Ninth assignment SIO203B/MAE294B, 2017

For recitation on Friday June 9th

Problem10.1: Consider the eigenvalue problem

$$y'' = -\lambda y\,, \qquad \text{with BCs:} \qquad y(0) = 0\,, \qquad y'(1) = y(1)\,. \tag{1}$$

(i) Prove that all the eigenvalues are real. *(ii)* Find the transcendental equation whose solutions determine the eigenvalues λ_n. *(iii)* Find an explicit expression for the smallest eigenvalue λ_0 and the associated eigenfunction $y_0(x)$. *(iv)* Show that the eigenfunctions are orthogonal with respect to an appropriately defined inner product. *(v)* Attempt to solve the inhomogeneous boundary value problem

$$y'' = a(x), \qquad \text{with BCs:} \qquad y(0) = 0\,, \qquad\qquad y'(1) = y(1)\,, \tag{2}$$

via an expansion using the eigenmodes. Show that this expansion fails because the problem has no solution for an arbitrary $a(x)$. *(iv)* Find the solvability condition on $a(x)$ which ensures that the problem (2) does have a solution, and then obtain the solution using a modal expansion.

Problem11.4: Consider

$$y'' + \frac{a}{x^2} y = 0\,. \tag{3}$$

Take $a > 0$ and obtain the physical-optics approximation. Compare to the exact solution. Is the physical-optics approximation asymptotically valid as $x \to \infty$? As $x \to 0$? Is the physical-optics approximation ever valid?

A WKB eigenproblem: Find an approximation to the large eigenvlaues of the Sturm-Liouville problem

$$\phi'' + \lambda \mathrm{e}^{2x} \phi = 0\,, \qquad \text{posed on } 0 < x < 1, \text{ with BCs:} \qquad \phi(0) = 0\,, \qquad \phi'(1) = 0\,. \tag{4}$$

(Bonus for comparison with a numerical solution.)

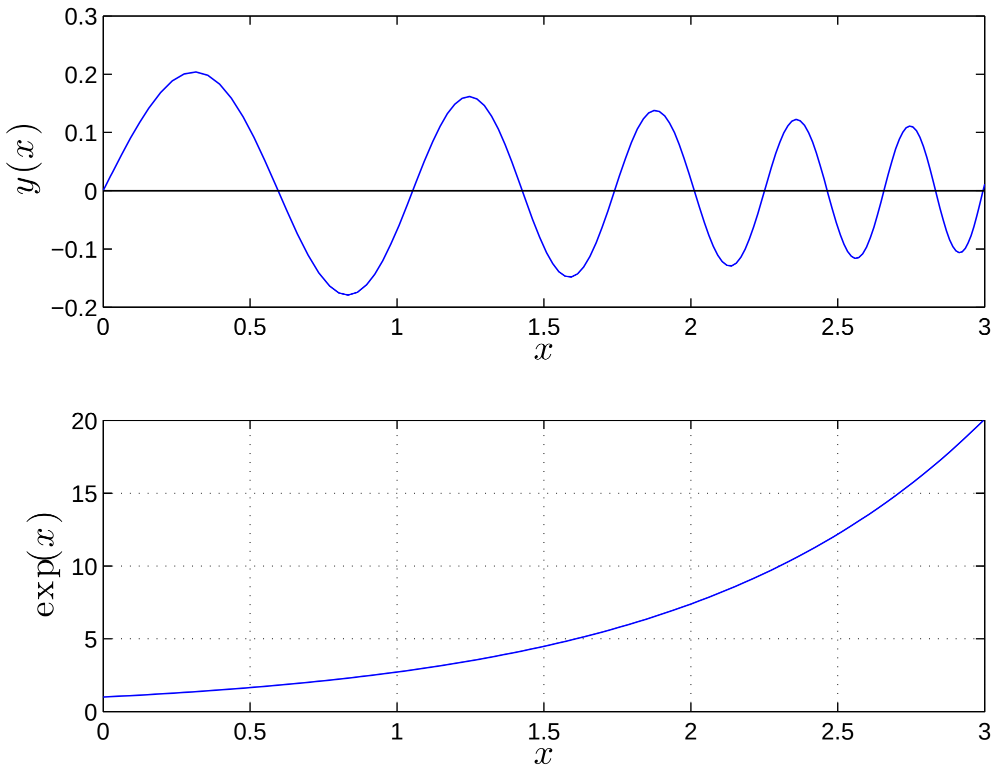

Figure 1: Figure for problem **6**.

From the 2015 final: The top panel of figure 1 shows the solution to one of the four initial value problems:

$$
\begin{aligned}
\epsilon^2 y_1'' - \mathrm{e}^{-x} y_1 = 0\,, &\quad y_1(0) = 0\,, \quad y_1'(0) = 1\,,\\
\epsilon^2 y_2'' - \mathrm{e}^{x} y_2 = 0\,, &\quad y_2(0) = 0\,, \quad y_2'(0) = 1\,,\\
\epsilon^2 y_3'' + \mathrm{e}^{-x} y_3 = 0\,, &\quad y_3(0) = 0\,, \quad y_3'(0) = 1\,,\\
\epsilon^2 y_4'' + \mathrm{e}^{x} y_4 = 0\,, &\quad y_4(0) = 0\,, \quad y_4'(0) = 1\,.
\end{aligned}
$$

(a) Which $y_n(x)$ is shown in the top panel figure 1? (b) Use the WKB approximation and the information in the bottom panel of Figure 1 to estimate the value of ϵ used in top panel. A one-significant figure numerical answer is good enough.