This exam is open notes, but no computers.

Problem 1
Find the \( x \to \infty \) leading-order behavior of the function \( J(x) \equiv \int_{x}^{\infty} e^{-t^3/3} \, dt \).

Problem 2
As \( \epsilon \to 0 \), the function
\[
F(\epsilon) \equiv \int_{0}^{\infty} \frac{dy}{(1 + y)^{1/2}(\epsilon^2 + y)}
\]
has an expansion
\[
F(\epsilon) \sim a \frac{\ln 1}{\epsilon} + b + O(\epsilon^2 \ln \epsilon).
\]
Find the numbers \( a \) and \( b \). The indefinite integral
\[
\int \frac{dy}{(1 + y)^{1/2}y} = \ln \left( \frac{\sqrt{1 + y} - 1}{\sqrt{1 + y} + 1} \right)
\]
may be useful.

Problem 3
(i) Find \( \lim_{t \to \infty} x(t) \), where \( x(t) \) is the solution of
\[
\dot{x} = -\frac{x^3}{20000} - \frac{x^2}{2} - 2 - \frac{1}{20000} x - 1,
\]
with the initial condition \( x(0) = -1 \). (ii) Evaluate \( \lim_{t \to \infty} x(t) \) with the initial condition \( x(0) = -1.02 \). Give two significant figures in both cases.

Problem 4
Consider the diffusion problem
\[
\psi_{xx} + \psi_{yy} = -e^{-y}
\]
in the “corrugated half-plane” \( -\infty < x < \infty \) and \( \epsilon \cos kx < y \). At the wavy boundary: \( \psi(x, \epsilon \cos kx) = 0 \), and the condition at infinity is
\[
\lim_{y \to \infty} \psi(x, y) = A(\epsilon, k),
\]
where \( A(\epsilon, k) \) is an unknown function. (i) Solve the problem with \( \epsilon = 0 \) and show that \( A(0, k) = 1 \). (ii) Use a perturbation expansion \( \epsilon \ll 1 \) to show that
\[
A(k, \epsilon) = 1 + \epsilon^2 A_2(k) + O(\epsilon^3).
\]
Find \( A_2(k) \) and check your answer by showing that \( A_2(1/2) = 0 \).