Seventh assignment SIO203B/MAE294B, 2017

For recitation on Friday May 26th

Problem 4.3 part B

Show that

$$\int_0^1 \mathrm{e}^t \left(\frac{t}{1+t^2}\right)^n \mathrm{d}t \sim \sqrt{\frac{\pi}{2n}}\,\frac{\mathrm{e}}{2^n}\,, \qquad \text{as } n \to \infty. \tag{1}$$

Problem 4.5 part B

The beta function is

$$B(x,y) \stackrel{\text{def}}{=} \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t\,. \tag{2}$$

With a change of variables show that

$$B(x,y) = \int_0^\infty \mathrm{e}^{-xv}(1-\mathrm{e}^{-v})^{y-1}\,\mathrm{d}v\,. \tag{3}$$

Suppose that y is fixed and $x \to \infty$. Use Laplace's method to obtain the leading order approximation

$$B(x,y) \sim \frac{\Gamma(y)}{x^y}\,. \tag{4}$$

Go to the *Digital Library of Special Functions*, chapter 5 and find the relation between the beta function and the gamma function. (You can probably also find this formula in **RHB**, or any text on special functions.) Use this relation to show that

$$\frac{\Gamma(x)}{\Gamma(x+y)} \sim \frac{1}{x^y}\,, \qquad \text{as } x \to \infty. \tag{5}$$

Remark: this result can also be deduced from Stirling's approximation, but its a rather messy calculation.

Problem 4.7 part B

Find leading order $x \to \infty$ asymptotic approximations to the integrals

$$E(x) = \int_0^\infty \mathrm{e}^{-xt-t^4/4}\,\mathrm{d}t\,, \qquad \text{and} \qquad F(x) = \int_0^\infty \mathrm{e}^{xt-t^4/4}\,\mathrm{d}t\,. \tag{6}$$

Problem 8.6 part A

A multiple scale ($0 < \epsilon \ll 1$) reduction of the system

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\epsilon y\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \tfrac{1}{2}\epsilon x^2\,, \tag{7}$$

begins with

$$x = \left[A(s)\mathrm{e}^{\mathrm{i}t} + A^*(s)\mathrm{e}^{-\mathrm{i}t}\right] + \epsilon x_1(t,s) + \cdots\,, \qquad y = B(s) + \epsilon y_1(t,s) + \cdots \tag{8}$$

where $s = \epsilon t$ is the slow time. *(i)* Find coupled evolution equations for $A(s)$ and $B(s)$. *(ii)* Show that the system from part *(i)* can be reduced to

$$B_s = E - B^2 , \tag{9}$$

where E is a constant of integration. *(iii)* Determine E if the initial conditions are

$$x(0) = 2 , \qquad \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0 , \qquad y(0) = 0 . \tag{10}$$

(iv) With the initial condition above, find $\lim_{s\to\infty} B(s)$.

Hand-in problems due in class on Tuesday May 30th

Problem 8.10 part A

The equation of motion of a pendulum with length ℓ in a gravitational field g is

$$\ddot{\theta} + \omega^2 \sin\theta = 0 , \qquad \text{with} \qquad \omega^2 \stackrel{\text{def}}{=} \frac{g}{\ell} . \tag{11}$$

Suppose that the maximum displacement is $\theta_{\max} = \phi$. *(a)* Show that the period P of the oscillation is

$$\omega P = 2\sqrt{2} \int_0^{\phi} \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\phi}} .$$

(b) Suppose that $\phi \ll 1$. By approximating the integral above, obtain the coefficient of ϕ^2 in the expansion:

$$\omega P = 2\pi \left[1 + ?\phi^2 + O(\phi^3)\right]$$

(c) Check this result by re-derving it via a multiple scale expansion applied to (11). *(d)* A grandfather clock swings to a maximum angle $\phi = 5^\circ$ from the vertical. How many seconds does the clock lose or gain each day if the clock is adjusted to keep perfect time when the swing is $\phi = 2^\circ$?

Problem 9.1 part A

Consider the *nonlinear* inverted pendulum

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} - \left[1 + \frac{\alpha}{\epsilon}\cos\left(\frac{t}{\epsilon}\right)\right] \sin\theta = 0 . \tag{12}$$

Apply the two-time method to this nonlinear equation and find the effective potential resulting from averaging the rapid oscillations. Calculate the phase-plane orbits in the effective potential and use MATLAB to plot the orbits.