Third assignment SIO203B/MAE294B, 2017

For recitation on Friday April 21st

Read section 4.3: Failure of RPS: singular perturbation problems

Problem 3.1

Integrate the fox-rabbit system (3.16) to show that the closed orbits in Figure 3.4 are given by

$$r + f - \ln(rf) = \text{constant} \tag{1}$$

Consider an initial condition $f(0) = r(0) = \epsilon$ with $0 < \epsilon \ll 1$. There is a boom and bust cycle: see figure 3.5. The initial condition is the bust. At some subsequent time there is a boom: the two populations are again equal and much greater than one. Estimate the size of the populations at this boom.

Problem 4.8

Find an approximate solution of the boundary value problem:

$$10^{-12} v_{xx} = \mathrm{e}^{x^4} v \,, \qquad \text{with BCs} \qquad v(\pm 1) = 1 \,. \tag{2}$$

This is an example of a boundary-layer problem and I haven't covered that material yet. But if you read the introductory discussion of boundary layers in section 4.3, and consider figure 4.3 (a), you might discover the boundary-layer method yourself.

Problem 5.1

Consider the partial differential equation

$$\kappa \left(C_{xx} + C_{zz} \right) - \mu C = 0 \tag{3}$$

in the region above $z = h_{\max} \cos kx$. The boundary conditions are $C(x, h_{\max} \cos kx) = C_*$ and $C(x, z) \to 0$ as $z \to \infty$. *(i)* Describe a physical situation governed by this boundary value problem. *(ii)* Solve the problem with $h_{\max} = 0$. *(iii)* Based on your exact solution, non-dimensionalize the problem with non-zero $h_{\max}$ and determine the non-dimensional control parameters. *(iv)* Use perturbation theory to find the first effects of small non-zero $h_{\max}$ on the "inventory"

$$A \stackrel{\text{def}}{=} \int_{h(x)}^{\infty} \int_0^{2\pi/k} C(x, z) \, \mathrm{d}x \, \mathrm{d}z \,. \tag{4}$$

(I think you'll have to go to second order in $h_{\max}$.)

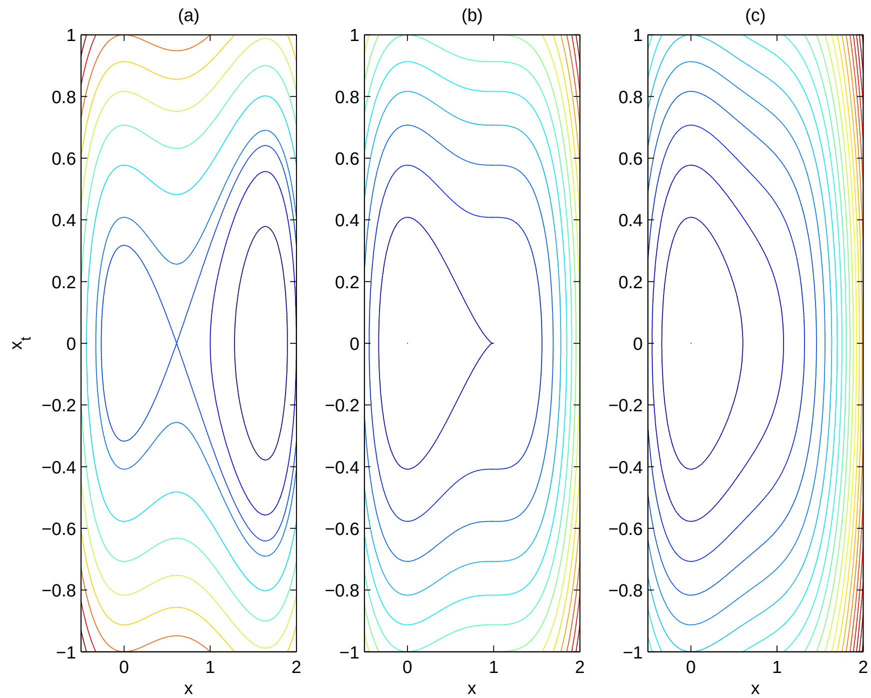

Figure 1: Which phase-plane corresponds to (5)?

Problem 3.15

The nonlinear oscillator

$$\ddot{x} + x - 2x^2 + x^3 = 0\,, \tag{5}$$

has an energy integral of the form

$$E = \tfrac{1}{2}\dot{x}^2 + V(x)\,. \tag{6}$$

(a) Find the potential function $V(x)$ and sketch this function on the range $-\frac{1}{2} < x < 2$. Label your axes so that your sketch of $V(x)$ is quantitative. *(b)* Figure 1 shows three possible phase plane diagrams. In ten or twenty words explain which diagram corresponds to the oscillator in (5).

Problem 4.7

This difficult problem is for discussion rather than solution — it may be assigned as "hand-in" later. Finding the "best" way to non-dimensionalize the problem is tricky. Don't spend a lot of time on this at the expense of the other problems.

Let's make a small change to the formulation of the belligerent-drunks example in (4.25) and (4.26). Suppose that we model the bars using a Neumann boundary condition. This means that the flux of drunks, rather than the concentration, is prescribed at $x = 0$ and ℓ: the boundary condition in (4.26) is changed to

$$\kappa u_x(0,t) = -F\,, \qquad \text{and} \qquad \kappa u_x(\ell,t) = F\,, \tag{7}$$

where F, with dimensions drunks per second, is the flux entering the domain from the bars. Try to repeat *all calculations* in section 4.2, including the analog of the $\beta \ll 1$ perturbation expansion. You'll find that it is not straightforward and that a certain amount of ingenuity is required to understand the weakly interacting limit with fixed-flux boundary conditions.

Hand-in problems due in class on Thursday April 20th

Problem 2.5

Find two- or three- term approximations to all real solutions of

$$x^2 - 1 = \mathrm{e}^{\epsilon x}\,, \qquad \text{as } \epsilon \to 0\,. \tag{8}$$

(Two or three depending on your energy level.) Using figure 2.1 as an example, and considering the largest positive root, use MATLAB to compare your approximation with the exact relation.

Problem 4.5

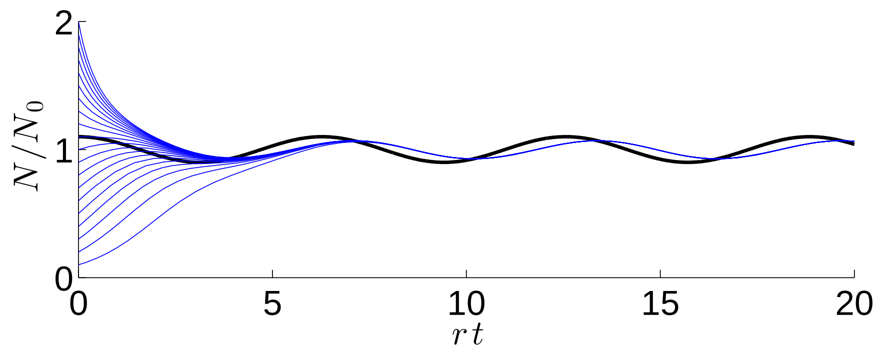

Figure 2: Numerical solution of (9) with various initial conditions. At large time all initial conditions convergence to a periodic solution that lags the carrying capacity.

Consider the logistic equation with a periodically varying carrying capacity:

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)\,, \qquad \text{with} \qquad K = K_0 + K_1 \cos\omega t\,. \tag{9}$$

The initial condition is $N(0) = N_0$. *(i)* Based on the $K_1 = 0$ solution, non-dimensionalize this problem. Show that there are three control parameters. *(ii)* Suppose that K_1 is a perturbation i.e., $K_1/K_0 \ll 1$ and that $N(t) \approx K_0$. Find the periodic-in-time solution of the perturbed problem (e.g., see Figure 2). *(iii)* Discuss the phase lag between the population, $N(t)$, and the carrying capacity $K(t)$ e.g., in figure 2 which curve is the carrying capacity?

Problem 4.10

Consider a medium $-\ell < x < \ell$ in which the temperature $\theta(x,t)$ is determined by

$$\theta_t - \kappa\theta_{xx} = \alpha \mathrm{e}^{\beta\theta}\,, \tag{10}$$

with boundary conditions $\theta(\pm\ell, t) = 0$. The right hand side is a heat source due to an exothermic chemical reaction. The simple form in (10) is obtained by linearizing the Arrhenius law. The medium is cooled by the cold walls at $x = \pm\ell$. *(i)* Put the problem into the non-dimensional form

$$\Theta_T - \Theta_{XX} = \epsilon \mathrm{e}^{\Theta} \qquad \text{with BCs} \qquad \Theta(\pm 1, \epsilon) = 0\,. \tag{11}$$

Your answer should include a definition of the dimensionless control parameter ϵ in terms of κ, α, β and ℓ. *(ii)* Assuming that $\epsilon \ll 1$, calculate the *steady* solution $\Theta(X,\epsilon)$ using a regular perturbation expansion. Obtain two or three non-zero terms and check your answer by showing that the "central temperature" is

$$C(\epsilon) \stackrel{\text{def}}{=} \Theta(0,\epsilon)\,, \tag{12}$$

$$= \frac{\epsilon}{2} + \frac{5\epsilon^2}{24} + \frac{47\epsilon^3}{360} + \text{ord}\left(\epsilon^4\right)\,. \tag{13}$$

(iii) Integrate the steady version of (11) exactly and deduce that:

$$\underbrace{\mathrm{e}^{-C/2}\tanh^{-1}\sqrt{1-\mathrm{e}^{-C}}}_{\stackrel{\text{def}}{=}F(C)} = \sqrt{\frac{\epsilon}{2}}\,. \tag{14}$$

(Use MATHEMATICA to do the integral.) Plot the function $F(C)$ and show that there is no steady solution if $\epsilon > 0.878$. *(iv)* Based on the graph of $F(C)$, if $\epsilon < 0.878$ then there are *two* solutions. There is the "cold solution", calculated perturbatively in (13), and there is a second "hot solution" with a large central temperature. Find an asymptotic expression for the hot central temperature as $\epsilon \to 0$.