xt^2/2}e^{-xt^4/4!-xt^6/6!} \ldots \approx e^{-x-xt^2/2} \left( 1 - \frac{xt^4}{24} - \frac{xt^6}{720} + O \left( x^2 t^8 \right) \right).
\]

Notice the \( x^2 \) in the big Oh error estimate above — this \( x^2 \) will bite us below. We now substitute the expansion (6.13) into the integral (6.10) and integrate term-by-term using
\[
\int_{0}^{\infty} t^p e^{-at^2} \, dt = \frac{1}{2} a^{-\frac{p+1}{2}} \Gamma \left( \frac{p+1}{2} \right).
\]
Thus we have

\[ U(x, y) = e^{-x} \int_{0}^{\infty} e^{-\frac{1}{2}xt^2} \left[ \frac{1}{x^{-1/2}} - \frac{1}{4} x t^4 x^{-3/2} - \frac{1}{720} x^6 t^6 + O \left( x^2 t^8 \right) \right] \, dt \] (6.15)

The underbraces indicate the order of magnitude of each term after using (6.14) to evaluate the integral. Notice that a term of order \( x^2 t^6 \) is of order \( x^{-5/2} \) after integration. Thus, if we desire a systematic expansion, we should not keep the term \( x^6 \) and drop \( x^2 t^8 \). After integration both these terms are order \( x^{-5/2} \), and we should keep them both, or drop them both.

Proceeding with the integration

\[ U(x, y) \sim e^{-x} \sqrt{\frac{2}{\pi x}} \int_{0}^{\infty} e^{-v^2} \left[ 1 - \frac{1}{6x} \times 3 - \frac{8}{720x^2} \times \frac{15}{8} + O \left( x^{-2} \right) \right] \, dv, \] (6.16)

\[ = e^{-x} \sqrt{\frac{\pi}{2x}} \left[ 1 - \frac{1}{6x} \times 3 - \frac{8}{720x^2} \times \frac{15}{8} + O \left( x^{-2} \right) \right], \] (6.17)

\[ \sim e^{-x} \sqrt{\frac{\pi}{2x}} \left[ 1 - \frac{1}{8x} + O \left( x^{-2} \right) \right]. \] (6.18)

Discretion is the better part of valor, so I’ve dropped the inconsistent term and written \( O(x^{-2}) \) above.
Another way to generate more terms in the expansion is to convert $U(x, y)$ into a Laplace transform via $u = \cosh t - 1$:

\[
U(x, y) \sim e^{-x} \int_0^\infty \frac{e^{-xu}}{\sqrt{2u + u^2}} \, du,
\]

\[
\sim e^{-x} \int_0^\infty e^{-xu} \left[ 1 - \frac{u}{4} + \frac{3u^2}{32} - \frac{5u^3}{128} + O\left(u^4\right) \right] \, du,
\]

\[
= e^{-x} \sqrt{\frac{1}{2x}} \left[ \Gamma\left(\frac{1}{2}\right) - \frac{1}{4x} \Gamma\left(\frac{3}{2}\right) + \frac{3}{32x^2} \Gamma\left(\frac{5}{2}\right) + O\left(x^{-3}\right) \right],
\]

\[
= e^{-x} \sqrt{\frac{\pi}{2x}} \left[ 1 - \frac{1}{8x} + \frac{9}{128x^2} + O\left(x^{-3}\right) \right].
\]

The Laplace-transform approach is more systematic because the coefficients in the series expansion (6.20) are not functions of $x$, and the expansion is justified using Watson's lemma. However, the argument about the dominance of the peak provides insight and is all one needs to quickly obtain the leading-order asymptotic expansion.

### 6.2 Another Laplacian example

Consider

\[
I_n \overset{\text{def}}{=} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} (\cos t)^n \, dt.
\]

With a little integration by parts one can show that

\[
I_n = \left(1 - \frac{1}{n}\right) I_{n-2}.
\]

Then, since $I_0 = 1$ and $I_1 = 2/\pi$, it is easy to compute the exact integral at integer $n$ recursively.

Let's use Laplace's method to find an $n \to \infty$ asymptotic approximation. We write the integral as

\[
I_n = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} e^{n \ln \cos t} \, dt,
\]

and then make the small $t$-approximation

\[
\ln \cos t = \ln \left(1 - \frac{t^2}{2}\right) \approx -\frac{t^2}{2}.
\]

Thus the leading order is obtained by evaluating a gaussian integral

\[
I_n \sim \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-nt^2/2} \, dt,
\]

\[
= \sqrt{\frac{2}{\pi n}}.
\]

Figure 6.2 compares this approximation to the exact integral. Suppose we're disappointed with the performance of this approximation at $n = 5$, and want just one more term. The
The easiest way to bash out an extra term is
\[
\ln \cos t = \ln \left(1 - \frac{t^2}{2} + \frac{t^4}{24} + \text{ord}(t^6)\right),
\]
(6.29)
\[
= \left(\frac{t^2}{2} - \frac{t^4}{24} + \text{ord}(t^6)\right) + \frac{1}{2} \left(\frac{t^2}{2} + \text{ord}(t^4)\right)^2 + \text{ord}(t^6),
\]
(6.30)
\[
= -\frac{t^2}{2} - \frac{t^4}{12} + \text{ord}(t^6),
\]
(6.31)
and then
\[
I_n \sim \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-nt^2/2} e^{-nt^4/12} \, dt,
\]
(6.32)
\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-nt^2/2} \left(1 - \frac{nt^4}{12}\right) \, dt,
\]
(6.33)
\[
= \sqrt{\frac{2}{\pi n}} \left(1 - \frac{1}{4n}\right).
\]
(6.34)
This works very well at \(n = 5\). In the unlikely event that more terms are required, then it is probably best to be systematic: change variables with \(v = -\ln \cos t\) and use Watson’s lemma.

### 6.3 Laplace’s method with moving maximum

**Large \(s\) asymptotic expansion of a Laplace transform**

Not all applications of Laplace’s method fall into the form (6.1). For example, consider the Laplace transform
\[
\mathcal{L}\left[ e^{-1/t} \right] = \int_0^\infty e^{-\frac{1}{t} - st} \, dt, \quad \text{as } s \to \infty.
\]
(6.35)
Watson’s lemma is defeated by this example.
In the exponential in (6.35) have \(\chi \overset{\text{def}}{=} t^{-1} + st\), and
\[
\frac{d\chi}{dt} = 0, \quad \Rightarrow \quad -\frac{1}{t^2} + s = 0.
\]
(6.36)
Thus the integrand is biggest at \( t_* = s^{-1/2} \) — the peak is approaching \( t = 0 \) as \( s \) increases. Close to the peak

\[
\chi = \chi(t_*) + \frac{1}{2} \chi''(t_*)(t - t_*)^2 + O(t - t_*)^3, \tag{6.37}
\]

\[
= 2s^{1/2} + s^{-3/2}(t - s^{-1/2})^2 + O(t - t_*)^3. \tag{6.38}
\]

The width of the peak is \( s^{-3/4} \ll s^{-1/2} \), so it helps to introduce a change of variables

\[
v \equiv s^{3/4}(t - s^{-1/2}). \tag{6.39}
\]

In terms of the original variable \( t \) the peak of the integrand is moving as \( s \) increases. We make the change of variable in (6.39) so that the peak is stationary at \( v = 0 \). The factor \( s^{3/4} \) on the right of (6.39) ensures that the width of the \( v \)-peak is not changing as \( s \to \infty \).

Notice that \( t = 0 \) corresponds to \( v = -s^{1/4} \to -\infty \). But the integrand has decayed to practically to zero once \( v \gg 1 \). Thus the lower limit can be taken to \( v = -\infty \). The Laplace transform is therefore

\[
\mathcal{L} \left[ e^{-t/t} \right] \sim s^{-3/4} e^{-2s^{1/2}} \int_{-\infty}^{\infty} e^{-tv^2} \, dt, \quad \text{as } s \to \infty. \tag{6.40}
\]

This Laplace transform is exponentially small as \( s \to \infty \), and of course the original function was also exponentially small as \( t \to 0 \). I trust you’re starting to appreciate that there is an intimate connection between the small-\( t \) behaviour of \( f(t) \) and the large-\( s \) behaviour of \( \tilde{f}(s) \).

Remark: the Laplace transform of any function must vanish as \( s \to \infty \). So, if you’re asked to find the inverse Laplace transform of \( s \), the answer is there is no function with this transform.

**Stirling’s approximation**

A classic example of a moving maximum is provided by *Stirling’s approximation* to \( n! \). Starting from

\[
\Gamma(n + 1) = \int_0^\infty n^t e^{-t} \, dt, \tag{6.41}
\]

let’s derive the fabulous result

\[
\Gamma(n + 1) \sim \sqrt{2\pi n} n^n \left( \frac{n}{e} \right)^n \left( 1 + \frac{1}{12n} + O \left( n^{-2} \right) \right), \quad \text{as } n \to \infty. \tag{6.42}
\]

At \( x = 1 \), we have from the leading order \( 1 \approx \sqrt{2\pi}/e = 0.9221 \), which is not bad! And with the next term \( \sqrt{2\pi}/e \times (13/12) = 0.99898 \). It only gets better as \( x \) increases.

We begin by moving everything in (6.41) upstairs into the exponential:

\[
\Gamma(n + 1) = \int_0^\infty e^{-\chi} \, d\chi, \tag{6.43}
\]

where

\[
\chi \equiv n \ln(n) - t. \tag{6.44}
\]

The maximum of \( \chi \) is at \( t_* = x \) — the maximum is moving as \( x \) increases. We can expand \( \chi \) around this moving maximum as

\[
\chi = x \ln x - x + \frac{(t - x)^2}{2x} + O(t - x)^3, \tag{6.45}
\]

\[
= x \ln x - x - v^2, \tag{6.46}
\]

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where \( v \overset{\text{def}}{=} (t - x) / \sqrt{2x} \) is the new variable of integration. With this Gaussian approximation we have

\[
\Gamma(x + 1) = e^{x \ln x - x} \sqrt{\frac{2}{\pi x}} \int_{-\infty}^{\infty} e^{-v^2} \, dv.
\] (6.47)

This is the leading order term in (6.41).

**Exercise:** Obtain the next term, \( 1/12x \), in (6.41).

### 6.4 Uniform approximations

Consider a function of two variables defined by:

\[
J(x, \alpha) \overset{\text{def}}{=} \int_{0}^{\infty} e^{-x(\sinh t - \alpha t)} \, dt, \quad \text{with } x \to \infty, \text{ and } \alpha \text{ fixed.} \tag{6.48}
\]

In this case

\[
\phi = \sinh t - \alpha t, \quad \text{and} \quad \frac{d\phi}{dt} = \cosh t - \alpha. \tag{6.49}
\]

The location of the minimum of \( \phi \) crucially depends on whether \( \alpha \) is greater or less than one. If \( \alpha < 1 \) then the minimum of \( \phi \) is at \( t = 0 \) and

\[
J(x, \alpha < 1) \sim \int_{0}^{\infty} e^{-x(1-\alpha)t} \, dt, \quad \text{as } x \to \infty, \text{ and } \alpha < 1 \text{ fixed.} \tag{6.50}
\]

If \( \alpha > 1 \), the minimum of \( \phi(t) \) moves away from \( t = 0 \) and enters the interior of the range of integration. Let’s call the location of the minimum \( t_\ast(\alpha) \):

\[
cosh t_\ast(\alpha) = \alpha, \quad \text{and therefore} \quad t_\ast = \ln \left( \alpha + \sqrt{\alpha^2 - 1} \right). \tag{6.52}
\]

If \( \alpha > 1 \) then \( t_\ast \) is real and positive. Notice that

\[
\phi(t_\ast) = \sinh t_\ast - \alpha t_\ast = \sqrt{\alpha^2 - 1} - \alpha t_\ast(\alpha), \tag{6.53}
\]

and

\[
\phi''(t_\ast) = \sinh t_\ast = \sqrt{\alpha^2 - 1}. \tag{6.54}
\]

Then we expand \( \phi(t) \) in a Taylor series round \( t_\ast \):

\[
\phi(t) = \phi(t_\ast) + \frac{1}{2}(t - t_\ast)^2 \phi''(t_\ast) + O((t - t_\ast)^3). \tag{6.55}
\]

To leading order

\[
J(x, \alpha > 1) \sim e^{-x\phi_\ast} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(t - t_\ast)^2 \phi''(t_\ast)} \, dt, \tag{6.56}
\]

Notice we’ve extended the range of integration to \( t = -\infty \) above. The error is small, and this enables us to evaluate the integral exactly

\[
J(x, \alpha > 1) \sim e^{-\phi_\ast(\alpha)} \sqrt{\frac{2\pi}{x\phi''_\ast(\alpha)}}, \quad \text{as } x \to \infty. \tag{6.57}
\]
If we use the expressions for \( t_\ast(\alpha) \) and \( \phi''_\ast(\alpha) \) above then we obtain an impressive function of the parameter \( \alpha \):

\[
J(x, \alpha > 1) \sim \frac{2\pi}{x^{\frac{1}{2}}(\alpha^2 - 1)^{\frac{1}{2}}} \exp \left( -x\sqrt{\alpha^2 - 1} \right) \left( \alpha + \sqrt{\alpha^2 - 1} \right)^{\frac{\alpha x}{2}} , \quad \text{as } x \to \infty. \tag{6.58}
\]

Comparing (6.51) with (6.58), we wonder what happens if \( \alpha = 1 \)? And how does the asymptotic expansion change continuously from the simple form in (6.51) to the complicated expression in (6.58) as \( \alpha \) passes continuously through 1?

Notice that as \( x \to \infty \):

\[
J(x, 1) = \int_0^\infty e^{-x(sinh - t)} \, dt , \tag{6.59}
\]

\[
\sim \int_0^\infty e^{-xt^{3/6}} \, dt , \tag{6.60}
\]

\[
= 2^{1/3} 3^{-2/3} \Gamma \left( \frac{1}{3} \right) x^{-1/3} . \tag{6.61}
\]

So, despite the impression given by (6.51) and (6.58), \( J(x, 1) \) is not singular.

We’re interested in the transition where \( \alpha \) is close to 1, so we write

\[
\alpha = 1 + \epsilon \tag{6.62}
\]

where \( \epsilon \) is small. Then

\[
J(x, \alpha) \sim \int_0^\infty e^{x\xi t - \frac{1}{3}xt^3} \, dt = x^{-1/3} \int_0^\infty e^{\xi \tau - \frac{1}{3} \tau^3} \, d\tau , \tag{6.63}
\]

where \( \xi \) is a similarity variable:

\[
\xi \overset{\text{def}}{=} (\alpha - 1)x^{2/3} . \tag{6.64}
\]

The transition from (6.51) to (6.58) occurs when \( \alpha - 1 = O(x^{-2/3}) \), and \( \xi = O(1) \). The transition is described uniformly by a special function

\[
J(\xi) \overset{\text{def}}{=} \int_0^\infty e^{\xi \tau - \frac{1}{3} \tau^3} \, d\tau . \tag{6.65}
\]

Our earlier results in (6.51), (6.58) and (6.61) are obtained as special cases by taking \( \xi \to -\infty \), \( \xi \to +\infty \) and \( \xi = 0 \) in \( J(\xi) \).

### 6.5 Problems

**Problem 6.1.** Considering \( U(x, y) \) in (6.10), show that

\[
x^2 U_{xx} + xU_x - x^2 U = U_{yy} . \tag{6.66}
\]

Evaluate \( U(x, \infty) \) in terms of modified Bessel functions.

**Problem 6.2.** Consider

\[
V(x, k, p) \overset{\text{def}}{=} \int_0^{kx^{-p}} e^{-x \cosh t} \, dt , \quad \text{as } x \to \infty. \tag{6.67}
\]

Find a leading-order approximation to (i) \( V(x, k, 1) \); (ii) \( V(x, k, 1/2) \) and (iii) \( V(x, k, 1/4) \). Hint: In one of the three cases you’ll need to use the error function.
Problem 6.3. Show that
\[ \int_{0}^{1} e^t \left( \frac{t}{1 + t^2} \right)^n \, dt \sim \sqrt{\frac{\pi}{2n}} e^{\frac{e}{2n}}, \quad \text{as } n \to \infty. \] (6.68)

Problem 6.4. Show that
\[ \int_{0}^{\pi} t^n \sin t \, dt \sim \frac{\pi^{n+2}}{n^2}, \quad \text{as } n \to \infty. \] (6.69)

Problem 6.5. The beta function is
\[ B(x, y) \overset{\text{def}}{=} \int_{0}^{1} t^{x-1} (1 - t)^{y-1} \, dt. \] (6.70)

With a change of variables show that
\[ B(x, y) = \int_{0}^{\infty} e^{-xv} (1 - e^{-v})^{y-1} \, dv. \] (6.71)

Suppose that \( y \) is fixed and \( x \to \infty \). Use Laplace’s method to obtain the leading order approximation
\[ B(x, y) \sim \frac{\Gamma(y)}{x^y}. \] (6.72)

Go to the Digital Library of Special Functions, chapter 5 and find the relation between the beta function and the gamma function. (You can probably also find this formula in RHB, or any text on special functions.) Use this relation to show that
\[ \frac{\Gamma(x)}{\Gamma(x + y)} \sim \frac{1}{x^y}, \quad \text{as } x \to \infty. \] (6.73)

Remark: this result can also be deduced from Stirling’s approximation, but it’s a rather messy calculation.

Problem 6.6. Find an asymptotic approximation of
\[ \int_{0}^{\infty} \int_{0}^{\infty} e^{-n(x^2 + y^2)} \frac{dx}{(1 + x + y)^n} \, dy \quad \text{as } n \to \infty. \] (6.74)

Problem 6.7. Find the \( x \to \infty \) leading-order behaviour of the integrals
\[ A(x) = \int_{-1}^{1} e^{-xt^3} \, dt, \quad B(x) = \int_{-1}^{1} e^{+xt^3} \, dt, \] (6.75)
\[ C(x) = \int_{-1}^{1} e^{-xt^4} \, dt, \quad D(x) = \int_{-1}^{1} e^{+xt^4} \, dt, \] (6.76)
\[ E(x) = \int_{0}^{\infty} e^{-xt - t^4/4} \, dt, \quad F(x) = \int_{0}^{\infty} e^{+xt - t^4/4} \, dt, \] (6.77)
\[ G(x) = \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{(1 + t^2)^x}, \quad H(x) = \int_{-\infty}^{\infty} e^{t^2} \frac{dt}{(1 + t^2)^x}, \] (6.78)
\[ I(x) = \int_{0}^{\pi/2} e^{-x \sec t} \, dt, \quad J(x) = \int_{0}^{\pi/2} e^{-x \sin^2 t} \, dt, \] (6.79)
\[ K(x) = \int_{-1}^{1} (1 - t^2) e^{-x \cosh t} \, dt, \quad L(x) = \int_{-1}^{1} (1 - t^2) e^x \cosh t \, dt. \] (6.80)
Problem 6.8. Find the leading order asymptotic expansion of

\[ M(x) \defeq \int_0^\infty e^{xt} t^{-t} \, dt \]  

(6.81)
as \( x \to \infty \) and as \( x \to -\infty \).

Problem 6.9. Find the first two terms in the asymptotic expansion of

\[ N(x) \defeq \int_0^\infty t^n e^{-t^2 - \frac{x}{t}} \, dt \]  

(6.82)
as \( x \to \infty \).

Problem 6.10. Show that

\[ \int_0^\infty e^{-x} \left( \frac{1}{1 + e^{-x}} \right)^n \, dx \sim \sqrt{2\pi \frac{(n-1)^{-3/2}}{n^{n+1}}} \]  

(6.83)
as \( n \to \infty \).

(I am 80% sure this is correct.)

Problem 6.11. (i) Draw a careful graph of \( \phi(t) = (1 - 2t^2)^2 \) for \(-2 \leq t \leq 2\). (ii) Use Laplace's method to show that as \( x \to \infty \)

\[ \int_0^{1/2} \sqrt{1 + t} e^{x\phi} \, dt \sim e^x \left( \frac{1}{4} \sqrt{\frac{\pi}{x}} + \frac{p}{x^{3/2}} + \frac{q}{x^{5/2}} + \cdots \right) , \]  

(6.84)
and determine the constants \( p \) and \( q \). Find asymptotic expansion as \( x \to \infty \) of

\( (ii) \quad \int_0^1 \sqrt{1 + t} e^{x\phi} \, dt , \quad (iii) \quad \int_{-1}^1 \sqrt{1 + t} e^{x\phi} \, dt . \)  

(6.85)
Calculate the expansion up to and including terms of order \( x^{-3/2} e^x \).
Problem 6.12. Consider the function

\[ F(x) \equiv \int_{0}^{\infty} \exp \left( -\frac{t^3}{3} + xt \right) \, dt. \]  \hspace{1cm} (6.86)

(i) \( F(x) \) satisfies a second-order linear inhomogeneous differential equation. Find the ODE and give the initial conditions \( F(0) \) and \( F'(0) \) in terms of the \( \Gamma \)-function. (ii) Perform a local analysis of this ODE round the irregular singular point at \( x = \infty \) and say what you can about the large \( x \) behaviour of \( F(x) \). (iii) Use Laplace's method on (6.86) to obtain the complete \( x \to \infty \) leading-order approximation to \( F(x) \). (iv) Numerically evaluate (6.86) and make a graphical comparison with Laplace's approximation on the interval \( 0 \leq x \leq 3 \) (see figure 6.3).

%% MATLAB script for Laplace's method.
%% You'll have to supply the ??'s and code \tt\{myfun\}.
clear
xx = [0:0.05:3];
nloop = length(xx);
FF = zeros(1,nloop);  \hspace{1cm} % Store function values in FF
uplim = 10; \hspace{1cm} % 10=\infty for the upper limit of quad?
lowlim = realmin;  \hspace{1cm} % avoid a divide-by-zero error
for n=1:nloop
    F = quad(@(t)myfun(t,xx(n)),lowlim,uplim);
    FF(n) = F;
end
plot(xx,FF)
hold on
approx = sqrt(??)*xx.^(??).*exp(2*xx.^(??)/3);
plot(xx,approx,'--')
hold off
xlabel('x')
ylabel('F(x)')
Problem 6.13. Find the first few terms in the $x \to \infty$ asymptotic expansion of

$$F(x) \overset{\text{def}}{=} \int_0^1 \exp \left( -\frac{xt^2}{1+t} \right) \, dt.$$  \hfill (6.87)

Improve figure 6.4 by adding the higher-order approximations to the lower panel.

Problem 6.14. Find the first two terms in the $x \to \infty$ expansion of

$$Y(x) \overset{\text{def}}{=} \int_0^e e^{-xt^2/(1+t^2)} \, dt.$$  \hfill (6.88)

Problem 6.15. Show that as $x \to \infty$

$$\int_0^\infty \frac{e^{-t}}{tx} \, dt \sim e^{-x} \left[ \frac{1}{2x} + \frac{1}{8x^2} + \text{ord} \left( x^{-3} \right) \right].$$  \hfill (6.89)
Lecture 7

Autonomous differential equations

This long lecture has too much material. But a lot of it is stuff you should have learnt in school e.g., how to solve the simple harmonic oscillator. What’s not covered in lectures is assigned as reading.

7.1 The phase line

As an application of algebraic perturbation theory we’ll discuss the “phase line” analysis of first-order autonomous differential equations. That is, equations of the form:

$$\dot{x} = f(x). \quad (7.1)$$

These equations are separable: see Chapter 14 of RHB, Chapter 1 of BO.

Separation of variables followed by integration often leads to an opaque solution with $x$ given only as an implicit function of $t$. A typical example is

$$\dot{x} = \sin x \quad \text{with initial condition} \quad x(0) = x_0. \quad (7.2)$$

We can separate variables and integrate

$$t = \int_{x_0}^{x} \frac{dx'}{\sin x'}, \quad \Rightarrow \quad \tan \left( \frac{x}{2} \right) = e^t \tan \left( \frac{x_0}{2} \right). \quad (7.3)$$

You can check by substitution that the solution above satisfies the differential equation and the initial condition. Suppose the initial condition that $x(0) = 17\pi/4$. Can you use the solution in (7.3) to find $\lim_{t \to \infty} x(t)$? It’s not so easy because the inverse of tan is multivalued.

Fortunately it is much simpler to analyze (7.1) on the phase line: see Figure 7.1. With this construction it is very easy to see that

$$x_0 = \frac{9}{4} \pi, \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = 3\pi. \quad (7.4)$$

The solution of (7.2) moves monotonically along the $x$-axis and, in the case above, approaches the fixed point at $x = 3\pi$, where $\dot{x} = 0$.

If we consider (7.1) with a moderately complicated $f(x)$ given graphically — for example in Figure 7.2 — then we can predict the long-time behaviour of all initial conditions with no effort at all. The solutions either trek off to $+\infty$, or to $-\infty$, or evolve towards fixed points defined by $f(x) = 0$. Moreover the evolution of $x(t)$ is monotonic.
Figure 7.1: The arrows indicate the direction of the motion along the line produced by \( \dot{x} = \sin x \).

Figure 7.2: There are three fixed points indicated by the *’s on the x-axis. The fixed point at \( x = -1 \) is unstable and point at \( x = 2 \) is stable. The point at \( x = 1 \) is stable to negative displacements and unstable to positive displacements.

**Example:** Sketch the 1D vector field corresponding to

\[
\dot{x} = e^{-x^2} - x. \tag{7.5}
\]

The main point of this example is that it is easier to separately draw the graphs of \( e^{-x^2} \) and \( x \) (rather than the difference of the two functions). This makes it clear that there is one stable fixed point at \( x \approx 0.65 \), and that this fixed point attracts all initial conditions.

### 7.2 Population growth — the logistic equation

Malthus (1798) in *An essay on the principle of population* argued that human populations increase according to

\[
\dot{N} = rN. \tag{7.6}
\]

If \( r > 0 \) then the population increases without bound. Verhulst (1838) argued that Malthusian growth must be limited by a nonlinear saturation mechanism, and proposed the simplest model of this saturation:

\[
\dot{N} = rN \left(1 - \frac{N}{K}\right). \tag{7.7}
\]

A phase-line analysis of the Verhulst equation (7.7) quickly shows that for all \( N(0) > 0 \):

\[
\lim_{t \to \infty} N(t) = K. \tag{7.8}
\]

In ecology the Verhulst equation (7.7) is known as the \( r-K \) model; \( K \) is the "carrying capacity" and \( r \) is the growth rate. Yet another name for (7.7) is the "logistic equation".

To solve (7.7) we could use separation of variables, or alternatively we might recognize a Bernoulli equation.\(^1\) For a change of pace, let’s use the trick for solving Bernoulli equations:

\[\text{That is, an equation of the form}\]

\[
\frac{dy}{dx} = a(x)y + b(x)y^n.
\]

\(^1\)
Figure 7.3: Solutions of the logistic equation (7.7). The curves which start with small \( N(0) \) are S-shaped ("sigmoid"). Can you show that the inflection point \( \dot{N} = 0 \) is at time \( t_* \) defined by \( N(t_*) = K/2 \)?

Divide (7.7) by \(-N^2\):

\[
\frac{d}{dt} \frac{1}{N} = -\frac{r}{N} + \frac{r}{K}.
\]  

(7.9)

This is a linear differential equation for \( X \equiv 1/N \), with integrating factor \( e^{rt} \), and solution

\[
N(t) = \frac{N_0K}{(K - N_0)e^{-rt} + N_0}.
\]

(7.10)

Above, the initial condition is \( N_0 = N(0) \). This solution with various values of \( N_0/K \) produces the “sigmoid curves” shown in Figure 7.3.

The logistic equation is notable because the exact solution is not an opaque implicit formula like (7.3) — the solution in (7.10) exhibits \( N \) as an explicit function of \( t \). This is one of the few cases in which the explicit solution is useful.

**Exercise:** (i) Solve the logistic equation by separation of variables. (ii) Show that the population is increasing most rapidly when \( N = K/2 \). (Hint: only a very small calculation is required in (ii).)

### 7.3 The phase plane

A two-dimensional autonomous system has the form

\[
\dot{x} = u(x, y), \quad \dot{y} = v(x, y).
\]

(7.11)

(The dot indicates a time derivative.) The phase plane, \((x, y)\), is the two-dimensional analog of the phase line. The state of the system at some time \( t_0 \) is specified by giving the location of a point \((x, y)\) and at every point there is an arrow indicating the instantaneous direction in which the system moves. The collection of all these arrows is a “quiver”. The set of arrows is also called a direction field, but quiver is the relevant MATLAB command.

The simplest example is the harmonic oscillator

\[
\ddot{x} + x = 0.
\]

(7.12)

We begin by writing this second-order equation as a system with the form in (7.11):

\[
\dot{x} = y \quad \dot{y} = -x.
\]

(7.13)
Thus at each point in the $(x,y)$-plane there is a velocity vector,
\[ \mathbf{q} = y\mathbf{\hat{x}} - x\mathbf{\hat{y}}, \] (7.14)
and in a small time $\delta t$ the system moves along this vector through a distance $\delta t\mathbf{q}$ to the next point in the plane. Thus the system moves along an orbit in the phase plane; the vector $\mathbf{q}$ is tangent to every point on the orbit.

The harmonic oscillator example is so simple that you should be able to draw the sketch vector field without the aid of MATLAB. The orbits are just circles centered on the origin,
\[ x^2 + y^2 = \frac{1}{2}E, \] (7.15)
where the energy of the oscillator, $E$, is constant.

Here is a list of things we should do with this example, and with other phase-plane differential equations

1. Locate the fixed points;
2. Perform a linear stability analysis of the fixed points;
3. Admire the “nullclines”: $\dot{x} = 0$ or $\dot{y} = 0$.
4. Calculate the divergence of the two-dimensional phase fluid i.e., $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$.

This linear example is too simple to illustrate the power of this technique so let’s move on.

**Example:** As a slightly more complicated example of phase-plane analysis we consider the Volterra equations:

\[ \begin{align*}
\dot{r} &= r - fr, \\
\dot{f} &= -f + fr
\end{align*} \] (7.16)

This is a simple “predator-prey” model in which $f(t)$ is the population of foxes and $r(t)$ is the population of rabbits. In the absence of foxes, and with unlimited grass, the rabbit population grows exponentially. The growth of the fox population requires rabbits, else foxes starve.

The state of the system is specified by giving the location of a point in the phase plane $(r,f)$. In this example the arrow at the point $(r,f)$ is

\[ \mathbf{q}(r,f) \overset{\text{def}}{=} (r - fr)\mathbf{\hat{r}} + (-f + fr)\mathbf{\hat{f}}. \] (7.17)

where $\mathbf{\hat{r}}$ and $\mathbf{\hat{f}}$ are unit vectors along the rabbit axis and the fox axis respectively. The collection of all solutions is visualized as a collection of phase-plane orbits, with the vector $\mathbf{q}$ tangent to every point of the orbit. Figure 7.4 shows three phase-space orbits, and the associated quiver.

We can easily locate the fixed points. There are just two:
\[ (r,f) = (0,0), \quad \text{and} \quad (r,f) = (1,1). \] (7.18)

First consider the linear stability analysis of the fixed point at the origin. We’re interested in small displacements away from the origin, so we simply drop the nonlinear terms in (7.16) to obtain the associated linear system

\[ \begin{align*}
\dot{r} &= r, \\
\dot{f} &= -f
\end{align*} \] (7.19)

The solution is
\[ r = r_0 e^t, \quad f = f_0 e^{-t}. \] (7.20)

The origin is an unstable fixed point: a small rabbit population grows exponentially (e.g., the invasion of Australia by 24 rabbits released in 1859). Moreover, we can eliminate $t$ between $r$ and $f$ in (7.20) to obtain
\[ r f = r_0 f_0. \] (7.21)

Thus near the origin the phase space orbits are hyperbolas. This type of fixed point, with exponential-time-time growth in one direction and exponential-in-time decay in another direction, is called a saddle point, or an x-point.
Now turn to the fixed point at \((1,1)\). To look at displacements from \((1,1)\) we introduce new variables \((a,b)\) defined by

\[ r = 1 + a, \quad f = 1 + b. \]  

(7.22)

In this simple example we can rewrite the system exactly in terms of \((a,b)\):

\[ \dot{a} = -b - ab, \quad \dot{b} = a + ab. \]  

(7.23)

Neglecting the quadratic term \(ab\), the associated linear system is

\[ a = -b, \quad \dot{b} = a. \]  

(7.24)

We could solve (7.24) by eliminating \(b\) or \(a\) to obtain

\[ \dot{a} + a = 0, \quad \text{or} \quad \dot{b} + b = 0. \]  

(7.25)

The general solution is a linear combination of \(\cos t\) and \(\sin t\), and the constants of integration are determined by the initial conditions \((a_0,b_0)\). This is simple, but there is an alternative based on a trick that will come in handy later: introduce

\[ z \overset{\text{def}}{=} a + ib. \]  

(7.26)

With this “complexification”, the system (7.24) is

\[ \dot{z} = iz \quad \text{with solution} \quad \frac{a + ib}{z_0} = (a_0 + ib_0) (\cos t + i \sin t). \]  

(7.27)

Notice that

\[ |z|^2 = a^2 + b^2 = a_0^2 + b_0^2. \]  

(7.28)

Thus, according to the linear approximation\(^2\) if the system is slightly displaced from the fixed point \((1,1)\) it simply orbits around at a fixed distance from \((1,1)\) — this type of fixed point is called a center or an \(o\)-point.

\(^2\)In this case we have to be concerned that the neglected nonlinear terms have a long-term impact e.g., the radius of the circle could grow slowly as a result of weak nonlinearity.
Figure 7.4: Three solutions of the Volterra system (7.16). The vector field is tangent to these phase space orbits. Notice the kinks in the blue trajectory.

```matlab
function foxRabbit
    % phase portrait of the Volterra predator-prey system
    tspan = [0 10];
    aZero = [ 0.25, 0.25 ]; bZero = [ 0.5, 0.5 ]; cZero = [ 0.75, 0.75 ];
    [ta, xa] = ode45(@dfr,tspan,aZero);
    [tb, xb] = ode45(@dfr,tspan,bZero);
    [tc, xc] = ode45(@dfr,tspan,cZero);
    plot(xa(:,1), xa(:,2),xb(:,1),xb(:,2),xc(:,1),xc(:,2))
    axis equal
    hold on
    xlabel('$r$','interpreter','latex','fontsize',20)
    ylabel('$f$','interpreter','latex','fontsize',20)
    axis([0 4 0 4])
    % now the quiver
    [R F] = meshgrid(0:0.2:4);
    U = R - F.*R;
    V = -F + F.*R;
    quiver(R,F,U,V)
end

%------- nested function -------% 
function dxdt = dfr(t,x)
    dxdt = [ x(1) - x(1)*x(2); - x(2)+x(1)*x(2) ];
end
```
7.4 Matlab ODE tools

The MATLAB code foxRabbit that produces figure 7.4 is shown in the associated verbatim box. The code is encapsulated as a function foxRabbit, with neither input nor output arguments. This construction enables the function dfr — which is called by ode45 with the handle @dfr — to be included inline. The command axis equal is used so that circles look like circles.

One problem with figure 7.4 is that solution curves are not smooth. There are kinks in the biggest orbit — the one that corresponds to initial condition a. The problem is that ode45 aggressively uses large time steps if possible. The command \[ [\text{ta, xa}] = \text{ode45}(@\text{dfr}, \text{tspan}, \text{aZero}) \] outputs the solution at times determined by the internal logic of ode45 and those times are too coarsely spaced to make a smooth plot of the solution.

To get a smooth solution curve, at closely spaced times controlled by you, rather than by ode45, there are several modifications of the script, indicated in the code smoothFoxRabbit in the verbatim box below figure 7.5. First, create a vector that contains the desired output times: \[ \text{t} = \text{linspace}(0, \text{max(\text{tspan})}, 200) \]. Next, ode45 is called with a single output argument:

\[ \text{sola} = \text{ode45}(@\text{dfr}, \text{tspan}, \text{aZero}, \text{options}); \]

This creates a MATLAB structure, called sola in this example. The structure sola contains all the information required to interpolate the solution between the times determined by ode45. The MATLAB function deval performs that interpolation. We access the solution at the times specified in t via the command \[ \text{xa} = \text{deval}(\text{sola}, \text{t}) \]. This creates a matrix xa with two columns and \text{length(t)} rows. The first column is the dfr rabbit variable, \( x(1) \), and the second column of xa is the foxes \( x(2) \).

Note that in the upper panel Figure 7.5, the rotation of ordinate label created by ylabel is set to zero. More importantly perhaps, the tolerances for ode45 are set with the MATLAB command odeset. The command

\[ \text{options} = \text{odeset}('\text{AbsTol}',1e-7, '\text{RelTol}',1e-4); \]

creates a MATLAB structure called options. ode45 will accept this structure as an optional input argument. I must confess that I don't understand how these tolerances work. You'll note that if you use the default tolerances then the phase space orbit computed by smoothFoxRabbit doesn't close. This is a numerical error: the orbits are closed — see problem 7.10. When I saw this problem I decreased the tolerances using odeset and the picture improved. This adventure shows that numerical solutions are not the same as exact solutions.
Figure 7.5: Another version of figure 7.4. The trajectory in the top panel is evaluated at densely sampled times so that the plot is smoother than in figure 7.4. The lower panel shows the two populations as functions of time. Which is the fox and which is the rabbit?

```matlab
function smoothFoxRabbit
% phase portrait of the Volterra predator-prey system
tspan = [0 20]; t = linspace(0,max(tspan),200);
options = odeset('AbsTol',1e-7, 'RelTol',1e-4);
aZero = [ 0.25, 0.25 ]; sola = ode45(@dfr,tspan,aZero,options);
xa = deval(sola,t);
subplot(2,1,1)
plot(xa(1,:), xa(2,:))
axis equal
hold on
subplot(2,1,2)
plot(t,xa(1,:),t,xa(2,:), 'g--')
xlabel('$t$','interpreter','latex','fontsize',20)
ylabel('$r(t)$ and $f(t)$','interpreter','latex','fontsize',20)
%------- nested function --------%
function dxdt = dfr(t,x)
dxdt = [ x(1) - x(1)*x(2); - x(2)+x(1)*x(2)];
end
end
```
7.5 The linear oscillator

Consider the damped and forced oscillator equation,

\[ m\ddot{x} + \alpha \dot{x} + kx = f, \quad (7.29) \]

with an initial condition such as

\[ x(0) = x_0, \quad \frac{dx}{dt}(0) = u_0. \quad (7.30) \]

You can think of this as the mass-spring system in the Figure 7.6 with damping provided by low-Reynolds number air resistance so that the drag is linearly proportional to the velocity.

We can obtain the energy equation if we multiply (7.29) by \( \dot{x} \) and write the result as

\[ \frac{d}{dt} \left( \frac{1}{2} mx^2 + \frac{1}{2} kx^2 \right) = -\alpha \dot{x}^2 + \dot{x} f. \quad (7.31) \]

This expresses the rate of change of energy as the difference between the rate at which the force \( f \) does work, \( \dot{x} f \), and the dissipation of energy by drag \(-\alpha \dot{x}^2\).

Resonance

Begin by considering an harmonically forced oscillator with no damping:

\[ \ddot{x} + \omega^2 x = \cos \sigma t. \quad (7.32) \]

Suppose that the oscillator is at rest at \( t = 0 \):

\[ x(0) = 0, \quad \dot{x}(0) = 0. \quad (7.33) \]

The solution is

\[ x = \frac{\cos \sigma t - \cos \omega t}{\omega^2 - \sigma^2}. \quad (7.34) \]

We can check this answer by taking \( t \to 0 \), and showing that

\[ x \to \frac{t^2}{2} \quad (7.35) \]

both by expanding the solution in (7.34) or by identifying a small-\( t \) dominant balance between two of the three terms in (7.32).

There is a problem if the oscillator is resonantly forced i.e., if the forcing frequency \( \sigma \) is equal to the natural frequency \( \omega \). Then the solution is

\[ x(t) = \lim_{\omega \to \sigma} \frac{\cos \sigma t - \cos \omega t}{\omega^2 - \sigma^2} = \frac{t}{2\sigma} \sin \omega t. \quad (7.36) \]

(You can use l'Hôpital's rule to evaluate the limit.) If the oscillator is resonantly forced, then the displacement grows linearly with time. We'll use this basic result later many times in the sequel.

**Exercise:** Solve the initial value problem

\[ \ddot{x} + \omega^2 x = \sin \sigma t, \quad x(0) = \dot{x}(0) = 0. \quad (7.37) \]

What happens if \( \omega = \sigma \)?
An initial value problem for a damped oscillator

Now consider an unforced oscillator \( f = 0 \) with initial conditions \( x(0) = 0 \) and \( \dot{x}(0) = u_0 \). The “natural” frequency of the undamped \((\alpha = 0)\) and unforced \((f = 0)\) oscillator is

\[
\omega \overset{\text{def}}{=} \sqrt{\frac{k}{m}}. \tag{7.38}
\]

This suggests a non-dimensionalization

\[
t \overset{\text{def}}{=} \omega t, \quad \text{and} \quad \bar{x} = \frac{u_0}{\omega} x. \tag{7.39}
\]

The scaled problem is

\[
\frac{d^2 \bar{x}}{d\bar{t}^2} + \frac{\alpha}{m\omega} \frac{d\bar{x}}{d\bar{t}} + \bar{x} = 0, \tag{7.40}
\]

with initial conditions

\[
\bar{x}(0) = 0, \quad \text{and} \quad \frac{d\bar{x}}{d\bar{t}}(0) = 1. \tag{7.41}
\]

We’ve also taken \( x_0 = 0 \) so that there is a single non-dimensional control parameter, \( \beta \). We proceed dropping the bars.

If \( \beta < 2 \), then the exact solution of the initial value problem posed above is

\[
x = v^{-1} e^{-\beta t/2} \sin vt, \quad \text{with} \quad v \overset{\text{def}}{=} \sqrt{1 - \frac{\beta^2}{4}}. \tag{7.42}
\]

Figure 7.7 shows the phase-space portrait of the damped oscillator. Because of damping, all trajectories spiral into the origin. If the damping is weak the spiral is wound tightly i.e., it takes many periods for the energy to decay to half of its initial value.

The main effect of small damping is to reduce the amplitude of the oscillation exponentially in time, with an e-folding time \( 2/\beta \). Damping also slightly shifts the frequency of the oscillation:

\[
v = 1 - \frac{\beta^2}{8} + \text{ord} \left( \beta^4 \right). \tag{7.43}
\]

The frequency shift is only important once \( \beta^2 t \sim 1 \), and on that long time the amplitude of the residual oscillation is exponentially small \( (\sim e^{-1/2\beta}) \). So we don’t worry too much about the frequency shift. A good \( \beta \ll 1 \) approximation to the exact solution in (7.42) is

\[
x \approx e^{-\beta t/2} \sin t. \tag{7.44}
\]

**Exercise:** When does the approximation in (7.44) first differ in sign from the exact \( x(t) \)?
The method of averaging

If $\beta = 0$ then the solution of the oscillator equation (7.40) is

$$x = a \sin(t + \chi), \quad (7.45)$$

where the amplitude $a$ and the phase $\chi$ are constants set by initial conditions. The undamped oscillator conserves energy

$$E \overset{\text{def}}{=} \frac{1}{2} \dot{x}^2 + \frac{1}{2} x^2. \quad (7.46)$$

Moreover, $E$ is “equipartitioned” between kinetic and potential. Thus, averaged over a cycle,

$$E = \langle \dot{x}^2 \rangle = \langle x^2 \rangle = \frac{a^2}{2}. \quad (7.47)$$

These results are exact if $\beta = 0$.

But because of dissipation the energy decays:

$$\frac{dE}{dt} = -\beta \dot{x}^2. \quad (7.48)$$

If $\beta$ is non-zero, but small, then we might guess that the solution has the form in (7.45) except that the amplitude $a$ is slowly deceasing i.e., $a$ is a function of slow time. We also assume that the phase $\chi$ is slowly changing. Thus

$$\dot{x} = a \cos(t + \chi) + \dot{a} \sin(t + \chi) + \dot{\chi} a \cos(t + \chi). \quad (7.49)$$

Because only a little energy is lost in each cycle we can average (7.45) to obtain

$$\frac{dE}{dt} = -\beta \langle \dot{x}^2 \rangle, \quad (7.50)$$

$$= -\beta \langle E \rangle. \quad (7.51)$$

Thus

$$E = E_0 e^{-\beta t}. \quad (7.52)$$

The amplitude therefore varies as $a = a_0 e^{-\beta t/2}$, which is in agreement with the exact solution (7.44). This argument does not determine the evolution of the phase $\chi$.

7.6 Nonlinear oscillators

The nonlinear oscillator equation for $x(t)$ is

$$\ddot{x} = -U_x, \quad (7.53)$$

where $U(x)$ is the potential. The linear oscillator is the special case $U = \omega^2 x/2$.

We can obtain a good characterization of the solutions of (7.53) using conservation of energy: multiply (7.53) by $\dot{x}$ and integrate to obtain

$$\frac{1}{2} \dot{x}^2 + U(x) = E, \quad (7.54)$$
where the constant energy \( E \) is determined by the initial condition

\[
E = \left[ \frac{1}{2} \dot{x}^2 + U(x) \right]_{t=0}.
\]  

(7.55)

Let’s consider the mass-spring system in Figure 7.6 as an example. Suppose that the spring gets stronger as the extension \( x \) increases. We can model this “stiff” spring by adding nonlinear terms to Hooke’s law:

\[
\text{spring force} = -k_1 x - k_3 x^3 + \cdots
\]  

(7.56)

where the \( \cdots \) indicate the possible presence of additional terms as the displacement \( x \) increases further. If the spring is stiff then \( k_3 > 0 \) i.e., the first non-Hookean term increases the restoring force above Hooke’s law.

Note that in (7.56) are assuming that the force depends symmetrically on the displacement \( x \) i.e., the series in (7.56) contains only odd terms. Don’t worry too much about that assumption — the problems offer plenty of scope to investigate asymmetric restoring forces.

The equation of motion of the mass \( m \) on a non-Hookean spring is therefore

\[
m \ddot{x} = -k_1 x - k_3 x^3 + \cdots
\]  

(7.57)

This is equivalent to (7.54) with

\[
U = \frac{k_1}{m} \frac{x^2}{2} + \frac{k_3}{m} \frac{x^4}{4} + \cdots
\]  

(7.58)

Now we can simply contour the energy \( E \) in the phase plane \((x, \dot{x})\). We don’t need to draw the quiver of direction field arrows because we know that the orbits are confined to curves of constant energy. The arrows are tangent to the curves of constant \( E \) and you can easily visualize them if so inclined.

**The Duffing oscillator**

For example, suppose we truncate the series in (7.57) after the \( k_3 x^3 \). Then, after some scaling, we have the **Duffing oscillator**

\[
\ddot{x} + x \pm x^3 = 0,
\]  

(7.59)
Figure 7.8: The phase plane of the Duffing oscillator. Can you tell which panel corresponds to the $+$ sign in (7.60)? Does a low-energy solution orbit the origin in a clockwise or a counter clockwise direction?

where $\pm$ depends on the sign of $k_3$. Please make sure you understand how all the coefficients have been normalized to either 1 or $-1$ without loss of generality.

The energy of the Duffing oscillator is

$$E = \frac{1}{2} \dot{x}^2 + \frac{1}{2} x^2 \pm \frac{1}{4} x^4. \quad (7.60)$$

Figure 7.8 shows the curves of constant energy drawn with the MATLAB routines \texttt{meshgrid} and \texttt{contour}. 

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Figure 7.9: The top panel shows the Morse potential and the bottom panel shows three phase space trajectories corresponding to $E = 0.1, 0.3$ and 1.

**The Morse oscillator - turning points**

Although it is easy to draw energy contours with MATLAB there is an educational construction that enables one to sketch the energy curves by hand. I’ll explain this construction using the Morse potential,

$$U = \frac{1}{2} \left(1 - e^{-x}\right)^2,$$

as an example. The top panel of figure 7.9 shows the Morse potential and the bottom panel shows three phase space trajectories corresponding to $E = 0.1, 0.3$ and 1. The construction involves:

1. Drawing an energy level $E$ in the top panel;
2. Locating the turning points, $x(E)$’s defined by $E = U(x)$, in the top panel;
3. Dropping down to the bottom panel, and locating the turning points in the phase plane;
4. Sketching the curve of constant energy $E$ — keep in mind that it is symmetric about the $x$-axis.

This is best explained on a blackboard.

**The projectile problem reloaded:** Let’s use energy conservation to obtain another solution of the projectile problem which is superior to (2.17). From the non-dimensional equation of motion (2.5), and the initial condition, we quickly obtain the energy integral

$$\frac{1}{2}z^2 - \frac{1}{\epsilon} \frac{1}{1 + \epsilon z} = \frac{1}{2} - \frac{1}{\epsilon}.$$  \hspace{1cm} (7.62)

The maximum height of the projectile is therefore

$$z_{\text{max}} = \frac{1}{2 - \epsilon}.$$  \hspace{1cm} (7.63)
Now consider the first half of the trajectory, when \(0 \leq t < \tau/2\), so that the particle is going up with velocity
\[
\dot{z} = +\sqrt{1 - \frac{2}{\epsilon} + \frac{1}{1 + \epsilon z}}.
\]  
(7.64)

We can integrate this differential equation by separation of variables to obtain
\[
\frac{1}{2} \tau = \int_{0}^{z_{\text{max}}} \sqrt{1 + \epsilon z} \frac{dz}{1 - (2 - \epsilon)z},
\]  
(7.65)
\[
= z_{\text{max}} \int_{0}^{1} \frac{1 + a \xi}{1 - \xi} d\xi,
\]  
(7.66)

where
\[
a(\epsilon) \overset{\text{def}}{=} \frac{\epsilon}{2 - \epsilon}.
\]  
(7.67)

We resort to MATHEMATICA to do the integral above:

Assuming\([0 < a\), Integrate\([\sqrt{(1 + a x)/(1 - x)}\), \{x, 0, 1\}]\]

Note the use of Assuming to avoid the bother of sorting out various conditional expressions. Thus we find
\[
\tau(\epsilon) = \frac{2}{2 - \epsilon} \left(1 + \frac{(1 + a)\arctan(\sqrt{a})}{\sqrt{a}}\right).
\]  
(7.68)

We have a not-very-complicated exact solution so we don’t really need perturbation theory. Nonetheless we press on and use mathematica we obtain the expansion
\[
\tau = \frac{4}{2 - \epsilon} \left(1 + \frac{a}{3} + \frac{a^2}{15} + \frac{a^3}{35} + \frac{a^4}{63} + \frac{a^5}{99} + \frac{a^6}{143} + \text{ord}\left(a^7\right)\right).
\]  
(7.69)

If we insist on powers of \(\epsilon\) then we have to replace \(a(\epsilon)\) using (7.67), leading to a more complicated series
\[
\tau = 2 + \frac{4 \epsilon}{3} + \frac{4 \epsilon^2}{5} + \frac{16 \epsilon^3}{35} + \frac{16 \epsilon^4}{63} + \frac{32 \epsilon^5}{231} + \frac{32 \epsilon^6}{429} + \text{ord}\left(\epsilon^7\right).
\]  
(7.70)

The two series above have the same formal order of accuracy. But I’m pretty sure that (7.69) is superior.

Exercise: Make sure you understand (7.62) and (7.63), and discuss the physical significance of \(\epsilon = 2\).

## 7.7 Problems

**Problem 7.1.** (a) Find \(\lim_{t \to \infty} x(t)\), where \(x(t)\) is the solution of
\[
\dot{x} = (x - 1)^2 - \frac{x^3}{100}, \quad x(0) = 1.
\]

(b) Find the \(t \to \infty\) limit if the initial condition is changed to \(x(0) = 1.2\). In both cases give a numerical answer with two significant figures.

**Problem 7.2.** Consider
\[
\dot{x} = x^p, \quad \text{with initial condition } x(0) = 1.
\]  
(7.71)

If \(p = 1\), the solution \(x(t)\) grows exponentially and takes an infinite time to reach \(x = \infty\). On the other hand, if \(p = 2\), then \(x(t)\) reaches \(\infty\) in finite time. Draw a graph of the time to \(\infty\) as a function of \(p\).

**Problem 7.3.** Back in the day, students were taught to evaluate trigonometric integrals like (7.3) with the substitution \(\theta = \tan x'/2\). Show that \(d\theta'/\sin x' = d\theta/\theta\) and do the integral.
Problem 7.4. The velocity of a skydiver falling to the ground is given by

\[ m \dot{v} = mg - kv^2, \quad (7.72) \]

where \( m \) is the mass, \( g = 32.2 \text{ feet/(second)^2} \) is gravity and \( k \) is an empirical constant related to air resistance. (a) Obtain an analytic solution assuming that \( v(0) = 0 \). (b) Use your solution to find the terminal velocity in terms of \( m, g \) and \( k \). (c) Check your answer by analyzing the problem on the phase line. (d) An experimental study with skydivers in 1942 was conducted by dropping men from 31,400 feet to an altitude of 2,100 feet at which point the skydivers opened their chutes. This long freefall took 116 seconds on average and the average weight of the men plus their equipment was 261.2 pounds. Calculate the average velocity. (e) Use the data above to estimate the terminal velocity and the drag constant \( k \). A straightfoward approach requires solving a transcendental equation either graphically or numerically. But you can avoid this labor by making an approximation that the average velocity is close to the terminal velocity. If you do make this approximation, then you should check it carefully and identify the non-dimensional parameter that controls the validity of the approximation.

Problem 7.5. Consider the logistic equation with a periodically varying carrying capacity:

\[ \dot{N} = rN \left( 1 - \frac{N}{K} \right), \quad \text{with} \quad K = K_0 + K_1 \cos \omega t. \quad (7.73) \]

The initial condition is \( N(0) = N_0 \). (i) Based on the \( K_1 = 0 \) solution, non-dimensionalize this problem. Show that there are three control parameters. (ii) Suppose that \( K_1 \) is a perturbation i.e., \( K_1/K_0 \ll 1 \) and that \( N(t) \approx K_0 \). Find the periodic-in-time solution of the perturbed problem (e.g., see Figure 7.10). (iii) Discuss the phase lag between the population, \( N(t) \), and the carrying capacity \( K(t) \) e.g., in figure 7.10 which curve is the carrying capacity?

Problem 7.6. As a model of combustion triggered by a small perturbation, consider

\[ \dot{x} = x^2 (1 - x), \quad x(0) = \epsilon. \quad (7.74) \]

(i) Start with the simpler problem

\[ \dot{y} = y^2, \quad y(0) = \epsilon. \quad (7.75) \]

Explain why problem (7.75) is a small-time approximation to problem (7.74). (ii) Use separation of variables to find the exact solution of (7.75) and show that \( y(t) \) reaches \( \infty \) in a finite time.
Let’s call this the “blow-up” time, \( t_\ast (\epsilon) \). Determine the function \( t_\ast (\epsilon) \). (iii) Use a phase-line analysis to show that the solution of (7.74) never reaches \( \infty \) — in fact:

\[
\lim_{t \to \infty} x(t; \epsilon) = 1.
\]  

(7.76)

(iv) Use separation of variables to find the exact solution of (7.74); make sure your solution satisfies the initial condition. (I encourage you to do the integral with Mathematica or Maple.)

(v) At large times \( x(t, \epsilon) \), is somewhere close to 1. Simplify the exact solution from (iv) to obtain an explicit (i.e., exhibit \( x \) as a function of \( t \)) large-time solution. Make sure sure you explain how large \( t \) must be to ensure that this approximate solution is valid. (vi) Summarize your investigation with a figure such as 7.11.

**Problem 7.7.** Consider the differential equation

\[
\dot{x} = r - x - e^{-x}.
\]  

(7.77)

Sketch all the qualitatively different vector fields on the \( x \)-axis that occur as the parameter \( r \) is varied between \(-\infty\) and \(+\infty\). Show that something interesting happens as \( r \) passes through one. Suppose \( r = 1 + \epsilon \), with \( 0 < \epsilon \ll 1 \). Determine the location of the fixed points as a function of \( \epsilon \) and decide their stability. Obtain an approximation to the differential equation (7.77), valid in the limit \( \epsilon \to 0 \) and \( x = \text{ord}(\sqrt{\epsilon}) \). (Make sure you explain why \( x = \text{ord}(\sqrt{\epsilon}) \) is interesting.)

**Problem 7.8.** Kermack & McKendrick [Proc. Roy. Soc. A 115 A, 700 (1927)] proposed a model for the evolution of an epidemic. The population is divided into three classes:

- \( x(t) = \) number of healthy people,
- \( y(t) = \) number of infected people,
- \( z(t) = \) number of dead people.

Assume that the epidemic evolves very rapidly so that slow changes due to births, emigration, and the 'background death rate', are negligible. (Kermack & McKendrick argue that bubonic plague is so virulent that this assumption is valid.) The other model assumptions are that healthy people get sick at a rate proportional to the product of \( x \) and \( y \). This is plausible if healthy people and sick people encounter each other at a rate proportional to their numbers, and if there is a constant probability of transmission. Sick people die at a constant rate. Thus, the model is

\[
\dot{x} = -\alpha xy, \quad \dot{y} = \alpha xy - \beta y, \quad \dot{z} = \beta y.
\]
(i) Show that $N = x + y + z$ is constant. (ii) Use the $\dot{x}$ and $\dot{z}$ equations to express $x(t)$ in terms of $z(t)$. (iii) Show that $z(t)$ satisfies first order equation:

$$\dot{z} = \beta \left[ N - z - x_0 \exp(-\alpha z/\beta) \right]$$

where $x_0 = x(0)$. Use non-dimensionalization to put the equation above into the form:

$$u_\tau = a - bu - e^{-u},$$

and show that $a \geq 1$ and $b > 0$. (iv) Determine the number of fixed points and decide their stability. (v) Show that if $b < 1$, then the death rate, $\dot{z} \propto u_\tau$, is increasing at $t = 0$ and reaches its maximum at some time $0 < t_\ast < \infty$. Show that the number of infectives, $y(t)$, reaches its maximum at the same time, $t_\ast$, that the death rate peaks. The term epidemic is reserved for this case in which things get worse before they get better. (vi) Show that if $b > 1$ then the maximum value of the death rate is at $t = 0$. Thus, there is no epidemic if $b > 1$. (vii) The condition that $b = 1$ is the threshold for the epidemic. Can you give a biological interpretation of this condition? That is, does the dependence of $b$ on $\alpha$, $\beta$, and $x_0$ seem ‘reasonable’?

**Problem 7.9.** How is the Kermack-McKendrick model modified if the infected people are flesh-eating zombies?

**Problem 7.10.** Integrate the system (7.4) and show that the closed orbits in Figure 7.4 are given by

$$r + f - \ln(rf) = \text{constant} \quad (7.78)$$

**Problem 7.11.** As a model of competition (for grass) between rabbits and sheep consider the autonomous system

$$\dot{r} = r(3 - r - 2s) \quad \text{and} \quad \dot{s} = s(2 - r - s). \quad (7.79)$$

In the absence of one species, the population of the other species is governed by a logistic model. The competition is interesting because rabbits reproduce faster than sheep. But sheep can gently nudge rabbits out of the way, so the negative sheep-feedback is stronger on the rabbits than visa versa. Find the four fixed points of this system and analyze their stability. Compute some solutions and draw a phase-space figure analogous to Figure 7.4.

**Problem 7.12.** The red army, with strength $R(t)$, fights the green army, with strength $G(t)$. The conflict starts from an initial condition $G(0) = 2R(0)$ and proceeds according to

$$\dot{R} = -G, \quad \dot{G} = -3R. \quad (7.80)$$

The war stops when one army is extinct. Which army wins, and how many soldiers at left at this time? (You can solve this problem without solving a differential equation.)

**Problem 7.13.** Consider the system

$$\dot{x} = -x + y^2, \quad \dot{y} = x - 2y + y^2 \quad (7.81)$$

Use MATLAB to compute a few orbits visualize the direction field. Locate the fixed points and analyze their stability. Sketch the orbits near the fixed points. Show that $x = y$ is an orbit and that $|x - y| \to 0$ as $t \to \infty$ for all other orbits.
Figure 7.12: Solution of problem 7.15. The dashed curve is the envelope predicted by multiple scale theory and the solid curve is the ode45 solution.

**Problem 7.14.** A theoretically inclined vandal wants to break a steam radiator away from its foundation. She steadily applies a force of $F = 100$ Newtons and discovers that the top of the radiator is displaced by 2 cm. Unfortunately this is only one tenth of the displacement required. But the vandal can apply an unsteady force $f(t)$ according to the schedule

$$f(t) = \frac{1}{2}F(1 - \cos \omega t), \quad F = 100N.$$

The mass of the radiator is 50 kilograms and the foundation resists movement with a force proportional to displacement. At what frequency and for how long must the vandal exert the force above to succeed?

**Problem 7.15.** Consider the nonlinearly damped oscillator

$$\ddot{x} + \epsilon \dot{x}^3 + x = 0, \quad \text{with ICs} \quad x(0) = 1, \quad \dot{x}(0) = 0. \quad (7.82)$$

Assuming that $\epsilon \ll 1$, use the energy equation and the method of averaging to determine the slow evolution of the amplitude $a$ in the approximate solution (7.45). Take $\epsilon = 1$ and use ode45 to compare a numerical solution of the cubically damped oscillator with the method of averaging (see Figure 7.12).

**Problem 7.16.** Consider a medium $-\ell < x < \ell$ in which the temperature $\theta(x, t)$ is determined by

$$\theta_t - \kappa \theta_{xx} = \alpha e^{\beta \theta}, \quad (7.83)$$

with boundary conditions $\theta(\pm \ell, t) = 0$. The right hand side is a heat source due to an exothermic chemical reaction. The simple form in (7.83) is obtained by linearizing the Arrhenius law. The medium is cooled by the cold walls at $x = \pm \ell$. (i) Put the problem into the non-dimensional form

$$\Theta_T - \Theta_{XX} = \epsilon e^\Theta \quad \text{with BCs} \quad \Theta(\pm 1, \epsilon) = 0. \quad (7.84)$$

Your answer should include a definition of the dimensionless control parameter $\epsilon$ in terms of $\kappa$, $\alpha$, $\beta$ and $\ell$. (ii) Assuming that $\epsilon \ll 1$, calculate the *steady* solution $\Theta(X, \epsilon)$ using a regular perturbation expansion. Obtain two or three non-zero terms and check your answer.
by showing that the “central temperature” is

\[ C(\epsilon) \overset{\text{def}}{=} \Theta(0, \epsilon), \quad (7.85) \]

\[ = \frac{\epsilon}{2} + \frac{5\epsilon^2}{24} + \frac{47\epsilon^3}{360} + \text{ord}(\epsilon^4). \quad (7.86) \]

(iii) Develop an approximate solution with iteration. (iv) Integrate the steady version of (7.84) exactly and deduce that:

\[ e^{-C/2} \tanh^{-1} \sqrt{1 - e^{-C}} = \frac{\epsilon}{\sqrt{2}}. \quad (7.87) \]

(Use Mathematica to do the integral.) Plot the function \( F(C) \) and show that there is no steady solution if \( \epsilon > 0.878 \). (v) Based on the graph of \( F(C) \), if \( \epsilon < 0.878 \) then there are two solutions. There is the “cold solution”, calculated perturbatively in (7.86), and there is a second “hot solution” with a large central temperature. Find an asymptotic expression for the hot central temperature as \( \epsilon \to 0 \).

**Problem 7.17.** Consider an oscillator with a slowly changing frequency \( \omega(\epsilon t) \):

\[ \ddot{x} + \omega^2 x = 0. \quad (7.88) \]

Use the method of averaging to show that the action \( A \overset{\text{def}}{=} E/\omega \) is approximately constant. Test this result with ode45 using the frequency

\[ \omega(t) = 3 + 2 \tanh(\epsilon t), \quad (7.89) \]
and the initial condition $x(-40) = 0$ and $\dot{x}(-40) = 1$ e.g., see Figure 7.13. Use several values of $\epsilon$ to test action conservation e.g., try to break the constant-action approximation with large $\epsilon$.

**Problem 7.18.** The nonlinear oscillator

$$\ddot{x} + x - 2x^2 + x^3 = 0, \quad (7.90)$$

has an energy integral of the form

$$E = \frac{1}{2} \dot{x}^2 + V(x). \quad (7.91)$$

**(a)** Find the potential function $V(x)$ and sketch this function on the range $-\frac{1}{2} < x < 2$. Label your axes so that your sketch of $V(x)$ is quantitative. **(b)** Figure 7.14 shows three possible phase plane diagrams. In ten or twenty words explain which diagram corresponds to the oscillator in (7.90).

**Problem 7.19.** The top panel in figure 7.15 shows a potential and the bottom panel shows four constant energy curves in the phase plane. Match the curves in the bottom panel to the indicated energy levels.

**Problem 7.20.** In section 2.2 we encountered the concentration $u(x, \beta)$ defined by boundary value problem

$$u_{xx} = \beta u^2, \quad \text{with BCs} \quad u(\pm 1) = 1, \quad (7.92)$$

Let $c(\beta) \overset{\text{def}}{=} u(0, \beta)$ be the concentration at the center of the domain. Show that

$$\sqrt{\frac{2\beta c}{3}} = \int_1^{c^{-1}} \frac{du}{\sqrt{u^3 - 1}}. \quad (7.93)$$
Determine $c$ as $\beta \to \infty$. 

Figure 7.15: Match the curves to the energy level.
Lecture 8

Multiple scale theory

8.1 Introduction to two-timing

In the previous lecture we solved the damped oscillator equation

$$\frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + x = 0, \quad (8.1)$$

with initial conditions

$$x(0) = 0, \quad \text{and} \quad \frac{dx}{dt}(0) = 1. \quad (8.2)$$

You should recall that the exact solution is

$$x = \nu^{-1} e^{-\beta t/2} \sin vt, \quad \text{with} \quad \nu \overset{\text{def}}{=} \sqrt{1 - \frac{\beta^2}{4}}. \quad (8.3)$$

A good or useful $\beta \ll 1$ approximation to this exact solution is

$$x \approx e^{-\beta t/2} \sin t. \quad (8.4)$$

Let’s use this example to motivate the multiple-scale method.

Failure of the regular perturbation expansion

If $\beta \ll 1$ we might be tempted to try an RPS on (8.1):

$$x(t, \beta) = x_0(t) + \beta x_1(t) + \beta^2 x_2(t) + \cdots \quad (8.5)$$

A reasonable goal is to produce the good approximation (8.4). The RPS will not be successful and this failure will drive us towards the method of multiple time scales, also known as “two timing”.

The leading-order problem is

$$\ddot{x}_0 + x_0 = 0, \quad \text{with IC} \quad x_0 = 0, \quad \dot{x}_0(0) = 1. \quad (8.6)$$

The solution is

$$x_0 = \sin t. \quad (8.7)$$

The first-order problem is

$$\frac{d^2x_1}{dt^2} + x_1 = -\cos t, \quad \text{with IC} \quad \dot{x}_1(0) = 0, \quad \frac{dx_1}{dt}(0) = 0. \quad (8.8)$$
This is a resonantly forced oscillator equation, with solution

\[ x_1 = -\frac{t}{2} \sin t. \]  

Thus the developing RPS is

\[ x(t, \beta) = \sin t - \frac{\beta t}{2} \sin t + \beta^2 x_2(t) + \cdots \]  

At this point we recognize that the RPS is misleading: the exact solution damps to zero on a time scale \( 2/\beta \), while the RPS suggests that the solution is growing linearly with time. With hindsight we realize that the RPS is producing the Taylor series expansion of the exact solution in (7.42) about \( \beta = 0 \). Using \textsc{mathematica}, this series is

\[ x(t, \beta) = \sin t - \frac{\beta t}{2} \sin t + \frac{\beta^2}{8} \left[ t^2 \sin t + \sin t - t \cos t \right] + O(\beta^3). \]  

Calculating more terms in the RPS will not move us closer to the useful approximation in (8.4); instead we’ll grind out the useless approximation in (8.11). In this example the small term in (7.40) is small relative to the other terms at all times. Yet the small error slowly accumulates over long times \( \sim \beta^{-1} \). This is a secular error.

Two-timing

Looking at the good approximation in (8.4) we are inspired to introduce a slow time:

\[ s \overset{\text{def}}{=} \beta t. \]  

We assume that \( x(t, \beta) \) has a perturbation expansion of the form

\[ x(t, \beta) = x_0(t, s) + \beta x_1(t, s) + \beta^2 x_2(t, s) + \cdots \]  

Notice how this differs from the RPS in (8.5).

At each order \( x_n \) is a function of both \( s \) and \( t \) a function of both \( t \) and \( s \). To keep track of all the terms we use the rule

\[ \frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + \beta \partial_s, \]  

Figure 8.1: Comparison of the exact solution in (8.3) (the solid black curve), with the two-term RPS in (8.10) (the blue dotted curve) and the two-time approximation in (8.23) (the dashed red curve). It is difficult to distinguish the two-time approximation from the exact result.
and the equation of motion is
\[(\partial_t + \beta \partial_s)^2 x + \beta (\partial_t + \beta \partial_s) x + x = 0.\] (8.15)

At leading order

\[\beta^0 : \quad \partial_t^2 x_0 + x_0 = 0, \quad \text{with general solution} \quad x_0 = A(s) e^{it} + A^*(s) e^{-it}.\] (8.16)

Notice that the “constant of integration” is actually a function of the slow time \(s\). We determine the evolution of this function \(A(s)\) at next order \(1\).

At next order

\[\beta^1 : \quad \partial_t^2 x_1 + x_1 = -2x_{0s} - x_{0t},\] (8.18)
\[= -2iA_s e^{it} - iA e^{it} + c.c.\] (8.19)

Again we have a resonantly forced oscillator. but this time we can prevent the secular growth of \(x_1\) on the fast time scale by requiring that

\[2A_s + A = 0.\] (8.20)

Thus the leading-order solution is

\[x_0(s, t) = A_0 e^{-s/2} e^{it} + A_0^* e^{-s/2} e^{-it}.\] (8.21)

The constant of integration \(A_0\) is determined to satisfy the initial conditions. This requires

\[0 = A_0 + A_0^*, \quad 1 = iA_0 - iA_0^*, \quad \Rightarrow \quad A_0 = \frac{1}{2} i.\] (8.22)

Thus we have obtained the good approximation

\[x_0 = e^{-\beta t/2} \sin t.\] (8.23)

### 8.2 The Duffing oscillator

We consider an oscillator with a nonlinear spring

\[m \ddot{x} + k_1 x + k_3 x^3 = 0,\] (8.24)

and an initial condition

\[x(0) = x_0, \quad \dot{x}(0) = 0.\] (8.25)

If \(k_3 > 0\) then the restoring force is stronger than linear — this is a **stiff spring**.

We can non-dimensionalize this problem into the form

\[\ddot{x} + x + \epsilon x^3 = 0,\] (8.26)

with the initial condition

\[x(0) = 1, \quad \dot{x}(0) = 0.\] (8.27)

---

\(^1\)We could alternatively write the general solution of the leading order problem as

\[x_0 = R \cos(t + \phi),\] (8.17)

where the amplitude \(R\) and the phase \(\phi\) are as yet undetermined functions of \(s\). I think the complex notation in (8.16) is a little simpler.
We use this Duffing oscillator as an introductory example of the multiple time scale method.

Energy conservation,
\[
\frac{1}{2} \dot{x}^2 + \frac{1}{2} x^2 + \frac{1}{4} \epsilon x^4 = \frac{E}{\frac{1}{2} + \frac{1}{4} \epsilon},
\] (8.28)
immediately provides a phase-plane visualization of the solution and shows that the oscillations are bounded.

Exercise: Show that in (8.26), \( \epsilon = k_3 x_0^3 / k_1 \).

Exercise: Derive (8.28).

The naive RPS
\[
x = x_0(t) + \epsilon x_1(t) + \cdots
\] (8.29)
leads to
\[
\dot{x}_0 + x_0 = 0, \quad \Rightarrow \quad x_0 = \cos t,
\] (8.30)
and at next order
\[
\ddot{x}_1 + x_1 = -\cos^3 t, \quad \Rightarrow \quad \dot{x}_1 + x_1 = -\frac{1}{8} (e^{3it} + 3e^{it} + c.c.),
\] (8.31)
\[
\dot{x}_1 + x_1 = -\frac{1}{4} \cos 3t - \frac{3}{4} \cos t.
\] (8.32)
The \( x_1 \)-oscillator problem is resonantly forced and the solution will grow secularly, with \( x_1 \propto t \sin t \). Thus the RPS fails once \( t \sim \epsilon^{-1} \).

Two-timing

Instead of an RPS we use the two-time expansion
\[
x = x_0(s,t) + \epsilon x_1(s,t) + \cdots
\] (8.34)
where \( s = \epsilon t \) is the slow time. Thus the expanded version of (8.26) is
\[
(\partial_t + \epsilon \partial_s)^2 (x_0(s,t) + \epsilon x_1(s,t) + \cdots) + (x_0(s,t) + \epsilon x_1(s,t) + \cdots) + \epsilon (x_0(s,t) + \epsilon x_1(s,t) + \cdots)^3 = 0
\] (8.35)
The leading order is
\[
\partial^2_s x_0 + x_0 = 0,
\] (8.36)
with general solution
\[
x_0 = A(s)e^{it} + A^*(s)e^{-it}.
\] (8.37)
The amplitude \( A \) is a function of the slow time \( s \). At next order, \( \epsilon^1 \), we have
\[
\partial^2_t x_1 + x_1 = -2 \partial_t \partial_s x_0 - x_0^3,
\] (8.38)
\[
= -2iA_s e^{it} - A^3 e^{3it} - 3A^2 A^* e^{it} + c.c.
\] (8.39)
To prevent the secular growth of \( x_1 \) we must remove the resonant terms, \( e^{\pm it} \) on the right of (8.39) — this prescription determines the evolution of the slow time:
\[
2iA_s + 3|A|^2 A = 0.
\] (8.40)
The remaining terms in (8.39) are
\[
\partial^2_t x_1 + x_1 = -A^3 e^{3it} + c.c.
\] (8.41)
The solution will be \( x_1 \propto e^{\pm 3it} \) and will remain bounded.
Polar coordinates

To solve (8.40) it is best to transform to polar coordinates

\[ A = r(s) e^{i\theta(s)} \quad \text{and} \quad A_s = (rs + ir \theta_s) e^{i\theta}; \quad (8.42) \]

substituting into the amplitude equation (8.40)

\[ r_s = 0, \quad \text{and} \quad \theta_s = \frac{3}{2} r^2. \quad (8.43) \]

The energy of this nonlinear oscillator is constant and thus \( r \) is constant, \( r(s) = r_0 \). The phase \( \theta(s) \) therefore evolves as \( \theta = \theta_0 + 3r_0^2 s/2 \).

The reconstituted solution is

\[ x = r_0 \exp \left[ i \left( 1 + \frac{3}{2} \epsilon r_0^2 \right) t + i \theta_0 \right] + \text{c.c.} + \text{ord}(\epsilon). \quad (8.44) \]

The velocity of the oscillator is

\[ \frac{dx}{dt} = ir_0 \exp \left[ i \left( 1 + \frac{3}{2} \epsilon r_0^2 \right) t + i \theta_0 \right] + \text{c.c.} + \text{ord}(\epsilon). \quad (8.45) \]

To satisfy the initial condition in (8.27) at leading order, we take \( \theta_0 = 0 \) and \( r_0 = 1/2 \). Thus, with this particular initial condition,

\[ x = \cos \left[ \left( 1 + \frac{3}{8} \epsilon \right) t \right] + \text{ord}(\epsilon). \quad (8.46) \]

The frequency of the oscillator in (8.44),

\[ \nu = 1 + \frac{3}{2} \epsilon r_0^2, \quad (8.47) \]

depends on the amplitude \( r_0 \) and the sign of \( \epsilon \). If the spring is stiff (i.e., \( k_3 > 0 \)) then \( \epsilon \) is positive and bigger oscillations have higher frequency.

Higher-order corrections to the frequency

8.3 The quadratic oscillator

The quadratic oscillator is

\[ \ddot{x} + x + \epsilon x^2 = 0. \quad (8.48) \]

The conserved energy is

\[ E = \frac{1}{2} \dot{x}^2 + \frac{1}{2} x^2 + \frac{1}{3} \epsilon x^3, \quad (8.49) \]

and the curves of constant energy in the phase plane are shown in Figure 8.2.

Following our experience with the Duffing oscillator we try the two-time expansion

\[ (\partial_t + \epsilon \partial_s)^2 (x_0(s,t) + \epsilon x_1(s,t) + \cdots) + (x_0(s,t) + \epsilon x_1(s,t) + \cdots) + \epsilon (x_0(s,t) + \epsilon x_1(s,t) + \cdots)^2 = 0. \quad (8.50) \]

The leading order solution is again

\[ x_0 = A(s)e^{it} + A^*(s)e^{-it}, \quad (8.51) \]
and at next order
\[ \partial_t^2 x_1 + x_1 = -2 \left( iA_s e^{it} - iA_s e^{-it} \right) - \frac{\left( A^2 e^{2it} + 2|A|^2 + A^* e^{-2it} \right)}{x_0^2} . \] (8.52)

Elimination of the resonant terms \( e^{\pm it} \) requires simply \( A_s = 0 \), and then the solution of the remaining equation is
\[ x_1 = \frac{1}{3} A^2 e^{2it} - 2|A|^2 + \frac{1}{3} A^* e^{-2it} . \] (8.53)

This is why the quadratic oscillator is not used as an introductory example: there is no secular forcing at order \( \epsilon \).

To see the effects of nonlinearity in the quadratic oscillator, we must press on to higher orders, and use a slower slow time:
\[ s = \epsilon^2 t . \] (8.54)

Thus we revise (8.50) to
\[ \left( \partial_t + \epsilon^2 \partial_s \right)^2 \left( x_0(s, t) + \epsilon x_1(s, t) + \cdots \right) + \left( x_0(s, t) + \epsilon x_1(s, t) + \cdots \right) \]
\[ + \epsilon \left( x_0(s, t) + \epsilon x_1(s, t) + \cdots \right)^2 = 0 . \] (8.55)

The solutions at the first two orders are the same as (8.51) and (8.53). At order \( \epsilon^2 \) we have
\[ \partial_t^2 x_2 + x_2 = -2 \partial_t \partial_s x_0 + 2x_0\partial_s x_1 , \] (8.56)
\[ = -2 \left( iA_s e^{it} - iA_s^* e^{-it} \right) - 2 \left( A e^{it} + A^* e^{-it} \right) \left( \frac{1}{3} A^2 e^{2it} - 2|A|^2 + \frac{1}{3} A^* e^{-2it} \right) \] (8.57)

Eliminating the \( e^{\pm it} \) resonant terms produces the amplitude equation
\[ iA_s = \frac{5}{3} |A|^2 A . \] (8.58)
Although the nonlinearity in (8.48) is quadratic, the final amplitude equation in (8.58) is cubic. In fact, despite the difference in the original nonlinear term, the amplitude equation in (8.58) is essentially the same as that of the Duffing oscillator in (8.40).

**Example:** the Morse oscillator. Using dimensional variables, the Morse oscillator is

\[ \ddot{x} + \frac{dV}{dx} = 0 \quad \text{with the potential} \quad V = \frac{v}{2} (1 - e^{-\alpha x})^2. \]  

(8.59)

The phase plane is shown in figure ?? — the orbits are curves of constant energy

\[ E = \frac{1}{2} x^2 + \frac{v}{2} (1 - e^{-\alpha x})^2. \]  

(8.60)

There is a turning point at \( x = \infty \) corresponding to the “energy of escape” \( E_{\text{escape}} = v/2 \).

A “natural” choice of non-dimensional variables is

\[ \bar{x} \overset{\text{def}}{=} \alpha x, \quad \text{and} \quad \bar{t} = \alpha \sqrt{\nu} t. \]  

(8.61)

In these variables, \( \nu \to 1 \) and \( \alpha \to 1 \) in the barred equations. Thus, the non-dimensional equation of motion is

\[ \ddot{\bar{x}} + e^{-\bar{x}} (1 - e^{-\bar{x}}) = 0. \]  

(8.62)

If we’re interested in small oscillations around the minimum of the potential at \( x = 0 \), then the small parameter is supplied by an initial condition such as

\[ x(0) = \epsilon, \quad \text{and} \quad \dot{x}(0) = 0. \]  

(8.63)

We rescale with

\[ x = \epsilon X, \]  

(8.64)

so that the equation is

\[ \epsilon \ddot{X} + e^{-\epsilon x} \left(1 - e^{-\epsilon x}\right) = 0, \]  

(8.65)

or

\[ \ddot{X} + X - \frac{3}{2} \epsilon^2 X^2 + \frac{7}{6} \epsilon^3 X^3 = \text{ord} \left(\epsilon^4\right). \]  

(8.66)

The multiple time scale expansion is now

\[ X = X_0(s, t) + \epsilon X_1(s, t) + \epsilon^2 X_2(s, t) + \cdots \]  

(8.67)

with slow time \( s = \epsilon^2 t \).

The main point of this example is that it is necessary to proceed to order \( \epsilon^2 \), and therefore to retain the term \( 7\epsilon^2 X^3/6 \), to obtain the amplitude equation. One finds

\[ iA_t = \]  

(8.68)

### 8.4 Symmetry and the universality of the Landau equation

So far the two-time expansion always leads to the Landau equation

\[ A_s = pA + q|A|^2A. \]  

(8.69)

If you dutifully solve some of the early problems in this lecture you’ll obtain (8.69) again and again. Why is that? If we simply list all the terms up to cubic order that *might* occur in an amplitude equation we have


(8.70)

The coefficients are denoted by “?” and we’re not interested in the precise value of these numbers, except in so far as most of them turn out to be zero. The answer in (8.69) is simple because we have two terms on the right, instead of the nine in (8.70). We’ve been down in the
weeds calculating, but we have not asked the big question why do we have to calculate only the two coefficients $p$ and $q$?

We have been considering only autonomous differential equations, such as

$$\frac{dx}{dt} + x + \varepsilon x^5 = 0.$$  

(8.71)

This means that if $x(t)$ is a solution of (8.71) then so is $x(t - \alpha)$, where $\alpha$ is any constant. In other words, the equations we’ve been considering are unchanged (“invariant”) if

$$t \to t + \alpha.$$  

(8.72)

Now if we try to solve (8.69) with a solution of the form

$$x(t) = A(s)e^{it} + A^*(s)e^{-it} + \varepsilon x_1(t, s) + \cdots$$  

(8.73)

then

$$x(t + \alpha) = A e^{i\alpha} e^{it} + A^* e^{-i\alpha} e^{-it} + \varepsilon x_1 + \cdots$$  

(8.74)

Thus the time-translation symmetry of the original differential equation implies that the amplitude equation should be invariant under the rotation

$$A \to A e^{i\alpha},$$  

(8.75)

where $\alpha$ is any constant. Only the underlined terms in (8.70) respect this symmetry and therefore only the underlined terms can appear in the amplitude equation.

Exercise: many of our examples have time reversal symmetry i.e., the equation is invariant under $t \to -t$. For example, the nonlinear oscillator (with no damping) is invariant under $t \to -t$. Show that this implies that $p$ and $q$ in (8.69) must be pure imaginary.

8.5 The resonantly forced Duffing oscillator

The linear oscillator

First consider the forced linear oscillator

$$\ddot{x} + \mu \dot{x} + \omega^2 x = f \cos \sigma t.$$  

(8.76)

We can find the “steady solution” with

$$x = X e^{i\sigma t} + X^* e^{-i\sigma t}$$  

(8.77)

After some easy algebra

$$X = \frac{f}{2} \frac{1}{\omega^2 - \sigma^2 + i\mu\omega},$$  

(8.78)

and the squared amplitude of the response is

$$|X|^2 = \frac{f^2}{4} \frac{1}{(\omega^2 - \sigma^2)^2 + \mu^2 \omega^2}.$$  

(8.79)

We view $|X|^2$ as a function of the forcing frequency $\sigma$ and notice there is a maximum at $\sigma = \omega$ i.e., when the oscillator is resonantly forced. The maximum response, namely

$$\max_{\forall \sigma} |X| = \frac{f}{2\mu\omega},$$  

(8.80)
is limited by the damping \( \mu \).

In the neighbourhood of this peak the amplitude in (8.79) can approximated by the Lorentzian
\[
|X|^2 \approx \frac{f^2}{4\omega^2} \frac{1}{4(\omega - \sigma)^2 + \mu^2}.
\]
(8.81)
The difference between \( \omega \) and \( \sigma \) is de-tuning.

**Nondimensionalization of the nonlinear oscillator**

Now consider the forced and damped Duffing oscillator:
\[
\ddot{x} + \mu \dot{x} + \omega^2 x + \beta x^3 = f \cos \sigma t.
\]
(8.82)
We’re interested in the weakly damped and nearly resonant problem. That is \( \mu/\omega \) is small and \( \sigma \) is close to \( \omega \). Inspired by the linear solution we define non-dimensional variables
\[
\bar{t} = \omega t, \quad \text{and} \quad \bar{x} = \frac{\mu \omega x}{f}.
\]
(8.83)
The non-dimensional equation is then
\[
\dddot{x} + \epsilon \ddot{x} + x + \epsilon \beta_1 x^3 = \epsilon \cos \left[ (1 + \epsilon \sigma_1) \bar{t} \right].
\]
(8.84)
The non-dimensional parameters above are
\[
\epsilon \overset{\text{def}}{=} \frac{\mu}{\omega} \ll 1,
\]
(8.85)
and
\[
\beta_1 \overset{\text{def}}{=} \frac{\beta f^2}{\mu^2 \omega^3}, \quad \sigma_1 \overset{\text{def}}{=} \frac{1}{\epsilon} \left( \frac{\sigma}{\omega} - 1 \right) = \frac{\sigma - \omega}{\mu}.
\]
(8.86)
We refer to \( \sigma_1 \) as the “detuning”.

Notice how we have used the solution of the linear problem to make a non-obvious definition of non-dimensional variables. We now take the *distinguished limit* \( \epsilon \to 0 \) with \( \beta_1 \) and \( \sigma_1 \) fixed.

**The amplitude equation and its solution**

We attack (8.84) with our multiple-scale expansion
\[
x = A(s)e^{it} + A^*(s)e^{-it} + \epsilon x_1(t, s) + \cdots
\]
(8.87)
The amplitude equation that emerges at order \( \epsilon^1 \) is
\[
A_s + \frac{1}{2}A - \frac{3i}{2} \beta_1 |A|^2 A = -\frac{i}{4} e^{i\sigma_1 s}.
\]
(8.88)
We can remove the \( s \)-dependence by going to a rotating frame
\[
A(s) = B(s)e^{i\sigma_1 s}.
\]
(8.89)
In terms of \( B \) we have
\[
B_s + \frac{1}{2}B + i \left[ \sigma_1 - \frac{3}{2} \beta_1 |B|^2 \right] B = -\frac{i}{4}.
\]
(8.90)
The term in the square brackets is an amplitude-dependent frequency.

Now look for steady solutions — we find

\[
|B|^2 = \frac{1}{4} \frac{1}{1 + (2\sigma_1 - 3\beta_1 |B|^2)^2}, \tag{8.91}
\]

If we set \(\beta_1 = 0\) we recover a non-dimensional version of our earlier Lorentzian approximation to the response curve of a linear oscillator. With non-zero \(\beta_1\) we might have to solve a cubic equation to determine the steady state amplitude: see figure 8.3. There are "multiple solutions" i.e., for the same detuning \(\sigma_1\) there as many as three solutions for \(|B|^2\). The middle branch is unstable — the system ends up on either the lower or upper branch, depending on initial conditions. Figure 8.4 illustrates the two different attracting solutions.

Notice that the scaled problem has three non-dimensional parameters, \(\epsilon, \beta_1\) and \(\sigma_1\). But in (8.88) only \(\beta_1\) and \(\sigma_1\) appear. (Of course \(\epsilon\) is hidden in the definition of the slow time \(s\).) These perturbation expansions are called reductive because they reduce the number of non-dimensional parameters by taking a distinguished limit.
Figure 8.4: Energy $E = (x^2 + \omega^2 x^2)/2 + \beta x^4/4$ as a function of time for five ode45 solutions of the forced Duffing equation (8.82) differing only in initial conditions. There is a high energy attractor that collects two of the solutions, and a low energy attractor that gets the other three solutions. The MATLAB code is below. Note how the differential equation is defined in the nested function oscill so that the parameters $\omega$, $\mu$ defined in the main function ForcedDuffing are passed.

```matlab
function ForcedDuffing
% Multiple solutions of the forced Duffing equation
% Slightly different initial conditions fall on different limit cycles
% tspan = [0 300]; $\omega$ =1; $\mu$ =0.05; $\beta$ = 0.1; $f$ = 0.25;
sig = 1.2*$\omega$; $y_{init}$ = [0 1 1.0188 1.0189 2];
for n=1:length($y_{init}$)
    $y_0$=[$y_{init}$(n) 0];
    [t,$y$] = ode45(@oscill,tspan,$y_0$);
    %Use the linear energy $E$ as an index of amplitude
    $E$ = 0.5*( $\omega$*$\omega$* $y(:,1).^2$ + 0.5*$\beta$*$y(:,1).^4$ + $y(:,2).^2$ );
    subplot(2,1,1) plot(t,$E(:)$)
    xlabel('$t$','interpreter','latex','fontsize',16)
    ylabel('$E(t)$','interpreter','latex','fontsize',16)
    hold on
end
%------------- nested function --------------------%
function dydt = oscill(t,$y$)
    dydt = zeros(2,1);
    dydt(1) = $y(2)$;
    dydt(2) = - $\mu$*$y(2) - \omega^2*y(1) - \beta*y(1)^3 + f*cos( sig*t );
end
end
```
8.6 Problems

Problem 8.1. In an early lecture we compared the exact solution of the initial value problem
\[ \ddot{f} + (1 + \epsilon)f = 0, \quad \text{with ICs} \quad f(0) = 1, \quad \text{and} \quad \dot{f}(0) = 0, \quad (8.92) \]
with an approximation based on a regular perturbation expansion — see the discussion surrounding (2.50). Redo this problem with a two-time expansion. Compare your answer with the exact solution and explain the limitations of the two-time expansion.

Problem 8.2. Consider
\[ \frac{d^2 g}{dt^2} + \left[ 1 + \epsilon \left( \frac{d g}{dt} \right)^2 \right] g = 0, \quad \text{with ICs} \quad g(0) = 1, \quad \text{and} \quad \frac{dg}{dt}(0) = 0. \quad (8.93) \]
(i) Show that a RPS fails once \( t \sim \epsilon^{-1} \). (ii) Use the two-timing method to obtain the solution on the long time scale.

Problem 8.3. Consider the initial value problem:
\[ \frac{d^2 u}{dt^2} + u = 2 + 2\epsilon u^2, \quad \text{with ICs} \quad u(0) = \frac{du}{dt}(0) = 0. \quad (8.94) \]
(i) Supposing that \( \epsilon \ll 1 \), use the method of multiple time scales \( (s = \epsilon t) \) to obtain an approximate solution valid on times of order \( \epsilon^{-1} \). (ii) Consider
\[ \frac{d^2 v}{dt^2} + v = u, \quad \text{with ICs} \quad v(0) = \frac{dv}{dt}(0) = 0, \quad (8.95) \]
where \( u(t, \epsilon) \) on the right is the solution from part (i). Find a leading-order approximation to \( v(t, \epsilon) \), valid on the long time scale \( t \sim \epsilon^{-1} \).

Problem 8.4. Consider the initial value problem:
\[ \frac{d^2 w}{dt^2} + w = 2 \cos(\epsilon t) + 2\epsilon w^2, \quad \text{with ICs} \quad w(0) = \frac{dw}{dt}(0) = 0. \quad (8.96) \]
Supposing that \( \epsilon \ll 1 \), use the method of multiple time scales \( (s = \epsilon t) \) to obtain an approximate solution valid on times of order \( \epsilon^{-1} \).

Problem 8.5. Use multiple scale theory to find an approximate solution of
\[ \frac{d^2 x}{dt^2} + x = e^{\epsilon^2 t} + \epsilon e^{-\epsilon t} x^2, \quad \text{with ICs} \quad x(0) = \frac{dx}{dt}(0) = 0, \quad (8.97) \]
valid on the time scale \( t \sim \epsilon^{-1} \ll \epsilon^{-2} \).

Problem 8.6. A multiple scale \( (0 < \epsilon \ll 1) \) reduction of the system
\[ \frac{d^2 x}{dt^2} + 2\epsilon y \frac{dx}{dt} + x = 0, \quad \frac{dy}{dt} = \frac{1}{2}\epsilon^2 x^2, \quad (8.98) \]
begins with
\[ x = \left[ A(s) e^{it} + A^*(s) e^{-it} \right] + \epsilon x_1(t, s) + \cdots, \quad y = B(s) + \epsilon y_1(t, s) + \cdots \quad (8.99) \]
where \( s = \epsilon t \) is the slow time. (i) Find coupled evolution equations for \( A(s) \) and \( B(s) \). (ii) Show that the system from part (i) can be reduced to
\[
B_s = E - B^2,
\]
where \( E \) is a constant of integration. (iii) Determine \( E \) if the initial conditions are
\[
x(0) = 2, \quad \frac{dx}{dt}(0) = 0, \quad y(0) = 0.
\]
(iv) With the initial condition above, find
\[
\lim_{s \to \infty} B(s).
\]
Note: even if you can’t do part (ii), you can still answer parts (iii) and (iv) by assuming (8.100).

**Problem 8.7.** (a) Use multiple scales to derive a set of amplitude equations for the two coupled, linear oscillators:
\[
\begin{align*}
\ddot{x} + 2\epsilon\alpha\dot{x} + (1 + k\epsilon)x &= 2\epsilon\mu(x - y), \\
\ddot{y} + 2\epsilon\beta\dot{y} + (1 - k\epsilon)y &= 2\epsilon\mu(y - x).
\end{align*}
\]
(b) Consider the special case \( \alpha = \beta = k = 0 \). Solve both the amplitude equations and the exact equation with the initial condition \( x(0) = 1, y(0) = \dot{y}(0) = \dot{x}(0) = 0 \). Show that both methods give
\[
x(t) \approx \cos[(1 - \epsilon\mu)t] \cos(\epsilon\mu t).
\]

**Problem 8.8.** Consider two nonlinearly coupled oscillators:
\[
\begin{align*}
\ddot{x} + 4x &= \epsilon y^2, \\
\ddot{y} + y &= -\epsilon\alpha xy,
\end{align*}
\]
where \( \epsilon \ll 1 \). (a) Show that the nonlinearly coupled oscillators in (1) have an energy conservation law. (b) The multiple scale method begins with
\[
x(t) = A(s)e^{2it} + \text{c.c.}, \quad y(t) = B(s)e^{it} + \text{c.c.},
\]
where \( s \overset{\text{def}}{=} \epsilon t \) is the “slow time” and \( A \) and \( B \) are “amplitudes”. Find the coupled evolution equations for \( A \) and \( B \) using the method of multiple scales. (c) Show that the amplitude equations have a conservation law
\[
|B|^2 - 2\alpha |A|^2 = E,
\]
and use this result to show that
\[
4A_{ss} - \alpha EA - 2\alpha^2 |A|^2 A = 0.
\]
Obtain the analogous equation for \( B(s) \). (d) Describe the solutions of (8.108) in qualitative terms. Does the sign of \( \alpha \) have a qualitative impact on the solution?

**Problem 8.9.** The equation of motion of a pendulum with length \( \ell \) in a gravitational field \( g \) is
\[
\ddot{\theta} + \omega^2 \sin \theta = 0, \quad \text{with} \quad \omega^2 \overset{\text{def}}{=} \frac{g}{\ell}.
\]
Suppose that the maximum displacement is \( \theta_{\text{max}} = \phi \). (a) Show that the period \( P \) of the oscillation is
\[
\omega P = 2\sqrt{2} \int_0^\phi \frac{d\theta}{\sqrt{\cos \theta - \cos \phi}}.
\]
(b) Suppose that $\phi \ll 1$. By approximating the integral above, obtain the coefficient of $\phi^2$ in the expansion:

$$\omega P = 2\pi \left[ 1 + ? \phi^2 + O(\phi^3) \right]$$

(c) Check this result by re-deriving it via a multiple scale expansion applied to (8.109). (d) A grandfather clock swings to a maximum angle $\phi = 5^\circ$ from the vertical. How many seconds does the clock lose or gain each day if the clock is adjusted to keep perfect time when the swing is $\phi = 2^\circ$?

**Problem 8.10.** (H) Find a leading order approximation to the general solution $x(t, \epsilon)$ and $y(t, \epsilon)$ of the system

$$\frac{d^2 x}{dt^2} + 2\epsilon y \frac{dx}{dt} + x = 0 \quad \text{and} \quad \frac{dy}{dt} = \frac{1}{2}\epsilon \ln x^2,$$

which is valid for $t = \text{ord}(\epsilon^{-1})$. You can quote the result

$$\frac{1}{2\pi} \int_0^{2\pi} \ln \cos^2 \theta \, d\theta = -\ln 4.$$

**Problem 8.11.** (H) Find the leading order approximation, valid for times of order $\epsilon^{-1}$, to the solution $x(t, \epsilon)$ and $y(t, \epsilon)$ of the system

$$\ddot{x} + \epsilon y \dot{x} + x = y^2, \quad \text{and} \quad \dot{y} = \epsilon (1 + x - y - y^2),$$

with initial conditions $x = 1$, $\dot{x} = 0$ and $y = 0$. 

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Lecture 9

Rapid fluctuations

In this lecture we consider some unusual examples of the two-timing method.

9.1 A Lotka-Volterra Example

As another example of multiple scale theory, let’s consider the Lotka-Volterra equation with a sinusoidally varying carrying capacity:

\[
\frac{dn}{dt} = n \left( 1 - \frac{n}{1 + \kappa \cos \omega t} \right). \tag{9.1}
\]

Problem 7.3 asked you to analyze this equation with \( \kappa \ll 1 \) i.e., small fluctuations in the carrying capacity. Here we consider the case of rapid fluctuations: \( \omega \to \infty \), with \( \kappa \) fixed and order unity. In this limit the small parameter is

\[
\epsilon \overset{\text{def}}{=} \frac{1}{\omega}. \tag{9.2}
\]

Figure 9.1 shows a numerical solution: the population fluctuates about an average value which is close to \( \sqrt{9/16} = 0.6614 \). This average population is quite different from the average carrying capacity, namely 1.

Define a fast time

\[
\tau \overset{\text{def}}{=} \omega t = \frac{t}{\epsilon}, \tag{9.3}
\]

and assume that the solution depends on both \( t \) and \( \tau \) so that

\[
\left( \frac{1}{\epsilon} \partial_\tau + \partial_t \right) n = n \left( 1 - \frac{n}{1 + \kappa \cos \tau} \right). \tag{9.4}
\]

Figure 9.1: An ode45 solution of the Lotka-Volterra equation (9.1) with \( \omega = 20 \) and \( \kappa = 3/4 \).
Now we attempt to solve this equation with the multiple time scale expansion

\[ n = n_0(t, \tau) + \epsilon n_1(t, \tau) + \cdots \]  

(9.5)

The leading order is

\[ \frac{1}{\epsilon} : \quad \partial_\tau n_0 = 0, \quad \text{with solution} \quad n_0 = f(t). \]  

(9.6)

Although the carrying capacity is varying rapidly, the leading-order population \( f(t) \) does not vary on the fast time scale \( \tau \). The environment is fluctuating so rapidly that the population does not react e.g., clouds blowing overhead lead to modulations in sunlight on the scale of minutes. But plants don’t die when the sun is momentarily obscured by a cloud.

At next order

\[ \epsilon^0 : \quad \partial_\tau n_1 + \frac{df}{dt} = f \left( 1 - \frac{f}{1 + \kappa \cos \tau} \right). \]  

(9.7)

We average the equation above over the fast time scale to obtain

\[ \frac{df}{dt} = f \left( 1 - \frac{f}{\sqrt{1 - \kappa^2}} \right). \]  

(9.8)

Thus the long time limit is \( f \to \sqrt{1 - \kappa^2} \). This is in agreement with the MATLAB solution in Figure 9.1. To determine the fluctuations about this average we can subtract (9.8) from (9.7) to obtain

\[ \partial_\tau n_1 = \left( \frac{1}{\sqrt{1 - \kappa^2}} - \frac{1}{1 + \kappa \cos \tau} \right) \frac{f^2}{|1 - \kappa^2|} \]  

(9.9)

9.2 Stokes drift

Consider the motion along the \( x \)-axis of a fluid particle in a simple compressive wave e.g., a sound wave. The position of the particle is determined by solving the nonlinear differential equation

\[ \frac{dx}{dt} = u \cos(kx - \omega t), \]  

(9.10)

with an initial condition \( x(0) = a \). We non-dimensionalize this problem by defining

\[ \tilde{x} \overset{\text{def}}{=} kx \quad \text{and} \quad \tilde{t} \overset{\text{def}}{=} \omega t. \]  

(9.11)

The non-dimensional problem is

\[ \frac{d\tilde{x}}{d\tilde{t}} = \epsilon \cos(\tilde{x} - \tilde{t}), \quad \text{with IC} \quad \tilde{x}(0) = \tilde{a}. \]  

(9.12)

The non-dimensional wave amplitude,

\[ \epsilon \overset{\text{def}}{=} \frac{uk}{\omega}, \]  

(9.13)

is the ratio of the maximum particle speed \( u \) to the phase speed \( \omega/k \). We proceed dropping all bars.

\[ ^1 \text{The time average of the reciprocal carrying capacity is computed using:} \]

\[ \frac{1}{2\pi} \int_0^{2\pi} \frac{d\tau}{1 + \kappa \cos \tau} = \frac{1}{\sqrt{1 - \kappa^2}}. \]

This integral is a favorite textbook example of the residue theorem.
Figure 9.2 shows some numerical solutions of (9.12) with $\epsilon = 0.3$. Even though the time-average velocity at a fixed point is zero there is a slow motion of the particles along the $x$-axis with constant average velocity. If one waits long enough then a particle will move very far from its initial position and travel through many wavelengths.

**Method 1: RPS to obtain a one-period map**

Fortunately the velocity is a periodic function of time. Given the initial condition $x(0)$, we use a straightforward RPS to determine the one-period map $F$:

$$x(2\pi) = F[x(0), \epsilon].$$

The position of the particle over many periods follows by iteration of this map. That is

$$x(4\pi) = F[x(2\pi), \epsilon], \quad x(6\pi) = F[x(4\pi), \epsilon], \quad \text{etc.}$$

We don't have to worry about secular errors because $F$ is determined by solving the differential period over a finite time $0 < t < 2\pi$. The second period is just the same as the first.

**Method 2: two-timing**

To analyze this problem with multiple scale theory we introduce

$$s \overset{\text{def}}{=} \epsilon^2 t.$$ (9.16)

Why $\epsilon^2$ above? Because we tried $\epsilon^1$ and found that there were no secular terms on this time scale.

**Exercise:** Assume that $s = \epsilon t$ and repeat the following calculation. Does it work?

With the slow time $s$, the dressed-up problem is

$$\epsilon^2 \dot{x}_s + x_t = \epsilon \cos(x - t).$$

We now go to town with the RPS:

$$x = x_0(s, t) + \epsilon x_1(s, t) + \epsilon^2 x_2(s, t) + \cdots$$

(9.18)
Notice that
\[\cos(x - t) = \cos(x_0 - t) - \sin(x_0 - t) \left[ \epsilon x_1(s, t) + \epsilon^2 x_2(s, t) + \cdots \right] - \cos(x_0 - t) \left[ \epsilon x_1(s, t) + \epsilon^2 x_2(s, t) + \cdots \right]^2 + \cdots \] (9.19)

We cannot assume that \(x_0\) is smaller than one, so must keep \(\cos(x_0 - t)\) and \(\sin(x_0 - t)\). We are assuming the higher order \(x_n\)'s are bounded, and since \(\epsilon \ll 1\) we can expand the sinusoids as above.

At leading order, \(\epsilon^0\):
\[x_{0t} = 0, \quad \Rightarrow \quad x_0 = f(s). \] (9.20)

The function \(f(s)\) is the slow drift. At next order, \(\epsilon^1\):
\[x_{1t} = \cos(f - t) \Rightarrow x_1 = \sin f - \sin(f - t). \] (9.21)

We determined the constant of integration above so that \(x_1\) is zero initially i.e., we are saying that \(f(0)\) is equal to the initial position of the particle.

At \(\epsilon^2\)
\[f_s + x_{2t} = -\sin(f - t) \left[ \sin f - \sin(f - t) \right]. \] (9.22)

Averaging over the fast time \(t\) we obtain
\[f_s = \langle \sin^2(f - t) \rangle = \frac{1}{2}. \] (9.23)

Thus the average position of the particle is
\[f = \frac{s}{2} + a = \frac{\epsilon^2}{2} t + a. \] (9.24)

The prediction is that the averaged velocity in figure 9.2 is \((0.3)^2/2 = 0.045\). You can check this by noting that the final time is \(20\pi\).

Subtracting 9.23 from 9.22 we have the remaining oscillatory terms:
\[x_{2t} = -\sin(f - t) \sin f - \sin(2f - 2t). \] (9.25)

Integrating and applying the initial condition we have
\[x_2 = -\cos(f - t) \sin f + \frac{1}{4} \cos(2f - 2t) + \cos f \sin f - \frac{1}{4} \cos 2f. \] (9.26)

This is bounded and all is well.

The solution we’ve constructed consists of a slow drift and a rapid oscillation about this slowly evolving mean position. Note however that the mean position of the particle is
\[\langle x \rangle = f + \epsilon \underbrace{\sin f}_{\langle x_1 \rangle} + \epsilon^2 \underbrace{\left[ \frac{1}{2} \sin 2f - \frac{1}{4} \cos 2f \right]}_{\langle x_2 \rangle} + \text{ord}(\epsilon^2). \] (9.27)

In other words, the mean position is not the same as the leading-order term.
Method 3: two-timing and the “guiding center"

In this variant we use the two-timing but insist that the leading-order term is the mean position of the particle. This means that the leading-order solution no longer satisfies the initial condition, and that constants of integration at higher orders are determined by insisting that

\[ \forall n \geq 1 : \quad \langle x_n \rangle = 0. \] (9.28)

OK, let’s do it, starting with the scaled two-time equation in (9.17). The leading order is

\[ x_{0t} = 0, \quad \Rightarrow \quad x_0 = g(s). \] (9.29)

The function \( g(s) \) is the “guiding center” — it’s different from \( f(s) \) in the previous method.

At next order, \( \epsilon^1 \):

\[ x_{1t} = \cos(f - t) \quad \Rightarrow \quad x_1 = -\sin(g - t). \] (9.30)

This is not the same as the first-order term in (9.21); in (9.30) we have determined the constant of integration so that \( \langle x_1 \rangle = 0 \).

At order \( \epsilon^2 \) we have

\[ g_s + x_{2t} = \sin^2(g - t) = \frac{1}{2} - \frac{1}{2} \cos(2g - 2t). \] (9.31)

The average of (9.31) is the motion of the guiding center:

\[ g_s = \frac{1}{2} \quad \Rightarrow \quad g = \frac{\epsilon^2}{2} t + g(0). \] (9.32)

The oscillatory part of the solution, with zero time average, is

\[ x_2 = \frac{1}{4} \sin(2g - 2t). \] (9.33)

Now we must satisfy the initial conditions by requiring that

\[ a = g(0) - \epsilon \sin(g(0)) + \epsilon^2 \frac{1}{4} \sin(2g(0)) + \cdots \] (9.34)

We can invert this series to obtain

\[ g(0) = a + \epsilon \sin a + \cdots \] (9.35)

I prefer this guiding center method. But in either case the essential point is that the leading-order drift velocity is \( \epsilon^2/2 \).

9.3 Problems

Problem 9.1. Investigate Stokes drift in the two-dimensional incompressible velocity field with streamfunction \( \psi(x, y, t) \). The velocity is obtained as

\[ \dot{x} = -\psi_y, \quad \dot{y} = \psi_x. \] (9.36)

Supposing that the streamfunction has the form

\[ \psi = a(x, y) \cos(t/\epsilon) + b(x, y) \sin(t/\epsilon), \] (9.37)

obtain an expression for the streamfunction of the Stokes flow. Study the special case \( a = \cos x \) and \( b = \cos y \) numerically.
Lecture 10

Regular perturbation of partial differential equations

10.1 Potential flow round a slightly distorted cylinder

10.2 Gravitational field of a slightly distorted sphere

10.3 Problems

Problem 10.1. Consider Laplace’s equation,

\[ \phi_{xx} + \phi_{yy} = 0, \quad (10.1) \]

in a domain which is a periodic-in-\( x \) channel with walls at \( y = \pm(1 + \epsilon \cos kx) \). The boundary condition on the walls is

\[ (\nabla \phi + \hat{i}) \cdot \hat{n} = 0, \quad (10.2) \]

where \( \hat{n} \) is the outward normal and \( \hat{i} \) is the unit vector in the \( x \)-direction. Obtain two terms in the expansion of

\[ J(\epsilon) \overset{\text{def}}{=} \iint \phi_x \, dx \, dy. \quad (10.3) \]
Lecture 11

Eigenvalue problems

11.1 Regular Sturm-Liouville problems

The second-order differential equation associated with a Sturm-Liouville eigenproblem has the form

\[(p \phi')' - q \phi + \lambda w \phi = 0, \quad (11.1)\]

where \(p(x), q(x)\) and \(r(x)\) are real functions. The associated boundary value problem is posed on \(a < x < b\) with BCs which we write as

\[\alpha \phi(a) - \alpha' \phi'(a) = 0, \quad \text{and} \quad \beta \phi(b) + \beta' \phi'(b) = 0, \quad (11.2)\]

where \([\alpha, \alpha', \beta, \beta']\) are real constants.

The sign convention in (11.2) is so that (11.27) below looks pretty. In physical problems, such as radiation of heat through the boundaries at \(x = a\) and \(b\), the mixed boundary condition is

\[\frac{d\phi}{dn} + \text{(a positive constant)} \times \phi = 0, \quad (11.3)\]

where \(n\) is the outwards normal at the boundary. At \(x = a < b\) the outwards normal implies \(n = -x\). Thus if \([\alpha, \alpha', \beta, \beta']\) are all positive then the boundary conditions in (11.2) conform to (11.3) at \(x = a\) and \(b\).

Example: The archetypical SL eigenproblem with Dirichlet boundary conditions is:

\[- \phi'' = \lambda \phi, \quad \text{with BCs} \quad \phi(0) = \phi(\pi) = 0. \quad (11.4)\]

Prove that \(\lambda \geq 0\) before attempting a solution.

Example: The archetypical SL eigenproblem with Neuman boundary conditions is:

\[- \phi'' = \lambda \phi, \quad \text{with BCs} \quad \phi'(0) = \phi'(\pi) = 0. \quad (11.5)\]

Prove that \(\lambda \geq 0\) before attempting a solution.

Example: A soluble example with non-constant coefficients:

\[- \phi'' = \lambda x^{-2} \phi, \quad \text{with BCs} \quad \phi(1) = \phi(\ell) = 0. \quad (11.6)\]

First we prove that \(\lambda \geq 0\). The differential equation is an Euler equation which can be solved with \(\phi = x^\nu\). Thus we find that

\[\nu^2 - \nu + \lambda = 0, \quad \text{or} \quad \nu = \frac{1}{2} \pm \sqrt{\lambda - \frac{1}{4}}. \quad (11.7)\]

We have to consider two cases: (a) \(\lambda < 1/4\) and (b) \(\lambda > 1/4\). In case (a) we have two real \(\nu\)'s, but there is no way to combine these into a solution that satisfies the boundary conditions. So we turn to case (b), for which

\[\lambda = \frac{1}{4} \pm i\sqrt{\lambda - \frac{1}{4}}. \quad (11.8)\]
where $\eta \overset{\text{def}}{=} \ell \lambda^{1/3}$. If you look at the graphs of the Airy and Bairy function in the upper panel of Figure 11.2 then you can anticipate that this eigenrelation has an infinite number of solutions. The first five are

$$\ell \lambda^{1/3} = [2.6664, 4.3425, 5.7410, 6.9861, 8.1288],$$

and the lower panel of Figure 11.2 shows the corresponding eigenfunctions. Again, notice that the $n$‘th eigenfunction has $n - 1$ interior zeros.

Figure 11.1: The first five Sturm-Liouville eigenfunctions in (11.12) with $\ell = 3$. 

and

$$x^\nu = x^{1/2} \left[ \cos \left( \sqrt{\lambda - \frac{1}{4}} \ln x \right) \pm i \sin \left( \sqrt{\lambda - \frac{1}{4}} \ln x \right) \right]$$

(11.9)

We can linearly combine the two solutions above to obtain a solution satisfying the BC at $x = 1$

$$\phi = x^{1/2} \sin \left( \sqrt{\lambda - \frac{1}{4}} \ln x \right)$$

(11.10)

The second boundary condition at $x = \ell > 1$ determines the eigenvalue

$$\lambda = \frac{1}{4} + \left( \frac{n\pi}{\ln \ell} \right)^2,$$

(11.11)

and therefore

$$\phi = x^{1/2} \sin \left( n\pi \frac{\ln x}{\ln \ell} \right).$$

(11.12)

The first five eigenfunctions are shown in figure 11.1. Notice that the $n$‘th eigenfunction has $n - 1$ interior zeros.

Example: We’ll use the SL problem

$$-\phi'' = \lambda x \phi$$

(11.13)

as an Airy-function reminder. The problem is posed on the interval $0 < x < \ell$ with Dirichlet boundary conditions

$$\phi(0) = \phi(\ell) = 0.$$  

(11.14)

This SL problem is a little irregular because the weight function is zero at $x = 0$. But, once again, we can show that $\lambda$ is positive. The analytic solution of (11.13) is

$$\phi = \frac{\text{Ai}(-x\lambda^{1/3})}{\text{Ai}(0)} - \frac{\text{Bi}(-x\lambda^{1/3})}{\text{Bi}(0)}.$$  

(11.15)

It is certainly reassuring to know that $\lambda^{1/3}$ in the equation above is a real positive number. The construction above satisfies both the differential equation and the boundary condition at $x = 0$. The boundary condition at $x = \ell$ produces the eigenrelation

$$\frac{\text{Ai}(-\eta)}{\text{Ai}(0)} = \frac{\text{Bi}(-\eta)}{\text{Bi}(0)},$$  

(11.16)

where $\eta \overset{\text{def}}{=} \ell \lambda^{1/3}$.

Figure 11.2: The upper panel shows the two Airy functions on the right of (11.16); each intersection corresponds to an eigenvalue. The lower panel shows the first five eigenfunctions corresponding to the eigenvalues in (11.17).
11.2 Properties of Sturm-Liouville eigenproblems

We write the differential equation in (11.1) as
\[ L \phi = \lambda w \phi, \quad (11.18) \]
where \( L \) is the second-order differential operator
\[ L \overset{\text{def}}{=} -\frac{d}{dx} p \frac{d}{dx} + q. \quad (11.19) \]

L is defined with the minus sign so that if \( p \) is a positive function then \( L \) is a positive operator.

The operator \( L \) is hungry, looking for a function to differentiate. If \( p(x) > 0 \) and \( w(x) > 0 \)
on \([a,b]\) and \( p, q \) and \( w \) are all free from bad singularities, then the problem is regular.

The main results for regular problems are:

1. The eigenvalues are real, countable and ordered so that there is a smallest one
\[ \lambda_0 < \lambda_1 < \lambda_2 < \cdots \quad (11.20) \]
Moreover \( \lim_{n \to \infty} \lambda_n = \infty \).

2. The zeroth eigenfunction (also known as the gravest mode, or the ground state) has no
interior zeroes. Eigenfunction \( \phi_{n+1}(x) \) has one more interior zero than \( \phi_n(x) \).

3. Eigenfunctions with different eigenvalues are orthogonal with respect to the weight function \( w \):
\[ \int_a^b \phi_m(x) \phi_n(x) w(x) \, dx = \xi_n \delta_{mn}. \quad (11.21) \]

Often, but not always, we normalize the eigenfunctions so that \( \xi_n = 1 \).

4. **Within** \((a,b)\), any square integrable\(^1\) function \( f(x) \) can be represented as
\[ f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x), \quad (11.22) \]
with
\[ f_n = \xi_n^{-1} \int_a^b \phi_n(x) f(x) w(x) \, dx. \quad (11.23) \]

5. Considering the truncated sum
\[ \hat{f}_N(x) = \sum_{n=1}^{N} f_n \phi_n(x), \quad (11.24) \]
as an approximation to the target function \( f(x) \), we define the error in \( \hat{f}_N(x) \) as
\[ e(f_1,f_2,\cdots,f_N) \overset{\text{def}}{=} \int_a^b \left[ f - \sum_{n=1}^{N} f_n \phi_n \right]^2 w \, dx. \quad (11.25) \]

If we adjust \( f_1 \) through \( f_N \) to minimize the error \( e \) then we recover the expression for \( f_n \)
in (11.23). Moreover
\[ \lim_{N \to \infty} e = 0. \quad (11.26) \]

\(^1\)Square integrable means with respect to the weight function. That is
\[ \int_a^b f^2 w \, dx < \infty. \]
Eigenvalue are real

To prove that all eigenvalues are real, multiply (11.1) by $\phi^*(x)$ and integrate over the interval. With IP we find

$$\lambda = \left[ \begin{array}{c} p |\phi'|^2 + q |\phi|^2 dx + \hat{\beta} p(b) |\phi'(b)|^2 + \hat{\alpha} q(a) |\phi'(a)|^2 \\ \int_a^b |\phi|^2 w dx \end{array} \right],$$

(11.27)

where $\hat{\beta} \equiv \beta'/\beta$ and $\hat{\alpha} \equiv \alpha'/\alpha$. The right hand side — which is manifestly real — is known as the Rayleigh quotient. If $q(x)$, $\hat{\alpha}$ and $\hat{\beta}$ are all non-negative then the Rayleigh quotient also shows that the eigenvalues are positive.

Exercise: how does (11.27) change if $\beta = 0$ and/or $\alpha = 0$?

Eigenfunctions are orthogonal

To prove orthogonality above we first prove

$$uLv - vLu = [p(vu' - uv')]', \quad (11.28)$$

and therefore

$$\int_a^b uLv \, dx - \int_a^b vLu \, dx = [p(vu' - uv')]_a^b, \quad (11.29)$$

Exercise: use the identity above to prove (11.21).

The eigenfunction expansion minimizes the squared error

Exercise: prove that $f_n$ in (11.23) minimizes $e(f_1, f_2, \cdots, f_N)$ in (11.25).

Example: expand $f(x) = 1$ in terms of $\sin nx$ (again).

11.3 Trouble with BVPs

Not all BVPs have solutions. An example is the thermal diffusion problem

$$\psi_t = \psi_{xx} + 1, \quad (11.30)$$

posed on $0 < x < 1$ with insulating boundary conditions

$$\psi_x(0, t) = \psi_x(1, t) = 0. \quad (11.31)$$

For obvious reasons there are no steady solutions to this partial differential equation: the BVP

$$-\psi_{xx} = 1, \quad \text{with} \quad \psi_x(0) = \psi_x(1) = 0, \quad (11.32)$$

has no solutions. To see this mathematically, integrate the differential equation in (11.32) from $x = 0$ to $x = 1$:

$$- [\psi_x]_{x=0}^{x=1} = 1. \quad (11.33)$$

The contradiction tells us not to bother looking for a steady solution to (11.30). Instead, we should solve the diffusion equation (11.30) with the ansatz $\psi = t + \bar{\psi}(x, t)$, where $\bar{\psi}$ is required to satisfy the initial conditions.
Exercise: show that \( \lim_{t \to -\infty} \hat{\psi} = 0 \).

In other examples, the non-existence of solutions is less obvious. Consider the example

\[ y'' + y = \ln x, \quad \text{with} \quad y(0) = y'(\pi) = 0. \quad (11.34) \]

If we multiply the differential equation by \( \sin x \) and integrate from \( x = 0 \) to \( \pi \) then, with some IP, we quickly obtain

\[
\int_0^\pi \sin x (y'' + y) \, dx = \int_0^\pi \sin x \ln x \, dx = 0.641182. \quad (11.35)
\]

Again, the contradiction tells us that there is no point in looking for a solution.

We can write (11.34) using fancy operator notation as

\[ L y = \ln \frac{1}{x}, \quad \text{where} \quad L \overset{\text{def}}{=} \frac{d^2}{dx^2} + 1. \quad (11.36) \]

Now let's do the calculation using some Sturm-Liouville machinery: Multiply by \( \sin x \), integrate over the interval, use the identity (11.28) and use the fact that \( \sin x \) and \( y \) are zero on both boundaries. Thus again we obtain the contradiction

\[
\int_0^\pi y L \sin x \, dx = \int_0^\pi \sin x \ln x \, dx \quad (11.37)
\]

It's the same calculation as before. You should get used to seeing that

\[
\int u L v \, dx = \int v L u \, dx. \quad (11.38)
\]

If you use the identity above be sure to check that the boundary conditions imply that all the terms falling outside the integral are zero! Go back to (11.28) and see what's required for that to happen.....

More generally we might confront

\[ y'' + y = f(x), \quad \text{with} \quad y(0) = y'(\pi) = 0, \quad (11.39) \]

where \( f(x) \) is some function. If we multiply the differential equation by \( \sin x \) and integrate from \( x = 0 \) to \( \pi \) we have

\[
\int_0^\pi \sin x f(x) \, dx = 0. \quad (11.40)
\]

In order for a solution of (11.39) to exist, the solvability condition in (11.40) must be satisfied. And if (11.40) is satisfied then (11.39) has an infinity of solutions: if \( y(x) \) is a solution of (11.39) then \( y(x) + a \sin x \) is also a solution for every value of the constant \( a \).

To understand what's going on here, we turn to the eigenfunction expansion method.

### 11.4 The eigenfunction expansion method

Suppose that somehow we have solved the eigenvalue problem

\[ L \phi = \lambda w \phi, \quad (11.41) \]
with BCs $\phi_n(a) = \phi_n(b) = 0$. This means we possess the full spectrum $\lambda_n$ and $\phi_n(x)$ with $n = 0, 1, \cdots$

The solution of the boundary value problem

$$L y = f,$$

with BCs $y(a) = 0$ and $y(b) = 0$, is represented as

$$y(x) = \sum_{n=0}^{\infty} y_n \phi_n(x), \quad \text{with} \quad y_m = \xi_m^{-1} \int_a^b \phi_m y w \, dx.$$  \hspace{1cm} (11.43)

Multiply (11.42) by $\phi_m(x)$ and integrate over the interval. Using the identity in (11.28), and the boundary conditions, the left hand side of (11.42) produces

$$\int_a^b \phi_m L y \, dx = \int_a^b y L \phi_m \, dx,$$

$$= \lambda_m \int_a^b y \phi_m w \, dx,$$

$$= \lambda_m \xi_m y_m.$$  \hspace{1cm} (11.44)

Thus

$$y_m = \frac{1}{\lambda_m \xi_m} \int_a^b \phi_m f \, dx.$$  \hspace{1cm} (11.45)

This is OK provided that all the eigenvalues are non-zero. If there is a zero eigenvalue then this problem has no solution.

**Example:** Solve

$$y'' = \ln x,$$  \hspace{1cm} (11.46)

with $y(0) = y(\pi) = 0$. In this example the weight function is $w(x) = 1$ and

$$L = -\frac{d^2}{dx^2}, \quad \phi_n = \sin nx, \quad \lambda_n = n^2,$$  \hspace{1cm} (11.47)

with $n = 1, 2, \cdots$. There are no zero modes, and therefore there is no problem with existence of the solution. The expansion coefficients are

$$y_n = -\frac{2}{\pi n^2} \int_0^\pi \sin nx \ln x \, dx.$$  \hspace{1cm} (11.48)

Evaluating the integral in (11.48) numerically, we find that the first six terms in the series solution are

$$y_1 = -0.408189 \sin x + 0.193982 \sin 2x + 0.029721 \sin 3x$$

$$+ 0.0052863 \sin 4x + 0.0103642 \sin 5x + \cdots$$  \hspace{1cm} (11.49)

Figure 11.3 compares the exact solution of this BVP, namely

$$y = \frac{1}{2}x^2 \ln x - \frac{1}{2} \pi x \ln \pi + \frac{3}{4} \left( \pi x - x^2 \right),$$  \hspace{1cm} (11.50)

with (11.49) truncated with 2, 4 and 6 terms.

**Example:** Reconsider our earlier example

$$y'' + y = \ln x,$$  \hspace{1cm} (11.51)

with BCs $y(0) = y(\pi) = 0$.

We first find the eigenspectrum by solving

$$L \phi = \lambda \phi \quad \phi(0) = \phi(\pi) = 0.$$  \hspace{1cm} (11.52)

The eigenfunctions are

$$\phi_n(x) = \sin nx, \quad n = 1, 2, 3, \cdots$$  \hspace{1cm} (11.53)

with eigenvalues

$$\lambda_n = n^2 - 1 = 0, 1, 3, 8 \cdots$$  \hspace{1cm} (11.54)

Oh oh, there is a zero eigenvalue. So there is no solution. 

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Figure 11.3: Comparison of the exact solution (11.50) (the solid curve) with truncations of the series (11.49). The two-term truncation is the green dashed curve, the four-term truncation is the red dash-dot curve and the six term truncation is the dotted cyan curve.

Example: Find \( a \) so that the problem
\[
y'' + y = \ln x + a, \quad \text{with BCs} \quad y(0) = y(\pi) = 0, \tag{11.55}
\]
has a solution. Construct the most general solution as an eigenfunction expansion.

We know that \( \sin x \) is a zero-eigenmode, and therefore the solvability condition is
\[
0 = \int_0^\pi (\ln x + a) \sin x \, dx \quad \text{implying that} \quad a = \frac{\int_0^\pi \sin x \ln \frac{1}{x} \, dx}{\int_0^\pi \sin x \, dx} = -0.320591. \tag{11.56}
\]

Now we can solve the problem using the eigenmodes defined by the SL problem
\[
-\left( \frac{d^2}{dx^2} + 1 \right) \phi = \lambda \phi, \quad \text{with} \quad \phi(0) = \phi(\pi) = 0. \tag{11.57}
\]
The eigenvalues are
\[
\lambda_n = n^2 - 1, \quad \text{with eigenfunction} \quad \phi_n(x) = \sin nx. \tag{11.58}
\]
Operationally, we multiply (11.55) by \( \sin mx \) and integrate over the interval from \( x = 0 \) to \( \pi \). Thus
\[
\frac{1}{2} \sum_{n=2}^\infty \int_0^\pi y \sin mx \, dx = \int_0^\pi (\ln x + a) \sin mx \, dx, \quad \text{with} \quad m = 2, 3, \ldots \tag{11.59}
\]
The solution of the BVP is
\[
y = A \sin x - \frac{2}{\pi} \sum_{n=2}^\infty \frac{\int_0^\pi (\ln x + a) \sin nx \, dx}{n^2 - 1} \sin nx, \tag{11.60}
\]
where \( A \) is an arbitrary constant.

Exercise: Show that the BVP
\[
y'' = \ln x, \quad \text{with BCs} \quad y'(0) = y'(\pi) = 0, \tag{11.61}
\]
has no solution. Determine \( a \) so that the related BVP
\[
y'' = \ln x + a, \quad \text{with BCs} \quad y'(0) = y'(\pi) = 0, \tag{11.62}
\]
has a solution. Find the most general solution of this problem by direct integration, and by the eigenfunction expansion method.
11.5 Eigenvalue perturbations

Consider a string with slightly non-uniform mass density

$$\rho(x) = \rho_0 [1 + \epsilon b(x)] .$$

(11.63)

We compute the change in the eigenfrequencies induced by the small non-uniformity in density.

Using nondimensional variables, the eigenproblem is

$$\psi'' + \lambda (1 + \epsilon b) \psi = 0 ,$$

(11.64)

with $$\psi(0) = \psi(\pi) = 0 .$$

First suppose that $$\epsilon = 0 .$$ We quickly see that the solution of the $$\epsilon = 0$$ eigenproblem is

$$\lambda = k^2 , \quad \text{with corresponding eigenfunction} \quad s_k(x) \overset{\text{def}}{=} \sqrt{\frac{2}{\pi}} \sin kx ,$$

(11.65)

where $$k = 1, 2 \cdots$$ is the mode number.

Let’s focus on mode number $$m$$ and ask how the eigenvalue $$m^2$$ changes when $$\epsilon \neq 0 ?$$ We use the RPS

$$\psi = s_m + \epsilon \psi^{(1)} + \cdots \quad \text{and} \quad \lambda = m^2 + \epsilon \lambda^{(1)} + \cdots$$

(11.66)

Although it is not strictly necessary, it helps to normalize the perturbed eigenfunction so that

$$\int_0^\pi \psi^2 \, dx = 1 .$$

(11.67)

Expansion of the normalization above implies

$$\int_0^\pi s_m^2 \, dx = 1 , \quad \int_0^\pi s_m \psi^{(1)} \, dx = 0 , \quad \int_0^\pi 2 s_m \psi^{(2)} + (\psi^{(1)})^2 \, dx , \quad \text{etc.}$$

(11.68)

The expressions above will simplify the appearance of subsequent formulas.

The first three orders of the perturbation hierarchy are

$$\mathcal{L}s_m = 0 ,$$

(11.69)

$$\mathcal{L}\psi^{(1)} = \left( \lambda^{(1)} + m^2 b \right) s_m ,$$

(11.70)

$$\mathcal{L}\psi^{(2)} = \left( \lambda^{(2)} + \lambda^{(1)} b \right) s_m + \left( \lambda^{(1)} + m^2 b \right) \psi^{(1)} .$$

(11.71)

Above,

$$\mathcal{L} \overset{\text{def}}{=} - \frac{d^2}{dx^2} - m^2 ,$$

(11.72)

is the “unperturbed operator”.

Notice that (11.69) says that $$\mathcal{L}$$ has a zero eigenvalue with the zero-mode $$s_m .$$ Thus there is a problem with solving boundary value problems of the form

$$\mathcal{L}f = \text{some function of } x ,$$

(11.73)
with \( f(0) = f(\pi) = 0 \). For a solution to exist, the right hand side must be orthogonal to \( \phi^{(0)} \). Thus to ensure that \( (11.70) \) and \( (11.71) \) have solutions, one requires

\[
\lambda^{(1)} = -m^2 \int b s_m^2 \, dx, \tag{11.74}
\]

\[
\lambda^{(2)} = -\lambda^{(1)} \int b s_m^2 \, dx - m^2 \int b s_m \psi^{(1)} \, dx, \tag{11.75}
\]

and so on. (All integrals are from 0 to \( \pi \).) Notice how the expansion of the eigenvalue is obtained by enforcing the solvability condition order by order.

The first-order shift in the \( m \)'th eigenvalue is given by \( (11.74) \). If the density in the middle of the string is increased then the eigenfrequency is decreased.

**Exercise:** At one point in the calculation above \( (11.68) \) was used to simplify an expression. Where was that?

### The second-order terms

The second-order correction from \( (11.73) \) can be written as

\[
\lambda^{(2)} = \left( \frac{\lambda^{(1)}}{m^2} \right)^2 - m^2 \int b s_m \psi^{(1)} \, dx. \tag{11.76}
\]

To get \( \lambda^{(2)} \), we have to solve \( (11.70) \) for \( \psi^{(1)} \) and then substitute into \( (11.76) \). Using the modal projection method, the solution of \( (11.70) \) is

\[
\psi^{(1)}(x) = m^2 \sum_{n=1}^{\infty} \frac{J_{mn} s_n(x)}{n^2 - m^2}, \tag{11.77}
\]

where the sum above does not include the singular term, \( m = n \), and

\[
J_{mn} \overset{\text{def}}{=} \int_0^\pi b s_m s_n \, dx. \tag{11.78}
\]

Thus the second-order shift in the eigenvalue is

\[
\lambda^{(2)} = \left( \frac{\lambda^{(1)}}{m^2} \right)^2 - m^4 \sum_{n=1}^{\infty} \frac{J_{mn}^2}{n^2 - m^2}. \tag{11.79}
\]

### 11.6 The vibrating string

A physical example is a string with uniform tension \( T \) (in Newtons) and mass density \( \rho(x) \) (kilograms per meter) stretched along the axis of \( x \). The wave speed is

\[
c \overset{\text{def}}{=} \sqrt{\frac{T}{\rho}}. \tag{11.80}
\]

After linearization the transverse displacement \( \eta(x, t) \) satisfies

\[
\rho \eta_{tt} - T \eta_{xx} = -\rho g + f. \tag{11.81}
\]

Above \( f(x, t) \) is an externally imposed force (in addition to gravity \( g \)). If the string is stretched between two supports at \( x = 0 \) and \( \ell \) then the BCs are

\[
\eta(0) = 0, \quad \text{and} \quad \eta(\ell) = 0. \tag{11.82}
\]
We remove the “static sag” via
\[ \eta(x, t) = \eta_s(x) + u(x, t), \]  
(11.83)
where the static solution \( \eta_s(x) \) is determined by solving
\[ \eta_{sxx} = \frac{\rho g}{I} \]  
(11.84)
with \( \eta_s = 0 \) at the boundaries.

Exercise: Show that if \( \rho \) is constant
\[ \eta_s(x) = -\frac{\rho g}{2T} x(\ell - x). \]  
(11.85)

Thus the disturbance, \( u(x, t) \), satisfies
\[ \rho u_{tt} - Tu_{xx} = f. \]  
(11.86)

If \( f = 0 \) there is a special class of “eigensolutions” which we find with the substitution
\[ u(x, t) = \phi(x)e^{-i\omega t}. \]  
(11.87)

This produces a typical SL eigenproblem
\[ T\phi_{xx} + \omega^2 \rho \phi = 0, \]  
(11.88)
with BCs \( \phi(0) = \phi(\ell) = 0 \) inherited from (11.82).

Example: Suppose that \( \rho \) is uniform so that the wave speed, \( c = \sqrt{T/\rho} \), is a constant. the spectrum is therefore
\[ \phi_n(x) = \sin k_n x, \quad \text{with eigenfrequency} \quad \omega_n = \frac{n\pi c}{\ell}. \]  
(11.89)

The wavenumber above is \( k_n = \omega_n/c = n\pi/\ell \).

Exercise: Solve the problem above with a free boundary at \( x = \ell \): i.e., the boundary condition is changed to
\[ u_x(\ell) = 0. \]

After we solve (11.88) we possess a set of complete orthogonal functions, \( \phi_n(x) \) each with a corresponding eigenfrequency. The orthogonality condition is
\[ \int_0^\ell \phi_p(x)\phi_q(x)\rho(x)dx = \beta_p\delta_{pq}, \]  
(11.90)
where \( \beta_p \) is a normalization constant. (Often, but not always, we make \( \beta_p = 1 \).)

Returning to the forced problem in (11.86) we build the solution of this partial differential equation as linear superposition of the eigenmodes
\[ u(x, t) = \sum_{m=1}^\infty \hat{u}_m(t)\phi_m(x). \]  
(11.91)

Using orthogonality, the modal amplitude is given by
\[ \beta_n \hat{u}_n(t) = \int_0^\ell u(x, t)\phi_n(x)\rho(x)dx. \]  
(11.92)

Evolution equations for the modal amplitudes follow via projection of the equation (11.86) onto the modes i.e., multiply (11.86) by \( \phi_n(x) \) and integrate from \( x = 0 \) to \( x = \ell \). After some integration by parts one finds
\[ \frac{d^2\hat{u}_n}{dt^2} + \omega_n^2 \hat{u}_n = \beta_n^{-1} \int_0^\ell \phi_n(x)f(x, t)dx. \]  
(11.93)
If we also represent the forcing \( f \) as
\[
f(x, t) = \rho(x) \sum_{m=0}^{\infty} \hat{f}(t) \phi_m(x),
\]
with
\[
\beta_n \hat{f}_n(t) = \int_0^\ell \phi_n(x) f(x, t) \, dx,
\]
then we have
\[
\frac{d^2 \hat{u}_n}{dt^2} + \omega_n^2 \hat{u}_n = \hat{f}_n.
\]

Each modal amplitude, \( \hat{u}_n(t) \), satisfies a forced harmonic oscillator equation.

We declare victory after reducing the partial differential equation (11.86) to a big set of uncoupled ordinary differential equations in (11.95). This victory is contingent on solving the eigenproblem (11.88) and understanding basic properties of SL problems such as the orthogonality condition in (11.90).

**Example:** In some problems the tension \( T \) is non-uniform and the wave equation is then
\[
\rho u_{tt} - (Tu_x)_x = f.
\]

A nice example is a dangling chain of length \( \ell \) suspended from the point \( x = 0 \), with \( x \) positive downwards. The tension is then
\[
T(x) = \rho g \int_x^\ell \rho(x') \, dx'.
\]
The BCs \( \eta(0) = 0 \) and \( \eta_x(\ell) = 0 \).

### 11.7 Problems

**Problem 11.1.** (i) Transform
\[
a(x) y'' + b(x) y' + c(x) y + d(x) E y(x) = 0,
\]
with boundary conditions \( y(\alpha) = y(\beta) = 0 \), into the Sturm-Liouville form
\[
[p(x) y']' + [q(x) + E w(x)] y = 0.
\]

Hint: multiply by an integrating factor \( I(x) \); determine \( I(x) \) by matching up terms. Your answer should include clear expressions for \( [p, q, w] \) in terms of \( [a, b, c, d] \). (ii) Write Bessel’s equation
\[
y'' + \frac{x^{-1}}{x} y' + \left[ E - \nu^2 x^{-2} \right] y = 0
\]
in Sturm-Liouville form. (iii) Prove that the eigenfunctions \( y_n(x) \) associated with (11.99) satisfy
\[
(E_n - E_m) \int_\alpha^\beta y_n(x) y_m(x) w(x) \, dx = 0.
\]

Thus if \( E_m \neq E_n \) then the eigenfunctions are orthogonal with respect to the weight function \( r(x) \).

**Problem 11.2.** Consider the eigenvalue problem
\[
y'' = -\lambda y,
\]
with BCs: \( y(0) = 0 \), \( y'(1) = y(1) \).

(i) Prove that all the eigenvalues are real. (ii) Find the transcendental equation whose solutions determine the eigenvalues \( \lambda_n \). (iii) Find an explicit expression for the smallest eigenvalue \( \lambda_0 \) and the associated eigenfunction \( y_0(x) \). (iv) Show that the eigenfunctions are orthogonal with
respect to an appropriately defined inner product. (v) Attempt to solve the inhomogeneous boundary value problem
\[ y'' = a(x), \quad y(0) = 0, \quad \text{with BCs:} \quad y'(1) = y(1), \] (11.103)
via an expansion using the eigenmodes. Show that this expansion fails because the problem has no solution for an arbitrary \( a(x) \). (iv) Find the solvability condition on \( a(x) \) which ensures that the problem (11.103) does have a solution, and then obtain the solution using a modal expansion.

**Problem 11.3.** Consider the eigenproblem
\[ -(\phi')' = \lambda x^{-2+\epsilon} \phi', \quad \text{with BCs} \quad \phi(1) = \phi(\ell) = 0. \] (11.104)
In the lecture we solved the \( \epsilon = 0 \) problem and showed that the smallest eigenvalue is
\[ \lambda_1 = \frac{1}{4} + \frac{\pi}{\ln \ell}. \] (11.105)
Find the change in \( \lambda_1 \) induced by the perturbation \( \epsilon \ll 1 \).

**Problem 11.4.** There is a special value of \( \alpha \) for which the boundary value problem
\[ y'' = \sin x - \alpha, \quad y'(0) = y'(\pi) = 0 \] (11.106)
has a solution. Find the special value of \( \alpha \) and in that case solve the boundary value problem by: (i) expansion using a set of eigenfunctions; (ii) explicit solution using a combination of homogeneous and particular solutions. (iii) Use MATLAB to compare the explicit solution with a three-term truncation of the series solution.

**Problem 11.5.** Consider the eigenproblem
\[ (a(x)y')' + \lambda y = 0, \quad y'(0) = y'(1) = 0, \] (11.107)
where \( a(x) > 0 \) for \( 0 \leq x \leq 1 \). (i) Verify that \( y(x) = 1 \) and \( \lambda = 0 \) is an eigensolution. (ii) Show that \( \lambda = 0 \) is the smallest eigenvalue. (iii) Now consider the perturbed problem in which the boundary condition at \( x = 1 \) is changed to
\[ \epsilon y(1) + y'(1) = 0, \]
where \( \epsilon \) is a positive real number. With \( 0 < \epsilon \ll 1 \), use perturbation theory to determine the \( O(\epsilon) \) shift of the \( \lambda = 0 \) eigenvalue. (iii) Obtain the \( O(\epsilon^2) \) term in the expansion of the \( \lambda = 0 \) eigenvalue.

**Problem 11.6.** (i) Verify that \( y(x) = 1 \) and \( \lambda = 0 \) is an eigensolution of
\[ (xy')' + \lambda y = 0, \quad \lim_{x \to 0} xy' = 0, \quad y'(1) = 0. \] (11.108)
(ii) Show that \( \lambda = 0 \) is the smallest eigenvalue. (iii) Now consider the perturbed problem in which the boundary condition at \( x = 1 \) is changed to
\[ \epsilon y(1) + y'(1) = 0, \]
where \( \epsilon \) is a positive real number. With \( 0 < \epsilon \ll 1 \), use perturbation theory to determine the \( O(\epsilon) \) shift of the \( \lambda = 0 \) eigenvalue. (iii) Obtain the \( O(\epsilon^2) \) term in the expansion of the \( \lambda = 0 \) eigenvalue.
Problem 11.7. When $\varepsilon = 0$ the eigenproblem

$$y'' + \varepsilon y' + \lambda y = 0, \quad y(0) = y(\pi) = 0,$$

has the solution $\lambda = 1$ and $y = a \sin x$. Use perturbation theory ($\varepsilon \ll 1$) to investigate the dependence of the eigenvalue $\lambda$ on $a$ and $\varepsilon$.

Problem 11.8. Consider a diffusion problem defined on the interval $0 \leq x \leq \ell$:

$$u_t = \kappa u_{xx}, \quad u(0,t) = 0, \quad u_x(\ell,t) = 0,$$

with initial condition $u(x,t) = f(x)$. (i) If you use separation of variables then it is easy to anticipate that you’ll find a Sturm-Liouville eigenproblem with sinusoidal solutions. Sketch the first two eigenfunctions before doing this algebra. Explain why you are motivated to nondimensionalize so that $0 \leq x \leq \pi/2$. (ii) With $\ell \to \pi/2$ and $\kappa \to 1$, work out the Sturm-Liouville algebra and find the eigenfunctions and eigenvalues. (iii) With $f(x) = 1$ find the solution as a Fourier series and use MATLAB to visualize the answer.

Problem 11.9. A rod occupies $1 \leq x \leq 2$ and the thermal conductivity depends on $x$ so that diffusion equation is

$$u_t = (x^2 u_x)_x.$$

The boundary and initial conditions are

$$u(1,t) = u(2,t) = 0, \quad u(x,0) = 1.$$

(i) The total amount of heat in the rod is

$$H(t) = \int_1^2 u(x,t) \, dx.$$ 

Show that $H(0) = 1$ and

$$\frac{dH}{dt} = 4u_x(2,t) - u_x(1,t).$$

Physically interpret the two terms on the right hand side above. What is the sign of the $u_x(2,t)$ and the sign $u_x(1,t)$? (ii) Before solving the PDE, show that roughly 61% of the heat escapes through $x = 2$. (There is a simple analytic expression for the fraction 0.61371 $\cdots$ which you should find.) (iii) Use separation of variables to show that the eigenfunctions are

$$\phi_n(x) = \frac{1}{\sqrt{x}} \sin \left( \frac{n \pi \ln x}{\ln 2} \right).$$

Find the eigenvalue which is associated with the $n$'th eigenfunction. (iv) Use modal orthogonality to find the series expansion of the initial value problem.

Problem 11.10. Solve the BVP

$$y'' + \varepsilon y + \epsilon f(x)y = 0, \quad y(0) = 1, \quad y(\pi) = 0,$$

with a perturbation expansion in $\varepsilon$; $f(x)$ is some $O(1)$ function. Hint: solve the problem exactly in the special case $f(x) = 1$ to infer the form of the perturbation expansion. Is this a regular or singular problem?
Problem 11.11. Consider the inhomogeneous diffusion equation

\[ u_t - u_{xx} = \frac{1}{\sqrt{x(\pi - x)}}, \quad u(0, t) = u(\pi, t) = 0. \]

(i) Using a modal expansion

\[ u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin nx, \]

find the ordinary differential equations satisfied by the amplitudes \( u_n(t) \). (ii) Your answer to (i) will involve the integral

\[ h_n \equiv \frac{2}{\pi} \int_{0}^{\pi} \sin nx \sqrt{x(\pi - x)} \, dx, \]

which you probably can’t evaluate off-the-cuff. (Try looking this one up?) However you should deduce that \( h_n \) is zero if \( n \) is even and

\[ h_n \equiv \frac{4}{\pi} \int_{0}^{\pi/2} \frac{\sin nx}{\sqrt{x(\pi - x)}} \, dx, \quad \text{if } n \text{ is odd.} \]

Find a leading-order \( n \to \infty \) asymptotic approximation of \( h_n \). (iii) I believe that \( u_n \sim n^{-q} \) as \( n \to \infty \). Find \( q \).

Problem 11.12. Find the eigenfunction expansion of \( f(t) = 1 - e^{-\alpha x} \) in \( 0 < x < \pi \) in terms of the eigenfunctions defined by

\[ u'' + \lambda u = 0, \quad \text{with BCs} \quad u(0) = u'(\pi) = 0. \quad (11.111) \]

For which values of \( \alpha \) is this eigenfunction expansion most effective?

Problem 11.13. Solve the eigenproblem

\[ (xu')' + \frac{3 + \lambda}{x} u = 0, \quad \text{with BCs} \quad y(1) = y(2) = 0. \quad (11.112) \]

Use these eigenfunctions to solve the inhomogeneous problem

\[ (xy')' + \frac{3 + \lambda}{x} y = \sin x \quad \text{with BCs} \quad y(1) = y(2) = 0. \quad (11.113) \]

Problem 11.14. Use an eigenfunction expansion to solve

\[ \frac{d^2 g}{dx^2} = \delta(x - x'), \quad \text{with BCs} \quad g(0, x') = g(1, x') = 0, \quad (11.114) \]

on the interval \( 0 < x < 1 \) for the Green's function \( g(x, x') \). You can also solve this problem using the patching method. **Remark:** considering the inhomogeneous BVP on the interval \( 0 < x < 1 \) for \( y(x) \):

\[ y'' = f, \quad \text{with BCs} \quad y(0) = y(1) = 0, \quad (11.115) \]

we now have two different solution methods: Green's functions and eigenfunction expansions. The eigenfunction solution of (11.114) is the connection between these two methods. Can you extend this to the general SL problem

\[ (pg')' + qg = \delta(x - x') \quad \text{with our usual homogenous BCs?} \quad (11.116) \]
**Problem 11.15.** Use separation of variables to solve the eigenproblem

$$\phi_{xx} + \phi_{yy} = -\lambda \phi .$$  \hspace{1cm} (11.117)

The domain is the $\pi \times \pi$ square, $0 < (x,y) < \pi$, with the Dirichlet boundary condition $u = 0$. The two-dimensional Lagrange identity for the operator $\nabla^2$ (aka $\mathcal{L}$) is a well-known vector identity. Use this to prove that the eigenfunctions are orthogonal. Use Galerkin projection and an eigenfunction expansion to solve the inhomogeneous problem

$$u_{xx} + u_{yy} = 1, \quad \text{with BCs } u = 0 \text{ on the square.}$$  \hspace{1cm} (11.118)
Lecture 12

WKB

12.1 The WKB approximation

Suppose we need to solve
\[ \epsilon^2 y'' = Q(x) y, \quad \text{as } \epsilon \to 0. \tag{12.1} \]

The approximate WKB solution to this singular perturbation problem is
\[ y \approx A Q^{-1/4} \exp \left[ \frac{1}{\epsilon} \int_1^x \sqrt{Q(t)} \, dt \right] + B Q^{-1/4} \exp \left[ -\frac{1}{\epsilon} \int_1^x \sqrt{Q(t)} \, dt \right]. \tag{12.2} \]

The approximation above fails in the neighbourhood of \( x^* \) where \( Q(x^*) = 0 \). The point \( x^* \) is called a turning point. We’ll need a different approximation in the vicinity of a turning point. But everywhere else (12.2) is a spectacular approximation. Note the lower limits of integration in (12.2) are unspecified — different choices for these amounts lower limits to redefinition the constants of integration \( A \) and \( B \).

**Exercise:** Check the special cases \( Q = 1 \) and \( Q = -1 \) and commit (12.2) to memory.

**Example:** Solve
\[ y'' + \sqrt{x} y = 0, \quad \text{with } y(1) = 0 \text{ and } y'(1) = 1. \tag{12.3} \]

In this example \( \epsilon = 1 \) and \( Q = -\sqrt{x} \). Therefore
\[ \pm \int_1^x \sqrt{Q(t)} \, dt = \pm \epsilon \frac{4}{5} \left( x^{5/4} - 1 \right) \tag{12.4} \]

We’re going to apply the initial conditions at \( x = 1 \) so it is wise use \( t = 1 \) as the lower limit in the integral above. Thus the solution satisfying \( y(1) = 0 \) is
\[ y' \approx \frac{A}{x^{1/8}} \sin \left[ \frac{1}{\epsilon} \frac{4}{5} \left( x^{5/4} - 1 \right) \right]. \tag{12.5} \]

The cosine is eliminated by the requirement that \( y'(1) = 0 \). We’ve included the factor \( \epsilon^{-1} \) because when we take the derivative of the we have
\[ y' \approx \frac{A}{\epsilon x^{1/8}} \cos \left[ \frac{1}{\epsilon} \frac{4}{5} \left( x^{5/4} - 1 \right) \right]. \tag{12.6} \]

When we take the derivative we only differentiate the cosine — the derivative of the amplitude \( x^{-1/8} \) is order \( \epsilon \) smaller. Requiring that \( y'(1) = 1 \) we see that \( A = \epsilon = 1 \). Thus the WKB approximation is
\[ y^{WKB}(x) = x^{-1/8} \sin \left[ \frac{4}{5} \left( x^{5/4} - 1 \right) \right]. \tag{12.7} \]

Figure ?? shows that this is an excellent approximation to the solution to the initial value problem — unless we get too close to the turning point at \( x = 0 \).
Following BO, the most efficient route to \((12.2)\) is to make the exponential substitution
\[
y = e^{\frac{x}{\epsilon}}
\] 
(12.8)
in \((12.1)\). One finds that the phase function \(S(x)\) satisfies the Ricatti equation
\[
\epsilon S'' + S'^2 = Q.
\] 
(12.9)
We’ve “nonlinearized” the linear equation \((12.1)\). But \((12.9)\) can be solved using a regular perturbation series
\[
S = S_0(x) + \epsilon S_1(x) + \epsilon^2 S_2(x) + \cdots
\] 
(12.10)

### 12.2 Applications of the WKB approximation

#### Example:
Let’s assess the accuracy of WKB applied to Airy’s equation
\[
y'' = xy
\] 
(12.11)
with \(x \gg 1\). There is no explicit small parameter. But if we’re interested in \(x \approx 10\) we might write
\[
x = 10 \cdot X
\]
where \(X \sim 1\). With this rescaling, Airy’s equation acquires a small WKB parameter:
\[
10^{-3} y_{XX} = X y.
\] 
(12.12)
This argument correctly suggests that the WKB approximation works if \(x \gg 1\).

#### Example:
Consider
\[
\epsilon^2 y'' - x^{-1} y = 0.
\] 
(12.13)
How small must \(\epsilon\) be in order for the physical optics approximation to within 5\% when \(x \geq 1\)?

#### Example:
Consider
\[
y'' + k x^{-\alpha} y = 0, \quad y(1) = 0, \quad y'(1) = 1.
\] 
(12.14)
Is WKB valid as \(x \to \infty\)?

With \(k = 1\), I found
\[
S_0 = \pm \frac{2}{2 - \alpha} \left( x^{1 - \frac{\alpha}{2}} - 1 \right), \quad \text{and} \quad S_1 = \frac{\alpha}{4} \ln x, \quad \text{and} \quad S_2 = \frac{\alpha(\alpha - 4)}{16(\alpha - 2)} \left( x^{\frac{\alpha}{2} - 1} - 1 \right).
\] 
(12.15)
The calculation of \(S_2\) should be checked (and should do general \(k\)). But the tentative conclusion is that WKB works if \(\alpha < 2\). Note \(\alpha = 2\) is a special case with an elementary solution.

#### Example:
Let’s use the WKB approximation to estimate the eigenvalues of the Sturm-Liouville eigenproblem
\[
y'' + \lambda \left( x + x^{-1} \right) y = 0, \quad \text{with BCs} \quad y'(1) = 0, \quad y(L) = 0.
\] 
(12.16)
The physical optics approximation is
\[
y = w^{-1/4} \sin \left( \lambda^{1/2} \int_{\text{phase}}^{L} \sqrt{w(x')} \, dx' \right),
\] 
(12.17)
and the leading-order derivative is
\[
y' = -\lambda^{1/2} w^{-1/4} \cos \left( \lambda^{1/2} \int_{\text{phase}}^{L} \sqrt{w(x')} \, dx' \right).
\] 
(12.18)
Notice how the phase in \((12.17)\) has been constructed so that the boundary condition at \(x = L\) is already satisfied. To apply the derivative boundary condition at \(x = 1\) we have from \((12.18)\)
\[
\sqrt{\lambda_n^{\text{WKB}}} J(L) = \pi \left( n + \frac{1}{2} \right), \quad n = 0, 1, \ldots
\] 
(12.19)
where
\[ J(L) \overset{\text{def}}{=} \int_{1}^{L} \sqrt{x + x^{-1}} \, dx. \] (12.20)

Notice that we have only used the geometric optics approximation to obtain the eigencondition in (12.19).

In Figure 12.1 we take \( L = 5 \) and compare the WKB eigenvalue with those obtained from the MATLAB routine \texttt{bvp4c}. It is not easy to analytically evaluate \( J(L) \), so instead we calculate \( J(L) \) using \texttt{quad}. Figure 12.1 shows the relative percentage error,
\[ e \equiv 100 \times \frac{\lambda_{\text{bvp4c}} - \lambda_{\text{WKB}}}{\lambda_{\text{bvp4c}}}, \] (12.21)
as a function of \( n = 0, 2, \cdots, 5 \). The WKB approximation has about 18% error for \( \lambda_0 \), but the higher eigenvalues are accurate.

To use \texttt{bvp4c} we let \( y_1(x) = y(x) \) and write the eigenproblem as the first-order system
\begin{align*}
    y'_1 &= y_2, \quad (12.22) \\
    y'_2 &= -\lambda \left( x + x^{-1} \right) y_1, \quad (12.23) \\
    y'_3 &= \left( x + x^{-1} \right) y_1^2. \quad (12.24)
\end{align*}

This Sturm-Liouville boundary value problem always has a trivial solution viz., \( y(x) = 0 \) and \( \lambda \) arbitrary. We realize that this is trivial, but perhaps \texttt{bvp4c} isn’t that smart. So with (12.24) we force \texttt{bvp4c} to look for a nontrivial solution by adding an extra equation with the boundary conditions
\[ y_3(0) = 0, \quad \text{and} \quad y_3(L) = 1. \] (12.25)

We also have \( y_2(1) = 0 \) and \( y_1(L) = 0 \), so there are four boundary conditions on a third-order problem. This is OK because we also have the unknown parameter \( \lambda \). The addition of \( y_3(x) \) also ensures that \texttt{bvp4c} returns a normalized solution:
\[ \int_{1}^{L} y^2 \left( x + x^{-1} \right) \, dx = 1. \] (12.26)

An alternative that avoids the introduction of \( y_3(x) \) is to use \( y_1(1) = 1 \) as a normalization, and as an additional boundary condition. However the normalization in (12.26) is standard.

In summary, the system for \( \{y_1, y_2, y_3\} \) now only has nontrivial solutions at special values of the eigenvalue \( \lambda \).
The function `bllzWKBeig`, with neither input nor output arguments, solves the eigenproblem with \( L = 5 \). The code is written as an argumentless function so that three nested functions can be embedded. This is particularly convenient for passing the parameter \( L \) — avoid global variables. Notice that all functions are concluded with `end`. In this relatively simple application of `bvp4c` there are only three arguments:

1. a function `odez` that evaluates the right of (12.22) through (12.24);
2. a function `bcz` for evaluating the residual error in the boundary conditions;
3. a MATLAB structure `solinit` that provides a guess for the mesh and the solution on this mesh.

`solinit` is set up with the utility function `bvpinit`, which calls the nested function `initz`. `bvp4c` returns a MATLAB structure that I've imaginatively called `sol`. In this structure, `sol.x` contains the mesh and `sol.y` contains the solution on that mesh. `bvp4c` uses the smallest number of mesh points it can. So, if you want to make a smooth plot of the solution, as in the lower panel of Figure 12.1, then you need the solution on a finer mesh, called `xx` in this example. Fortunately `sol` contains all the information needed to compute the smooth solution on the fine mesh, which is done with the auxiliary function `deval`.

**Example:** Compute the next WKB correction to the \( n = 0 \) eigenvalue and compare both (12.19) and the improved eigenvalue to the numerical solution for \( 1 \leq L \leq 10 \).
function billzWKBeig
L = 5;
J = quad(@(x)sqrt(x+x.^(-1)),1,L);

%The first 6 eigenvalues; n = 0 is the ground state.
nEig = [0 1 2 3 4 5];
lamWKB = (nEig+0.5).^2*(pi/J)^2;
lamNum = zeros(1,length(lamWKB));

for N = 1:length(nEig)
lamGuess = lamWKB(N);
x = linspace(1,L,10);
solinit = bvpinit(x,@initz,lamGuess);
sol = bvp4c(@odez,@bcz,solinit);
lambda = sol.parameters;
lamNum(N) = lambda;
end

err = 100*(lamNum - lamWKB)./lamNum;
figure
subplot(2,1,1)
plot(nEig,err,'*-')
xlabel('Mode Number','interpreter','latex')
ylabel('$\epsilon$','interpreter','latex','fontsize',16)

% Plot the last eigenfunction
xx = linspace(1,L);
ssol = deval(sol,xx);
subplot(2,1,2)
plot(xx,ssol(1,:))
xlabel('$x$','interpreter','latex','fontsize',16)
ylabel('$y_5(x)$','interpreter','latex','fontsize',16)

%---------------- Nested Functions ----------------%
function dydx = odez(x,y,lambda)

%ODEZ evaluates the derivatives
dydx = [ y(2); -lambda*(x+x^(-1))*y(1);
       (x+x^(-1))*y(1)*y(1)];
end

% BCs applied
function res = bcz(ya, yb, lambda)

res = [ ya(2) ; yb(1); ya(3) ; yb(3) - 1];
%Four BCs: solve three first-order
%equations and also determine lambda.
end

% Use a simple guess for the Nth eigenmode
function yinit = initz(x)

alpha = (N + 1/2)*pi/(L-1);
yinit = [ sin(alpha*(L - x))
          alpha*cos(alpha*(L - x))
        (x - 1)/(L - 1) ];
end
12.3 An eigenproblem with a turning point

Let's apply the WKB approximation to estimate the large eigenvalues of the Sturm-Liouville eigenproblem
\[ \phi'' + \lambda \sin x \phi = 0, \quad \phi(0) = \phi \left( \frac{\pi}{2} \right) = 0. \tag{12.27} \]
There is a turning point at \( x = 0 \) so the WKB approximation does not apply close to the boundary.

Hope is eternal and we begin by ignoring the turning point and constructing a physical optics approximation:
\[ \phi^\text{hope} = x^{-1/4} \sin \left( \sqrt{\lambda} \int_0^x \sqrt{\sin v} \, dv \right). \tag{12.28} \]
The construction above satisfies the boundary condition at \( x = 0 \) and then the other boundary condition at \( \pi/2 \) determines our hopeful approximation to the eigenvalue. To ensure that \( \phi^\text{hope}(\pi/2) = 0 \), the argument of the \( \sin \) must be \( n\pi \) and thus the approximate eigenvalue is
\[ \lambda_n^\text{hope} = \left( \frac{n\pi}{J} \right)^2, \quad n = 1, 2, \ldots \tag{12.29} \]
In the expression above the integral of the phase function is
\[ J \overset{\text{def}}{=} \int_0^{\pi/2} \sqrt{\sin v} \, dv = \sqrt{\frac{2}{\pi}} \Gamma^2 \left( \frac{3}{4} \right) = 1.19814 \cdots \tag{12.30} \]
We’ll see later that the approximation in (12.29) is not very accurate — we can’t just ignore the turning point and hope for the best.

Let’s use a combination of WKB and asymptotic matching to account for the turning point and obtain a better approximation to the eigenvalues.

The outer solution — use WKB

We apply the WKB approximation where it is guaranteed to work. This is in the outer region defined by \( \lambda^{1/3} x \gg 1 \). The construction that satisfies the boundary condition at \( x = \pi/2 \) is
\[ \phi^{\text{WKB}} = x^{1/4} \sin \left( \sqrt{\lambda} \int_x^{\pi/2} \sqrt{\sin \theta} \, d\theta \right). \tag{12.31} \]
To perform the match we will need the “inner limit” of the approximation above. In the region where
\[ \lambda^{-1/3} \ll x \ll 1 \tag{12.32} \]
the WKB approximation is valid and we can simplify the phase function in (12.31):
\[ \phi^{\text{WKB}} \sim x^{-1/4} \sin \left( \sqrt{\lambda} J - \int_0^x \sqrt{\sin v} \, dv \right), \quad \sim x^{-1/4} \sin \left( \sqrt{\lambda} J - \frac{2}{3} \sqrt{\lambda} x^{3/2} + \text{ord} \left( x^{7/2} \right) \right). \tag{12.34} \]
The inner solution — an Airy approximation

Close to \( x = 0 \) — specifically in the region where \( x \lambda^{1/3} \) is order unity — we can approximate the differential equation by

\[
\phi_{xx} + \lambda \left( x + \text{ord}(x^3) \right) \phi = 0 \tag{12.35}
\]

As an inner variable we use

\[
X = \lambda^{1/3}x, \tag{12.36}
\]

so that the leading-order inner approximation is a variant of Airy's equation

\[
\Phi_{XX} + X \Phi = 0. \tag{12.37}
\]

The solution that satisfies the boundary condition at \( X = 0 \) is

\[
\Phi = Q \left[ \frac{\text{Ai}(-X)}{\text{Ai}(0)} - \frac{\text{Bi}(-X)}{\text{Bi}(0)} \right]. \tag{12.38}
\]

Matching

To take the outer limit of the inner solution in (12.38) we look up the relevant asymptotic expansions of the Airy functions. Then we write the outer limit of (12.38) as

\[
\Phi \sim \frac{2Q}{\sqrt{3\pi \text{Ai}(0)}} \frac{1}{X^{1/4}} \left[ \frac{\sqrt{3}}{\cos \frac{\pi}{6}} \sin \left( \frac{2}{3} X^{3/2} + \frac{\pi}{4} \right) - \frac{1}{\sin \frac{\pi}{6}} \cos \left( \frac{2}{3} X^{3/2} + \frac{\pi}{4} \right) \right], \tag{12.39}
\]

\[
= \frac{2Q}{\sqrt{3\pi \text{Ai}(0)}} \frac{1}{X^{1/4}} \sin \left( \frac{2}{3} X^{3/2} + \frac{\pi}{12} \right). \tag{12.40}
\]

We now match the phase in (12.34) with that in (12.40), this requires

\[
\sqrt{\lambda} J - n \pi = \frac{\pi}{12}, \tag{12.41}
\]

or

\[
\lambda^{\text{WKB}} = \left( \frac{12n - 1}{12} \right)^2 \pi, \quad n = 1, 2, 3 \cdots \tag{12.42}
\]

Notice that with \( n = 1 \) the hopeful approximation in (12.30) is about 18% larger than the correct WKB-Airy approximation in (12.42). The numerical comparison below shows that (12.42) is good even for \( n = 1 \):

\[
\begin{align*}
\lambda_{\text{bvp4c}} & = 5.7414 \, 25.2094 \, 58.4349 \, 105.4114 \, 166.1422 \, 240.6232 \\
\lambda_{\text{WKB}} & = 5.7771 \, 25.2568 \, 58.4341 \, 105.4673 \, 166.1456 \, 240.6793
\end{align*}
\]

Note that numerical results above fluctuate in the final decimal place as I change the resolution and the initial guess in \texttt{bvp4c}.

\footnote{Airy factoids used here are

as \( t \to -\infty \):

\[
\text{Ai}(t) \sim \frac{1}{\sqrt{\pi}(-t)^{1/4}} \sin \left( \frac{2}{3} (-t)^{3/2} + \frac{\pi}{4} \right), \quad \text{and} \quad \text{Bi}(t) \sim \frac{1}{\sqrt{\pi}(-t)^{1/4}} \cos \left( \frac{2}{3} (-t)^{3/2} + \frac{\pi}{4} \right).
\]

And

\[
\text{Ai}(0) = \frac{\text{Bi}(0)}{\sqrt{3}} = \frac{1}{3^{2/3} \Gamma(2/3)} = 0.355028.
\]
Figure 12.2: A figure for problem [12.1]

12.4 Problems

Problem 12.1. Figure [12.2] shows the solution of one of the four initial value problems:

\[ \epsilon^2 y_1'' + (1 + x)y_1 = 0, \quad y_1(0) = 0, \quad y_1'(0) = 1, \quad \text{(12.43)} \]
\[ \epsilon^2 y_2'' + (1 + x)^{-1} y_2 = 0, \quad y_2(0) = 0, \quad y_2'(0) = 1, \quad \text{(12.44)} \]
\[ \epsilon^2 y_3'' - (1 + x)y_3 = 0, \quad y_3(0) = 0, \quad y_3'(0) = 1, \quad \text{(12.45)} \]
\[ \epsilon^2 y_4'' - (1 + x)^{-1} y_4 = 0, \quad y_4(0) = 0, \quad y_4'(0) = 1. \quad \text{(12.46)} \]

(i) Which \( y_n(x) \) is shown in Figure [12.2]? (ii) Use the WKB approximation to estimate the value of \( \epsilon \) used to draw figure [12.2].

Problem 12.2. Consider the IVP

\[ \ddot{x} + 256e^{4t}x = 0, \quad x(0) = 0, \quad \dot{x}(0) = 1. \quad \text{(12.47)} \]

Estimate the position and magnitude of the first positive maximum of \( y(t) \). Compare the WKB approximation with a numerical solution on the interval \( 0 < t \leq 1 \).

Problem 12.3. Consider the differential equation

\[ y'' + \frac{400}{Q(x) + x^2} y = 0. \quad \text{(12.48)} \]

How can we apply the WKB approximation to this equation? Compare the physical optics approximation to a numerical solution with the initial conditions \( y(0) = 1 \) and \( y'(0) = 0 \).

Problem 12.4. Consider the differential equation

\[ y'' + x^2 y = 0, \quad \text{(12.49)} \]
with $x \gg 1$. Solve the differential equation using the WKB approximation and following our discussion of Airy’s equation, assess the accuracy of the large-$x$ WKB approximation to (12.49). Go to Chapter 10 of the DLMF (or some other reference, such as the appendix of BO) and find the exact solution of the differential equation

$$y'' + a^2 x^4 y = 0$$

in terms of Bessel functions. Compare the asymptotic expansion of the exact Bessel function solution with your WKB approximation.

**Problem 12.5.** Use the exponential substitution $y = \exp(S/\epsilon)$ to construct a WKB approximation to the differential equation

$$\epsilon^2 (p^2 y')' + q y = 0.$$  

Above $p(x)$ and $q(x)$ are coefficient functions, independent of the small parameter $\epsilon$.

**Problem 12.6.** Consider the eigenproblem

$$\phi'' + \lambda w \phi = 0, \quad \phi(0) = 0, \quad \phi'(1) + \phi(1) = 0.$$  

The weight function, $w(x)$ above, is positive for $0 \leq x \leq 1$. (i) Show that the eigenvalues $\lambda_n$ are real and positive. (ii) Show that eigenfunctions with distinct eigenvalues are orthogonal

$$(\lambda_n - \lambda_m) \int_0^1 \phi_n \phi_m w \, dx = 0.$$  

(iii) With $w = 1$, find the first five eigenvalues and plot the first five eigenfunctions. You should obtain transcendental equation for $\lambda$, and then solve that equation with \textsc{matlab}. (iv) Next, with non-constant $w(x)$, use the WKB approximation to obtain a formula for $\lambda_n$. (v) Consider

$$w = (a + x)^2.$$  

Take $a = 1$ and use \textsc{bvp4c} to calculate the first five eigenvalues and compare $\lambda^{WKB}$ with $\lambda^{bvp4c}$. (vi) Is the WKB approximation better or worse if $a$ increases?

**Problem 12.7.** Consider the Sturm-Liouville problem

$$(w y')' + \lambda y = 0,$$  

with boundary conditions

$$\lim_{x \to 0} w y' = 0, \quad y(1) = 0.$$  

Assume that $w(x)$ increases monotonically with $w(0) = 0$ and $w(1) = 1$ e.g., $w(x) = \sin \pi x/2$. Further, suppose that if $x \ll 1$ then

$$w(x) = w_1 x + \frac{w_2}{2} x^2 + \frac{w_3}{6} x^3 + \cdots.$$  

There is a regular singular point at $x = 0$, and thus we require only that $y(0)$ is not infinite. Show that the transformation $y = w^{-1/2} Y$ puts the equation into the Schrödinger form

$$Y'' + \left[ \frac{\lambda}{w} - \frac{w''}{2w} + \frac{w'^2}{4w^2} \right] Y = 0.$$  

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Use the WKB method and matching to find an approximation for the large eigenvalues \( \lambda = \epsilon^2 \gg 1 \) in terms of the \( w_n \)'s and the constant

\[
q \equiv \int_0^1 \frac{dx}{\sqrt{w(x)}}. \tag{12.59}
\]

Some useful information from DLMF: The solution of

\[
u'' + \left[ \frac{a^2}{4z} + \frac{1 - \nu^2}{4z^2} \right] u = 0 \tag{12.60}
\]
is

\[
u(z) = A\sqrt{z}J_\nu(a\sqrt{z}) + B\sqrt{z}Y_\nu(a\sqrt{z}), \tag{12.61}
\]
where \( J_\nu \) and \( Y_\nu \) are Bessel functions. You will need to look up basic properties of Bessel functions.

**Problem 12.8.** Consider the epsilonless Schrödinger equation

\[
y'' + p^2 y = 0, \tag{12.62}
\]
where \( p(x) > 0 \).

(i) Try to solve the equation by substituting

\[
Y \equiv \exp \left( \pm i \int_0^x p(t) \, dt \right) \tag{12.63}
\]
Unfortunately this doesn’t work: \( Y(x) \) is not an exact solution of (12.62) unless \( p \) is constant. Instead, show that \( Y \) satisfies

\[
Y'' + \left( p^2 \mp ip' \right) Y = 0. \tag{12.64}
\]
(ii) Compare (12.64) with (12.62), and explain why \( Y(x) \) is an approximate solution of (12.62) if

\[
\left| \frac{d}{dx} \frac{1}{p} \right| \ll 1. \tag{12.65}
\]
(iii) Prove that if \( y_1 \) and \( y_2 \) are two linearly independent solutions of (12.62) then the Wronskian

\[
W \equiv y_1 y'_2 - y'_1 y_2 \tag{12.66}
\]
is constant. (iv) Show that the Wronskian of

\[
Y_1 \equiv \exp \left( +i \int_0^x p(t) \, dt \right) \quad \text{and} \quad Y_2 \equiv \exp \left( -i \int_0^x p(t) \, dt \right) \tag{12.67}
\]
is equal to \( 2ip \). This suggests that if we modify the amplitude of \( Y(x) \) like this:

\[
Y_3 \equiv \frac{1}{\sqrt{p}} \exp \left( +i \int_0^x p(t) \, dt \right) \quad \text{and} \quad Y_4 \equiv \frac{1}{\sqrt{p}} \exp \left( -i \int_0^x p(t) \, dt \right), \tag{12.68}
\]
then we might have a better approximation. (v) Show that the Wronskian of \( Y_3 \) and \( Y_4 \) is a constant. (vi) Find a Schrödinger equation satisfied by \( Y_3 \) and \( Y_4 \) and discuss the circumstances in which this equation is close to (12.62).

**Problem 12.9.** Consider

\[
y'' + xy = 0, \tag{12.69}
\]
and suppose that

\[
y(x) \sim x^{-1/4} \cos(2x^{3/2}/3) \quad \text{as} \quad x \to +\infty. \tag{12.70}
\]
Solve this problem exactly in terms of well known special functions. Find the asymptotic behaviour of \( y(x) \) as \( x \to -\infty \). Check your answer with MATLAB (see Figure 12.9).
Figure 12.3: Figure for the problem 12.9 showing a comparison of the exact solution (the solid black curve) with the asymptotic expansions as $x \rightarrow -\infty$ (the dot-dash blue curve) and as $x \rightarrow +\infty$ (the dashed red curve).

Problem 12.10. (i) Consider the eigenproblem

$$y'' + E(1 + \eta x)y = 0, \quad y(0) = 0, \quad y(\pi) = 0,$$

where $\eta$ is a parameter and $E$ is an eigenvalue. With $\eta = 0$ the gravest mode is

$$y = \sin x, \quad E = 1.$$  \hspace{1cm} (12.72)

(ii) Suppose $|\eta| \ll 1$. Find the $O(\eta)$ shift in the eigenvalue using perturbation theory. If you're energetic, calculate the $O(\eta^2)$ term for good measure (optional). (iii) In equation (10.1.31) of BO, there is a WKB approximation to the eigenvalue $E(\eta)$. Take $n = 1$, and expand this formula for $E$ up to and including terms of order $\eta^2$; compare this with your answer to part (i). (iii) Use bvp4c in MATLAB to calculate $E(\eta)$, with $0 < \eta < 2$, numerically. Compare the WKB approximation in (10.1.31) with your numerical answer by plotting $E_{\text{bvp4c}}(\eta)$ and $E_{\text{WKB}}(\eta)$ in the same figure.

Problem 12.11. Consider the Sturm-Liouville eigenproblem

$$y'' + \lambda (1 + a \sin x)^2 y = 0, \quad y(0) = y(\pi) = 0.$$  \hspace{1cm} (12.73)

(a) Using bvp4c, compute the first two eigenvalues, $\lambda_1$ and $\lambda_2$, as a functions of $a$ in the range $-3/4 < a < 3$. (b) Estimate $\lambda_1(a)$ and $\lambda_2(a)$ using the WKB approximation. (c) Assuming $|a| \ll 1$ use perturbation theory to compute the first two nonzero terms in the expansion of $\lambda_1(a)$ and $\lambda_2(a)$ about $a = 0$. Compare these approximations with the WKB solution — do they agree? (d) Compare the WKB approximation to those from bvp4c by plotting the various results for $\lambda_n(a)/n^2$ on the interval $-3/4 < a < 3$.

Remark: If $a = -1$ the differential equation has a turning point at $x = \pi/2$. This requires special analysis — so we're staying well away from this ticklish situation by taking $a > -3/4$. 

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Figure 12.4: Figure for the problem with (12.71).

Figure 12.5: Figure for the problem containing (12.73).
Evaluating integrals by matching

13.1 Singularity subtraction

Considering
\[ F(\epsilon) = \int_0^\pi \frac{\cos x}{\sqrt{x^2 + \epsilon^2}} \, dx , \quad \text{as } \epsilon \to 0 , \quad (13.1) \]
we cannot set \( \epsilon = 0 \) because the resulting integral is logarithmically divergent at \( x = 0 \). An easy way to make sense of this limit is to write
\[ F(\epsilon) = -\int_0^\pi \frac{1 - \cos x}{\sqrt{x^2 + \epsilon^2}} \, dx + \int_0^\pi \frac{dx}{\sqrt{x^2 + \epsilon^2}} , \quad (13.2) \]
\[ \sim -\int_0^\pi \frac{1 - \cos x}{x} \, dx + \ln \left( \pi + \sqrt{\pi^2 + \epsilon^2} \right) - \ln \epsilon , \quad (13.3) \]
\[ \sim \ln \frac{1}{\epsilon} - \int_0^\pi \frac{1 - \cos x}{x} \, dx + \ln 2 \pi , \quad (13.4) \]
with errors probably \( \text{ord}(\epsilon) \). This worked nicely because we could exactly evaluate the elementary integral above. This method is called singularity subtraction — to evaluate a complicated nearly-singular integral one finds an elementary integral with the same nearly-singular structure and subtracts the elementary integral from the complicated integral. To apply this method one needs a repertoire of elementary nearly singular integrals.

Exercise: generalize the example above to
\[ F(\epsilon) = \int_0^a f(x) \frac{dx}{\sqrt{x^2 + \epsilon^2}} . \quad (13.5) \]

Example: find the small \( x \) behaviour of the exponential integral
\[ E(x) = \int_x^\infty \frac{e^{-t}}{t} \, dt . \quad (13.6) \]

Notice that
\[ \frac{dE}{dx} = -\frac{e^{-x}}{x} = -\frac{1}{x} + 1 - \frac{x}{2} + \cdots \quad (13.7) \]
If we integrate this series we have
\[ E(x) = -\ln x + C + x - \frac{x^2}{4} + \text{ord}(x^3) . \quad (13.8) \]
The problem has devolved to determining the constant of integration \( C \). We do this by subtracting the singularity. We use elementary nearly-singular-as \( x \to 0 \) integral:
\[ \ln x = -\int_1^x \frac{dt}{t} . \quad (13.9) \]
We use this elementary integral to subtract the logarithmic singularity from \((13.6)\):

\[ E(x) + \ln x = -\int_x^1 \frac{1 - e^{-t} - t e^{-t}}{t} dt + \int_1^\infty \frac{e^{-t}}{t} dt. \] (13.10)

Now we take the limit \(x \to 0\) and encounter only convergent integrals:

\[ C = \lim_{x \to 0} (E(x) + \ln x), \] (13.11)

\[ = -\int_1^0 \frac{1 - e^{-t} - t e^{-t}}{t} dt + \int_1^\infty \frac{e^{-t}}{t} dt, \] (13.12)

\[ = -\gamma_E. \] (13.13)

Above, we've used the result from problem 14.7 to recognize Euler's constant \(\gamma_E \approx 0.57721\). To summarize, as \(x \to 0\)

\[ E(x) \sim -\ln x - \gamma_E + x - \frac{x^2}{4} + \text{ord} \left( x^3 \right). \] (13.14)

### 13.2 Local and global contributions

Consider

\[ A(\epsilon) \overset{\text{def}}{=} \int_0^1 \frac{e^x}{\sqrt{\epsilon + x}} dx. \] (13.15)

The integrand is shown in Figure 13.1. How does the function \(A(\epsilon)\) behave as \(\epsilon \to 0\)? The leading order behaviour is perfectly pleasant:

\[ A(0) = \int_0^1 \frac{e^x}{\sqrt{x}} dx. \] (13.16)

This integral is well behaved and we can just evaluate it, for example as

\[ A(0) = \int_0^1 \frac{1}{\sqrt{x}} + x^{1/2} + \frac{x^{3/2}}{2!} + \frac{x^{5/2}}{3!} + \cdots dx, \]

\[ \approx 2 + \frac{2}{3} + \frac{1}{5} + \frac{2}{21}, \]

\[ = 2.91429. \] (13.17)

Alternatively, with the MATHEMATICA command NIntegrate, we find \(A(0) = 2.9253\).

To get the first dependence of \(A\) on \(\epsilon\), we try taking the derivative:

\[ \frac{dA}{d\epsilon} = -\frac{1}{2} \int_0^1 \frac{e^x dx}{(\epsilon + x)^{3/2}}. \] (13.18)

But now setting \(\epsilon = 0\) we encounter a divergent integral. We've just learnt that the function \(A(\epsilon)\) is not differentiable at \(\epsilon = 0\). Why is this?

Referring to Figure 13.1 we can argue that the peak contribution to the integral in (13.15) is

\[ \text{peak width, ord}(\epsilon) \times \text{peak height, ord}(\epsilon^{-1/2}) = \text{ord} \left( \epsilon^{1/2} \right). \] (13.19)

Therefore the total integral is

\[ A(\epsilon) = \text{an ord}(1) \text{ global contribution} + \text{an ord}(\epsilon^{1/2}) \text{ contribution from the peak} + \text{higher-order terms} - \text{probably a series in } \sqrt{\epsilon}. \] (13.20)

The \(\text{ord}(\epsilon^{1/2})\) is not differentiable at \(\epsilon = 0\) — this is why the integral on the right of (13.18) is divergent. This argument suggests that

\[ A(\epsilon) = 2.9253 + c \sqrt{\epsilon} + \text{higher-order terms}. \] (13.21)

How can we obtain the constant \(c\) above?
Figure 13.1: The integrand in (13.15) with $\epsilon = 0.01$. There is a peak with height $\epsilon^{-1/2} \gg 1$ and width $\epsilon \ll 1$ at $x = 0$. The peak area scales as $\epsilon^{1/2}$, while the outer region makes an $O(1)$ contribution to the integral.

**Method 1: subtraction**

We have

$$A(\epsilon) - A(0) = \int_0^1 e^x \left( \frac{1}{\sqrt{\epsilon + x}} - \frac{1}{\sqrt{x}} \right) \, dx,$$

(13.22)

$$\sim \int_0^\infty \frac{1}{\sqrt{\epsilon + x}} - \frac{1}{\sqrt{x}} \, dx,$$

(13.23)

$$= \sqrt{\epsilon} \int_0^\infty \frac{1}{\sqrt{1 + t}} - \frac{1}{\sqrt{t}} \, dt,$$

(13.24)

$$= -2\sqrt{\epsilon}.$$  

(13.25)

**Exercise:** Explain the transition from (13.22) to (13.23).

Although this worked very nicely, it is difficult to get further terms in the series with subtraction.

**Method 2: range splitting and asymptotic matching**

We split the range at $x = \delta$, where

$$\epsilon \ll \delta \ll 1,$$

(13.26)

and write the integral as

$$A(\epsilon) \equiv \int_0^\delta \frac{e^x \, dx}{\sqrt{\epsilon + x}} + \int_\delta^1 \frac{e^x \, dx}{\sqrt{\epsilon + x}}.$$

(13.27)

We can simplify $A_1(\epsilon, \delta)$ and $A_2(\epsilon, \delta)$ and add the results together to recover $A(\epsilon)$. Of course, the artificial parameter $\delta$ must disappear from the final answer. This cancellation provides a good check on the consistency of our argument and the correctness of algebra.

To simplify $A_1$ note

$$A_1 = \int_0^\delta \frac{1 + x + \text{ord}(x^2)}{\sqrt{\epsilon + x}} \, dx.$$  

(13.28)

This is a splendid approximation because $x$ is small everywhere in the range of integration.
The integrals are elementary, and we obtain

\[
A_1 = 2\sqrt{\epsilon + \delta} - 2\sqrt{\epsilon} + \frac{1}{3}\epsilon^{3/2} + \frac{2}{3}\delta^{3/2} + \epsilon - \frac{1}{3}\epsilon\sqrt{\delta + \epsilon} + \text{ord}(\delta^{5/2}),
\]

(13.29)

\[
= 2\sqrt{\delta} + \frac{\epsilon}{\sqrt{\delta}} - 2\sqrt{\epsilon} + \frac{1}{3}\epsilon^{3/2} + \frac{2}{3}\delta^{3/2} + \text{ord}\left(\frac{\epsilon^2}{\delta^{3/2}}, \delta^{1/2} \epsilon\right).
\]

(13.30)

To be consistent about which terms are discarded we gear by saying that \(\delta = \text{ord}(\epsilon^{1/2})\). Then all the terms in \(\text{ord}\) garbage heap in (13.30) are of order \(\epsilon^{5/4}\).

To simplify \(A_2\) we use the approximation

\[
A_2 = \int_0^1 e^x \left( \frac{1}{\sqrt{x}} - \frac{\epsilon}{2x^{3/2}} + \text{ord}(\epsilon^2 x^{-5/2}) \right) \, dx.
\]

(13.31)

This approximation is good because \(x \geq \delta \gg \epsilon\) everywhere in the range of integration. Now we can evaluate some elementary integrals:

\[
A_2 = \int_0^1 e^x \frac{e}{\sqrt{x}} \, dx - \int_0^\delta e^x \frac{e}{\sqrt{x}} \, dx + \epsilon \int_0^1 e^x \frac{d}{dx} x^{-1/2} + \text{ord}\left(\frac{\epsilon^2}{\delta^{3/2}}\right),
\]

(13.32)

\[
= A(0) - \int_0^\delta x^{-1/2} + x^{1/2} \, dx + \epsilon [x^{-1/2} e^x]_0^\delta - \epsilon \int_\delta^1 x^{-1/2} e^x \, dx + \text{ord}\left(\frac{\epsilon^2}{\delta^{3/2}}, \delta^{5/2}\right),
\]

(13.33)

\[
= A(0) - 2\sqrt{\delta} - \frac{2}{3}\delta^{3/2} + \epsilon e - \frac{\epsilon}{\sqrt{\delta}} - \epsilon A(0) + \text{ord}\left(\frac{\epsilon^2}{\delta^{3/2}}, \delta^{5/2}, \epsilon \delta^{1/2}\right).
\]

(13.34)

The proof of the pudding is when we sum (13.30) and (13.34) and three terms containing the arbitrary parameter \(\delta\), namely

\[
2\sqrt{\delta}, \quad \frac{2}{3}\delta^{3/2}, \quad \text{and} \quad \frac{\epsilon}{\sqrt{\delta}},
\]

(13.35)

all cancel. We are left with

\[
A(\epsilon) = A(0) - 2\epsilon^{1/2} + [e - A(0)] \epsilon + \frac{1}{3}\epsilon^{3/2} + \text{ord}(\epsilon^2).
\]

(13.36)

The terms of order \(\epsilon^0\) and \(\epsilon^1\) come from the outer region, while the terms of order \(\epsilon^{1/2}\) and \(\epsilon^{3/2}\) came from the inner region (the peak).

**Another example of matching**

Let us complete problem 4.3 by finding a few terms in the \(t \to 0\) asymptotic expansion of

\[
\dot{x}(t) = \int_0^\infty \frac{v e^{-vt}}{1 + v^2} \, dv.
\]

(13.37)

If we simply set \(t = 0\) then the integral diverges logarithmically. We suspect \(\dot{x} \sim \ln t\). Let’s calculate \(\dot{x}(t)\) at small \(t\) precisely by splitting the range at \(v = a\), where

\[
1 \ll a \ll \frac{1}{t}.
\]

(13.38)

For instance, we could take \(a = \text{ord}(t^{-1/2})\). Then we have

\[
\dot{x} = \int_0^a v - v^2 t + \cdots \, dv + \int_a^\infty e^{-vt} \left( \frac{1}{v} - \frac{1}{v^3} + \cdots \right) \, dv
\]

(13.39)
Now we have a variety of integrals that can be evaluated by elementary means, and by recognizing the exponential integral:

\[
\begin{align*}
\dot{x} &\sim \frac{1}{2} \ln(1 + a^2) - at + t \tan^{-1} a + E(at) + \cdots \quad (13.40) \\
\sim \ln a - at + \frac{\pi t}{2} - \ln(at) - \gamma + at + \text{ord} \left( a^2 t^2, \frac{t}{a}, a^{-2} \right), \quad (13.41) \\
= \ln \frac{1}{t} - \gamma - \ln(at) - \frac{\pi t}{2} + \text{ord} (t^2). \quad (13.42)
\end{align*}
\]

### 13.3 An electrostatic problem — H section 3.5

Here is a crash course in the electrostatics of conductors:

\[
\nabla \cdot \vec{e} = \rho, \quad \text{and} \quad \nabla \times \vec{e} = 0. \quad (13.43)
\]

Above \(\vec{e}(\vec{x})\) is the electric field at point \(\vec{x}\) and \(\rho(\vec{x})\) is the density of charges (electrons per cubic meter). Both equations can be satisfied at once by introducing the electrostatic potential \(\phi\):

\[
\vec{e} = -\nabla \phi, \quad \text{and therefore} \quad \nabla^2 \phi = -\rho. \quad (13.44)
\]

To obtain the electrostatic potential \(\phi\) we must solve Poisson’s equation above.

This is accomplished using the Green’s function

\[
\nabla^2 g = -\delta(\vec{x}), \quad \Rightarrow \quad g = \frac{1}{4\pi r}, \quad (13.45)
\]

where \(r \defeq |\vec{x}|\) is the distance from the singularity (the point charge). Hence if there are no boundaries

\[
\phi(\vec{x}) = \frac{1}{4\pi} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d\vec{x}'. \quad (13.46)
\]

So far, so good: in free space, given \(\rho(\vec{x})\), we must evaluate the three dimensional integral above. The charged rod at the end of this section is a non-trivial example.

If there are boundaries then we need to worry about about boundary conditions e.g., on the surface of a charged conductor (think of a silver spoon) the potential is constant, else charges would flow along the surface. In terms of the electric field \(\vec{e}\), the boundary condition on the surface of a conducting body \(\mathcal{B}\) is that

\[
\vec{e} \cdot \vec{t}_\mathcal{B} = 0, \quad \text{and} \quad \vec{e} \cdot \vec{n}_\mathcal{B} = \sigma \quad (13.47)
\]

where \(\vec{t}_\mathcal{B}\) is any tangent to the surface of \(\mathcal{B}\), \(\vec{n}_\mathcal{B}\) is the unit normal, pointing out of \(\mathcal{B}\), and \(\sigma\) is the charge density (electrons per square meter) sitting on the surface of \(\mathcal{B}\).

**Example: a sphere.** The simplest example is sphere of radius \(a\) carrying a total charge \(q\), with surface charge density

\[
\sigma = \frac{q}{4\pi a^2}. \quad (13.48)
\]

Outside the sphere \(\rho = 0\) and the potential is

\[
\phi = \frac{q}{4\pi r}, \quad \text{so that} \quad \vec{e} = \frac{qr}{4\pi r^2}, \quad (13.49)
\]

where \(\vec{r}\) is a unit vector pointing in the radial direction (i.e., our notation is \(\vec{x} = r\vec{r}\)). The solution above is the same as if all the charge is moved to the center of the sphere.
For a non-spherical conducting body $B$ things aren’t so simple. We must solve $\nabla^2 \phi = 0$ outside the body with $\phi = \phi_B$ on the surface $B$ of the body, where $\phi_B$ is an unknown constant. (We are considering an isolated body sitting in free space so that $\phi(x) \to 0$ as $r \to \infty$.)

We don’t know the surface charge density $\sigma(x)$, but only the total charge $q$, which is the surface integral of $\sigma(x)$:

$$q = \int_B \sigma \, dS = \int_B e \cdot n_B \, dS.$$  \hspace{1cm} (13.50)

This is a linear problem, so the solution $\phi(x)$ will be proportional to the total charge $q$. We define the capacity $C_B$ of the body as

$$q = C_B \phi_B.$$  \hspace{1cm} (13.51)

The capacity is an important electrical property of $B$.

**Example**  The electrostatic energy is defined via the volume integral

$$E \overset{\text{def}}{=} \frac{1}{2} \int |e|^2 \, dV,$$  \hspace{1cm} (13.52)

where the integral is over the region outside of $B$. Show that

$$E = \frac{1}{2} C_B \phi_B.$$  \hspace{1cm} (13.53)

If you have a sign error, consider that the outward normal to body, $n_B$, is the inward normal to free space.

**Example: a charged rod.** Find the potential due to a line distribution of charge with density $\eta$ (electrons per meter) along $-a < z < a$.

In this example the charge density is

$$\rho = \eta \frac{\delta(s) \chi(z)}{2\pi s},$$  \hspace{1cm} (13.54)

where $s = \sqrt{x^2 + y^2}$ is the cylindrical radius. The signature function $\chi(z)$ is one if $-a < z < +a$, and zero otherwise.

We now evaluate the integral in (13.46) using cylindrical coordinates, $(\theta, s, z)$ i.e., $d\mathbf{x} = d\theta ds dz$. The $s$ and $\theta$ integrals are trivial, and the potential is therefore

$$\phi(s, z) = \frac{\eta}{4\pi} \int_{-a}^a \frac{d\xi}{\sqrt{(z - \xi)^2 + s^2}},$$  \hspace{1cm} (13.55)

$$= \frac{\eta}{4\pi} \int_{-a+z/s}^{a+z/s} \frac{dt}{\sqrt{1 + t^2}},$$  \hspace{1cm} (13.56)

$$= \frac{\eta}{4\pi} \ln \left[ \frac{r_+ - z + a}{r_- - z - a} \right],$$  \hspace{1cm} (13.57)

where

$$r_\pm \equiv \sqrt{s^2 + (a \mp z)^2}.$$  \hspace{1cm} (13.59)

$r_\pm$ is the distance between $x$ and the end of the rod at $z = \pm a$.

Using

$$z = \frac{r_+^2 - r_-^2}{4a},$$  \hspace{1cm} (13.60)

the expression in (13.58) can alternatively be written as

$$\phi = \frac{\eta}{4\pi} \ln \left( \frac{r_+ + r_- + 2a}{r_+ + r_- - 2a} \right).$$  \hspace{1cm} (13.61)

If you dutifully perform this algebra you’ll be rewarded by some remarkable cancellations. The expression in (13.61) shows that the equipotential surfaces are confocal ellipsoids — the foci are at $z = \pm a$. The solution is shown in Figure 13.2.
A slender body

In section 5.3, H considers an axisymmetric body \( B \) defined in cylindrical coordinates by

\[
\sqrt{x^2 + y^2} = \epsilon B(z). \tag{13.62}
\]

(I’m using different notation from H: above \( s \) is the cylindrical radius.) The integral equation in H is then

\[
1 = \frac{1}{4\pi} \int_{-1}^{1} \frac{f(\xi; \epsilon) \, d\xi}{\sqrt{(z - \xi)^2 + \epsilon^2 B(z)^2}}. \tag{13.63}
\]

I think it is easiest to attack this integral equation by first asymptotically estimating the integral

\[
\phi(s, z) = \frac{1}{4\pi} \int_{-1}^{1} \frac{f(\xi; \epsilon) \, d\xi}{\sqrt{(z - \xi)^2 + s^2}}, \quad \text{as } s \to 0. \tag{13.64}
\]

Notice that we can’t simply set \( s = 0 \) in (13.64) because the “simplified” integral,

\[
\frac{1}{4\pi} \int_{-1}^{1} \frac{f(\xi; \epsilon) \, d\xi}{|z - \xi|},
\]

is divergent. Instead, using the example below, we can show that

\[
\phi(s, z) = \frac{f(z)}{2\pi} \ln \left( \frac{2\sqrt{1 - z^2}}{s} \right) + \frac{1}{4\pi} \int_{-1}^{1} \frac{f(\xi; \epsilon) - f(z; \epsilon)}{|\xi - z|} \, d\xi + O(s). \tag{13.65}
\]

Thus the integral equation (13.63) is approximated by

\[
1 \approx \frac{f(z; \epsilon)}{2\pi} \ln \left( \frac{2\sqrt{1 - z^2}}{\epsilon B(z)} \right) + \frac{1}{4\pi} \int_{-1}^{1} \frac{f(\xi; \epsilon) - f(z; \epsilon)}{|\xi - z|} \, d\xi. \tag{13.66}
\]

Figure 13.2: Equipotential contours, \( \phi(x, z) \) in (13.61), surrounding a charged rod that lies along the \( z \)-axis between \( z = -a \) and \( z = a \). The surfaces are confocal ellipsoids.
As \( \epsilon \to 0 \) there is a dominant balance between the left hand side and the first term on the right, leading to

\[
f(z; \epsilon) \approx \frac{2\pi}{\ln \left( \frac{2\sqrt{1-z^2}}{\epsilon B(z)} \right)},
\]

\[
\approx \frac{2\pi}{L - \ln \left( \frac{B(z)}{2\sqrt{1-z^2}} \right)},
\]

(13.67)

where \( L \equiv \ln \frac{1}{\epsilon} \gg 1 \). Thus expanding the denominator in (13.68) we have

\[
f(z; \epsilon) \approx \frac{2\pi}{L} + \frac{2\pi}{L^2} \ln \left( \frac{B(z)}{2\sqrt{1-z^2}} \right) + \text{ord} \left( \frac{L^{-3}}{L} \right).
\]

(13.69)

This is the solution given by H. For many purposes we might as well stop at (13.67), which provides the sum to infinite order in \( L^{-n} \). However if we need a nice explicit result for the capacity,

\[
C(\epsilon) = \int_{-1}^{1} f(z; \epsilon) \, dz,
\]

(13.70)

then the series in (13.69) is our best hope.

**Example:** obtain the approximation (13.66). This is a good example of singularity subtraction. We subtract the nearly singular part from (13.64):

\[
\phi(s, z) = \frac{1}{4\pi} \int_{-1}^{1} \frac{f(\xi) - f(z)}{|z - \xi|^2 + s^2} \, d\xi + \frac{f(z)}{2\pi} \int_{-1}^{1} \frac{d\xi}{|z - \xi|^2 + s^2}.
\]

(13.71)

In the first integral on the right of (13.71) we can set \( s = 0 \) without creating a divergent integral: this move produces the final term in (13.66), with the denominator \( |z - \xi| \).

The final term in (13.71) is the potential of a uniform line density of charges on the segment \(-1 < z < 1\) i.e., the potential of a charged rod back in (13.58) (but now with \( a = 1 \)). We don’t need (13.58) in its full glory — we’re taking \( s \to 0 \) with \(-1 < z < 1\). In this limit (13.61) simplifies to

\[
\frac{1}{4\pi} \int_{-1}^{1} \frac{d\xi}{|z - \xi|^2 + s^2} \approx \frac{1}{2\pi} \ln \left( \frac{2\sqrt{1-z^2}}{s} \right).
\]

(13.72)

Thus we have

\[
\phi(s, z) = \frac{1}{4\pi} \int_{-1}^{1} \frac{f(\xi) - f(z)}{|z - \xi|} \, d\xi + \frac{f(z)}{2\pi} \ln \left( \frac{2\sqrt{1-z^2}}{s} \right).
\]

(13.73)

### 13.4 Problems

**Problem 13.1.** Find the leading-order behavior of

\[
H(\epsilon) = \int_{0}^{\pi} \frac{\cos x}{x^2 + \epsilon^2} \, dx, \quad \text{as} \ \epsilon \to 0.
\]

(13.74)

**Problem 13.2.** Consider the integral

\[
I(x) \equiv \int_{x}^{\infty} \frac{W(z)}{z} \, dz,
\]

(13.75)

where \( W(x) \) is a smooth function that decays at \( x = \infty \) and has a Taylor series expansion about \( x = 0 \):

\[
W(x) = W_0 + xW_0' + \frac{1}{2}x^2W_0'' + \cdots
\]

(13.76)
Some examples are: \( W(z) = \exp(-z) \), \( W(z) = \text{sech}(z) \), \( W(z) = (1 + z^2)^{-1} \) etc. (i) Show that the integral in (4) has an expansion about \( x = 0 \) of the form

\[
I(x) = W_0 \ln \left( \frac{1}{x} \right) + C - W_0' x - \frac{1}{4} W_0'' x^2 + O(x^3),
\]

where the constant \( C \) is

\[
C = \int_0^1 \left[ W(z) + W(1/z) - W_0 \right] \frac{dz}{z}.
\]

(ii) Evaluate \( C \) if \( W(z) = (1 + z^2)^{-1} \). (iii) Evaluate the integral exactly with \( W(z) = (1 + z^2)^{-1} \), and show that the expansion of the exact solution agrees with the formula above.

**Problem 13.3.** Find useful approximations to

\[
F(x) \overset{\text{def}}{=} \int_0^\infty \frac{du}{\sqrt{x^2 + u^2 + u^4}},
\]

as (i) \( x \to 0 \); (ii) \( x \to \infty \).

**Problem 13.4.** Find the first two terms in the \( \epsilon \to 0 \) asymptotic expansion of

\[
F(\epsilon) \overset{\text{def}}{=} \int_0^\infty \frac{dy}{(1 + y)^{1/2}(\epsilon^2 + y)}.
\]

**Problem 13.5.** Consider

\[
H(r) \overset{\text{def}}{=} \int_0^\infty \frac{x \, dx}{(r^2 + x)^{3/2}(1 + x)}.
\]

(i) First, with \( r \to 0 \), find the first two non-zero terms in the expansion of \( H \). (ii) With \( r \to \infty \), find the first two non-zero terms, counting constants and \( \ln r \) as the same order.

**Problem 13.6.** Find two terms in the expansion the elliptic integral

\[
K(m) \overset{\text{def}}{=} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m^2 \sin^2 \theta}},
\]

as \( m \uparrow 1 \).

**Problem 13.7.** Find three terms (counting \( \epsilon^n \) and \( \epsilon^n \ln \epsilon \) as different orders) in the expansion of the elliptic integral

\[
J(m) \overset{\text{def}}{=} \int_0^{\pi/2} \frac{\sqrt{1 - m \cos^2 \theta} \, d\theta}{d\theta},
\]

as \( m \uparrow 1 \).

**Problem 13.8.** This is H exercise 3.8. Consider the integral equation

\[
x = \int_{-1}^1 \frac{f(t; \epsilon)}{\epsilon^2 + (t - x)^2} \, dt,
\]

posed in the interval \(-1 \leq x \leq 1\). Assuming that \( f(x; \epsilon) \) is \( O(\epsilon) \) in the end regions where \( 1 - |t| = \text{ord}(\epsilon) \), obtain the first two terms in an asymptotic expansion of \( f(x; \epsilon) \) as \( \epsilon \to 0 \).

**Problem 13.9.** Show that as \( \epsilon \to 0 \):

\[
\int_0^1 \ln \frac{x}{\epsilon + x} \, dx = -\frac{1}{2} \ln^2 \left( \frac{1}{\epsilon} \right) - \frac{\pi^2}{6} + \epsilon \left( 1 - \frac{\epsilon}{4} + \frac{\epsilon^2}{9} - \frac{\epsilon^3}{16} + \cdots \right).
\]
Lecture 14

Boundary Layers

14.1 Stommel’s dirt pile

Consider a pile of dirt formed by a rain of sediment falling onto a conveyor belt. The belt stretches between \( x = 0 \) and \( x = \ell \) and moves to the left with speed \(-c\): see the figure. If \( h(x,t) \) denotes the height of a sandpile, then a very simple model is

\[
h_t - ch_x = s + \kappa h_{xx},
\]

with boundary conditions

\[
h(0,t) = 0, \quad \text{and} \quad h(\ell,t) = 0.
\]

The term \( s(x) \) on the right of (14.1) is the rate (meters per second) at which sand is accumulating on the belt.

We can make a sanity check by integrating (14.1) from \( x = 0 \) to \( x = \ell \):

\[
\frac{d}{dt} \int_0^\ell h(x,t) \, dx = \int_0^\ell s(x,t) \, dx + \kappa h_x(\ell,t) - \kappa h_x(0,t).
\]

Notice that the advective term, \( ch_x \), does not contribute to the budget above — advection is moving dirt but because \( h = 0 \) at the boundaries advection is not directly contributing to the fall of dirt over the edges.

The steady solution with a uniform source

If the sedimentation rate, \( s(x,t) \), is a constant then we can easily solve the steady state (\( t \to \infty \)) problem exactly:

\[
h(x, \infty) = \frac{s \ell}{c} \frac{1 - e^{-c x / \kappa}}{1 - e^{-c \ell / \kappa}} - \frac{sx}{c}.
\]

If the diffusion is very weak, meaning that

\[
\epsilon \overset{\text{def}}{=} \frac{\kappa}{c \ell} \ll 1,
\]

then there is a region of rapid variation, the boundary layer, at \( x = 0 \). This is where all the sand accumulated on the conveyor belt is pushed over the edge. Obviously if we reverse the direction of the belt, then the boundary layer will move to \( x = \ell \). Note we’re assuming that \( c > 0 \) so that \( \epsilon > 0 \).
Figure 14.1: Stommel’s boundary-layer problem.

Figure 14.2: The solution in \(14.7\). The solid curve is \(\epsilon = 0.1\) and the dashed curve is \(\epsilon = 0.025\).
If we introduce non-dimensional variables
\[ \bar{x} \overset{\text{def}}{=} \frac{x}{\ell}, \quad \text{and} \quad \bar{h} = \frac{ch}{s\ell}, \] (14.6)
then the solution in (14.4) is
\[ h(x, \epsilon) = \frac{1 - e^{-x/\epsilon}}{1 - e^{-1/\epsilon}} - x. \] (14.7)
This solution is shown in figure 14.2. We can consider two different limiting processes

1. **The outer limit:** \( \epsilon \to 0, \) with \( x \) fixed. Under this limit, the exact solution in (14.7) is \( h \to 1 - x. \) The outer limit produces a good approximation to the exact \( h(x, \epsilon) \), except close to \( x = 0 \) where the boundary condition is not satisfied.

2. **The inner limit:** \( \epsilon \to 0 \) with \( X \overset{\text{def}}{=} x/\epsilon \) fixed. Under this limit the exact solution in (14.7) is \( h \to 1 - e^{-X}. \) The inner limit produces a good approximation to the solution within the boundary layer. This is a small region in which \( x \) is order \( \epsilon. \) It is vital to understand that the term \( \epsilon h_{xx} \) is leading order within the boundary layer, and enables the solution to satisfy the boundary condition at \( x = 0. \)

Thus the function in (14.7) has two different asymptotic expansions. Each expansion is limited by non-uniformity as \( \epsilon \to 0. \)

### 14.2 Leading-order solution of the dirt-pile model

We want to take the inner and outer limits directly in the differential equation, **before** we have a solution. Understanding how to do this in the Stommel problem is one goal this lecture.

To make the problem a little more interesting, suppose that the sedimentation rate is some function of \( x \):
\[ s = s_{\text{max}} \bar{s} \left( \frac{x}{\ell} \right). \] (14.8)

We use \( s_{\text{max}} \) to define the non-dimensional \( \bar{h} \) back in (14.6). Dropping the bars, the non-dimensional problem is
\[ \epsilon h_{xx} + h_x = -s, \quad \text{with BCs:} \quad h(0) = h(1) = 0. \] (14.9)

We’re going to use boundary layer theory to obtain a quick and dirty leading-order solution of this problem. We’ll return later to a more systematic discussion.

**The outer expansion**

Start the attack on (14.9) with a regular perturbation expansion
\[ h(x, \epsilon) = h_0(x) + \epsilon h_1(x) + \epsilon^2 h_2(x) + \cdots \] (14.10)

We’re assuming that as \( \epsilon \to 0 \) with fixed \( x \) — the **outer limit** — that the solution has the structure in (14.17). Note that in the outer limit the \( h_n \)’s in (14.10) are independent of \( \epsilon. \)

**Exercise:** Consider the special case \( s = 1, \) with the exact solution in (14.7). Does the outer limit of that exact solution agree with the assumption in (14.10)?
The leading order is
\[ h_{0x} = -s, \]  
and we can solve this problem as
\[ h_0(x) = \int_s^x s(x') \, dx', \quad \text{or perhaps as} \quad h_0(x) = -\int_0^x s(x') \, dx'. \]  
Looking at the exact solution we know that the correct choice satisfies the BC at \( x = 1 \). We proceed with this inner solution and return later to show why the alternative ends in tears.

The inner expansion, and a quick-and-dirty matching argument

Now we can also define
\[ X \overset{\text{def}}{=} \frac{x}{\delta}, \quad \text{so that} \quad \frac{d}{dx} = \frac{1}{\delta} \frac{d}{dX}. \]  
\( \delta \) is the boundary layer thickness — we’re pretending that \( \delta \) is unknown. Using the inner variable \( X \), the problem (14.9) becomes is
\[ \epsilon \delta^{-2} h_{XX} + \delta^{-1} h_X = -s(\delta X). \]  
We get a nice two-term balance if
\[ \delta = \epsilon. \]  
With this definition of \( \delta \) we have the rescaled problem
\[ h_{XX} + h_X = -\epsilon s(\epsilon X). \]  
Now attack (14.16) with a regular perturbation expansion
\[ h(x, \epsilon) = H_0(X) + \epsilon H_1(X) + \epsilon^2 H_2(X) + \cdots \]  
Notice that in (14.17) we’re assuming that the \( H_n \)’s are independent of \( \epsilon \).

At leading order
\[ H_{0XX} + H_{0X} = 0, \quad \text{with solution} \quad H_0 = A_0 \left( 1 - e^{-X} \right). \]  
We’ve satisfied the BC at \( X = 0 \). But we still have an unknown constant \( A_0 \).

To determine \( A_0 \) we insist that “the inner limit of the outer solution is equal to the outer limit of the inner limit solution”. This means that there is a region of overlap in which
\[ A_0 \left( 1 - e^{-X} \right) \approx \int_s^x s(x') \, dx'. \]  
For instance, if \( x = \text{ord}(\epsilon^{1/2}) \ll 1 \) then \( X = \text{ord}((\epsilon^{-1/2}) \gg 1 \), and (14.19) tells us that
\[ A_0 = \int_0^1 s(x') \, dx'. \]
Construction of a uniformly valid solution

With $A_0$ determined by (14.20) we have completed the leading-order solution. We can combine our two asymptotic expansions into a single uniformly valid solutions using the recipe

$$\text{uniformly valid} = \text{outer} + \text{inner} - \text{match},$$

$$= \int_x^1 s(x') \, dx' + \int_0^1 s(x') \, dx' \left(1 - e^{-X}\right) - \int_0^1 s(x') \, dx',$$

$$= \int_x^1 s(x') \, dx' - \int_0^1 s(x') \, dx' e^{-x/\epsilon}.$$  

This is also known as the composite expansion.

Why can't we have a boundary layer at $x = 1$?

Now we return to (14.12) and discuss what happens if we make the incorrect choice

$$h_0(x) \equiv -\int_0^x s(x') \, dx'.$$

This outer solution satisfies the BC at $x = 0$. So we try to put a boundary layer at $x = 1$. Again we introduce a boundary-layer coordinate:

$$X \overset{\text{def}}{=} \frac{x - 1}{\delta},$$

so that $\frac{d}{dx} = \frac{1}{\delta} \frac{d}{dX}$.

The dominant balance argument convinces us that $\delta = \epsilon$, and using (14.17) we find exactly the same leading-order solution as before:

$$H_0 = A_0 \left(1 - e^{-X}\right),$$

except that now $X = \frac{x - 1}{\delta}$.

$H_0(X)$ above satisfies the BC at $X = 0$, which is the same as $x = 1$. But now when we attempt to match the outer solution in (14.24) it all goes horribly wrong: we take the limit $X \to -\infty$ and the exponential explodes. It is impossible to match the outer solution (14.24) with the inner solution in (14.26).

14.3 A nonlinear Stommel problem

Consider the Stommel model with nonlinear diffusivity:

$$\left(\frac{1}{2} h^2\right)_{xx} + h_x = -1, \quad h(0) = h(1) = 0.$$  

Where the pile is deeper there is more height for diffusion to move dirt around.

If we assume that the boundary layer is at $x = 0$ then easy calculations show that the leading-order interior solution is

$$h_0 = 1 - x,$$

and that the series continues as

$$h = \left(1 + \epsilon + 2\epsilon^2 + \cdots\right) (1 - x).$$
This perturbation series indicates that there is a simple exact solution that satisfies the $x = 1$ boundary condition:

$$h = A(\epsilon) (1 - x), \quad \text{where} \quad \epsilon A^2 - A + 1 = 0.$$  \hspace{1cm} (14.30)

This is pleasant, but it does not help with the boundary condition at $x = 0$. Introducing the boundary layer variable

$$X \overset{\text{def}}{=} x / \epsilon,$$  \hspace{1cm} (14.31)

we have the re-scaled equation

$$\left(\frac{1}{2} h^2\right)_{XX} + h_X = -\epsilon.$$  \hspace{1cm} (14.32)

We try for a solution with $h = H_0(X) + \epsilon H_1(X) + \cdots$. The leading-order equation is

$$\left(\frac{1}{2} H_0^2\right)_{XX} + H_{0X} = 0,$$  \hspace{1cm} (14.33)

which integrates to

$$H_0 H_{0X} + H_0 = C.$$  \hspace{1cm} (14.34)

This leading-order solution must satisfy both the $X = 0$ boundary condition and the matching condition

$$H_0(0) = 0, \quad \text{and} \quad \lim_{X \to \infty} H(X) = 1.$$  \hspace{1cm} (14.35)

If we apply the $x = 0$ boundary condition to (14.33), and assume that

$$\lim_{X \to 0} H_0 H_{0X} \overset{?}{=} 0,$$  \hspace{1cm} (14.36)

then we conclude that $C = 0$. But $C = 0$ in (14.34) quickly leads to $H_0 = -X$. This satisfies the boundary condition at $x = 0$, but not the matching condition. We are forced to consider that the limit above is non-zero. In that case we can determine the constant $C$ in (14.34) by matching to the interior. Thus $C = 1$ and

$$H_{0X} = \frac{1}{H_0} - 1.$$  \hspace{1cm} (14.37)

We solve (14.37) via separation of variables

$$\frac{H_0 dH_0}{1 - H_0} = dX,$$  \hspace{1cm} (14.38)

integrating to

$$-H_0 + \ln \frac{1}{1 - H_0} = X + A.$$  \hspace{1cm} (14.39)

Applying the boundary condition at $X = 0$ shows that $A = 0$, and thus $H_0(X)$ is determined implicitly by

$$H_0 = 1 - e^{-X - H_0}.$$  \hspace{1cm} (14.40)

This implicit solution is shown in figure 14.3. As $X \to \infty$ we use iteration to obtain the large-$X$ behaviour of the boundary layer solution

$$H_0(X) \sim 1 - e^{-X-1} + e^{-2X-2} + \cdots \quad \text{as} \quad X \to \infty.$$  \hspace{1cm} (14.41)

This demonstrates matching to the leading-order interior solution.
Might we find another solution of (14.27) with a boundary layer at \(x = 1\)? The answer is yes: (14.27) has both the reflection symmetry
\[
h \to -h, \quad \text{and} \quad x \to -x, \tag{14.42}
\]
and the translation symmetry
\[
x \to x + a. \tag{14.43}
\]
Thus we can define \(x_1 = x - 1/2\) so that the boundary conditions are applied at \(x_1 = \pm 1/2\). The reflection symmetry then implies that if \(h(x_1)\) is a solution then so is \(-h(-x_1)\). With this trickery the solution we’ve just described is transformed into a perfectly acceptable solution but with a boundary layer at the other end of the domain.

Exercise: assume that the boundary layer is at \(x = 1\), so that the leading-order outer solution is now \(h_0 = -x\). Construct the boundary-layer solution using the inner variable \(X = (x - 1)/\epsilon\) — you’ll be able to satisfy both the \(x = 1\) boundary condition and match onto the inner limit of the outer solution. Notice that this solution has \(h(x) \leq 0\).

Reformulation of the nonlinear diffusion model

But as a solution of the dirt-pile model the second solution above makes no sense: dirt piles can’t have negative height. And the physical intuition that put the boundary layer at \(x = 0\) can’t be wrong simply because we use a more complicated model of diffusion. The problem is that the nonlinear diffusion equation in (14.27) should be
\[
\frac{1}{2} |h| h_{xx} + h_x = -1, \quad h(0) = h(1) = 0. \tag{14.44}
\]
In other words, the diffusivity should vary with \(|h|\), not \(h\). Back in (14.27), our translation of the physical problem into mathematics was faulty. Changing \(h\) to \(|h|\) in destroys the symmetry in (14.42).

Now let’s use the correct model in (14.44) and show that the boundary layer cannot be at \(x = 1\). If we try to put the boundary layer at \(x = 1\) then the leading-order interior solution is
\[
h_0 = -x. \tag{14.45}
\]
Using the boundary layer coordinate
\[
X \overset{\text{def}}{=} \frac{x - 1}{\epsilon}, \tag{14.46}
\]
the leading-order boundary layer equation is
\[- \left( \frac{1}{2} H_0^2 \right)_{XX} + H_0X = 0, \tag{14.47}\]

Above we have assumed that $H_0(X) < 0$ so that $|H_0| = -H_0$. The differential equation in (14.50) must be solved with boundary and matching conditions

$$H_0(0) = 0, \quad \text{and} \quad \lim_{X \to -\infty} H_0 = -1. \tag{14.48}$$

The second condition above is matching onto the inner limit of the outer solution. We can integrate (14.50) and apply the matching condition to obtain

$$\frac{dH_0}{dX} = \frac{H_0 + 1}{H_0}. \tag{14.49}$$

Now if $-1 < H_0 < 0$ then the equation above implies that

$$\frac{dH_0}{dX} < 0. \tag{14.50}$$

The sign in (14.50) is not consistent with a solution that increases monotonically from $H_0(-\infty) = -1$ to $H_0(0) = 1$. Moreover if we integrate (14.49) with separation of variables we obtain an implicit solution

$$X = H_0 - \ln(1 + H_0), \quad \text{or equivalently} \quad H_0 = -1 + e^{-X + H_0}. \tag{14.51}$$

But as $X \to -\infty$ we do not get a match — the boundary layer cannot be at $x = 1$. Thus we cannot construct a solution of the $|h|$-model in (14.44) with a boundary layer at $x = 1$.

### 14.4 Stommel’s problem at infinite order

**Example:** To save chalk, in the lecture we use the particular source function $s = e^{a(x-1)}$ and assign the general $s(x)$ as reading.

The problem is

$$s h_{XX} + h_x = -e^{a(x-1)}, \quad \text{with BCs} \quad h(0) = h(1) = 0. \tag{14.52}$$

The interior solution, to infinite order, is

$$h = \frac{1}{1+e^{a}} \left[ 1 - e^{a(x-1)} \right] \frac{1}{a} \left[ 1 - e^{a(x-1)} \right]. \tag{14.53}$$

No matter how many terms we calculate, we will never satisfy the $x = 0$ boundary condition.

To expose the complete structure of higher-order boundary-layer problems let us discuss the form of the high-order terms in Stommel’s problem. Recall our model for the steady state sandpile is

$$s h_{XX} + h_x = -s. \tag{14.54}$$

We assume that the source $s(x)$ has the Taylor series expansion around $x = 0$:

$$s(x) = s_0 + s_0' x + \frac{1}{2} s_0'' x^2 + \cdots, \tag{14.55}$$

and around $x = 1$:

$$s(x) = s_1 + (x-1)s_1' + \frac{1}{2}(x-1)^2 s_1'' + \cdots \tag{14.56}$$
The outer solution

The leading-order outer problem is

\[ h_{0x} = -s, \quad \Rightarrow \quad h_0 = \int_1^x s(x') \, dx', \tag{14.57} \]

and the following orders are

\[
\begin{align*}
  h_{1x} &= -h_{0xx} = +s_x, \quad \Rightarrow \quad h_1 = s(x) - s_1, \\
  h_{2x} &= -h_{1xx} = -s_{xx}, \quad \Rightarrow \quad h_2 = s_1' - s_x(x), \\
  h_{3x} &= -h_{2xx} = +s_{xxx}, \quad \Rightarrow \quad h_3 = s_{xx}(x) - s_1''.
\end{align*}
\tag{14.58 - 14.60}

Notice that \( h_n(1) = 0 \). It is clear how this series continues to higher order. We can assemble the first three terms of the outer solution as

\[
  h = \int_0^1 s(x') \, dx' - \int_0^x s(x') \, dx' + \epsilon [s(x) - s_1] + \epsilon^2 [s_1' - s_x(x)] + O(\epsilon^3). \tag{14.61}
\]

The boundary-layer solution

In the boundary layer, we must expand the source in a Taylor series

\[
  s(\epsilon X) = s_0 + \epsilon X s_0' + \frac{1}{2} \epsilon^2 X^2 s_0'' + \cdots \tag{14.62}
\]

If we don’t expand the source then there is no way to collect powers of \( \epsilon \) and maintain our assumption that the \( H_n \)‘s in

\[
  h(x, \epsilon) = H_0(X) + \epsilon H_1(X) + \epsilon^2 H_2(X) + \cdots \tag{14.63}
\]
are independent of $\varepsilon$. The RPS above leads to

\begin{align}
H_{0XX} + H_{0X} &= 0, \quad \Rightarrow \quad H_0 = A_0 \left( 1 - e^{-X} \right), \tag{14.64} \\
H_{1XX} + H_{1X} &= -s_0, \quad \Rightarrow \quad H_1 = A_1 \left( 1 - e^{-X} \right) - s_0 \left( X - 1 + e^{-X} \right), \tag{14.65} \\
H_{2XX} + H_{2X} &= -s_0' X, \quad \Rightarrow \quad H_2 = A_2 \left( 1 - e^{-X} \right) - s_0' \left( \frac{1}{2} X^2 - X + 1 - e^{-X} \right). \tag{14.66}
\end{align}

At every order we’ve satisfied the boundary condition $H_n(0) = 0$. Matching determines the constants $A_n$.

**Matching**

In the matching region $X \gg 1$ and we simplify the boundary layer solution by neglecting all the exponentially small terms involving $e^{-X}$. This gives

\begin{align}
H &\approx \frac{A_0 + \varepsilon A_1 - \varepsilon s_0 (X - 1) + \varepsilon^2 A_2 - \varepsilon^2 s_0' \left( \frac{1}{2} X^2 - X + 1 \right) + O(\varepsilon^3)}{H_0 + \varepsilon H_1 + \varepsilon^2 H_2} + O(\varepsilon^3). \tag{14.67}
\end{align}

We rewrite the outer solution in terms of $X = x/\varepsilon$ and take the inner limit, keeping terms of order $\varepsilon^2$:

\begin{align}
h &\sim \int_0^1 s(x') \, dx' - \varepsilon s_0 X - \frac{1}{2} \varepsilon^2 s_0' X^2 + \varepsilon \left[ s_0 + \varepsilon X s_0' - s_1 \right] + \varepsilon^2 \left[ s_1 - s_0' \right] + O(\varepsilon^3). \tag{14.68}
\end{align}

The inner limit of $h_0(x)$ produces terms of all order in $\varepsilon$ — above we’ve explicitly written only terms up to $\text{ord}(\varepsilon^2)$.

A shotgun marriage between these different expansions (14.67) and (14.68) of the same function $h(x, \varepsilon)$ implies that

\begin{align}
A_0 &= \int_0^1 s(x') \, dx', \quad A_1 = -s_1, \quad A_2 = s_1'. \tag{14.69}
\end{align}

All the other terms in (14.67) and (14.68) match. Notice that terms from $h_0$ match terms from $H_1$ and $H_2$, and from $H_3$ if we continue to higher order. It is interesting that the boundary layer constants $A_1$ and $A_2$ are determined by properties of the source at $x = 1$ i.e., the BL near $x = 0$ is affected by the source near $x = 1$.

**The special case $s(x) = 1$**

This special case is very simple: the infinite-order, the boundary-layer solution is

\begin{align}
H &= \frac{1 - e^{-X}}{H_0} - \frac{\varepsilon X}{H_1}. \tag{14.70}
\end{align}

And the infinite-order outer solution is simply

\begin{align}
h &= \frac{1 - x}{h_0}. \tag{14.71}
\end{align}

All the higher-order terms are zero. With the recipe

\begin{align}
\text{uniform} = \text{outer} + \text{inner} - \text{match}, \tag{14.72}
\end{align}

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we assemble an infinite-order uniform approximation:

\[ h_{\text{uni}}(x) = 1 - x - e^{-x/\epsilon}. \] (14.73)

Notice that the exact solution is

\[ h(x) = \frac{1 - e^{-x/\epsilon}}{1 - e^{-1/\epsilon}} - x; \] (14.74)

this differs from the infinite-order approximation by the exponentially small \( e^{-1/\epsilon} \).
14.5 Problems

Problem 14.1. (i) Solve the boundary value problem

\[ h_x = \varepsilon h_{xx} + \sin x, \quad h(0) = h(\pi) = 0, \]

exactly. To assist communication, please use the notation

\[ X \overset{\text{def}}{=} \frac{x - \pi}{\varepsilon}, \quad \text{and} \quad E \overset{\text{def}}{=} e^{-\pi/\varepsilon}. \]

This should enable you to write the exact solution in a compact form. (ii) Find the first three terms in the regular perturbation expansion of the outer solution

\[ h(x) = h_0(x) + \varepsilon h_1(x) + \varepsilon^2 h_2(x) + O(\varepsilon^3). \]

(iii) There is a boundary layer at \( x = \pi \). "Rescale" the equation using \( X \) above as the independent variable and denote the solution in the boundary layer by \( H(X) \). Find the first three terms in the regular perturbation expansion of the boundary-layer equation:

\[ H = H_0(X) + \varepsilon H_1(X) + \varepsilon^2 H_2(X) + O(\varepsilon^3). \]

(iv) The \( H_n \)'s above will each contain an unknown constant. Determine the three constants by matching to the interior solution. (v) Construct a uniformly valid solution, up to an including terms of order \( \varepsilon^2 \). You can check your algebra by comparing your boundary layer solution with the expansion of the exact solution from part (i). (vi) With \( \varepsilon = 0.2 \) and 0.5, use MATLAB to compare the exact solution from part (i) with the approximation in part (v).

Problem 14.2. Find a leading-order boundary layer solution to the forced Burgers equation

\[ \varepsilon h_{xx} + \left( \frac{1}{2} h^2 \right)_x = -1, \quad h(0) = h(1) = 0. \]

Use bvp4c to solve this problem numerically, and compare your leading order solution to the numerical solution: see figure 14.5.

Problem 14.3. The result of problem 14.2 is disappointing: even though \( \varepsilon = 0.05 \) seems rather small, the approximation in Figure 14.5 is only so-so. Calculate the next correction and compare the new improved solution with the bvp4c solution. (The numerical solution seems to have finite slope at \( x = 1 \), while the leading-order outer solution has infinite slope as \( x \to 1 \): perhaps there a higher-order boundary layer at \( x = 1 \) is required to heal this singularity?)
Problem 14.4. Use boundary layer theory to find leading order solution of

\[ h_x = \varepsilon \left( \frac{1}{2} h^3 \right)_{xx} + 1, \tag{14.75} \]

on the domain \( 0 < x < 1 \) with boundary conditions \( h(0) = h(1) = 0 \). You can check your answer by showing that \( h = 1/2 \) at \( x \approx 1 - 0.057\varepsilon \).
Lecture 15

More boundary layer theory

15.1 Variable speed

Suppose the conveyor belt is a stretchy membrane which moves with non-uniform speed \(-c(x)\). With non-constant \(c\), the dirt conservation equation in (14.1) generalizes to

\[
h_t - (ch)_x = s + \kappa h_{xx},
\]

with boundary conditions unchanged: \(h(0, t) = 0\) and \(h(\ell, t) = 0\).

Exercise: Make sure you understand why it is \((ch)_x\), rather than \(ch_x\), in (15.1). Nondimensionalize \((15.1)\) so that the steady state problem is

\[
ch_{xx} - (ch)_x = -s, \quad h(0) = h(1) = 0,
\]

with max \(c(x) = 1\) and max \(s(x) = 1\).

Example: A slow down

Suppose that the belt slows near \(x = 0\). Specifically, let’s assume that the belt speed is

\[
c = \sqrt{x}.
\]

The speed is zero at \(x = 0\), so we expect that dirt will start to pile up. If the source is uniform then the steady-state problem is

\[
\epsilon h_{xx} + (\sqrt{x} h)_x = -1, \quad \text{with BCs } h(0) = h(1) = 0.
\]

Exercise: Show that a particle starting at \(x = 1\) and moving with \(\dot{x} = -x^\beta\), with \(\beta < 1\), reaches \(x = 0\) in a finite time. What happens if \(\beta \geq 1\)?

The first two terms in the interior solution are

\[
h(x, \epsilon) = \left(x^{-1/2} - x^{1/2}\right) + \epsilon \left(\frac{1}{2}x^{-2} + \frac{1}{2}x^{-1} - x^{-1/2}\right) + \text{ord}(\epsilon^2).
\]

We’ve satisfied the BC at \(x = 1\) and the pile-up at \(x = 0\) is evident via the divergence of the outer solution as \(x \to 0\). Notice that this divergence is stronger at higher orders, and the RPS above is disordered as \(x \to 0\).

Turning to the boundary layer at \(x = 0\), we introduce

\[
X \overset{\text{def}}{=} \frac{x}{\delta}
\]
so that
\[ \frac{\varepsilon}{\delta^2} h_{XX} + \frac{1}{\delta^{1/2}} \left( \sqrt{X} h \right)_X = -1. \] (15.7)

A dominant balance between the first two terms is achieved with \( \varepsilon = \delta^{3/2} \), or
\[ \delta = \varepsilon^{2/3}. \] (15.8)

With this definition of \( \delta \), and
\[ h = H(X, \varepsilon), \] (15.9)
the boundary layer equation is
\[ H_{XX} + \left( \sqrt{X} H \right)_X = -\varepsilon^{1/3}. \] (15.10)

We attack with an RPS: \( h = H_0(X) + \varepsilon^{1/3} H_1(x) + \cdots \)

At leading order
\[ H_{0XX} + \left( \sqrt{X} H_0 \right)_X = 0, \] (15.11)
with solution
\[ H_0(X) = A_0 e^{-\frac{2}{x^{3/2}}} \int_0^X e^{\frac{2t^{3/2}}{3}} dt. \] (15.12)

We’ve satisfied the boundary condition at \( x = 0 \), and we must determine the remaining constant of integration \( A_0 \) by matching to the interior solution.

To match the interior, we need the asymptotic expansion of (15.12) as \( X \to \infty \): this can be obtained by following our earlier discussion of Dawson’s integral:
\[ H_0(X) \sim \frac{A_0}{\sqrt{X}}, \quad \text{as} \quad X \to \infty, \] (15.13)
\[ = \frac{\varepsilon^{1/3} A_0}{x^{1/2}}. \] (15.14)

On the other hand the inner expansion of the outer solution in (15.5) is
\[ h = \frac{1}{x^{1/2}} + O \left( x^{-1/2}, \varepsilon x^{-2} \right). \] (15.15)

We almost have a match — it seems we should take \( A_0 = \varepsilon^{-1/3} \) in (15.14) so that both functions are equal to \( x^{-1/2} \) in the matching region. But remember that we assumed that \( H_0(x) \) is independent of \( \varepsilon \), so \( A_0 \) cannot depend on \( \varepsilon \). Our expansion has failed.

**Exercise**: How would you gear so that the term \( \varepsilon x^{-2} \) in (15.14) is asymptotically negligible relative to \( x^{-1/2} \) in the matching region?

Fortunately there is a simple cure: the correct definition of the boundary layer solution — which replaces (15.9) — is
\[ h = \varepsilon^{-1/3} h(X, \varepsilon). \] (15.16)

In retrospect perhaps the rescaling in (15.16) is obvious — the interior RPS in (15.5) is becoming disordered as \( x \to 0 \). The problem is acute once the second term in the expansion is comparable to the first term, which happens once
\[ x^{-1/2} \sim \varepsilon x^{-2} \quad \text{or} \quad x \sim \varepsilon^{2/3} = \delta. \] (15.17)

This the boundary layer scale, and as we enter this region the interior solution is of order \( x^{-1/2} \sim \varepsilon^{-1/3} \) — this is why the rescaling in (15.16) is required. If we’d been smart we would
have made this argument immediately after (15.5) and avoided the mis-steps in (15.9) and (15.10).

Using the rescaled variable in (15.14), the boundary layer equation that replaces (15.10) is

$$H_{XX} + \left(\sqrt{X} H\right)_X = -\frac{\epsilon}{2}^{2/3}. \quad (15.18)$$

Now we can try the RPS

$$H(X, \epsilon) = H_0(X) + \epsilon^{2/3} H_1(X) + \cdots \quad (15.19)$$

We quickly find the leading order solution

$$H_0 = e^{-2X^{3/2}/3} \int_0^X e^{2t^{3/2}/3} \, dt. \quad (15.20)$$

This satisfies the $x = 0$ boundary condition and also matches the $x^{-1/2}$ from the interior.

We can now construct a leading-order uniformly valid solution as

$$h_{uni}(x, \epsilon) = e^{-1/3} e^{-2X^{3/2}/3} \int_0^X e^{2t^{3/2}/3} \, dt - x^{1/2}. \quad (15.21)$$

Figure [15.1] compares the uniformly valid approximation (15.21) with an exact solution of (15.4).

**Exercise:** evaluate the integral \( \int_0^1 h(x, \epsilon) \, dx \) to leading order as \( \epsilon \to 0 \).

**Example: higher-order corrections**

To illustrate how to bash out higher order corrections let’s calculate the first two terms in the BL solution of the BVP

$$\epsilon h_{xx} + [e^x h]_x = -2e^{2x}, \quad \text{with BCs} \quad h(0) = h(1) = 0. \quad (15.22)$$

We suspect there is a BL at $x = 0$. So we first develop the interior solution

$$h(x, \epsilon) = h_0(x) + \epsilon h_1(x) + \epsilon h_2(x) + \cdots \quad (15.23)$$

by satisfying the boundary condition at $x = 1$ at every order.

The leading-order term is

$$[e^x h_0]_x = -2e^{2x}, \quad \Rightarrow \quad h_0 = e^{2-x} - e^x. \quad (15.24)$$
The next two orders are

\[
[e^x h_1]_x = -h_{0xx}, \quad \Rightarrow \quad h_1 = 1 - 2e^{1-x} + e^{2-2x}, \quad (15.25)
\]

\[
[e^x h_2]_x = -h_{1xx}, \quad \Rightarrow \quad h_2 = 2 \left( e^{2-3x} - e^{1-2x} \right). \quad (15.26)
\]

Later, to perform the match, we will need the inner limit of this outer solution. So in preparation for that, as \( x \to 0 \),

\[
\begin{align*}
&h_0 + \epsilon h_1 + \epsilon^2 h_2 = (e^2 - 1) - (e^2 + 1)x + \frac{1}{2}(e^2 - 1)x^2 \\
&\quad + \epsilon(1 - e^2) - \epsilon x 2(e^2 - e) \\
&\quad + \epsilon^2(2(e^2 - e) + \text{ord}(x^3, \epsilon x^2, \epsilon^2 x)). \quad (15.27)
\end{align*}
\]

Turning to the boundary layer, we use the inner variable \( X = x/\epsilon \) so that the rescaled differential equation is

\[
h_{XX} + [e^X h]_X = -2e^{2x}. \quad (15.28)
\]

We substitute the inner expansion

\[
h = H_0(X) + \epsilon H_1(X) + \epsilon^2 H_2(X) + \cdots \quad (15.29)
\]

into the differential equation and collect powers of \( \epsilon \). The first three orders of the boundary-layer problem are

\[
\begin{align*}
H_{0XX} + H_0X &= 0, \quad (15.30) \\
H_{1XX} + [H_1 + XH_0]_X &= -2, \quad (15.31) \\
H_{2XX} + \left[H_2 + XH_1 + \frac{1}{2}X^2H_0\right]_X &= -4X. \quad (15.32)
\end{align*}
\]

Note that it is necessary to expand the exponentials within the boundary layer, otherwise we cannot ensure that the \( H_n \)'s do not depend on \( \epsilon \).

The solution for \( H_0 \) that satisfies the boundary condition at \( x = 0 \), and also matches the first term on the right of (15.27), is

\[
H_0 = (e^2 - 1) \left( 1 - e^{-X} \right). \quad (15.33)
\]

The solution for \( H_1 \) that satisfies the boundary condition at \( x = 0 \) is

\[
H_1 = A_1(1 - e^{-X}) + (e^2 + 1) \left( 1 - X - e^{-X} \right) + \frac{1}{2}(e^2 - 1)X^2e^{-X}. \quad (15.34)
\]

The constant \( A_1 \) is determined by matching to the interior solution. We can do this by taking the limit as \( X \to \infty \) in the boundary layer solution \( H_0 + \epsilon H_1 \). Effectively this means that all terms involving \( e^{-X} \) are exponentially small and therefore negligible in the matching. To help with pattern recognition we rewrite the outer limit of the boundary-layer solution in terms of the outer variable \( x \). Thus, in the matching region where \( X \gg 1 \) and \( x \ll 1 \), the boundary-layer solution in (15.33) and (15.34) is:

\[
H_0 + \epsilon H_1 \to (e^2 - 1) + \epsilon A_1 + \epsilon(1 + e^2) - (1 + e^2)x. \quad (15.35)
\]

To match the first term on the second line of (15.27) with (15.35) we require

\[
\epsilon A_1 + \epsilon(1 + e^2) = \epsilon(1 - e)^2, \quad \Rightarrow \quad A_1 = -2e. \quad (15.36)
\]
The final term in (15.35), namely $-(1 + e^2)x$, matches against a term on the first line of (15.27). That’s interesting, because $-(1 + e^2)x$ comes from $H_1$ and matches against $h_0$.

There are many remaining unmatched terms in (15.27) e.g., $\frac{1}{2}(e^2 - 1)x^2$ on the first line. This term will match against terms from $H_2$ i.e., it will require an infinite number of terms in the boundary layer expansion just to match terms arising from the expansion of the leading-order interior solution.

Now we construct a uniformly valid approximation using the recipe

$$\text{uniform} = \text{outer} + \text{inner} - \text{match}.$$  \hfill (15.37)

This gives

$$h_{\text{uni}} = e^{2-x} - e^x - (e^2 - 1)e^{-x}$$

$$+ \epsilon \left[ 1 - 2e^{1-x} + e^{2-2x} + e^{-x} \left( \frac{1}{2}x^2(e^2 - 1) - (e - 1)^2 \right) \right].$$  \hfill (15.38)

This construction satisfies the $x = 0$ boundary condition exactly. But there is an exponentially small embarrassment at $x = 1$. Figure 15.2 compares the numerical solution of (15.22) with the approximation in (15.38). At $\epsilon = 0.2$ the two-term approximation is significantly better than just the leading-order term. We don’t get line-width agreement — the $\epsilon^2$ term would help.
function StommelBL
% Solution of epsilon h_{xx} + (exp(x) h)_x = - 2 \exp(2 x)
epsilon = 0.2;
solinit = bvpinit(linspace(0 , 1 , 10) , @guess);
sol = bvp4c(@odez,@bcs,solinit);
% My fine mesh
xx = linspace(0,1,100); hh = deval(sol,xx);
figure; subplot(2,1,1)
hold on
plot(xx , hh(1,:),'k')
xlabel('$x$','interpreter','latex','fontsize',16)
ylabel('$h$','interpreter','latex','fontsize',16,'rotation',0)
axis([0 1 0 3.5])
% The BL solution
XX = xx/epsilon; EE =exp(-XX);
hZero = exp(2-xx) - exp(xx) - (exp(2) - 1).*EE;
hOne = 1 - 2*exp(1-xx) + exp(2-2*xx)...
    + EE.*( 0.5*XX.^2*(exp(2) - 1) - (exp(1) - 1).^2);
plot(xx, hZero+epsilon*hOne,'-.r' , xx,hZero,'--g')
legend('bvp4c','two terms' , 'one term')
text(0.02,3.2,'$\epsilon =0.2$','interpreter','latex','fontsize',16)

% The differential equations
function dhdx = odez(x,h)
    dhdx = [h(2)/epsilon ; ...%
        - exp(x).*h(2)/epsilon - exp(x).*h(1) - 2*exp(2*x)];
end

% residual in the boundary condition
function res = bcs(um,up)
    res = [um(1) ; up(1) ];
end

% Initial guess at the solution
function hinit = guess(x)
    hinit = [(1-x^2) ; 2*x];
end
end
15.2 A second-order BVP with a boundary layer

At the risk of repetition, let’s discuss another elementary example of boundary layer theory, focussing on intuitive concepts and on finding the leading-order uniformly valid solution. We use the BVP

\[ \varepsilon y'' + a(x)y' + b(x)y = 0, \quad \text{(15.39)} \]

with BCs

\[ y(0) = p, \quad y'(1) = q, \quad \text{(15.40)} \]

as our model.

The case \( a(x) > 0 \)

In the outer region we can look for a solution with the expansion

\[ y(x, \varepsilon) = y_0(x) + \varepsilon y_1(x) + \cdots \quad \text{(15.41)} \]

The leading-order term satisfies

\[ ay_0'' + by_0 = 0. \quad \text{(15.42)} \]

Solutions of this first order differential equation cannot satisfy two boundary conditions. With the construction

\[ y_0(x) = q \exp \left( \int_1^x \frac{b(v)}{a(v)} dv \right) \quad \text{(15.43)} \]

we have satisfied the boundary condition at \( x = 1 \). We return later to discuss why this is the correct choice if \( a(x) > 0 \). (If \( a(x) < 0 \) the boundary layer is at \( x = 1 \) and the outer solution should satisfy the boundary condition at \( x = 0 \).)

Unless we’re very lucky, \( \text{(15.43)} \) will not satisfy the boundary condition at \( x = 0 \). Let’s try to fix this problem by building a boundary layer at \( x = 0 \). Begin by introducing a boundary layer coordinate

\[ X \overset{\text{def}}{=} \frac{x}{\varepsilon}, \quad \text{(15.44)} \]

and writing

\[ y(x, \varepsilon) = Y(X, \varepsilon). \quad \text{(15.45)} \]

Then “re-scale” the differential equation \( \text{(15.39)} \) using the boundary-layer variables:

\[ Y_{XX} + a(\varepsilon X)Y_X + \varepsilon b(\varepsilon X)Y = 0. \quad \text{(15.46)} \]

Notice that within the boundary layer, where \( X \overset{\text{ord}(1)}{=} 1 \),

\[ a(\varepsilon X) = a(0) + \varepsilon X a'(0) + \frac{1}{2} \varepsilon^2 X^2 a''(0) + \text{ord} \left( \varepsilon^3 \right). \quad \text{(15.47)} \]

There is an analogous expansion for \( b(\varepsilon X) \).

In the boundary layer we use the “inner expansion”:

\[ Y(X, \varepsilon) = Y_0(X) + \varepsilon Y_1(X) + \cdots \quad \text{(15.48)} \]

The leading-order term is

\[ \varepsilon^0 : \quad Y_{0XX} + a(0)Y_{0X} = 0, \quad \text{(15.49)} \]

and, for good measure, the next term is

\[ \varepsilon^1 : \quad Y_{1XX} + a(0)Y_{1X} + a'(0)XY_{0X} + b(0)Y_0 = 0. \quad \text{(15.50)} \]
Terms in the Taylor series \((15.47)\) will impact the higher orders.

The solution of \((15.49)\) that satisfies the boundary condition at \(X = 0\) is

\[
Y_0 = p + A_0 \left( 1 - e^{-a(0)X} \right), \quad (15.51)
\]

where \(A_0\) is a constant of integration. We are assuming that \(a(0) > 0\) so that the exponential in \((15.51)\) decays to zero as \(X \to \infty\). This is why the boundary layer must be at \(x = 0\). The constant \(A_0\) can then be determined by demanding that in the outer solution \((15.43)\) agrees with the inner solution \((15.51)\) in the matching region where \(X \gg 1\) and \(x \ll 1\). This requirement determines \(A_0\):

\[
p + A_0 = q \exp \left( \int_0^1 \frac{b(v)}{a(v)} dv \right). \quad (15.52)
\]

Hence the leading order boundary-layer solution is

\[
Y_0 = p + \left[ q \exp \left( \int_0^1 \frac{b(v)}{a(v)} dv \right) - p \right] \left( 1 - e^{-a(0)X} \right), \quad (15.53)
\]

\[
= p e^{-a(0)X} + q \exp \left( \int_0^1 \frac{b(v)}{a(v)} dv \right) \left( 1 - e^{-a(0)X} \right). \quad (15.54)
\]

We construct a uniformly valid solutions using the earlier recipe

\[
Y_{uni} = q \exp \left( \int_x^1 \frac{b(v)}{a(v)} dv \right) + \left[ p - q \exp \left( \int_0^1 \frac{b(v)}{a(v)} dv \right) \right] e^{-a(0)X}. \quad (15.56)
\]

### 15.3 Other BL examples

Not all boundary layers have thickness \(\epsilon\). Let’s quickly consider some examples.

**Example:**

\[
\epsilon y'' - y = -f(x), \quad y(-1) = y(1) = 0, \quad (15.57)
\]

If we solve the simple case with \(f(x) = 1\) exactly we quickly see that \(y \approx 1\), except that there are boundary layers with thickness \(\sqrt{\epsilon}\) at both \(x = 0\) and \(x = 1\).

Thus we might hope to construct the outer solution of \((15.57)\) via the RPS

\[
y = f + \epsilon f' + \epsilon^2 f'' + \text{ord}(\epsilon^3). \quad (15.58)
\]

The outer solution above doesn’t satisfy either boundary condition: we need boundary layers at \(x = -1\), and at \(x = +1\).

Turning to the boundary layer at \(x = -1\) we introduce

\[
X \overset{\text{def}}{=} \frac{x + 1}{\sqrt{\epsilon}}, \quad \text{and} \quad y(x, \epsilon) = Y(X, \sqrt{\epsilon}). \quad (15.59)
\]

The re-scaled differential equation is

\[
Y_{XX} - Y = f(-1 + \sqrt{\epsilon}X), \quad (15.60)
\]

and we look for a solution with

\[
Y = Y_0(X) + \sqrt{\epsilon}Y_1(X) + \epsilon Y_2(X) + \cdots \quad (15.61)
\]

The leading-order problem is

\[
Y_{0XX} - Y_0 = -f(-1), \quad (15.62)
\]
with solution
\[ Y_0 = f(-1) + A_0 e^{-X} + B_0 e^X. \] (15.63)

We quickly set the constant of integration \( B_0 \) to zero — the alternative would prevent matching with the interior solution. Then the other constant of integration \( A_0 \) is determined so that the boundary condition at \( X = 0 \) is satisfied:
\[ Y_0 = f(-1) \left( 1 - e^{-X} \right). \] (15.64)

The boundary condition at \( x = +1 \) is satisfied with an analogous construction using the coordinate \( X \) \( \overset{\text{def}}{=} \) \( (x - 1)/\sqrt{\epsilon} \). One finds
\[ Y_0 = f(1) \left( 1 - e^X \right). \] (15.65)

Notice that the outer limit of this boundary layer is obtained by taking \( X \to -\infty \).

Finally we can construct a uniformly valid solutions via
\[ y^{\text{uni}}(x) = f(x) - f(-1) e^{-(x+1)/\sqrt{\epsilon}} - f(1) e^{(x-1)/\sqrt{\epsilon}}. \] (15.66)

Example:
\[ \epsilon y'' + y = f(x), \quad y(0) = y(1) = 0, \] (15.67)

If we solve the simple case with \( f(x) = 1 \) exactly we quickly see that this is not a boundary layer problem. This belongs in the WKB lecture.

Example: Find the leading order BL approximation to
\[ \epsilon u'' - u = -\frac{1}{\sqrt{1-x^2}}, \quad \text{with BCs} \quad u(\pm 1) = 0. \] (15.68)

The leading-order outer solution is
\[ u_0 = \frac{1}{\sqrt{1-x^2}}. \] (15.69)

Obviously this singular solution doesn’t satisfy the boundary conditions. We suspect that there are boundary layers of thickness \( \sqrt{\epsilon} \) at \( x = \pm 1 \). Notice that the interior solution (15.69) is \( \sim \epsilon^{-1/4} \) as \( x \) moves into this BL. Moreover, considering the BL at \( x = -1 \), we use \( X = (1 + x)/\sqrt{\epsilon} \) as the boundary layer coordinate, so that
\[ \frac{1}{\sqrt{1-x^2}} = \frac{1}{\epsilon^{1/4} \sqrt{X(2-\sqrt{\epsilon})}}. \] (15.70)

Hence we try a boundary-layer expansion with the form
\[ u(x, \epsilon) = \epsilon^{-1/4} \left[ U_0(X) + \sqrt{\epsilon} U_1(X) + \text{ord}(\epsilon) \right]. \] (15.71)

A main point of this example is that it is necessary to include the factor \( \epsilon^{-1/4} \) above.

The leading-order term in the boundary layer expansion is then
\[ U_0'' - U_0 = -\frac{1}{\sqrt{X}}, \] (15.72)

which we solve using variation of parameters
\[ U_0(X) = \frac{1}{2} e^{-X} \int_0^X \frac{e^v}{\sqrt{v}} \, dv - \frac{1}{2} e^X \int_0^X \frac{e^{-v}}{\sqrt{v}} \, dv + P e^X + Q e^{-X}. \] (15.73)

The boundary condition at \( X = 0 \) requires
\[ P + Q = 0. \] (15.74)

To match the outer solution as \( X \to \infty \), we must use the \( X \to \infty \) asymptotic expansion of the integrals in (15.73), indicated via the underbrace. We determine \( P \) so that the exponentially growing terms are eliminated, which requires that \( P = \sqrt{\pi}/2 \). Thus the boundary layer solution is
\[ U_0(X) = \sqrt{\pi} \sinh X - \int_0^X \frac{\sinh(X - v)}{\sqrt{v}} \, dv. \] (15.75)

(Must check this, and then construct the uniformly valid solution!)
Example: Find the leading order BL approximation to
\[ \epsilon y'' + xy' + x^2 y = 0, \]  
with BCs \( y(0) = p, \quad y(1) = q. \) \hspace{1cm} (15.76)
We divide and conquer by writing the solutions as
\[ y = p f(x, \epsilon) + q g(x, \epsilon), \]  
where
\[ \epsilon f'' + x f' + x^2 f = 0, \]  
with BCs \( f(0) = 1, \quad f(1) = 0, \) \hspace{1cm} (15.77)
and
\[ \epsilon g'' + x g' + x^2 g = 0, \]  
with BCs \( g(0) = 0, \quad g(1) = 1, \) \hspace{1cm} (15.78)
The outer solution of the \( g \)-problem is
\[ g = e^{(1-x^2)/2} + \epsilon g_1 + \cdots. \] \hspace{1cm} (15.80)
We need a BL at \( x = 0 \). A dominant balance argument shows that the correct BL variable is
\[ X = \frac{x}{\sqrt{\epsilon}}, \]
and if \( g(x, \epsilon) = G(X, \sqrt{\epsilon}) \) then the rescaled problem is
\[ G_{XX} + XG_X + \epsilon X^2 G = 0. \] \hspace{1cm} (15.82)
The leading-order problem is
\[ G_{0XX} + XG_{0X} = 0, \]
with general solution
\[ G_0 = P + \frac{Q}{\sqrt{\epsilon/2\pi}} \int_0^X e^{-v^2/2} \, dv. \] \hspace{1cm} (15.84)
To satisfy the \( X = 0 \) boundary condition we take \( P = 0 \), and to match the outer solution we require
\[ Q \int_0^\infty e^{-v^2/2} \, dv = \sqrt{\epsilon}. \] \hspace{1cm} (15.85)
The uniformly valid solution is
\[ g_{uni}(x, \epsilon) = e^{(1-x^2)/2} + \frac{e^{x/\sqrt{\epsilon}}}{\sqrt{2\pi}} \int_0^{x/\sqrt{\epsilon}} e^{-v^2/2} \, dv - \sqrt{\epsilon}, \] \hspace{1cm} (15.86)
\[ = e^{(1-x^2)/2} - \frac{e^{x/\sqrt{\epsilon}}}{\sqrt{2\pi}} \int_{x/\sqrt{\epsilon}}^\infty e^{-v^2/2} \, dv. \] \hspace{1cm} (15.87)
Now turn to the \( f \)-problem. The outer solution is \( f_n(x) = 0 \) at all orders. The solution of the leading-order boundary-layer problem is
\[ F_0(X) = \frac{1}{\sqrt{2\pi}} \int_X^\infty e^{-v^2/2} \, dv. \] \hspace{1cm} (15.88)
This is a stand-alone boundary layer.

Example: Let’s analyze the higher-order terms in the BL solution of our earlier example
\[ \epsilon y''' - y = -f(x), \quad y(-1) = y(1) = 0. \] \hspace{1cm} (15.89)
Our provisional outer solution is
\[ y(x) = f(x) + \epsilon f''(x) + \epsilon^2 f'''(x) + \text{ord}(\epsilon^3). \] \hspace{1cm} (15.90)
Let’s rewrite this outer solution in terms of the inner variable \( \lambda \) \hspace{1cm} (15.91)
\[ \lambda \defeq \frac{x - 1}{\sqrt{\epsilon}} \]
\[ y(x) = f(1 + \sqrt{\epsilon} X) + \epsilon f''(1 + \sqrt{\epsilon} X) + \epsilon^2 f'''(1 + \sqrt{\epsilon} X) + \text{ord}(\epsilon^3). \] \hspace{1cm} (15.91)
Assuming that \( \sqrt{\epsilon} X \) is small in the matching region, we expand the outer solution:
\[ y(x) = f(1) + \sqrt{\epsilon} X f'(1) + \epsilon \left( \frac{1}{2} x X^2 + 1 \right) f''(1) + \epsilon^3 \left( X + \frac{1}{6} x X^3 \right) f'''(1) \]
\[ + \epsilon^2 \left( 1 + \frac{1}{2} x X^2 + \frac{1}{24} X^4 \right) f'''(1) + \text{ord} \left( \epsilon^{3/2} \right). \hspace{1cm} (15.92)\]
We hope that the outer expansion of the inner solution at \( x = 1 \) will match the series above. The rescaled inner problem is
\[
Y_{XX} - Y = -f(1 + \sqrt{\varepsilon}X), \quad (15.93)
\]
\[
= -f(1) - \sqrt{\varepsilon} f'(1) - \varepsilon \frac{1}{2} X^2 f''(1) + \text{ord}(\varepsilon^{3/2}). \quad (15.94)
\]
The RPS is
\[
Y = f(1) \left(1 - e^{X}\right) + \sqrt{\varepsilon} Y_1(X) + \varepsilon Y_2(X) + \varepsilon^{3/2} Y_3(X) \text{ord}(\varepsilon^2), \quad (15.95)
\]
with
\[
Y_1'' - Y_1 = -X f'(1), \quad (15.96)
\]
\[
Y_2'' - Y_2 = -\frac{1}{2} X^2 f''(1), \quad (15.97)
\]
\[
Y_3'' - Y_3 = -\frac{1}{6} X^3 f'''(1). \quad (15.98)
\]
We solve the equations above, applying the boundary condition \( Y_n(0) = 0 \), to obtain
\[
Y_1(X) = X f'(1), \quad Y_2(X) = \left(1 + \frac{1}{2} X^2 - e^X\right) f''(1), \quad (15.99)
\]
and
\[
Y_3(X) = \left(X + \frac{1}{6} X^3\right) f'''(1). \quad (15.100)
\]
Notice how the inner limit of the leading-order outer solution, \( y_0(x) = f(x) \), produces terms at all orders in the matching region. In order to match all of \( y_0(x) \) one requires all the \( Y_n(X) \)’s.

### 15.4 Problems

**Problem 15.1.** Find the leading-order uniformly valid boundary-layer solution to the Stommel problem
\[
- (e^x g)_x = \varepsilon g_{xx} + 1, \quad \text{with BCs} \quad g(0) = g(1) = 0. \quad (15.101)
\]
Do the same for
\[
(e^x f)_x = \varepsilon f_{xx} + 1, \quad \text{with BCs} \quad f(0) = f(1) = 0. \quad (15.102)
\]

**Problem 15.2.** Analyze the variable-speed Stommel problem
\[
\varepsilon h'' + (x^a h)_x = -1, \quad \text{with BCs} \quad h(0) = h(1) = 0, \quad (15.103)
\]
using boundary layer theory. (The case \( a = 1/2 \) was discussed in the lecture.) How thick is the boundary layer at \( x = 0 \), and how large is the solution in the boundary layer? Check your reasoning by constructing the leading-order uniformly valid solution when \( a = -1, a = 1 \) and \( a = 2 \).

**Problem 15.3.** Find the leading-order, uniformly valid solution of
\[
\varepsilon h'' + (\sin x h)_x = -1, \quad \text{with BCs} \quad h(0) = h\left(\frac{\pi}{2}\right) = 0. \quad (15.104)
\]

**Problem 15.4.** Find a leading-order boundary layer solution to
\[
\varepsilon h'' + (\sin x h)_x = -1, \quad \text{with BCs} \quad h(0) = h(\pi) = 0. \quad (15.105)
\]
(I think there are boundary layers at both \( x = 0 \) and \( x = 1 \).)

**Problem 15.5.** Considering the slow-down example [15.4], find the next term in the boundary-layer solution of this problem. Make sure you explain how the term \( \varepsilon x^{-2} \) in the outer expansion is matched as \( x \to 0 \).
Problem 15.6. Assuming that \( a(x) < 0 \), construct the uniformly valid leading-order approximation to the solution of

\[
\epsilon y'' + ay' + by = 0, \quad \text{with BCs} \quad y'(0) = p, \quad y'(1) = q. \quad (15.106)
\]

(Consider using linear superposition by first taking \((p, q) = (1, 0)\), and then \((p, q) = (0, 1)\).)

Problem 15.7. Consider

\[
\epsilon y'' + \sqrt{x} y' + y = 0, \quad \text{with BCs} \quad y(0) = p, \quad y(1) = q. \quad (15.107)
\]

(i) Find the rescaling for the boundary layer near \( x = 0 \), and obtain the leading order inner approximation. Then find the leading-order outer approximation and match to determine all constants of integration. (ii) Repeat for

\[
\epsilon y'' - \sqrt{x} y' + y = 0, \quad \text{with BCs} \quad y(0) = p, \quad y(1) = q. \quad (15.108)
\]

Problem 15.8. Find a leading order, uniformly valid solution of

\[
\epsilon y'' + \sqrt{x} y' + y^2 = 0, \quad \text{with BCs} \quad y(0, \epsilon) = 2, \quad y(1; \epsilon) = \frac{1}{3}. \quad (15.109)
\]

Problem 15.9. Find a leading order, uniformly valid solution of

\[
\epsilon y'' - (1 + 3x^2)y = x, \quad \text{with BCs} \quad y(0, \epsilon) = y(1; \epsilon) = 1. \quad (15.110)
\]

Problem 15.10. Find a leading-order, uniformly valid solution of

\[
\epsilon y'' - \frac{y'}{1 + 2x} - \frac{1}{y} = 0, \quad \text{with BCs} \quad y(0, \epsilon) = y(1; \epsilon) = 3. \quad (15.111)
\]

Problem 15.11. In an earlier problem you were asked to construct a leading order, uniformly valid solution of

\[
\epsilon y'' - (1 + 3x^2)y = x, \quad \text{with BCs} \quad y(0, \epsilon) = y(1; \epsilon) = 1. \quad (15.112)
\]

Now construct the uniformly valid two-term boundary layer approximation.

Problem 15.12. Consider

\[
\epsilon y'' + (1 + \epsilon)y' + y = 0, \quad y(0) = 0, \quad y(1) = e^{-1}, \quad m' = y, \quad m(1) = 0. \quad (15.113)
\]

Find two terms in the outer expansion of \( y(x) \) and \( m(x) \), applying only boundary conditions at \( x = 1 \). Next find two terms in the inner approximation at \( x = 0 \), applying the boundary condition at \( x = 0 \). Determine the constants of integration by matching. Calculate \( m(0) \) correct to order \( \epsilon \).
Lecture 16

Internal boundary layers

16.1 A linear example

BO section 9.6 has an extensive discussion of the boundary layer problem

\[ y'' + ay' + by = 0, \]  

in which \( a(x) \) has an internal zero. Suppose that we re-define the coordinate so that the domain is \(-1 \leq x \leq 1\) and the boundary conditions in (15.40) are

\[ y(-1) = p, \quad y(+1) = q. \]  

Further, suppose that \( a(x) \) has a simple zero at \( x = 0 \). The differential equation

\[ \epsilon y'' + \frac{\alpha x}{1 + x^2} y + \beta y = 0, \]  

is a typical example — figure 16.1 shows some bvp4c solutions.

Case 1: \( \alpha > 0 \)

Let’s consider (16.3) with \( \alpha > 0 \) and boundary conditions

\[ y(-1) = 1, \quad y(+1) = 0. \]  

This example will reveal all the main features of the general case. Our earlier arguments indicate that boundary layers not possible at either end of the domain. Thus there is a left interior solution, \( u(x) \), satisfying the boundary condition at \( x = -1 \):

\[ y = u_0 + \epsilon u_1 + \cdots \]  

with leading order

\[ \frac{\alpha x}{1 + x^2} u_{0x} + \beta u_0 = 0, \quad \Rightarrow \quad u_0 = |x|^{-\beta/\alpha} \exp\left[ -\frac{\beta}{2\alpha} \left( x^2 - 1 \right) \right]. \]  

There is also a right interior solution \( v(x) \), satisfying the boundary condition at \( x = +1 \):

\[ y = v_0 + \epsilon v_1 + \cdots \]  

In this case, with the homogeneous \( x = +1 \) boundary conditions in (16.4), the right interior solution is zero at all orders

\[ v_n = 0. \]
Figure 16.1: Internal boundary layer solution of (16.3) with $p = 1$ and $q = 0$, and $\epsilon = 0.05$ (green dotted) and 0.005 (red dashed) and 0.0005 (solid black).
We need a boundary layer at \( x = 0 \) to heal the \( x \to 0 \) singularity in (16.6) and to connect the right interior solution to the left interior solution. A distinguished-limit shows that the correct boundary-layer coordinate is
\[
X \overset{\text{def}}{=} \frac{x}{\sqrt{\epsilon}}. \tag{16.9}
\]

We must also re-scale the solution:
\[
y = \epsilon^{-\beta/2\alpha} Y(X). \tag{16.10}
\]

The scaling above is indicated because the interior solution in (16.6) is order \( \epsilon^{-\beta/2\alpha} \) once \( x \sim \sqrt{\epsilon} \). Without much work we have now determined the boundary layer thickness and the amplitude of the solution within the boundary layer. This is valuable information in interpreting the numerical solution in figure [16.1]—we now understand how the vertical axis must be rescaled if we reduce \( \epsilon \) further.

Using the boundary-layer variables, the BL equation is
\[
Y_{XX} + \frac{\alpha X}{1 + \epsilon X^2} Y_X + \beta Y = 0. \tag{16.11}
\]

We solve (16.11) with the RPS
\[
Y = Y_0(X) + \epsilon Y_1(X) + \cdots \tag{16.12}
\]

Leading order is the three-term balance
\[
Y_{0XX} + \alpha XY_{0X} + \beta Y_0 = 0, \tag{16.13}
\]

with matching conditions
\[
Y_0 \to |X|^{-\beta/\alpha} e^{\beta/2\alpha}, \quad \text{as } X \to -\infty, \tag{16.14}
Y_0 \to 0, \quad \text{as } X \to +\infty. \tag{16.15}
\]

We have to solve (16.13) exactly. When confronted with a second-order differential equation it is always a good idea to remove the first derivative term with the standard multiplicative substitution. In this case the substitution
\[
Y_0 = W e^{-\alpha X^2/4} \tag{16.16}
\]

into (16.13) results in
\[
W_{XX} + \left( \beta - \frac{1}{2} \alpha - \frac{1}{4} \alpha^2 X^2 \right) W = 0. \tag{16.17}
\]

Then, with \( Z \overset{\text{def}}{=} \sqrt{\alpha} X \), we obtain the parabolic cylinder equation
\[
W_{ZZ} + \left( \frac{\beta}{\alpha} - 1 - \frac{1}{4} Z^2 \right) W = 0, \tag{16.18}
\]

of order
\[
\nu \overset{\text{def}}{=} \frac{\beta}{\alpha} - 1. \tag{16.19}
\]

Provided that
\[
\frac{\beta}{\alpha} \neq 1, 2, 3, \ldots \tag{16.20}
\]
Figure 16.2: Solutions of (16.3) with $\alpha = -1 < 0$ and $\epsilon = 0.1$ (solid blue) and 0.02 (dashed black). There are boundary layers at $x = \pm 1$. The interior solution is zero to all orders in $\epsilon$. There is no internal boundary layer at $x = 0$.

the general solution of (16.13) is

$$Y_0 = e^{-\alpha X^2/4} \left[ AD_\nu (\sqrt{\alpha}X) + BD_\nu (-\sqrt{\alpha}X) \right].$$

(16.21)

We return to the exceptional case, in which $\nu = 0, 1, 2 \ldots$, later.

To take the outer limits, $X \to \pm \infty$, of the internal boundary layer solution in (16.21) we look up the asymptotic expansion of the parabolic cylinder functions e.g., in the appendix of BO, or in the DLMF:

$$D_\nu (t) \sim t^\nu e^{-t^2/4}, \quad \text{as } t \to \infty,$$

(16.22)

$$D_\nu (-t) \sim \frac{\sqrt{2\pi}}{\Gamma(-\nu)} t^{\nu-1} e^{t^2/4}, \quad \text{as } t \to \infty.$$

(16.23)

Matching in the right-hand outer limit, $X \to +\infty$, implies that $B = 0$. Matching in the left-hand outer limit $X \to -\infty$ requires that

$$A = (\alpha e)^{(\nu+1)/2} \frac{\Gamma(-\nu)}{\sqrt{2\pi}}.$$

(16.24)

Case 2: $\alpha < 0$

There is no internal boundary layer. Instead there are boundary layers at both $x = 1$ and $x = -1$. — see figure 16.2
Lecture 17

Initial layers

17.1 The over-damped oscillator

With our knowledge of boundary layer theory, let’s reconsider the over-damped harmonic oscillator from problem 2.6. With a change of notation, the problem in (2.68) is:

\[ \epsilon x_{tt} + x_t + x = 0, \quad \text{with the IC: } x(0) = 0, \quad x_t(0) = 1. \quad (17.1) \]

This scaling is convenient for the long-time solution, but not for satisfying the two initial conditions.

We are going to use a boundary-layer-in-time, also known as an initial layer, to solve this problem. To address the initial layer we introduce

\[ T \overset{\text{def}}{=} t/\epsilon, \quad \text{and} \quad X(T, \epsilon) = x(t, \epsilon). \quad (17.2) \]

The rescaled problem is

\[ X_{TT} + X_T + \epsilon X = 0, \quad \text{with the IC: } X(0) = 0, \quad X_T(0) = \epsilon. \quad (17.3) \]

Because \( X \) satisfies both the initial conditions it is convenient to attack this problem by first solving the initial-layer equation with

\[ X(T, \epsilon) = \epsilon X_1(T) + \epsilon^2 X_2(T) + \cdots \quad (17.4) \]

One finds

\[ X_{1TT} + X_{1T} = 0, \quad \Rightarrow \quad X_1 = 1 - e^{-T}, \quad (17.5) \]
\[ X_{2TT} + X_{2T} = -X_1, \quad \Rightarrow \quad X_1 = 2(1 - e^{-T}) - T - Te^{-T}. \quad (17.6) \]

All the constants of integration are determined because the initial-layer solution satisfies both initial conditions. Once \( T \gg 1 \), the initial-layer solution is

\[ X \rightarrow \epsilon + \epsilon^2 (2 - T) + \text{ord}(\epsilon^3), \quad (17.7) \]
\[ = \epsilon(1 - t) + 2\epsilon^2 + \text{ord}(\epsilon^3). \quad (17.8) \]

We must match the outer solution onto this function — to facilitate the match, we’ve written the solution in terms of the outer time \( t \). Notice how the term \( \epsilon^2 T \) switched orders. We can anticipate that there are further switchbacks from the \( \text{ord}(\epsilon^3) \) terms.
We obtain the outer solution by solving (17.1) (without the initial conditions!) with the RPS

\[ x(t, \epsilon) = \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots \]  

(17.9)

The first two terms are

\[ x_{1t} + x_1 = 0, \quad \Rightarrow \quad x_1 = A_1 e^{-t}, \]  

(17.10)

\[ x_{2t} + x_2 = -x_{1tt}, \quad \Rightarrow \quad x_1 = A_1 t e^{-t} + A_2 e^{-t}, \]  

(17.11)

and the reconstituted outer solution is

\[ x = \epsilon A_1 e^{-t} + \epsilon^2 \left( A_1 t e^{-t} + A_2 e^{-t} \right). \]  

(17.12)

In the matching region, \( \epsilon \ll t \ll 1 \), (17.12) is

\[ X \rightarrow \epsilon A_1 (1 - t) + \epsilon^2 A_1 t + \epsilon^2 A_2 + \text{ord}(\epsilon^3, \epsilon t^2, \epsilon^2 t^2) \]  

(17.13)

Comparing (17.13) with (17.8) we see that

\[ A_1 = 1, \quad \text{and} \quad A_2 = 2. \]  

(17.14)

The term \( \epsilon^2 A_1 t \) is orphaned because there is nothing in (17.8) to match it. We expect that switchbacks from the \( \text{ord}(\epsilon^3) \) terms in (17.8) will take care of the orphan. Notice that if we geared by taking \( t \sim \epsilon^{1/2} \) then the orphan is the smallest term in (17.13) — we can move it into the ord.

Finally we can construct a uniformly valid solution as

\[ x^{\text{uni}} = \epsilon \left( e^{-t} - e^{-T} \right) + \epsilon^2 \left( t e^{-t} - T e^{-T} + 2e^{-t} - 2e^{-T} \right) + \text{ord}(\epsilon^3). \]  

(17.15)

Figure 17.1 compares (17.15) with the exact solution

\[ x = \frac{2\epsilon}{\sqrt{1 - 4\epsilon}} e^{-t/2\epsilon} \sinh \left( \frac{\sqrt{1 - 4\epsilon} t}{2\epsilon} \right). \]  

(17.16)

Example: Consider

\[ \dot{x} = -x - xy + \epsilon \kappa y, \quad \epsilon y = x - xy - \epsilon \kappa y. \]  

(17.17)
17.2 Problems

Problem 17.1. Use boundary-layer theory to construct a leading-order solution of the IVP
\[ \epsilon x_{tt} + x_t + x = t e^{-t}, \quad \text{with} \quad x(0) = \dot{x}(0) = 0, \quad \text{as} \ \epsilon \to 0. \quad (17.18) \]

Problem 17.2. Find the leading order \( \epsilon \to 0 \) solution of
\[ \dot{u} = v, \quad \epsilon \dot{v} = -v - u^2, \quad (17.19) \]
for \( t > 0 \) with initial conditions \( u(0) = 1 \) and \( v(0) = 0 \).

Problem 17.3. Find the leading order \( \epsilon \to 0 \) solution of
\[ \epsilon \ddot{u} + (1 + t) \dot{u} + u = 1, \quad (17.20) \]
for \( t > 0 \) with initial conditions \( u(0) = 1 \) and \( \dot{u}(0) = -\epsilon^{-1} \).

Problem 17.4. A function \( y(t,x) \) satisfies the integro-differential equation
\[ \epsilon y_t = -y + f(t) + Y(t), \quad (17.21) \]
where
\[ Y(t) \overset{\text{def}}{=} \int_0^\infty y(t,x) e^{-\beta x} \, dx, \quad (17.22) \]
with \( \beta > 1 \). The initial condition is \( y(0,x) = a(x) \). (This is the Grodsky model for insulin release.) Use boundary layer theory to find the composite solution on the interval \( 0 < t < \infty \). Compare this approximate solution with the exact solution of the model. To assist communication, use the notation
\[ \alpha \overset{\text{def}}{=} 1 - \beta^{-1} \quad \text{and} \quad A \overset{\text{def}}{=} Y(0), \quad \text{and} \quad \tau \overset{\text{def}}{=} t/\epsilon. \quad (17.23) \]

Problem 17.5. Solve the previous problem with \( \beta = 1 \).

Problem 17.6. The Michaelis-Menten model for an enzyme catalyzed reaction is
\[ \dot{s} = -s + (s + k - 1)c, \quad \epsilon \dot{c} = s - (s + k)c, \quad (17.24) \]
where \( s(t) \) is the concentration of the substrate and \( c(t) \) is the concentration of the catalyst. The initial conditions are
\[ s(0) = 1, \quad c(0) = 0. \quad (17.25) \]
Find the first term in the: (i) outer solution; (ii) the “initial layer” (\( \tau \overset{\text{def}}{=} t/\epsilon \)); (iii) the composite expansion.
Lecture 18

Boundary layers in fourth-order problems

18.1 A fourth-order differential equation

Let us consider a fourth-order boundary value problem which is similar to problems occurring in the theory of elasticity:

\[- \epsilon^2 u_{xxxx} + u_{xx} = 1,\]  

(18.1)

with boundary conditions

\[u(-1) = u'(-1) = u(1) = u'(1) = 0.\]  

(18.2)

The outer solution might be obtained with the RPS such as

\[u(x, \epsilon) = u_0(x) + \epsilon u_1(x) + \cdots\]  

(18.3)

At leading order

\[u_{0xx} = 1, \quad \Rightarrow \quad u_0 = \frac{x^2 - 1}{2}.\]  

(18.4)

We've applied only two of the four boundary conditions above.

Before worrying about higher order terms in (18.3), let's turn to the boundary layer at \(x = -1\). We assume that the solution is an even function of \(x\) so the boundary layer at \(x = +1\) can be constructed by symmetry.

If we look for a dominant balance with \(X = (x + 1)/\delta\) we find that \(\delta = \epsilon\). Thus we consider a boundary layer rescaling

\[u(x, \epsilon) = U(X, \epsilon), \quad \text{where} \quad X \overset{\text{def}}{=} \frac{x + 1}{\epsilon}.\]  

(18.5)

The boundary layer problem is then

\[- U_{xxxx} + U_{xx} = \epsilon^2.\]  

(18.6)

Writing the leading-order outer solution in (18.4) in terms of \(X\), we have

\[u_0(x, \epsilon) = -\epsilon X + \frac{1}{2} \epsilon^2 X^2.\]  

(18.7)

Anticipating that we'll ultimately need to match the term \(-\epsilon X\) in (18.7), we pose the boundary-layer expansion

\[U(X, \epsilon) = \epsilon U_1(X) + \epsilon^2 U_2(X) + \epsilon^3 U_3(X) + \cdots\]  

(18.8)
There is no term $U_0(X)$ because the outer solution is $\text{ord}(\epsilon)$ in the matching region.

Thus we have the hierarchy

\begin{align*}
-U_{1XXXX} + U_{1XX} &= 0, \quad (18.9) \\
-U_{2XXXX} + U_{2XX} &= 1, \quad (18.10) \\
-U_{3XXXX} + U_{3XX} &= 0, \quad (18.11)
\end{align*}

and so on.

The general solution of (18.9) is

\[ U_1 = A_1 + B_1 X + C_1 e^{-X} + D_1 e^{X} \text{.} \quad (18.12) \]

Above we’ve anticipated that $D_0 = 0$ to remove the exponentially growing solution. Then applying the boundary conditions at $X = 0$ we find

\[ U_1 = A_1 \left( 1 - X - e^{-X} \right) \text{.} \quad (18.13) \]

To match (18.13) against the term $-\epsilon X$ in the interior solution in (18.7) we take

\[ A_1 = 1 \text{.} \quad (18.14) \]

Now we can construct a leading-order solution that is uniformly valid in the region near $x = -1$:

\[ u_{\text{uni}}(x) = \frac{x^2 - 1}{2} + \epsilon \left( 1 - e^{-(x+1)/\epsilon} \right) \text{.} \quad (18.15) \]

The derivative is

\[ u_{\text{uni}}(x) = x + e^{-(x+1)/\epsilon} \text{,} \quad (18.16) \]

which is indeed zero at $x = -1$.

**Higher order terms**

The equation for $U_2$, (18.10), has a solution

\[ U_2(X) = \frac{X^2}{2} + A_2 \left( 1 - X - e^{-X} \right) \text{.} \quad (18.17) \]

Above, we’ve satisfied both boundary conditions at $X = 0$. We’ve also matched the term $\epsilon^2 X^2/2$ in (18.7). To summarize, our boundary layer solution is

\[ U(X) = \epsilon \left( \underbrace{-X - e^{-X}}_{\text{orphan}} \right) + \epsilon^2 \frac{X^2}{2} + \epsilon^2 A_2 \left( 1 - X - e^{-X} \right) + \text{ord}(\epsilon^3) \text{.} \quad (18.18) \]

But we have unfinished business: we have not matched the orphan above with any term in the leading-order outer solution $u_0(x)$.

To take care of the orphan we must go to next order in the interior expansion:

\[ u(x, \epsilon) = \frac{x^2 - 1}{2} + \epsilon u_1(x) + \text{ord}(\epsilon^2) \text{.} \quad (18.19) \]

Thus

\[ u_{1XX} = 0, \quad \Rightarrow \quad u_1(x) = \underbrace{P_1}_{=1} + \underbrace{Q_1 x}_{=0} \text{.} \quad (18.20) \]
We take $Q_1 = 0$ because the solution is even, and $P_1 = 1$ to take care of the orphan. The solution $u_1(x)$ does not satisfy any of the four boundary conditions. To summarize, the outer solution is
\[ u(x, \epsilon) = \frac{x^2 - 1}{2} + \epsilon + \text{ord}(\epsilon^2). \] (18.21)

The \text{ord}(\epsilon) term above was accidently included in the uniform solution (18.16): in the outer region the expansion of (18.15) already agrees with all terms in (18.21).

Because $u_{0xxxx} = 0$, there are now no more non-zero terms in the outer region i.e., $u_2 = 0$, and therefore $A_2 = 0$ in (18.18). Moreover, all terms $U_3$, $U_4$ etcetera are also zero. Thus we have constructed an infinite-order asymptotic expansion. Using symmetry we can construct a uniformly valid solution throughout the whole domain
\[ u_{uni}(x) = \frac{x^2 - 1}{2} + \epsilon \left( 1 - e^{-(x+1)/\epsilon} - e^{(x-1)/\epsilon} \right). \] (18.22)

**18.2 Problems**

**Problem 18.1.** Solve (18.1) exactly and use MATLAB to compare the exact solution with the asymptotic solution in (18.22).

**Problem 18.2.** Find two terms in $\epsilon$ in the outer region and match to the inner solution at both boundaries for
\[ \epsilon^2 u''' - u'' = e^{ax}. \] (18.23)

The domain is $-1 \leq x \leq 1$ with BCs
\[ u(-1) = u'(-1) = 0, \quad \text{and} \quad u(1) = u'(1) = 0. \] (18.24)

**Problem 18.3.** Find two terms in $\epsilon$ in the outer region and match to the inner solution at both boundaries for
\[ \epsilon^2 u''' - u'' = 0. \] (18.25)

The domain is $0 \leq x \leq 1$ with BCs
\[ u(0) = 0, \quad u'(0) = 1, \quad \text{and} \quad u(1) = u'(1) = 0. \] (18.26)

**Problem 18.4.** Considering the eigenproblem
\[ -\epsilon^2 u''' + u'' = \lambda u, \] (18.27)
on the domain is $0 \leq x \leq \pi$ with BCs
\[ u(0) = u'(0) = 0, \quad \text{and} \quad u(\pi) = u'(\pi) = 0. \] (18.28)

(i) Prove that all eigenvalues are real and positive. (ii) Show that with a suitable definition of inner product, that eigenfunctions with different eigenvalues are orthogonal. (iii) Use boundary layer theory to find the shift in the unperturbed spectrum, $\lambda = 1, 2, 3 \ldots$, induced by $\epsilon$. 

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Lecture 19

Stationary phase

Fourier integrals are the imaginary analog of Laplace integrals. In analogy with (6.1) one has
\[ J(x) = \int_a^b f(t) e^{ix\psi(t)} \, dt, \tag{19.1} \]
where \( \psi(t) \) is a real phase function. As \( x \to \infty \) the integrand is very oscillatory. The asymptotic expansion of \( J(x) \) can be obtained via IP provided that \( f(t)/\psi'(t) \) is non-zero at either \( a \) or \( b \) and provided that \( f(t)/\phi'(t) \) is not singular for \( a \leq t \leq b \). If the second condition above is not met then IP fails. Instead we need the method of stationary phase.

Should explain how to define frequency and wavelength for a slowly varying wavetrain. Could use Lamb’s solution in 7.5 as an illustration.

19.1 The Airy function

An important example leading to asymptotic analysis of a Fourier integral is the \( x \to -\infty \) asymptotic expansion of the Airy function
\[ Ai(x) = \frac{1}{\pi} \int_0^\infty \cos \left( kx + \frac{k^3}{3} \right) \, dk. \tag{19.2} \]
We now use the method of stationary phase to determine the leading order asymptotic expansion of \( Ai(x) \) in this limit. We first make a cosmetic change in (4.8) by defining
\[ t \overset{\text{def}}{=} \frac{k}{|x|^{1/2}}, \tag{19.3} \]
so that
\[ Ai(x) = \frac{|x|^{1/2}}{\pi} \int_0^\infty \cos \left[ X\psi(t) \right] \, dt, \tag{19.4} \]
where
\[ X \overset{\text{def}}{=} |x|^{3/2}, \quad \text{and} \quad \psi(t) \overset{\text{def}}{=} t - \frac{t^3}{3}. \tag{19.5} \]
When \( X \) is large the phase of the cosine is changing rapidly and integrand is violently oscillatory. Successive positive and negative lobes almost cancel. The main contribution to the integral comes from the neighbourhood of the point \( t = t_* \) where the oscillations are slowest. This is the point of “stationary phase”, defined by
\[ \frac{d\psi}{dt} = 0, \quad \Rightarrow \quad t_* = 1. \tag{19.6} \]
Figure 19.1: The solid curve is $\text{Ai}(x)$ and the dashed curve is the asymptotic approximation in (19.11).

We get a leading-order approximation to the integral by expanding the phase function round $t_*$:

$$\psi \approx \psi_* + \frac{1}{2} (t - 1)^2 \psi''_*,$$

where $\psi_* \equiv \psi(t_*)$ and $\psi''_* \equiv \psi''(t_*)$. Thus as $x \to -\infty$:

$$\text{Ai}(x) \sim |x|^{1/2} \pi \int_{-\infty}^\infty \cos \left( X \left( \frac{2}{3} - (t - 1)^2 \right) \right) dt,$$

$$\sim |x|^{1/2} \pi X^{1/2} \Re e^{2iX/3} \int_{-\infty}^\infty e^{-iv^2} dv.$$

To ease the integration we have extended the range to $-\infty$. Recognizing the Fresnel integral,

$$\int_0^\infty e^{\pm i a v^2} dv = \sqrt{\frac{\pi}{2a}} e^{\pm i \pi/4},$$

we find that

$$\text{Ai}(x) \sim \frac{1}{\sqrt{\pi} x^{3/4}} \cos \left( \frac{2|x|^{3/2}}{3} - \frac{\pi}{4} \right), \quad \text{as} \quad x \to -\infty.$$  

This asymptotic approximation is compared with $\text{Ai}$ in Figure 19.1.

Exercise: Use contour integration to evaluate the Fresnel integrals in (19.10).

### 19.2 Group velocity

An important application of stationary phase is estimating the Fourier integrals that arise when we solve dispersive wave problems using the Fourier transform. The solution typically has the form

$$\psi(x, t) = \int_{-\infty}^\infty e^{i(kx - \omega(k)t)} \psi_0(k) \frac{dk}{2\pi},$$

where the function $\omega(k)$ is the *dispersion relation*. The integral can be written in the form

$$\psi(x, t) = \int_{-\infty}^\infty e^{it \phi(k, u)} \psi_0(k) \frac{dk}{2\pi},$$

where the phase function is

$$\phi(k, u) \equiv uk - \omega(k), \quad \text{with} \quad u \equiv \frac{x}{t}.$$
In the limit $t \to \infty$, with $u = x/t$ fixed, the stationary phase condition that $\partial_k \phi = 0$ leads to

$$u = v_g(k), \quad (19.15)$$

where the group velocity is

$$v_g(k) \overset{\text{def}}{=} \frac{d\omega}{dk}. \quad (19.16)$$

To apply the method of stationary phase to the integral in (19.12), we have to find all relevant solutions of (19.15), call them $k_1(u)$, $k_2(u)$, \ldots and $k_n(u)$. Usually $n$ is a modest number, like 1 or 2. Then we expand the phase functions around each $k_m(u)$ as

$$\phi(k,u) = \phi_m - \frac{1}{2}(k - k_m)^2 \omega_m'' + O((k - k_m)^3). \quad (19.17)$$

where $\phi_m \overset{\text{def}}{=} \phi(k_m)$ and $\omega_m'' \overset{\text{def}}{=} \frac{d^2\omega}{dk^2}(k_m)$. Thus

$$\psi(x,t) \sim \sum_{m=1}^{n} e^{i[k_m x - \omega(k_m)t]} \bar{\psi}_0(k_m) \int_{-\infty}^{\infty} e^{-\frac{u}{2}(k - k_m)^2 \omega_m''} \, dk. \quad (19.18)$$

Evaluating the Fresnel integrals we find

$$\psi(x,t) \sim \sum_{m=1}^{n} \frac{\bar{\psi}_0(k_m)}{\sqrt{2\pi t |\omega_m''|}} \exp \left[ i \left( k_m x - \omega(k_m)t - \text{sgn}(\omega_m'') \frac{\pi}{4} \right) \right]. \quad (19.19)$$

There is a fiddly $|\omega_m''|$ and sgn($\omega_m''$) in (19.19): this comes from the $\pm$ in the Fresnel integral (19.10).

Remark: if $\omega_m'' = 0$ the formula in (19.19) is invalid. This higher-order stationary point requires separate analysis.

### 19.3 An example: the Klein-Gordon equation

Let's use stationary phase to analyze the *Klein-Gordon* equation:

$$\psi_{tt} - \alpha^2 \psi_{xx} + \sigma^2 \psi = 0, \quad (19.20)$$

with the initial condition

$$\psi(x,0) = \frac{e^{-x^2/2\ell^2}}{\sqrt{2\pi \ell}}, \quad \text{and} \quad \psi_t(x,0) = 0. \quad (19.21)$$

First, let's introduce non-dimensional variables

$$\bar{x} \overset{\text{def}}{=} \frac{\sigma x}{a}, \quad \text{and} \quad \bar{t} = \sigma t, \quad \text{and} \quad \bar{\psi} = \frac{a}{\sigma} \psi. \quad (19.22)$$

With this change in notation, the non-dimensional problem is

$$\bar{\psi}_{\bar{t}\bar{t}} - \bar{\psi}_{\bar{x}\bar{x}} + \bar{\psi} = 0, \quad (19.23)$$

with the initial condition

$$\bar{\psi}(\bar{x},0) = \frac{e^{-\bar{x}^2/2\eta^2}}{\sqrt{2\pi \eta}}, \quad \text{and} \quad \psi_t(x,0) = 0. \quad (19.24)$$
There is a non-dimensional parameter,
\[ \eta \equiv \frac{\ell \sigma}{a}, \tag{19.25} \]
that controls the width of the initial disturbance relative to the intrinsic “Klein-Gordon length” \( a/\sigma \). We proceed dropping all the bars decorating the non-dimensional variables.

**Remark:** In the limit \( \eta \to 0 \), the initial condition is \( \psi(x,0) \to \delta(x) \). By taking \( \eta \to 0 \) we recover the Green’s function of the Klein-Gordon equation.

The Fourier transform of \( \psi \) is
\[ \tilde{\psi}(k,t) \overset{\text{def}}{=} \int_{-\infty}^{\infty} e^{-ikx} \psi(x,t) \, dx, \tag{19.26} \]
and with the operational rule \( \partial_x \to ik \) we find the transformed equation
\[ \tilde{\psi}_{tt} + (k^2 + 1)\tilde{\psi} = 0. \tag{19.27} \]
The solution that satisfies the transformed initial condition is
\[ \tilde{\psi} = \cos \omega t \, \tilde{\psi}_0(k) = \frac{1}{2} \left( e^{i\omega t} + e^{-i\omega t} \right) \tilde{\psi}_0(k). \tag{19.28} \]
Above, the Klein-Gordon dispersion relation is
\[ \omega(k) \overset{\text{def}}{=} \sqrt{k^2 + 1}, \tag{19.29} \]
and \( \tilde{\psi}_0(k) = e^{-\eta^2 k^2/2} \) is the transform of the initial condition.

Notice that the Klein-Gordon equation has two wave modes, one with dispersion relation \( \sqrt{k^2 + 1} \) and the other with \( -\sqrt{k^2 + 1} \). Our initial condition excites both waves. We record some consequences of the Klein-Gordon dispersion relation that we’ll need below:
\[ \frac{d\omega}{dk} = \frac{k}{\sqrt{k^2 + 1}} = \frac{k}{\omega}, \quad \text{and} \quad \frac{d^2\omega}{dk^2} = \frac{1}{\omega^3}. \tag{19.30} \]
The phase speed is \( c_p \overset{\text{def}}{=} \omega/k \), so for the Klein-Gordon equation notice that \( c_p v_g = 1 \).

From (19.28), the Fourier Integral Theorem now delivers the solution in the form
\[ \psi(x,t) = \int_{-\infty}^{\infty} e^{ikx} \cos(\omega t) \tilde{\psi}_0(k) \frac{dk}{2\pi}, \tag{19.31} \]
\[ = \frac{1}{2} \int_{-\infty}^{\infty} e^{ikx + i\omega t} \tilde{\psi}_0(k) \frac{dk}{2\pi} + \frac{1}{2} \int_{-\infty}^{\infty} e^{ikx - i\omega t} \tilde{\psi}_0(k) \frac{dk}{2\pi} \overset{\text{def}}{=} \psi_1 + \psi_2 \tag{19.32} \]

**Exercise:** Show that \( \psi_2^* = \psi_1 \).

Let’s consider \( \psi_2(x,t) \), written as :
\[ \psi_2(x,t) = \frac{1}{2} \int_{-\infty}^{\infty} e^{it\phi_2(k,u)} \tilde{\psi}_0(k) \frac{dk}{2\pi}, \tag{19.33} \]
with phase function
\[ \phi_2 \overset{\text{def}}{=} uk - \sqrt{k^2 + 1}. \tag{19.34} \]
The stationary-phase condition for \( \phi_2 \) is that
\[
u = \frac{k_\star}{\sqrt{k_\star^2 + 1}}; \quad (19.35)
\]
as \( k \) ranges from \( -\infty \) to \( +\infty \), \( u \) varies from \(-1\) to \(1\) i.e., there are no stationary wavenumbers if \(|u| > 1\). This is because our observer moving with speed \( u \) is out-running the waves — we return to this point later.

Solving (19.42) for \( k_\star \) as a function of \( u = x/t \):
\[
k_\star(u) = \frac{u}{\sqrt{1 - u^2}}, \quad (19.36)
\]
provided that \(-1 < u < 1\). Note that \( 1 + k_\star^2 = (1 - u^2)^{-1} \).

Close to the stationary point \( k_\star(u) \), the phase \( \phi_2 = uk - \omega(k) \) is therefore
\[
\phi_2(k) = -\frac{u}{\sqrt{1 - u^2}} \int_{-\infty}^{\infty} e^{-i\nu^2/2} \psi_0(k) \frac{dk}{2\pi}. \quad (19.38)
\]
Invoking the Fresnel formula
\[
\int_{-\infty}^{\infty} e^{\pm i a^2 v^2} dv = \sqrt{\frac{\pi}{a}} e^{\pm i\pi/4}, \quad (19.39)
\]
we obtain
\[
\psi_2(x,t) \sim \frac{1}{2} e^{-i\nu^2/2} \int_{-\infty}^{\infty} e^{-i\nu^2/2} \psi_0(k) \frac{dk}{2\pi}. \quad (19.40)
\]
provided that \( 0 < u \equiv x/t < 1 \). The total solution is \( \psi = \psi_2 + \psi_2^* \), or
\[
\psi(x,t) \sim \frac{\cos \left( t \sqrt{1 - u^2 + \frac{\pi}{4}} \right)}{\sqrt{2\pi t (1 - u^2)^{3/2}}} \tilde{\psi}_0 \left( \frac{u}{\sqrt{1 - u^2}} \right), \quad (19.41)
\]

**Visualization of the stationary phase solution**

Let's visualize this asymptotic solution using the initial condition
\[
\psi(x,0) = \frac{e^{-x^2/2\eta^2}}{\sqrt{2\pi \eta}}, \quad \text{or} \quad \tilde{\psi}_0(k) = e^{-k^2 \eta^2/2}. \quad (19.42)
\]
The parameter \( \eta \) controls the width of the initial disturbance.

First consider \( \eta \to 0 \), so that \( \psi(x,0) \to \delta(x) \). In this case the Klein-Gordon equation has an exact solution
\[
\psi(x,t) = -\frac{J_1 \left( t \sqrt{1 - u^2} \right)}{2 \sqrt{1 - u^2}} + \frac{1}{2} \left[ \delta(x-t) + \delta(x+t) \right], \quad (19.43)
\]
where \( J_1(z) \) is the first order Bessel function. Figure 19.2 compares the stationary phase solution in (19.40) (with \( \tilde{\psi}_0 = 1 \)) to the exact solution in (19.43). The approximation is OK
Figure 19.2: Comparison of the $\eta = 0$ stationary phase approximation (the blue solid curve) to the exact solution $\psi = -J_1(t\sqrt{1-u^2})/(2\sqrt{1-u^2})$ (the green dashed curve).

Figure 19.3: The stationary phase approximation with the Gaussian initial condition in with the Gaussian initial condition in (19.42).
provided we don’t get too close to \( x = t \). The stationary phase approximations says nothing about the \( \delta \)-pulses that herald the arrival of the signal.

Figure 19.3 shows the stationary phase approximation (19.40) with \( \eta \neq 0 \). The initial disturbance now has finite width and therefore contains no high-wavenumbers. Thus the rapid oscillations near the from at \( x = t \) have been removed. This is good: it probably means that the stationary phase approximation is accurately reproducing the waveform. But on the other hand, because of the \( \delta(x \pm t) \) in (19.43), there should be terms like
\[
e^{-\frac{(x\pm t)^2}{2\eta^2}} \sqrt{\frac{2\pi}{\eta}} \tag{19.44}
\]
in the exact solution. The stationary phase approximation fails when \( u = x/t \) is close to \( \pm 1 \), and therefore the approximation gives no hint of these pulses.

**Caustics: \( u \approx 1 \)**

To analyze the solution (19.32) close to front at \( x = t \), we write \( x = t + x' \) where \( x' \ll 1 \). The stationary-phase approximation, analogous to (19.40), is therefore
\[
\phi_2(k) = (t + x')k - \sqrt{1 + k^2} \frac{t}{\approx|k|(1 + \frac{1}{2}k^2)}
\]
\[
\approx x'k - \frac{t}{2k}, \quad \text{provided } k > 0. \tag{19.45}
\]
The approximation above is valid close to the stationary wavenumber, \( k = \infty \). The inverse transform is therefore
\[
\psi_2(x, t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{i(xk - \frac{t}{2k})} \hat{\psi}_0(k) \, dk,
\]
\[
= \frac{1}{4\pi} \sqrt{\frac{t}{2x'}} \int_{-\infty}^{\infty} e^{i\xi(k^{-1} - k)} \hat{\psi}_0 \left( \sqrt{\frac{t}{2x'}} \right) \, dk, \tag{19.46}
\]
where
\[
\xi \overset{\text{def}}{=} \sqrt{\frac{t}{2x'}}. \tag{19.47}
\]

### 19.4 Problems

**Problem 19.1.** The Bessel function of order zero is defined by
\[
J_0(x) \overset{\text{def}}{=} \frac{2}{\pi} \int_0^{\pi/2} \cos[x \cos t] \, dt. \tag{19.48}
\]

Show that \( J''_0 + x^{-1} J'_0 + J_0 = 0 \). Plot the integrand of (19.48) as a function of \( t \) at \( x = 10 \). Show that
\[
J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\pi}{4} \right). \tag{19.49}
\]

Use MATLAB (help besselj) to compare the leading order approximation with the exact Bessel function on the interval \( 0 < x < 5\pi \). The comparison is splendid: see Figure 19.4.
Problem 19.2. The Bessel function of order \( n \) has an integral representation

\[
J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin t - nt) \, dt.
\]

Show that

\[
J_n(n) \sim \frac{\Gamma\left(\frac{1}{3}\right)}{\pi^{2/3} 3^{1/6} n^{1/3}}, \quad \text{as } n \to \infty.
\]

Problem 19.3. Find leading-order, \( x \to \infty \) asymptotic expansion of

\[
(i) \quad \int_0^1 \sqrt{1 + t} e^{ix(1-2t^2)^2} \, dt,
\]

\[
(ii) \quad \int_1^\infty \sqrt{1 + t} e^{ix(1-2t^2)^2} \, dt.
\]

Problem 19.4. Staring at (19.40) suggests that in the limit \( \eta \to 0 \) (i.e., a \( \delta \)-function initial condition) the Klein-Gordon equation might have an exact similarity solution of the form

\[
\psi(x, t) = t\Psi\left(\sqrt{t^2 - x^2}\right).
\]

Find this solution in terms of Bessel functions, and verify the asymptotic expansion in (19.40) using well known Bessel function results.

Problem 19.5. According to article 238 in Lamb, the surface elevation produced by a two-dimensional splash is given by

\[
\eta(x, t) = \frac{1}{\pi} \int_0^\infty \cos(kx - \sqrt{\sqrt{g} k t}) \, dk.
\]

Show that as \( t \to \infty \)

\[
\eta(x, t) \sim \frac{\sqrt{gt}}{2^2 \sqrt{\pi x^2}} \left(\cos\frac{gt^2}{4x} + \sin\frac{gt^2}{4x}\right).
\]

Verify that the frequency and wavenumber from (19.55) are connected by the water-wave dispersion relation \( \omega = \sqrt{\sqrt{g} k} \).

Problem 19.6. Use a Fourier transform to solve

\[
\psi_t = \frac{i}{2} \psi_{xx} - \frac{1}{3} \psi_{xxx}, \quad \text{with IC } \psi(x, 0) = \delta(x).
\]

Use the method of stationary phase to estimate

\[
\lim_{t \to \infty} \psi(ut, t)
\]

with (a) \( u = 0 \); (b) \( u = \frac{1}{4} \); (c) all values of \( u \).
Lecture 20

Constant-phase (a.k.a. steepest-descent) contours

The method of steepest descents, and its big brother the saddle-point method, is concerned with complex integrals of the form

\[ I(\lambda) = \int_C h(z)e^{\lambda f(z)} \, dz, \quad \text{(20.1)} \]

where \( C \) is some contour in the complex \((z = x + iy)\) plane. The basic idea is to convert \( I(\lambda) \) to

\[ I(\lambda) = \int_D h(z)e^{\lambda f(z)} \, dz, \quad \text{(20.2)} \]

where \( D \) is a contour on which \( \Im f \) is constant. Thus if

\[ f = \phi + i\psi, \quad \text{(20.3)} \]

then on the contour \( D \)

\[ I(\lambda) = e^{i\psi_D} \int_D h(z)e^{i\phi} \, dt. \quad \text{(20.4)} \]

Above \( \psi_D \) is the constant imaginary part of \( f(z) \). We refer \( \psi \) as the phase function and \( D \) is therefore a constant-phase contour.

Contour \( D \) is orthogonal to \( \nabla\psi \) and, because

\[ \nabla\phi \cdot \nabla\psi = 0, \quad \text{(20.5)} \]

\( D \) is also tangent to \( \nabla\phi \). Thus as one moves along \( D \) one is always ascending or descending along the steepest direction of the surface formed by \( \phi(x, y') \) above the \((x, y')\)-plane. The main advantage to integrating along \( D \) is that the integral will be dominated by the neighbourhood of the point on \( D \) where \( \phi \) is largest.

Because \( \phi \) is changing most rapidly as one moves along \( D \), \( D \) is also a contour of steepest descent (or of steepest ascent).

**Exercise:** Prove (20.5) and the surrounding statements.

The method and its advantages are best explained via well chosen examples. In fact we’ve already used the method to calculate \( \text{Ai}(0) \) back in section 4.6 — you should review that example.
20.1 Asymptotic evaluation of an integral using a constant-phase contour

An example of the constant phase method is provided by

$$B(\lambda) \overset{\text{def}}{=} \int_0^1 e^{i\lambda x^2} \, dx.$$  \hfill (20.6)

If $\lambda \gg 1$ we quickly obtain a leading order approximation to $B$ using stationary phase. We improve on this leading-order approximation by considering

$$f(z) = iz^2 = -2xy + i(x^2 - y^2).$$  \hfill (20.7)

The end-points of (20.6) define two constant-phase contours — see Figure 20.1. The end-point at $(x,y) = (0,0)$ has $\psi = 0$, and the other end-point at $(x,y) = (1,0)$ has $\psi = 1$. The constant-phase contours through these endpoints are

$$D_0: z = re^{i\pi/4}, \quad \text{and} \quad D_1: z = \sqrt{1 + y^2} + iy.$$  \hfill (20.8)

(See the figure.) There are no singularities in the region enclosed by $C, D_0, D_1$ and $E$. Thus, as $E$ recedes to infinity, we have

$$B(\lambda) = \int_{D_0} e^{i\lambda z^2} \, dz + \int_{D_1} e^{i\lambda z^2} \, dz.$$  \hfill (20.9)

The first integral along $D_0$ in (20.9) is the standard Fresnel integral. In the second integral along $D_1$, the exponential function is

$$e^{i\lambda z^2} = e^{i\lambda} e^{-2\lambda y \sqrt{1 + y^2}}.$$  \hfill (20.10)

This verifies that on $D_1$ the integrand decreases monotonically away from the maximum at $z = 1$. On $D_1$ we strive towards a Laplace integral in (20.9) by making the change of variable

$$iz^2 = i - v, \quad \text{or} \quad z = \sqrt{1 - i v}.$$  \hfill (20.11)

Thus

$$\int_{D_1} e^{i\lambda z^2} \, dz = \frac{1}{2} e^{i\lambda} \int_0^\infty \frac{e^{-\lambda v}}{\sqrt{1 + iv}} \, dv.$$  \hfill (20.12)

The minus sign is because we integrate along $D_1$ starting at $v = \infty$.

Assembling the results above

$$B(\lambda) = \frac{1}{2} \sqrt{\frac{\pi}{\lambda}} e^{i\pi/4} - \frac{1}{2} e^{i\lambda} \int_0^\infty \frac{e^{-\lambda v}}{\sqrt{1 + iv}} \, dv.$$  \hfill (20.13)

Watson’s lemma delivers the full asymptotic expansion of the final integral in (20.13).

There is an alternative derivation that using the contour $\mathcal{F}$ in Figure ??, $\mathcal{F}$ is tangent to $D_1$ at $z = 1$, so that

$$\int_{D_1} e^{i\lambda z^2} \, dz \sim \int_{\mathcal{F}} e^{i\lambda z^2} \, dz \quad \text{as} \quad \lambda \to \infty.$$  \hfill (20.14)

Now in the neighbourhood of $z = 1$:

$$z = 1 + iy.$$  \hfill (20.15)

Thus

$$\int_{\mathcal{F}} e^{i\lambda z^2} \, dz \sim -e^{i\lambda} \int_0^\infty e^{-2\lambda y} e^{i\lambda y^2} \, dy.$$  \hfill (20.16)
$\psi = x^2 - y^2$

Figure 20.1: The contours $D_0$ and $D_1$. On $D_0$, $x = y$ and the phase is constant.

### 20.2 Problems

**Problem 20.1.** Show that if $p > 1$ and $0 < a$:

\[
\int_0^\infty e^{\pm ia^p v^p} \, dv = \frac{1}{a} \Gamma \left( 1 + \frac{1}{p} \right) e^{\pm i\pi/2p}. \tag{20.17}\]

The case $p = 2$ is the *Fresnel integral* and $p = 3$ is the example in (4.44).

**Problem 20.2.** Consider the integral

\[
P(a, b) \overset{\text{def}}{=} \int_0^\infty e^{-ax^2} \cos bx \, dx = \frac{1}{\pi} \int_C e^{-ax^2 + ibx} \, dx, \tag{20.18}
\]

where $C$ is the real axis, $-\infty < x < \infty$. Complete the square in the exponential and show that the line $y = b/2a$ is a contour $D$ on which the integrand is not oscillatory. Evaluate $P$ exactly by integration along $D$.

**Problem 20.3.** Use a constant phase contour to asymptotically evaluate

\[
Q(\lambda) \overset{\text{def}}{=} \int_0^\pi e^{i\lambda x} \ln x \, dx \tag{20.19}
\]

as $\lambda \to \infty$. 
Lecture 21

The saddle-point method

The saddle-point method is a combination of the method of steepest descents and Laplace's method in the complex plane.

In the previous lecture we considered integrals along constant-phase contours $\mathcal{D}$. In those earlier examples the maximum of $\phi$ on $\mathcal{D}$ was at the endpoints of the contour. But it is possible that the maximum of $\phi$ is in the middle of the contour. If $s$ is the arclength along the contour then

$$\frac{d\phi}{ds} = \nabla \phi \cdot \hat{r}_\mathcal{D}$$

(21.1)

where $\hat{r}_\mathcal{D}$ is the unit tangent to $\mathcal{D}$. In fact, from the Cauchy-Riemann equations,

$$\hat{r}_\mathcal{D} = \frac{\nabla \phi}{|\nabla \phi|}, \quad \text{so that} \quad \frac{d\phi}{ds} = |\nabla \phi|.$$  

(21.2)

Thus if there is an interior maximum on $\mathcal{D}$ at that point $z_*$ then

$$\left.\frac{d\phi}{ds}\right|_{z_*} = 0,$$

which requires $|\nabla \phi| = 0$ at $z_*$.  

(21.3)

However from the Cauchy-Riemann equations

$$|\nabla \phi| = |\nabla \psi| = \left|\frac{df}{dz}\right|.$$  

(21.4)

So at the point $z_*$, $|\nabla \phi|$, $|\nabla \psi|$ and $|df/dz|$ are all zero. In short, we can locate the saddle points by solving

$$\frac{df}{dz} = 0.$$  

(21.5)

21.1 The Airy function as $x \to \infty$

Let's recall our Fourier transform representation of the Airy function:

$$\text{Ai}(x) = \int_{-\infty}^{\infty} e^{i(kx+k^3/3)/2}\frac{dk}{2\pi}.$$  

(21.6)

In an earlier lecture we used stationary phase to investigate the $x \to -\infty$ behaviour of $\text{Ai}$. Now consider what happens if $x \to \infty$. In this case the stationary point is in the complex $k$-plane, at
$k = \pm \sqrt{x}i$. We’ll use this to illustrate the saddle-point method. But first we make the change of variable

$$k = \sqrt{x}\kappa$$

so that

$$A_i(x) = \frac{\sqrt{x}}{2\pi} \int_C e^{Xf(\kappa)} d\kappa,$$

where $C$ is the real axis and

$$X \overset{\text{def}}{=} x^{3/2}, \quad \text{and} \quad f(\kappa) \overset{\text{def}}{=} i \left( \kappa + \frac{\kappa^3}{3} \right).$$

If $\kappa = p + iq$ then in this problem we can write $f(\kappa)$ explicitly without too much work:

$$f(\kappa) = -q - p^2 q + \frac{q^3}{3} + i \left( p + \frac{p^3}{3} - pq^2 \right).$$

Contours of constant $\psi$ are shown in Figure 21.1. The saddle points at $\kappa = \kappa_*$ are located by

$$\frac{df}{dk} = 0, \quad \text{or} \quad \kappa_* = \pm i.$$

Notice that

$$f(i) = -\frac{2}{3},$$

and thus the constant-phase contours passing through $\kappa = i$ are determined by $\Im f = 0$, implying that

$$p = 0, \quad \text{and} \quad \frac{p^3}{3} - q^2 + 1 = 0.$$
Our attention is drawn to the saddle at $+i$, and we deform $C$ onto the contour $\mathcal{D}_+$ passing through $\kappa = +i$. We can parameterize the integral on $\mathcal{D}_+$ via
\[
p = \sqrt{3} \sinh t, \quad \text{and} \quad q = \cosh t.
\] (21.14)

Notice then that
\[
\phi = \frac{2}{3} \left( 3 \cosh t - 4 \cosh^3 t \right) = -\frac{2}{3} \cosh 3t,
\] (21.15)
and
\[
\frac{d\kappa}{dt} = \sqrt{3} \cosh t + i \sinh t.
\] (21.16)
Thus we have a new integral representation for the Airy function
\[
\text{Ai}(x) = \frac{\sqrt{3}}{2\pi} \int_{-\infty}^{\infty} \cos t \, e^{-\frac{2}{3} \cosh 3t} \, dt.
\] (21.17)
Laplace’s method applied to this integral quickly gives the leading-order result
\[
\text{Ai}(x) \sim \frac{e^{-2x^{3/2}}}{2\sqrt{\pi} x^{1/4}}.
\] (21.18)

Notice that if we had to numerically evaluate $\text{Ai}(x)$ for positive $x$ the representation in (21.14) would be very useful.

We were distracted by the exciting parameterization (21.14). A better way to obtain the asymptotic approximation of $\text{Ai}(x)$ is to use a path that is tangent to the true steepest-descent path at $\kappa = i$, i.e., the path
\[
\kappa = i + p, \quad \text{on which} \quad f = -\frac{2}{3} - p^2 + i\frac{p^3}{3}.
\] (21.19)
This is not a path of constant phase (notice the $ip^3/3$) but that doesn’t matter. We then have
\[
\text{Ai}(x) \sim \frac{\sqrt{X}}{2\pi} e^{iX} \int_{-\infty}^{\infty} e^{-xp^2} e^{ip^3/3} \, dp.
\] (21.20)
Watson’s lemma now delivers the full asymptotic expansion of the Airy function:
\[
\text{Ai}(x) \sim \frac{e^{-2x^{3/2}}}{2\sqrt{\pi} x^{1/4}} \sum_{n=0}^{\infty} \frac{\Gamma \left( \frac{3n+1}{2} \right)}{(2n)!} \left( -\frac{1}{9X} \right)^n, \quad \text{as } x \to \infty.
\] (21.21)

21.2 The Laplace transform of a rapidly oscillatory function

As an another example, consider the large-$s$ behaviour of the Laplace transform
\[
\mathcal{L} \left[ \sin \frac{1}{t} \right] = \int_0^{\infty} e^{-st} \sin \left( \frac{1}{t} \right) \, dt.
\] (21.22)
Although this is a real integral, we make an excursion into the complex plane by writing
\[
\mathcal{L} \left[ \sin \frac{1}{t} \right] = \mathfrak{I} \int_0^{\infty} e^{-st+it^{-1}} \, dt.
\] (21.23)
Our previous experience with Laplace's method suggests considering the function
\[
\phi \overset{\text{def}}{=} \frac{1}{t} - st,
\] (21.24)
in the complex \( t \)-plane. Our attention focusses on saddle points, which are located by

\[
\frac{d \phi}{d t} = 0, \quad \Rightarrow \quad t_* = \pm \frac{e^{-i \pi/4}}{\sqrt{s}}. \tag{21.25}
\]

Studying \( t_* \), we’re motivated to introduce a complex variable \( z = x + iy \) defined by

\[
z \overset{\text{def}}{=} \sqrt{s} t \quad \text{and therefore} \quad \phi = \sqrt{s} \left( \frac{i}{z} - z \right). \tag{21.26}
\]

In terms of \( z \), the Laplace transform in (21.23) is therefore

\[
\mathcal{L} \left[ \sin \frac{1}{t} \right] = \Im \left[ \frac{1}{\sqrt{s}} \int_0^\infty e^{\sqrt{s} \hat{\phi}} \, dz \right]. \tag{21.27}
\]

Back in (21.26), the \( \sqrt{s} \) in the definition of \( z \) ensures that the saddle points are at \( z = \omega \) and \( z = -\omega \), where

\[
\omega \overset{\text{def}}{=} e^{-i \pi/4} = -1 = \frac{1 - i}{\sqrt{2}}. \tag{21.28}
\]

That is, as \( s \to \infty \) the saddle points don’t move in the \( z \)-plane. \( \Re \hat{\phi} \) is shown in the \( z \)-plane as green contours in Figure 21.2. The black curves in Figure 21.2 are \( \Im \hat{\phi} = \pm \sqrt{2} \) and the saddle point at \( z = \omega \) is indicated by \( S \). The curve of steepest descent, which goes through \( S \), is \( \text{OSR} \).

Invoking Cauchy’s theorem on the closed curve \( \text{OURS} \): 

\[
\frac{1}{\sqrt{s}} \int_{\text{OURS}} e^{\sqrt{s} \hat{\phi}} \, dz = 0, \tag{21.29}
\]

and therefore, taking the points \( R \) and \( U \) to infinity,

\[
\mathcal{L} \left[ \sin \frac{1}{t} \right] = \Im \left[ \frac{1}{\sqrt{s}} \int_{\text{OSR}} e^{\sqrt{s} \hat{\phi}} \, dz \right]. \tag{21.30}
\]

When \( s \gg 1 \) we can evaluate the integral on the right of (21.30) using

\[
\hat{\phi}(z) = -2\omega + \omega^3 (z - \omega)^2 + O(z - \omega)^3. \tag{21.31}
\]

To go over the saddle point on \( \text{OSR} \)

\[
z - \omega = \omega^{1/2} \zeta \quad \text{so that} \quad \omega^3 (z - \omega)^2 = -\zeta^2. \tag{21.32}
\]
Thus

\[
\mathcal{L}\left[\sin \frac{1}{t}\right] \sim \frac{\sqrt{\pi}}{s^{3/4}} \omega^{1/2} e^{-2\omega \sqrt{s}} = \frac{\sqrt{\pi}}{s^{3/4}} e^{-\sqrt{2s}} \cos \left(\sqrt{2s} + \frac{\pi}{8}\right). \tag{21.33}
\]

**Must check!**

The saddle point method requires careful scrutiny of the real and imaginary parts of analytic functions. It is comforting to contour these functions with MATLAB. The script that generates Figure 21.2 is in the box, and it can be adapted to other examples we encounter.
Note the use of \texttt{1i} for $\sqrt{-1}$ in MATLAB, and the use of \texttt{meshgrid} to set up the complex matrix $z = xx + 1i*yy$. To get a good plot you have to supply some analytic information to \texttt{contour} e.g., we make MATLAB plot the saddle point contours $S\phi = \pm \sqrt{2s}$ in the second call to \texttt{contour}. And you can't trust MATLAB to pick sensible contour levels because the pole at $z = 0$ means that there are some very large values in the domain — so the vector \texttt{V} determines the contour levels using the convenient scale $\sqrt{2s}$. Since the real and imaginary parts of the analytic function $\phi$ intersect at right angles (recall $\nabla \phi_r \cdot \nabla \phi_i = 0$), we preserve angles with the command \texttt{axis equal}. 

```matlab
%% Saddle point plot
%% phi = 1i/t - s t with t = z/sqrt(s)
%% phi = sqrt(2s)(1i/z - z)
close all
clc
clear
s = 10; a=sqrt(2*s);
x = linspace(-1,8,200); y = linspace(-2,1,200);
[xx,yy] = meshgrid(x,y); z = xx + 1i*yy;
phi = sqrt(s)*(1i./z - z);
phReal = real(phi);
phImag = imag(phi);

% contour levels for real(phi):
V=[-6 :0.4: 2]*a;
subplot(2,1,1)
contour(xx,yy,phReal,V,'g')
hold on
% plot the saddle point contour
contour(xx,yy,phImag,[-a a],'k')
axis equal
hold on
xlabel('$x$','interpreter','latex')
ylabel('$y$','interpreter','latex')
axis([min(x) max(x) min(y) max(y) ])

% plot the x-axis:
plot([min(x) max(x)],[0 0],'k','linewidth',1.0)
% plot the y-axis:
plot([0 0],[min(y) max(y)],'k','linewidth',1.0)
title('real($\phi$) and imag($\phi$) = $\sqrt{2s}$','interpreter','latex')
% label the steepest descent contour
text(-0.05, 0, 'O')
b = 1/sqrt(2);
text(1*b, -1.2*b, 'S')
text(8.1, -1.3, 'R')
```
21.3 Inversion of a Laplace transform

We can further illustrate the saddle-point method by considering the problem of inverting a Laplace transforms using the Bromwich method. Recall again that the Laplace transform of a function \( f(t) \) is

\[
\tilde{f}(s) \overset{\text{def}}{=} \int_0^\infty f(t)e^{-st} \, dt.
\] (21.34)

We refuse to Laplace transform functions with nonintegrable singularities, such as \( f(t) = (t - 1)^{-2} \) and \( 1/\ln t \).

The inversion theorem says that

\[
f(t) = \frac{1}{2\pi i} \int_{B} e^{st} \tilde{f}(s) \, ds,
\] (21.35)

where the “Bromwich contour” \( B \) is a straight line parallel to the \( s \)-axis and to the right of all singularities of \( \tilde{f}(s) \). The theorem requires that \( f(t) \) is absolutely integrable over all finite integrals i.e., that

\[
\int_0^T |f(t)| \, dt < \infty, \quad \text{for } 0 \leq T < \infty.
\] (21.36)

Absolute integrability over the infinite interval is not required: \( e^{-st} \) takes care of that.

Now consider the Laplace transform

\[
\tilde{f}(x,s) = \frac{e^{x\sqrt{s}}}{s}.
\] (21.37)

According to the inversion theorem

\[
f(x,t) = \frac{1}{2\pi i} \int_{B} e^{st-x\sqrt{s}} \frac{ds}{s}.
\] (21.38)

The Bromwich contour must be in the RHP, to the right of the pole at \( s = 0 \).

Suppose we don’t have enough initiative to look up this inverse transform in a book, or to evaluate it exactly by deformation of \( B \) to a branch-line integral (see problems). Suppose further that we don’t notice that \( f(x,t) \) is a similarity solution of the highly regarded diffusion problem

\[
f_t = f_{xx}, \quad f(0,t) = 1, \quad f(x,0) = 0.
\] (21.39)

We’re dim-witted, and thus we try to obtain the \( t \to \infty \) asymptotic expansion of \( f(x,t) \).