Second assignment SIO203B/MAE294B, 2017

For recitation on Friday April 14th — Christiaan Huygens's birthday

Lecture 3 is assigned as reading

Problem 2.4

Find a three-term approximation to the real solutions of

$$\mathrm{e}^{x-x^2} = \epsilon x^2\,, \qquad \text{as } \epsilon \to 0\,. \tag{1}$$

Problem 3.1

(i) Find $\lim_{t\to\infty} x(t)$, where $x(t)$ is the solution of

$$\dot{x} = (x-1)^2 - \frac{x^3}{100}\,, \qquad x(0) = 1\,.$$

(ii) Find the $t \to \infty$ limit if the initial condition is changed to $x(0) = 1.2$. In both cases give a numerical answer with two significant figures.

Problem 3.11

The red army, with strength $R(t)$, fights the green army, with strength $G(t)$. The conflict starts from an initial condition $G(0) = 2R(0)$ and proceeds according to

$$\dot{R} = -G\,, \qquad \dot{G} = -3R\,. \tag{2}$$

The war stops when one army is extinct. Which army wins, and how many soldiers are left at this time? (You can solve this problem without solving a differential equation.)

Problem 4.4

(i) Solve the problem

$$\ddot{x} + \left(1 + \epsilon \mathrm{e}^{\alpha t}\right) x = 0\,, \qquad \text{with IC} \qquad x(0,\epsilon,\alpha) = 1\,, \qquad \dot{x}(0,\epsilon,\alpha) = 0\,, \tag{3}$$

with the RPS

$$x(t,\epsilon,\alpha) = x_0(t,\alpha) + \epsilon x_1(t,\alpha) + \cdots \tag{4}$$

Calculate x_0 and x_1. *(ii)* Bearing in mind that α might be positive or negative, discuss the utility of the RPS when t is large. *(iii)* If energetic, compare your perturbation solution to a numerical solution.

Hand-in problems due in class on Thursday April 20th

Problem 2.5

Find two- or three- term approximations to all real solutions of

$$x^2 - 1 = \mathrm{e}^{\epsilon x}\,, \qquad \text{as } \epsilon \to 0\,. \tag{5}$$

(Two or three depending on your energy level.) Using figure 2.1 as an example, and considering the largest positive root, use MATLAB to compare your approximation with the exact relation.

Problem 4.5

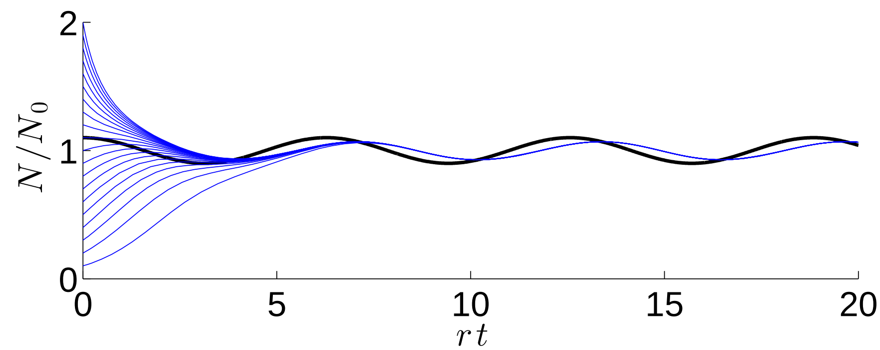

Figure 1: Numerical solution of (6) with various initial conditions. At large time all initial conditions convergence to a periodic solution that lags the carrying capacity.

Consider the logistic equation with a periodically varying carrying capacity:

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)\,, \qquad \text{with} \qquad K = K_0 + K_1 \cos\omega t\,. \tag{6}$$

The initial condition is $N(0) = N_0$. *(i)* Based on the $K_1 = 0$ solution, non-dimensionalize this problem. Show that there are three control parameters. *(ii)* Suppose that K_1 is a perturbation i.e., $K_1/K_0 \ll 1$ and that $N(t) \approx K_0$. Find the periodic-in-time solution of the perturbed problem (e.g., see Figure 1). *(iii)* Discuss the phase lag between the population, $N(t)$, and the carrying capacity $K(t)$ e.g., in figure 1 which curve is the carrying capacity?

Problem 4.10

Consider a medium $-\ell < x < \ell$ in which the temperature $\theta(x,t)$ is determined by

$$\theta_t - \kappa\theta_{xx} = \alpha e^{\beta\theta}\,, \tag{7}$$

with boundary conditions $\theta(\pm\ell, t) = 0$. The right hand side is a heat source due to an exothermic chemical reaction. The simple form in (7) is obtained by linearizing the Arrhenius law. The medium is cooled by the cold walls at $x = \pm\ell$. *(i)* Put the problem into the non-dimensional form

$$\Theta_T - \Theta_{XX} = \epsilon e^{\Theta} \qquad \text{with BCs} \qquad \Theta(\pm 1, \epsilon) = 0\,. \tag{8}$$

Your answer should include a definition of the dimensionless control parameter ϵ in terms of κ, α, β and ℓ. *(ii)* Assuming that $\epsilon \ll 1$, calculate the *steady* solution $\Theta(X,\epsilon)$ using a regular perturbation expansion. Obtain two or three non-zero terms and check your answer by showing that the "central temperature" is

$$C(\epsilon) \stackrel{\text{def}}{=} \Theta(0,\epsilon)\,, \tag{9}$$

$$= \frac{\epsilon}{2} + \frac{5\epsilon^2}{24} + \frac{47\epsilon^3}{360} + \text{ord}\left(\epsilon^4\right)\,. \tag{10}$$

(iii) Integrate the steady version of (8) exactly and deduce that:

$$\underbrace{e^{-C/2}\tanh^{-1}\sqrt{1-e^{-C}}}_{\stackrel{\text{def}}{=}F(C)} = \sqrt{\frac{\epsilon}{2}}\,. \tag{11}$$

(Use MATHEMATICA to do the integral.) Plot the function $F(C)$ and show that there is no steady solution if $\epsilon > 0.878$. *(iv)* Based on the graph of $F(C)$, if $\epsilon < 0.878$ then there are *two* solutions. There is the "cold solution", calculated perturbatively in (10), and there is a second "hot solution" with a large central temperature. Find an asymptotic expression for the hot central temperature as $\epsilon \to 0$.