SIO203B/MAE294B Final 2014

This exam is open notes, but no computers, iPhones or electronic assistance.

Problem 1

Consider the eigenproblem

$$y'' + \epsilon f(x) y + \lambda y = 0\,, \tag{1}$$

posed on $0 < x < \pi$, with Dirichlet boundary conditions $y(0) = y(\pi) = 0$. If $\epsilon = 0$, the gravest mode is $y_0 = \sin x$ with eigenvalue $\lambda_0 = \pi^2$. Calculate the first-order shift in the eigenvalue λ_0 induced by the perturbation $\epsilon f(x)$. (An unevaluated integral involving $f(x)$ is acceptable.)

Problem 2

Consider the boundary value problem

$$\epsilon y'' + 2\sigma x y' + 4x^2 y = 0\,, \tag{2}$$

posed on the interval $0 < x < 1$ with boundary conditions $y(0) = 0$ and $y(1) = 1$. *(i)* First, with

$$\sigma = +1\,, \tag{3}$$

construct a uniformly valid leading-order $\epsilon \to 0$ solution. *(ii)* Next, do the same with

$$\sigma = -1\,. \tag{4}$$

Problem 3

Find $\lim_{t\to\infty} x(t)$, where $x(t)$ is the solution of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{e}^x - 1.02 - x\,, \qquad \text{with initial condition} \qquad x(0) = 0\,. \tag{5}$$

Give a numerical answer with one significant figure.

Problem 4

Consider the eigenvalue problem

$$\epsilon y'' + 2xy' + \lambda x \int_0^1 y(t)\,\mathrm{d}t = 0\,, \tag{6}$$

posed on $0 < x < 1$ with boundary conditions $y(0) = y(1) = 0$. Notice that $y(x) = 0$ is a trivial solution for every value of λ. The problem has a nontrivial solution only if $\lambda(\epsilon)$ (the eigenvalue) has some special value. (a) Show that

$$2\epsilon\left[y'(1) - y'(0)\right] + (\lambda - 4)\int_0^1 y(t)\,\mathrm{d}t = 0\,. \tag{7}$$

(b) In the limit $\epsilon \to 0$, construct a nontrivial solution with boundary-layer theory and determine $\lambda(0)$. (c) Use the exact result (7) to deduce the next correction to the $\epsilon \to 0$ expansion of $\lambda(\epsilon)$.

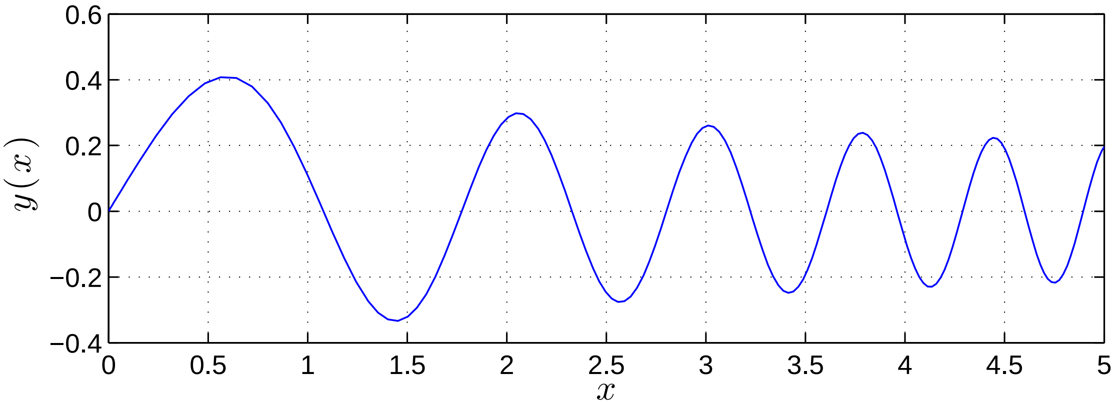

Figure 1: Figure for problem 5.

Problem 5

Figure 1 shows the solution of one of the following differential equations

$$\epsilon^2 y_1'' + (1+x)^2 y_1 = 0\,, \tag{8}$$

$$\epsilon^2 y_2'' - (1+x)^2 y_2 = 0\,, \tag{9}$$

$$\epsilon^2 y_3'' - (1+x)^{-2} y_3 = 0\,, \tag{10}$$

$$\epsilon^2 y_4'' + (1+x)^{-2} y_4 = 0\,. \tag{11}$$

Which $y_n(x)$ is show in the figure? Estimate the value of ϵ used in this calculation.