Problem 2.1 from part B

(i) Find a leading-order $x \to \infty$ asymptotic approximation to

$$A(x; p, q) \overset{\text{def}}{=} \int_x^\infty e^{-pt} \, dt.$$  \hspace{1cm} (1)

Show that the remainder is asymptotically negligible as $x \to \infty$. Above, $p$ and $q$ are both positive real numbers.

Problem 2.2 from part B

Find two terms in the $x \to \infty$ behaviour of $F(x) = \int_0^x \frac{e^{-v}}{\sqrt{v}} \, dv$.

Problem 3.1 from part B

(i) Obtain the leading-order asymptotic approximation for the integral

$$\int_{-1}^1 e^{xt^3} \, dt, \quad \text{as } x \to \pm \infty.$$ \hspace{1cm} (2)

(ii) Justify the asymptoticness of the expansion.

Problem 3.3 from part B

We previously obtained the full asymptotic series for $\text{erfc}(z)$ via IP:

$$\text{erfc}(x) \sim \frac{e^{-x^2}}{\sqrt{\pi x}} \sum_{n=0}^\infty (2n-1)!! \left( -\frac{1}{2x^2} \right)^n.$$ \hspace{1cm} (3)

Obtain this result by making a change of variables that converts $\text{erfc}(z)$ into a Laplace transform, and then use Watson’s lemma.

Problem 3.4 from part B

Find $x \to \infty$ leading-order asymptotic approximations of the integrals

$$A(x) = \int_0^x e^{-t^4} \, dt, \quad B(x) = \int_0^x e^{t^4} \, dt, \quad C(x) = \int_0^\infty e^{-xt} \ln(1 + t^2) \, dt.$$ \hspace{1cm} (4)
Hand-in, due in class on Thursday April 21st

Problem 5.5 from part A

Use boundary layer theory to find leading order solution of

\[ h_x = \epsilon \left( \frac{1}{3} h^3 \right)_{xx} + 1, \]

on the domain \( 0 < x < 1 \) with boundary conditions \( h(0) = h(1) = 0 \). You can check your answer by showing that \( h = 1/2 \) at \( x \approx 1 - 0.057 \epsilon \).

Problem 6.1 from part A

Analyze the variable-speed Stommel problem

\[ \epsilon h'' + (x^a h)_x = -1, \quad \text{with BCs} \quad h(0) = h(1) = 0, \]

using boundary layer theory. (The case \( a = 1/2 \) was discussed in the lecture.) How thick is the boundary layer at \( x = 0 \), and how large is the solution in the boundary layer? Check your reasoning by constructing the leading-order uniformly valid solution when \( a = -1 \), \( a = 1 \) and \( a = 2 \).

Bonus problem: asymptotic estimate of \( \text{erfc}(1) = 0.157299 \)

Estimate \( \text{erfc}(1) \) by applying the optimal stopping rule to the asymptotic series

\[ \text{erfc}(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}} \left[ 1 - \frac{1}{2x^2} + \frac{1 \times 3}{(2x^2)^2} - \frac{1 \times 3 \times 5}{(2x^2)^3} + \frac{1 \times 3 \times 5 \times 7}{(2x^2)^4} + O\left(x^{-10}\right) \right], \]

and show that the relative error is over 30%. Show that adding half of the smallest term in (7) significantly reduces the error. How many terms in the convergent Taylor series

\[ \text{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}} \left( z - \frac{1}{3} z^3 + \frac{1}{10} z^5 - \frac{1}{17} z^7 + \frac{1}{210} z^9 - \frac{1}{1585} z^{11} + \cdots \right) \]

are required to do better than the divergent asymptotic expansion at \( x = 1 \)?