

SIO203B/MAE294B Final 2015

No computers, calculators, iphones etc.

Problem 1

Find the leading-order, $x \rightarrow \infty$, asymptotic expansion of

$$E(x) \stackrel{\text{def}}{=} \int_0^\infty e^{-xt-t^4/4} dt, \quad \text{and} \quad F(x) \stackrel{\text{def}}{=} \int_0^\infty e^{xt-t^4/4} dt. \quad (1)$$

Problem 2

Use multiple scale theory to find an approximate solution of the initial value problem

$$u_{tt} + u = 2 [\cos(\epsilon t) + \epsilon u^2], \quad \text{with ICs} \quad u(0) = 0, \quad u_t(0) = 0. \quad (2)$$

Problem 3

The function $y(x, \epsilon)$ satisfies

$$\epsilon y'' + \sqrt{xy}' + e^{-y} = 0, \quad \text{in } 0 < x < 1, \quad (3)$$

and is subject to the BCs $y(0) = 0$ and $y(1) = 1$. (i) Find the rescaling for the boundary layer near $x = 0$, and obtain the leading-order inner approximation. (ii) Find the leading-order outer approximation and match the two approximations. (iii) Construct a uniform approximation. All definite integrals should be evaluated in terms of $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$.

Problem 4

If $\epsilon = 0$ the eigenproblem

$$(1 + \epsilon y) \frac{d^2 y}{dx^2} + \lambda y = 0, \quad y(0) = y(\pi) = 0, \quad (4)$$

has the solution $\lambda = 1$ and $y(x) = a \sin x$. Use perturbation theory ($0 < \epsilon \ll 1$) to investigate the dependence of the eigenvalue $\lambda(\epsilon)$ on a and ϵ . To check your answer, show that if $\epsilon = 1/7$ and $a = 1$ then $\lambda \approx 37/33$.

Problem 5

Find a leading-order $\epsilon \rightarrow 0$ approximation to

$$F(\epsilon) \stackrel{\text{def}}{=} \int_0^\infty \frac{x dx}{(\epsilon^2 + x^2)\sqrt{1+x^2}}. \quad (5)$$

Some of these indefinite integrals

$$\int \frac{dt}{1+t^2} = \arctan t, \quad \int \frac{dt}{\sqrt{1+t^2}} = \ln(t + \sqrt{1+t^2}), \quad (6)$$

$$\int \frac{dt}{t^2\sqrt{1+t^2}} = -\frac{\sqrt{1+t^2}}{t}, \quad \int \frac{dt}{t\sqrt{1+t^2}} = \ln\left(\frac{t}{1+\sqrt{1+t^2}}\right), \quad (7)$$

$$\int \frac{2t dt}{1+t^2} = \ln(1+t^2), \quad \int \frac{t dt}{\sqrt{1+t^2}} = \sqrt{1+t^2}. \quad (8)$$

might be useful.

TURN THE PAGE — THERE IS ANOTHER QUESTION

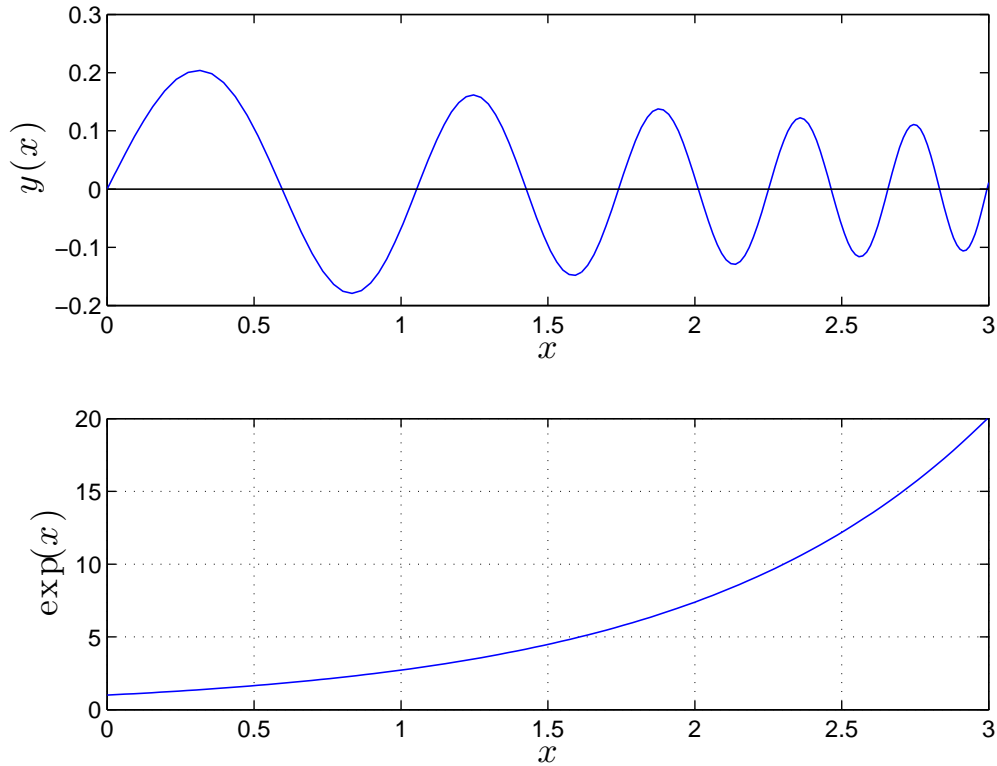


Figure 1: Figure for problem 6.

Problem 6

The top panel of figure 1 shows the solution to one of the four initial value problems:

$$\begin{aligned}
 \epsilon^2 y_1'' - e^{-x} y_1 &= 0, & y_1(0) &= 0, & y_1'(0) &= 1, \\
 \epsilon^2 y_2'' - e^x y_2 &= 0, & y_2(0) &= 0, & y_2'(0) &= 1, \\
 \epsilon^2 y_3'' + e^{-x} y_3 &= 0, & y_3(0) &= 0, & y_3'(0) &= 1, \\
 \epsilon^2 y_4'' + e^x y_4 &= 0, & y_4(0) &= 0, & y_4'(0) &= 1.
 \end{aligned}$$

(i) Which $y_n(x)$ is shown in the top panel figure 1? Lucky guesses don't count — explain your answer. (ii) Use the WKB approximation and the information in the bottom panel of Figure 1 to estimate the value of ϵ as either a decimal with one significant figure or as a simple fraction.