Eighth assignment SIO203B/MAE294B, 2017

For recitation on Friday June 2nd

Problem 1.11 part B: Solve the half-plane ($y > 0$) boundary value problem

$$y u_{xx} + u_{yy} = 0 \tag{1}$$

with $u(x, 0) = \cos qx$ and $\lim_{y\to\infty} u(x, y) = 0$.

Problem 5.9 part B: Find a leading order $x \to \infty$ asymptotic approximation to

$$B(x) = \int_0^{\pi} \mathrm{e}^{\mathrm{i}x(t+\cos t)}\, \mathrm{d}t\,. \tag{2}$$

Problem 5.11 part B: According to article 238 in Lamb, the surface elevation produced by a two-dimensional splash is given by

$$\eta(x, t) = \frac{1}{\pi}\int_0^{\infty} \cos\left(kx - \sqrt{gk}\, t\right) \mathrm{d}k\,. \tag{3}$$

Show that as $t \to \infty$

$$\eta(x, t) \sim \frac{\sqrt{g}t}{2^{\frac{3}{2}}\sqrt{\pi}x^{\frac{3}{2}}}\left(\cos\frac{gt^2}{4x} + \sin\frac{gt^2}{4x}\right)\,. \tag{4}$$

Verify that the frequency and wavenumber in (4) are connected by the water-wave dispersion relation $\omega = \sqrt{gk}$.

Problem 5.7 part B: The Bessel function of integer order n has an integral representation

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin t - nt)\, \mathrm{d}t\,. \tag{5}$$

Show that

$$J_n(n) \sim \frac{\Gamma\left(\frac{1}{3}\right)}{\pi 2^{2/3} 3^{1/6} n^{1/3}}\,, \qquad \text{as } n \to \infty. \tag{6}$$

Problem 11.2 part A: Consider the IVP

$$\ddot{x} + 256\mathrm{e}^{4t}x = 0\,, \qquad x(0) = 0\,, \qquad \dot{x}(0) = 1\,. \tag{7}$$

Estimate the position and magnitude of the first positive maximum of $y(t)$. Compare the WKB approximation with a numerical solution on the interval $0 < t \leq 1$.

Problem 11.3 part A: Consider the differential equation

$$y'' + \underbrace{\frac{400}{400 + x^2}}_{Q(x)}\, y = 0\,. \tag{8}$$

How can we apply the WKB approximation to this equation? Where is the small parameter in front of y''? Compare the physical optics approximation to a numerical solution with the initial conditions $y(0) = 1$ and $y'(0) = 0$.

Hand-in problems due in class on Tuesday May 30th

Problem 8.10 part A

The equation of motion of a pendulum with length ℓ in a gravitational field g is

$$\ddot{\theta} + \omega^2 \sin\theta = 0\,, \qquad \text{with} \qquad \omega^2 \stackrel{\text{def}}{=} \frac{g}{\ell}\,. \tag{9}$$

Suppose that the maximum displacement is $\theta_{\max} = \phi$. *(a)* Show that the period P of the oscillation is

$$\omega P = 2\sqrt{2}\int_0^{\phi} \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\phi}}\,.$$

(b) Suppose that $\phi \ll 1$. By approximating the integral above, obtain the coefficient of ϕ^2 in the expansion:

$$\omega P = 2\pi\left[1 + ?\phi^2 + O(\phi^3)\right]$$

(c) Check this result by re-derving it via a multiple scale expansion applied to (9). *(d)* A grandfather clock swings to a maximum angle $\phi = 5^\circ$ from the vertical. How many seconds does the clock lose or gain each day if the clock is adjusted to keep perfect time when the swing is $\phi = 2^\circ$?

Problem 9.1 part A

Consider the *nonlinear* inverted pendulum

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} - \left[1 + \frac{\alpha}{\epsilon}\cos\left(\frac{t}{\epsilon}\right)\right]\sin\theta = 0\,. \tag{10}$$

Apply the two-time method to this nonlinear equation and find the effective potential resulting from averaging the rapid oscillations. Calculate the phase-plane orbits in the effective potential and use MATLAB to plot the orbits.