Part B: Asymptotic Expansions and Integrals

W.R. Young

April 2017

1Scripps Institution of Oceanography, University of California at San Diego, La Jolla, CA 92093–0230, USA. wryoung@ucsd.edu
## Contents

1. **Why integrals?**
   - 1.1 The Airy function .................................................. 3
   - 1.2 Recursion relations: the example $n!$ .................................. 4
   - 1.3 Special functions defined by integrals .................................. 5
   - 1.4 Elementary methods for evaluating integrals .......................... 6
   - 1.5 Complexification ...................................................... 7
   - 1.6 Problems .................................................................. 10

2. **What is asymptotic?**
   - 2.1 An example: the erf function ........................................ 14
   - 2.2 The Landau symbols .................................................... 20
   - 2.3 The definition of asymptoticity ........................................ 22
   - 2.4 Manipulation of asymptotic series ..................................... 24
   - 2.5 Stokes lines .................................................................. 26
   - 2.6 Problems .................................................................. 26

3. **Integration by parts (IP)**
   - 3.1 The Taylor series, with remainder ..................................... 29
   - 3.2 Large-$s$ behaviour of Laplace transforms .......................... 31
   - 3.3 Watson’s Lemma ......................................................... 34
   - 3.4 Problems .................................................................. 34

4. **Laplace’s method**
   - 4.1 An example — the Gaussian approximation ....................... 37
   - 4.2 Another Laplacian example ............................................ 39
   - 4.3 Laplace’s method with moving maximum ............................ 41
   - 4.4 Uniform approximations ............................................... 42
   - 4.5 Problems .................................................................. 44

5. **Fourier Integrals and Stationary phase**
   - 5.1 Fourier Series ............................................................ 49
   - 5.2 Generalized Fourier Integrals ........................................... 54
   - 5.3 The Airy function ......................................................... 58
   - 5.4 Problems .................................................................. 59

6. **Dispersive wave equations**
   - 6.1 Group velocity ............................................................ 62
   - 6.2 The 1D KG equation ..................................................... 65
   - 6.3 Problems .................................................................. 70
Lecture 1

Why integrals?

Integrals occur frequently as the solution of partial and ordinary differential equations, and as the
definition of many “special functions”. The coefficients of a Fourier series are given as integrals
involving the target function etc. Green’s function technology expresses the solution of a differ-
cential equation as a convolution integral etc. Integrals are also important because they provide
the simplest and most accessible examples of concepts like asymptoticity and techniques such as
asymptotic matching.

1.1 The Airy function

Airy’s equation,

\[ y'' - xy = 0, \tag{1.1} \]

is an important second-order differential equation. The two linearly independent solutions, Ai(x) and Bi(x), are shown in figure 1.1. The Airy function, Ai(x), is defined as the solution that decays
as \( x \to \infty \), with the normalization

\[ \int_{-\infty}^{\infty} Ai(x) \, dx = 1. \tag{1.2} \]

We obtain an integral representation of Ai(x) by attacking (1.1) with the Fourier transform. Denote
the Fourier transform of Ai by

\[ \tilde{Ai}(k) = \int_{-\infty}^{\infty} Ai(x)e^{-ikx} \, dx. \tag{1.3} \]

Fourier transforming (1.1), we eventually find

\[ \tilde{Ai}(k) = e^{ik^3/3}. \tag{1.4} \]

Using the Fourier integral theorem

\[ Ai(x) = \int_{-\infty}^{\infty} e^{ikx+ik^3/3} \frac{dk}{2\pi}, \tag{1.5} \]

\[ = \frac{1}{\pi} \int_{0}^{\infty} \cos \left( kx + \frac{k^3}{3} \right) \, dk. \tag{1.6} \]

Notice that the integral converges at \( k = \infty \) because of destructive interference or catastrophic
cancellation.
Figure 1.1: The functions $Ai(x)$ and $Bi(x)$. The Airy function decays rapidly as $x \to \infty$ and rather slowly as $x \to -\infty$.

We’ll develop several techniques for extracting information from integral representations such as (1.6). We’ll show that as $x \to -\infty$:

$$Ai(x) \sim \frac{1}{\sqrt{\pi |x|^{1/4}}} \cos \left( \frac{2|x|^{3/2}}{3} - \frac{\pi}{4} \right),$$

(1.7)

and as $x \to +\infty$:

$$Ai(x) \sim e^{-\frac{2|x|^{3/2}}{3}} \frac{x^{1/4}}{2\sqrt{\pi x^{1/4}}}.$$

(1.8)

Exercise: Fill in the details between (1.3) and (1.4).

1.2 Recursion relations: the example $n!$

The factorial function

$$a_n = n!$$

(1.9)

satisfies the recursion relation

$$a_{n+1} = (n+1)a_n, \quad a_0 = 1.$$  

(1.10)

The integral representation

$$a_n = \int_0^\infty t^n e^{-t} \, dt$$

(1.11)

is equivalent to both the initial condition and the recursion relation. The proof is integration by parts:

$$\begin{align*}
\int_0^\infty t^{n+1} e^{-t} \, dt &= - \int_0^\infty \frac{d}{dt} t^{n+1} e^{-t} \, dt, \\
&= - \left[ t^{n+1} e^{-t} \right]_0^\infty + (n+1) \int_0^\infty t^n e^{-t} \, dt.
\end{align*}$$

(1.12)

(1.13)

Exercise: Memorize

$$n! = \int_0^\infty t^n e^{-t} \, dt.$$  

(1.14)

Later we will use the integral representation (1.14) to obtain Stirling’s approximation:

$$n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n, \quad \text{as } n \to \infty.$$  

(1.15)

Exercise: Compare Stirling’s approximation to $n!$ with $n = 1, 2$ and 3.
1.3 Special functions defined by integrals

We’ll soon encounter the error function and its complement:

\[ \text{erf}(z) \overset{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \, dt \quad \text{and} \quad \text{erfc}(z) \overset{\text{def}}{=} 1 - \text{erf}(z). \quad (1.16) \]

Another special function defined by an integral is the exponential integral of order \( n \):

\[ E_n(z) \overset{\text{def}}{=} \int_z^{\infty} e^{-t} \frac{t^n}{n!} \, dt. \quad (1.17) \]

We refer to the case \( n = 1 \) simply as the “exponential integral”.

**Example:** Singularity subtraction — small \( z \) behavior of \( E_n(z) \).

**The Gamma function:** \( \Gamma(z) \overset{\text{def}}{=} \int_0^{\infty} t^{z-1} e^{-t} \, dt, \text{ for } \Re z > 0. \)

There are many other examples of special functions defined by integrals. Probably the most important is the Gamma function, which is defined in the heading of this section — see Figure 1.2. If \( \Re z > 0 \) we can use integration by parts to show that \( \Gamma(z) \) satisfies the functional equation

\[ z \Gamma(z) = \Gamma(z + 1). \quad (1.18) \]

Using analytic continuation\(^1\) this result is valid for all \( z \neq 0, -1, -2 \cdots \). Thus the functional equation (1.18) is used to extend the definition of \( \Gamma \)-function throughout the complex plane. Notice that if \( z \) is an integer, \( n \), then

\[ \Gamma(n + 1) = n! \quad (1.19) \]

\(^1\)If \( f(z) \) and \( g(z) \) are analytic in a domain \( D \), and if \( f = g \) in a smaller domain \( E \subset D \), then \( f = g \) throughout \( D \).
The special value
\[ \Gamma \left( \frac{1}{2} \right) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}} \, dt = \int_{-\infty}^{\infty} e^{-u^2} \, du = \sqrt{\pi} \] (1.20)
is important.

**Exercise:** Use the functional equation (1.18) to obtain \( \Gamma(3/2) \) and \( \Gamma(-1/2) \).

**Exercise:** Use the functional equation (1.18) to find the leading order behaviour of \( \Gamma(z) \) near \( z = 0 \) and \( z = -1 \), and other negative integers. Work backwards and show that
\[ \Gamma(x) \sim \frac{(-1)^n}{n!} \frac{1}{x + n}, \quad \text{as } x \to -n. \]
Thus \( \Gamma(z) \) has poles at \( z = 0, -1, \ldots \) with residues \( (-1)^n/n! \).

### 1.4 Elementary methods for evaluating integrals

#### Change of variables

How can we evaluate the integral
\[ \int_0^\infty e^{-t^3} \, dt ? \] (1.21)

Try a change of variable
\[ v = t^3 \quad \text{and therefore} \quad dv = 3t^2 \, dt = 3v^{2/3} \, dt. \] (1.22)
The integral is then
\[ \frac{1}{3} \int_0^\infty e^{-v} v^{-2/3} \, dv = \frac{1}{3} \Gamma \left( \frac{1}{3} \right) = \Gamma \left( \frac{4}{3} \right). \] (1.23)

**Exercise:** Evaluate in terms of the \( \Gamma \)-function
\[ U(\alpha, p, q) \overset{\text{def}}{=} \int_0^\infty t^p e^{-\alpha t^q} \, dt. \]

**Exercise:** Show that
\[ L[p] = \int_0^\infty e^{-s t} \, dt = \frac{\Gamma(1+p)}{s^{1+p}}. \] (1.24)

#### Differentiation with respect to a parameter

Given that
\[ \frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-x^2} \, dx, \] (1.25)
we can make the change of variables \( x = \sqrt{t} x' \) and find that
\[ \frac{1}{2} \sqrt{\frac{\pi}{t}} = \int_0^\infty e^{-t x'^2} \, dx'. \] (1.26)

We now have an integral containing the parameter \( t \).
To evaluate
\[ \int_0^\infty x^2 e^{-t x^2} \, dx, \] (1.27)
we differentiate (1.26) with respect to \( t \) to obtain
\[ \frac{1}{4} \sqrt{\frac{\pi}{t^3}} = \int_0^\infty x^2 e^{-t x^2} \, dx, \quad \text{and again} \quad \frac{3}{8} \sqrt{\frac{\pi}{t^5}} = \int_0^\infty x^4 e^{-t x^2} \, dx. \] (1.28)
Differentiation with respect to a parameter is a very effective trick. For some reason it is not taught to undergraduates.

How would you calculate $L[t^p \ln t]$? No problem — just notice that

$$\partial_p t^p = \partial_p e^{p \ln t} = t^p \ln t,$$

and then take the derivative of (1.24) with respect to $p$

$$L[t^p \ln t] = \frac{\Gamma'(1 + p)}{s^{1+p}} - \frac{\Gamma(1 + p) \ln s}{s^{1+p}},$$

$$= \frac{\Gamma(1 + p)}{s^{1+p}} \left[ \psi(1 + p) - \ln s \right],$$

where the *digamma function*

$$\psi(z) \overset{\text{def}}{=} \frac{\Gamma'(z)}{\Gamma(z)}$$

is the derivative of $\ln \Gamma$.

### 1.5 Complexification

Consider

$$F(a, b) = \int_{0}^{\infty} e^{-at} \cos bt \, dt,$$

where $a > 0$. Then

$$F = \Re \int_{0}^{\infty} e^{-(a+ib)t} \, dt,$$

$$= \Re \frac{1}{a + ib} = \Re \frac{a - ib}{a^2 + b^2},$$

$$= \frac{a}{a^2 + b^2}.$$ 

As a bonus, the imaginary part gives us

$$\frac{b}{a^2 + b^2} = \int_{0}^{\infty} e^{-at} \sin bt \, dt.$$

Derivatives with respect to the parameters $a$ and $b$ generate further integrals.

**Contour integration**

The theory of contour integration, covered in part B, is an example of complexification. As revision we'll consider examples that illustrate important techniques.

**Example:** Consider the Fourier transform

$$f(k) = \int_{-\infty}^{\infty} e^{-ikx} \frac{1}{1 + x^2} \, dx.$$ 

We evaluate this Fourier transform using contour integration to obtain

$$f(k) = \pi e^{-|k|}.$$ 

Note particularly the $|k|$: if $k > 0$ we must close in the lower half of the $z = x + iy$ plane, and if $k < 0$ we close in the upper half plane.
Figure 1.3: The pie contour \( ABC \) in the complex plane \((z = x + iy = re^{i\theta})\) used to evaluate \( J(1) \) in (1.42). The ray \( AC \) is a contour of constant phase: \( z^3 = ir^3 \) and \( \exp(iz^3) = \exp(-r^3) \).

Example Let’s evaluate \( \text{Ai}(0) = \int_0^{\infty} \cos(kx^3) \, dk \) via contour integration. We consider a slightly more general integral

\[
J(\alpha) = \int_0^{\infty} e^{i\alpha x^3} \, dx,  
\]

\[
= |\alpha|^{-1/3} \int_0^{\infty} e^{i\text{sgn}(\alpha)x^3} \, dx.  
\]

Thus if we can evaluate \( J(1) \) we also have \( J(\alpha) \), and in particular \( \Re J(1/3) \), which is just what we need for \( \text{Ai}(0) \). But at the moment it may not even be clear that these integrals converge — we’re relying on the destructive cancellation of increasingly wild oscillations as \( x \to \infty \), rather than decay of the integrand, to ensure convergence.

To evaluate \( J(1) \) we consider the entire analytic function

\[
f(z) = e^{iz^3} = e^{i^3x^3} = e^{x^3 e^{3i\theta}} = \exp \left[ y^3 - 3x^2 y + i \left( x^3 - 3xy^2 \right) \right].  
\]

Notice from Cauchy’s theorem that the integral of \( f(z) \) over any closed path in the \( z \)-plane is zero. In particular, using the pie-shaped path \( ABC \) in the figure,

\[
0 = \int_{ABC} e^{iz^3} \, dz.  
\]

The pie-path \( ABC \) is cunningly chosen so that the segment \( CA \) (where \( z = re^{i\pi/6} \)) is a contour of constant phase, so called because

\[
f(z) = e^{-r^3} \quad \text{on } AC.  
\]

On \( CA \) phase of \( f(z) \) is a constant, namely zero.

Now write out (1.44) as

\[
0 = \int_0^R e^{ix^3} \, dx + \int_0^{\pi/6} e^{iR^3 e^{3i\theta}} \, iRe^{i\theta} \, d\theta + \int_0^0 e^{-r^3} \, dr.  
\]

Note that on the arc \( BC \), \( z = Re^{i\theta} \) and \( dz = iRe^{i\theta} \, d\theta \) — we’ve used this in \( M(R) \) above.

We consider the limit \( R \to \infty \). If we can show that the term in the middle, \( M(r) \), vanishes as \( R \to \infty \) then we will have

\[
J(1) = \int_0^{\infty} e^{-r^3} \, dr.  
\]

The right of (1.47) is a splendidly convergent integral and is readily evaluated in terms of our friend the \( \Gamma \)-function.
So we now focus on the troublesome \( M(R) \):

\[
|M(R)| = R \left| \int_0^{\pi/6} e^{i R^3 \cos 3\theta} e^{-R^3 \sin 3\theta} e^{i \theta} d\theta \right|,
\]

\[
\leq R \int_0^{\pi/6} e^{R^3 \cos 3\theta} e^{-R^3 \sin 3\theta} e^{i \theta} d\theta,
\]

\[
\leq R \int_0^{\pi/6} e^{-R^3 \sin 3\theta} d\theta,
\]

\[
< R \int_0^{\pi/6} e^{-R^3 (6\theta/\pi)} d\theta,
\]

\[
= \frac{\pi}{6R^2} \left( 1 - e^{-R^3} \right),
\]

\[\rightarrow 0, \quad \text{as } R \rightarrow \infty. \tag{1.48}\]

At (1.48) we’ve obtained a simple upper bound using the inequality

\[
\sin 3\theta > \frac{6\theta}{\pi}, \quad \text{for } 0 < \theta < \frac{\pi}{6}. \tag{1.50}\]

An alternative is to change variables with \( v = \sin 3\theta \) so that

\[
\int_0^{\pi/6} e^{-R^3 \sin 3\theta} d\theta = \frac{1}{3} \int_0^1 e^{-R^3 v} \frac{dv}{\sqrt{1-v^2}}, \tag{1.51}\]

and then use Watson’s lemma (from the next lecture). This gives a sharper bound on the arc integral. The final answer is

\[
\text{Ai}(0) = \frac{3^{1/3}}{\pi} \int_0^{\infty} e^{-r^3} dr = \frac{\Gamma(1/3)}{3^{1/3}\pi}. \tag{1.52}\]

In the example above we used a constant-phase contour to evaluate an integral exactly. A constant-phase contour is also a contour of steepest descent. The function in the exponential is

\[
iz^3 = (y^3 - 3xy^2) + i \left( x^3 - 3x^2y^2 \right). \tag{1.53}\]

On CA the phase is constant: \( \psi = 0 \). But from the Cauchy-Rieman equations

\[
\nabla \phi \cdot \nabla \psi = 0, \tag{1.54}\]

and therefore as one moves along CA one is moving parallel to \( \nabla \phi \). One is therefore always ascending or descending along the steepest direction of the surface formed by \( \phi(x, y) \) above the \((x, y)\)-plane. Thus the main advantage to integrating along the constant-phase contour CA is that the integrand is decreasing as fast as possible without any oscillatory behavior.

**Example:** Let’s prove the important functional equation

\[
\Gamma(z)\Gamma(1-z) = \int_0^{\infty} \frac{v^{z-1}}{1+v} dv = \frac{\pi}{\sin \pi z}. \tag{1.55}\]

**Example:** Later, in our discussion of the method of averaging, we’ll need the integral

\[
A(\kappa) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{1 + \kappa \cos t}. \tag{1.56}\]

We introduce a complex variable

\[
z = e^{it}, \quad \text{so that } dz = iz \, dt, \quad \text{and } \cos t = \frac{1}{2} z + \frac{i}{2} z^{-1}. \tag{1.57}\]

\[\text{This trick is a variant of Jordan’s lemma.}\]
Thus

\[ A(\kappa) = -\frac{i}{\pi} \int_C \frac{dz}{\kappa z^2 + 2z + \kappa}, \quad (1.58) \]

\[ = -\frac{i}{\pi\kappa} \int_C \frac{dz}{(z - z_+)(z - z_-)}, \quad (1.59) \]

where the path of integration, \( C \), is a unit circle centered on the origin. The integrand has simple poles at

\[ z_\pm = \kappa^{-1} \pm \sqrt{\kappa^{-2} - 1}. \quad (1.60) \]

The pole at \( z_+ \) is inside \( C \), and the other is outside. Therefore

\[ A(\kappa) = 2\pi i \left( -\frac{i}{\pi\kappa} \right) \frac{1}{z_+ - z_-}, \quad (1.61) \]

\[ = \frac{1}{\sqrt{1 - \kappa^2}}. \quad (1.62) \]

Mathematica, Maple and Gradshteyn & Ryzhik

*Tables of Integrals Series and Products* by I.S. Gradshteyn & I.M. Ryzhik is a good source for look-up evaluation of integrals. Get the seventh edition — it has fewer typos.

### 1.6 Problems

**Problem 1.1.** Use the elementary integral

\[ \frac{1}{n + 1} = \int_0^1 x^n \, dx, \quad (1.63) \]

to evaluate

\[ S(n) \overset{\text{def}}{=} \int_0^1 x^n \ln \left( \frac{1}{x} \right) \, dx \quad \text{and} \quad R(n) \overset{\text{def}}{=} \int_0^1 x^n \ln^2 \left( \frac{1}{x} \right) \, dx. \quad (1.64) \]

Do this problem two ways (i) integration by parts and (ii) differentiation with respect to the parameter \( n \).

**Problem 1.2.** Starting from

\[ \frac{a}{a^2 + \lambda^2} = \int_0^\infty e^{-ax} \cos \lambda x \, dx, \quad (1.65) \]

evaluate

\[ I(a, \lambda) = \int_0^\infty x e^{-ax} \cos \lambda x \, dx, \quad (1.66) \]

and for desert

\[ J(a) = \int_0^\infty e^{-ax} \frac{\sin x}{x} \, dx. \quad (1.67) \]

Notice that \( J(a) \) is an interesting Laplace transform.

**Problem 1.3.** Consider

\[ F(a, b) = \int_0^\infty e^{-a^2 u^2 - b^2 u^{-2}} \, du. \quad (1.68) \]

(i) Using a change of variables show that \( F(a, b) = a^{-1} F(1, ab) \). (ii) Show that

\[ \frac{\partial F(a, b)}{\partial b} = -2F(1, ab). \quad (1.69) \]

(iii) Use the results above to show that \( f \) satisfies a simple first order differential equation; solve the equation and show that

\[ F(a, b) = \frac{\sqrt{2\pi}}{2a} e^{-2ab}. \quad (1.70) \]
Problem 1.4. The harmonic sum is defined by

\[ H_N \equiv \sum_{n=1}^{N} \frac{1}{n}. \tag{1.71} \]

In this problem you’re asked to show that

\[ \lim_{N \to \infty} (H_N - \ln N) = \gamma_E, \tag{1.72} \]

where the Euler constant \( \gamma_E \) is defined in (1.80). (i) Prove that \( H_N \) diverges by showing that

\[ \ln(1 + N) \leq H_N \leq 1 + \ln N. \tag{1.73} \]

*Hint:* compare \( H_N \) with the area beneath the curve \( f(x) = x^{-1} \) — you’ll need to carefully select the limits of integration. Your answer should include a careful sketch. (ii) Prove that

\[ H_N = \int_{0}^{1} \frac{1 - x^N}{1 - x} \, dx. \tag{1.74} \]

*Hint:* \( x^{-1} = \int_{0}^{1} x^{-1} \, dx. \) (iii) Use MATLAB to graph

\[ F_N(x) \equiv \frac{1 - x^N}{1 - x}, \quad \text{for } 0 \leq x \leq 1, \tag{1.75} \]

with \( N = 100. \) This indicates that \( F_N(x) \) achieves its maximum value at \( x = 1. \) Prove that \( F_N(1) = N. \) These considerations should convince you that the integral in (1.74) is dominated by the peak at \( x = 1. \) (iv) With a change of variables, rewrite (1.74) as

\[ H_N = \int_{0}^{N} \left[ 1 - \left( \frac{1 - y}{N} \right)^N \right] \frac{dy}{y}. \tag{1.76} \]

(v) Deduce (1.72) by asymptotic evaluation, \( N \to \infty, \) of the integral in (1.76).

Problem 1.5. Consider a harmonic oscillator that is kicked at \( t = 0 \) by singular forcing

\[ \ddot{x} + x = \frac{1}{t}. \tag{1.77} \]

(i) Show that a particular solution of (1.77) is provided by the Stieltjes integral

\[ x(t) = \int_{0}^{\infty} e^{-st} \frac{1}{1 + s^2} \, ds. \tag{1.78} \]

(ii) Find the leading-order behaviour of \( x(t) \) as \( t \to \infty \) from the integral representation (1.78). (iii) Show that this asymptotic result corresponds to a two-term balance in (1.77). (iv) Evaluate \( x(0). \) (v) Can you find \( \dot{x}(0)? \) (vi) If your answer to (v) was “no”, what can you say about the form of \( x(t) \) as \( t \to 0? \) Do you get more information from the differential equation, or from the integral representation?

Problem 1.6. Evaluate the Fresnel integral

\[ F(\alpha) = \int_{0}^{\infty} e^{\alpha x^2} \, dx. \tag{1.79} \]
**Problem 1.7.** Euler’s constant is defined by
\[ \gamma_E \overset{\text{def}}{=} -\Gamma'(1). \] (1.80)

(i) Show by direct differentiation of the definition of the \(\Gamma\)-function that:
\[ \gamma_E = -\int_0^\infty e^{-t} \ln t \, dt. \] (1.81)

(ii) Judiciously applying IP to the RHS, deduce that
\[ \gamma_E = \int_1^0 \frac{1 - e^{-t} - e^{-t+1}}{t} \, dt. \] (1.82)

**Problem 1.8.** This problem uses many of the elementary tricks you’ll need for real integrals. (i) Show that
\[ \ln t = \int_0^\infty \frac{e^{-x} - e^{-xt}}{x} \, dx. \] (1.83)

(ii) From the definition of the \(\Gamma\)-function,
\[ \Gamma(z) \overset{\text{def}}{=} \int_0^\infty e^{-t} t^{z-1} \, dt, \quad \Re z > 0, \] (1.84)
show that the digamma function is
\[ \psi(z) \overset{\text{def}}{=} \frac{d \ln \Gamma}{dz} = \frac{\Gamma'(z)}{\Gamma(z)} = \int_0^\infty \left[ e^{-x} - \frac{1}{(x+1)^z} \right] \frac{dx}{x}, \quad \Re z > 0. \] (1.85)

Hint: Differentiate the definition of \(\Gamma(z)\) in (1.84), and use the result from part (i). (iii) Noting that (1.85) implies
\[ \psi(z) = \lim_{\delta \to 0} \left[ \int_\delta^\infty \frac{e^{-x}}{x} \, dx - \int_\delta^\infty \frac{1}{(x+1)^z} \frac{dx}{x} \right], \quad \Re z > 0, \] (1.86)
change variables with \(x + 1 = e^u\) in the second integral and deduce that:
\[ \psi(z) = \int_0^\infty \left( \frac{e^{-u}}{u} - \frac{e^{-zu}}{1 - e^{-u}} \right) \, du, \quad \Re z > 0. \] (1.87)

Explain in ten or twenty words why it is necessary to introduce \(\delta\) in order to split the integral on the RHS of (1.85) into the two integrals on the RHS of (1.86). (iv) We define Euler’s constant as
\[ \gamma_E \equiv -\psi(1) = -\Gamma'(1) = 0.57721 \cdots \] (1.88)

Show that
\[ \psi(z) = -\gamma_E + \int_0^\infty \frac{e^{-u} - e^{-ux}}{1 - e^{-u}} \, du, \]
\[ = -\gamma_E + \int_0^1 \frac{1 - v^{z-1}}{1 - v} \, dv. \]

(v) From the last integral representation, show that
\[ \psi(z) = -\gamma_E + \sum_{n=0}^\infty \left( \frac{1}{n+1} - \frac{1}{n+z} \right). \]

Notice we can now drop the restriction \(\Re z > 0\) — the beautiful formula above provides an analytic extension of \(\psi(z)\) into the whole complex plane.

---

\(^3\)But not all — there is no integration by parts.
**Problem 1.9.** Use pie-shaped contours to evaluate the integrals

\[ A = \int_0^\infty \frac{dx}{1 + x^3}, \quad \text{and} \quad B = \int_0^\infty \cos x^2 \, dx. \]  

(1.89)

**Problem 1.10.** Use the Fourier transform to solve the dispersive wave equation

\[ u_t = \nu u_{xxx}, \quad \text{with IC } u(x, 0) = \delta(x). \]  

(1.90)

Express the answer in terms of Ai.

**Problem 1.11.** Solve the half-plane \((y > 0)\) boundary value problem

\[ yu_{xx} + u_{yy} = 0 \]  

(1.91)

with \(u(x, 0) = \cos qx\) and \(\lim_{y \to \infty} u(x, y) = 0\).
Lecture 2

What is asymptotic?

2.1 An example: the erf function

We consider the error function

\[ \text{erf}(z) \overset{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \, dt. \] (2.1)

The upper panel of Figure 2.1 shows erf, and the complementary error function

\[ \text{erfc}(z) \overset{\text{def}}{=} 1 - \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} \, dt, \] (2.2)

on the real line.

The series on the right of

\[ e^{-t^2} = \sum_{n=0}^\infty \frac{(-t^2)^n}{n!} \] (2.3)

has infinite radius of convergence i.e., \( e^{-t^2} \) is an entire function in the complex \( t \)-plane. Thus we can simply integrate term-by-term in \( 2.1 \) to obtain a series for \( \text{erf}(z) \) that converges in the entire complex plane:

\[ \text{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-1)^n z^{2n+1}}{(2n+1)n!}, \] (2.4)

\[ = \frac{2}{\sqrt{\pi}} \left( z - \frac{1}{3} z^3 + \frac{1}{10} z^5 - \frac{1}{15} z^7 + \frac{1}{210} z^9 - \frac{1}{1320} z^{11} \right) + R_6, \] (2.5)

where \( \text{erf}_6(x) \) is the sum of the first six terms and \( R_6(z) \) is the remainder after 6 terms.

The lower panel of Figure 2.1 shows that \( \text{erf}_n \) (the sum of the first \( n \) nonzero terms) provides an excellent approximation to \( \text{erf} \) if \( |x| < 1 \). With matlab we find that

\[ \frac{\text{erf}(1) - \text{erf}_{10}(1)}{\text{erf}(1)} = 1.6217 \times 10^{-8}, \quad \text{and} \quad \frac{\text{erf}(2) - \text{erf}_{10}(2)}{\text{erf}(2)} = 0.0233. \] (2.6)

The Taylor series is useful if \( |z| < 1 \), but as \( |z| \) increases past 1 convergence is slow. Moreover some of the intermediate terms are very large and there is a lot of destructive cancellation between terms of different signs. Figure 2.2 shows that this cancellation is bad at \( z = 3 \), and it gets a lot worse as \( |z| \) increases. Thus, because of round-off error, a computer with limited precision cannot
Figure 2.1: Upper panel: the blue curve is erf($x$) and the red curve is erfc($x$). Lower panel shows erf($x$) and truncated Taylor series erf$_n(x)$, with $n = 1, 2 \cdots, 20$.

Figure 2.2: The terms in the Taylor series (2.5) with $x = 3$. The sum of the series — that is erf(3) — is very close to 1. But there are cancellations between terms of order $\pm 200$ before convergence takes hold. The problem quickly gets worse: at $x = 4$ the largest terms in the series exceed $10^5$. 

accurately sum the convergent Taylor series if $|z|$ is too large. Convergence is not as useful as one might think.

Now let’s consider an approximation to erf$(x)$ that’s good for large $x$. We work with the complementary error functions in (2.2) and use integration by parts

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \left( -\frac{1}{2t} \right) \frac{d}{dt} e^{-t^2} \, dt,$$

(2.7)

$$= \frac{2}{\sqrt{\pi}} \left( \frac{e^{-x^2}}{2x} - \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{e^{-t^2}}{2t^2} \, dt \right).$$

(2.8)

If we discard the final term in (2.8) we get a useful approximation:

$$\text{erfc}(x) \sim \frac{e^{-x^2}}{\sqrt{\pi}x}, \quad \text{as } x \to \infty. \quad (2.9)$$

The upper panel of Figure 2.3 shows that this leading-order asymptotic approximation is reliable once $x$ is greater than about 2 e.g., at $x = 2$ the error is 10.5%, and at $x = 4$ the error is less than 3%.

Exercise: If we try integration by parts on erf (as opposed erfc) something bad happens: try it and see.

Why does the approximation in (2.9) work? Notice that the final term in (2.8) can be bounded like this

$$\frac{2}{\sqrt{\pi}} \int_x^\infty \frac{e^{-t^2}}{2t^2} \, dt = \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{1}{4t^3} \times 2te^{-t^2} \, dt,$$

(2.10)

$$\leq \frac{2}{\sqrt{\pi}} \frac{1}{4x^3} \int_x^\infty 2te^{-t^2} \, dt,$$

(2.11)

$$= \frac{2}{\sqrt{\pi}} \frac{e^{-x^2}}{4x^3}.$$

(2.12)

The little trick we’ve used above in going from (2.10) to (2.11) is that

$$t \geq x, \quad \Rightarrow \quad \frac{1}{4t^3} \leq \frac{1}{4x^3}. \quad (2.13)$$

Pulling the $(4x)^{-3}$ outside, we’re left with an elementary integral. Variants of this maneuver appear frequently in the asymptotics of integrals (try the exercise below).

Using the bound in (2.23) in (2.8) we have

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \frac{e^{-x^2}}{2x} + \left[ \text{something which is much less than } \frac{2}{\sqrt{\pi}} \frac{e^{-x^2}}{2x} \text{ as } x \to \infty \right].$$

(2.14)

Thus as $x \to \infty$ there is a dominant balance in (2.14) between the left hand side and the first term on the right. The final term is smaller than the other two terms by a factor of at least $x^{-2}$.

Exercise: Prove that

$$\int_x^\infty \frac{e^{-t}}{t^N} \, dt < \frac{e^{-x}}{x^N}. \quad (2.15)$$

1 We restrict attention to the real line: $z = x + iy$. The situation in the complex plane is tricky — we’ll return to this later. We also defer the definition asymptotic approximation.

2 The $\sim$ in (2.9) denotes “asymptotic equivalence” and is defined in section 2.2. In (2.9) it means that

$$\lim_{x \to \infty} \sqrt{\pi}xe^{x^2} \text{erfc}(x) = 1.$$
Figure 2.3: Upper panel shows \( \text{erfc}(x) \) divided by the leading order asymptotic approximation on the right of (2.9); as \( x \to \infty \) the ratio approaches 1. The lower panel shows \( \text{erfc}(x) \) divided by an \( n \)-term truncation of (2.28) with \( n = 1, 2, 3 \) and 4.

One more term

We can develop an asymptotic series if we integrate by parts successively starting with (2.8):

\[
\text{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}} - \frac{1}{\sqrt{\pi}} \int_x^\infty \frac{1}{t^2} \left( -\frac{1}{2t} \right) \frac{d}{dt} e^{-t^2} \, dt,
\]

\( (2.16) \)

\[
= \frac{e^{-x^2}}{\sqrt{\pi x}} \left( 1 - \frac{1}{2x^2} \right) + \frac{3}{2\sqrt{\pi}} \int_x^\infty \frac{e^{-t^2}}{t^4} \, dt.
\]

\( (2.17) \)

We use the same trick to bound the remainder:

\[
R_2 = \frac{3}{4\sqrt{\pi}} \int_x^\infty \frac{2t e^{-t^2}}{t^5} \, dt < \frac{3}{4\sqrt{\pi x^5}} \int_x^\infty \frac{d}{dt} e^{-t^2} \, dt = \frac{3}{4\sqrt{\pi x^3}} e^{-x^2}.
\]

\( (2.18) \)

As \( x \to \infty \) the remainder \( R_2(x) \) is much less than the second term in the series, so we can suppress some information and write

\[
\text{erfc}(x) = \frac{e^{-x^2}}{\sqrt{\pi x}} \left[ 1 - \frac{1}{2x^2} + O \left( \frac{1}{x^4} \right) \right].
\]

\( (2.19) \)

The big O notation used above is explained in section 2.2 — it means that \( x^4 \) times the term \( O(x^{-4}) \) is bounded by some constant as \( x \to \infty \). You can see that the constant identified by the inequality (2.18) is in fact \( 3/4 \).
Yet more terms: the asymptotic series

Exercise: show that
\[
\int_0^\infty t^{-q}e^{-t^2} \, dt = \frac{1}{2}z^{-(q+1)}e^{-z^2} - \frac{1}{2}(q+1)J_{q+2}.
\] (2.20)

Using the result in the exercise above we integrate by parts \(N\) times and obtain an exact expression for \(\text{erfc}(x)\):
\[
\text{erfc}(x) = e^{-x^2} \sqrt{\frac{\pi}{x}} \sum_{n=0}^{N-1} \frac{(2n-1)!!}{x^{2n+1}} \left( -\frac{1}{2x^2} \right)^n + (-1)^N \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{e^{-t^2}}{(2t)^N} \, dt.
\] (2.21)

Above, \(R_N(x)\) is the remainder after \(N\) terms and the “double factorial” is \(7!! = 7 \cdot 5 \cdot 3 \cdot 1\) etc. To bound the remainder we use our trick again:
\[
|R_N| = \frac{2(2N-1)!!}{\sqrt{\pi}} \int_x^\infty \frac{(e^{-t^2})t}{2t \times (2t)^N} \, dt,
\] (2.22)
\[
\leq \frac{(2N-1)!!}{\sqrt{\pi}2^N x^{2N+1}} e^{-x^2}.
\] (2.23)

We have shown that
\[
\frac{|R_N|}{N\text{th term of the series}} \leq \frac{2N-1}{(2x)^2},
\] (2.24)
or equivalently
\[
|R_N| \leq \text{term } N+1 \text{ in the asymptotic series}.
\] (2.25)

Thus the first term we neglect in the expansion is an upper bound on the error as \(x \to \infty\). And if we fix \(N\) and increase \(x\) then the approximation to \(\text{erfc}(x)\) obtained by dropping the remainder gets better and better. But the limits
\[
x \to \infty \quad \text{and} \quad N \to \infty
\] (2.26)
do n’t “commute”. In other words, if we fix \(x\) at some large value, such as \(x = 3\), and increase \(N\) then the approximation gets better for a while, but then goes horribly wrong. This behaviour is illustrated in figure 2.4 which shows how
\[
\text{relative error} \overset{\text{def}}{=} \frac{\text{N-term approximation to } \text{erfc}(x)}{\text{erfc}(x)} - 1
\] (2.27)
depends on both \(N\) and \(x\) in our erf example.

Numerical use of asymptotic series — the optimal stopping rule

Suppose an unreasonable person insists on ignoring the simple limit \(x \to \infty\) and instead demands the best answer at a fixed value of \(x\), such as \(x = 2\). How many terms in the series
\[
\text{erfc}(x) \sim e^{-x^2} \sqrt{\frac{\pi}{x}} \left( 1 - \frac{1}{2x^2} + \frac{1 \times 3}{(2x^2)^2} - \frac{1 \times 3 \times 5}{(2x^2)^3} + \frac{1 \times 3 \times 5 \times 7}{(2x^2)^4} + O(x^{-10}) \right)
\] (2.28)
should one use to appease this tyrant? The numerators above are growing very quickly so at a fixed value of \(x\) this series for \(\text{erfc}(x)\) diverges as we add more terms. But Figure 2.4 shows that at
fixed $x$ there is an optimal value of $N$ at which the relative error is smallest. How do we find this best asymptotic estimate?

We showed above in (2.24) and (2.25) that as $x \to \infty$ the remainder $R_N(x)$ is less than the $(N + 1)$st term in the series. Thus a good place to stop summing is just before the smallest term in the series: we know the remainder is less than this smallest term. In practice we get good accuracy if we use the optimal stopping rule: locate the smallest term in the series and add all the previous terms. Do not include the smallest term in this sum.

The optimal stopping rule is a rule of thumb not a precise result — the remainder $R_N$ is less than the $(N + 1)$st term only when $x$ is sufficiently large i.e., in the limit $x \to \infty$. We have no assurance that this inequality applies at a particular value of $x$.

We illustrate the optimal stopping rule by estimating $\text{erfc}(2)$. With $x = 2$ the sum (2.28) is

$$\text{erfc}(2) \sim \frac{e^{-4}}{2\sqrt{\pi}} \left( 1 - \frac{1}{8} + \frac{3}{64} - \frac{15}{512} + \frac{105}{4096} - \frac{945}{32768} + \frac{10395}{262144} + \cdots \right).$$

The smallest term is $105/4096$. The optimal approximation is obtained by stopping before the smallest terms:

$$0.0051667 \left( 1 - \frac{1}{8} + \frac{3}{64} - \frac{15}{512} \right) = 0.00461172.$$  

(2.29)

The relative error is 0.0141116, or about 1.4%.

We get a much better answer by including half of the smallest term in the asymptotic series:

$$0.0051667 \left( 1 - \frac{1}{8} + \frac{3}{64} - \frac{15}{512} + \frac{1}{2} \frac{105}{4096} \right) = 0.00467795.$$  

(2.30)

With this mysterious improvement the relative error is now $-0.000046$. We should explain why adding half of the smallest term works so well. (Bender & Orszag and Hinch don’t mention this.....)
Exercise: \( \text{erfc}(1) = 0.157299 \) and the leading-order approximation is \( e^{-1}/\sqrt{\pi} = 0.207554 \). The relative error is therefore 0.31948 which seems unfortunately large. Show that according to the optimal stopping rule the leading-order approximation is optimal. Does adding half of the smallest term significantly reduce the error?

2.2 The Landau symbols

Let’s explain the frequently used “Landau symbols”. In asymptotic calculations the Landau notation is used to suppress information while still maintaining some precision.

Big Oh

We frequently use “big Oh” — in fact I may have accidentally done this without defining \( O \). One says \( f(\epsilon) = O(\phi(\epsilon)) \) as \( \epsilon \to 0 \) if we can find an \( \epsilon_0 \) and a number \( A \) such that

\[
|f(\epsilon)| < A|\phi(\epsilon)|, \quad \text{whenever } \epsilon < \epsilon_0.
\]

Both \( \epsilon_0 \) and \( A \) have to be independent of \( \epsilon \). Application of the big Oh notation is a lot easier than this definition suggests. Here are some \( \epsilon \to 0 \) examples

\[
\begin{align*}
\sin 32\epsilon &= O(\epsilon), \\
\sin 32\epsilon &= O(\epsilon^{1/2}), \\
\epsilon^5 &= O(\epsilon^2), \\
\cos \epsilon - 1 &= O(\epsilon^{1/2}), \\
\epsilon + \epsilon^2 \sin \frac{1}{\epsilon} &= O(\epsilon), \\
\sin \frac{1}{\epsilon} &= O(1), \\
e^{-1/\epsilon} &= O(\epsilon^n) \text{ for all } n.
\end{align*}
\]

The expression

\[
\cos \epsilon = 1 - \frac{\epsilon^2}{2} + O(\epsilon^3)
\]

(2.32)

means

\[
\cos \epsilon - 1 + \frac{\epsilon^2}{2} = O(\epsilon^3) .
\]

(2.33)

In some of the cases above

\[
\lim_{\epsilon \to 0} \frac{f(\epsilon)}{\phi(\epsilon)}
\]

(2.34)

is zero, and that’s good enough for \( O \). Also, according to our definition of \( O \), the limit in (2.34) may not exist — all that’s required is that ratio \( f(\epsilon)/\phi(\epsilon) \) is bounded by a constant independent of \( \epsilon \) as \( \epsilon \to 0 \). One of the examples above illustrates this case.

The big Oh notation can be applied to other limits in obvious ways. For example, as \( x \to \infty \)

\[
\sin x = O(1), \quad \sqrt{1 + x^2} = O(x^2), \quad \ln \cosh x = O(x).
\]

(2.35)

As \( x \to 1 \)

\[
\ln \left(1 + x + x^2\right) - x = O(x^2).
\]

(2.36)

Hinch’s ord

\( H \) uses the more precise notation \( \text{ord}(\phi(\epsilon)) \). We say

\[
f(\epsilon) = \text{ord}(\phi(\epsilon)) \iff \lim_{\epsilon \to 0} \frac{f(\epsilon)}{\phi(\epsilon)} \text{ exists and is nonzero.}
\]

(2.37)
For example, as $\epsilon \to 0$:

\[ \sinh(37\epsilon + \epsilon^3) = \text{ord}(\epsilon), \quad \text{and} \quad \frac{\epsilon}{\ln(1 + \epsilon + \epsilon^2)} = \text{ord}(1). \quad (2.38) \]

Notice that $\sinh(37\epsilon + \epsilon^3)$ is not $\text{ord}(\epsilon^{1/2})$, but

\[ \sinh(37\epsilon + \epsilon^3) = O(\epsilon^{1/2}), \quad \text{and} \quad \sin\left(\frac{1}{\epsilon}\right) \sinh(37\epsilon + \epsilon^3) = O(\epsilon^{1/2}) \quad (2.39) \]

Big Oh tells one a lot less than ord.

**Little Oh**

Very occasionally — almost never — we need “little Oh”. We say $f(\epsilon) = o(\phi(\epsilon))$ if for every positive $\delta$ there is an $\epsilon_0$ such that

\[ |f(\epsilon)| < \delta |\phi(\epsilon)|, \quad \text{whenever} \quad \epsilon < \epsilon_0. \]

Another way of saying this is that

\[ f(\epsilon) = o(\phi(\epsilon)) \quad \Leftrightarrow \quad \lim_{\epsilon \to 0} \frac{f(\epsilon)}{\phi(\epsilon)} = 0. \quad (2.40) \]

Obviously $f(\epsilon) = o(\phi(\epsilon))$ implies $f(\epsilon) = O(\phi(\epsilon))$, but not the reverse. Here are some examples

\[ \ln(1 + \epsilon) = o(\epsilon^{1/2}), \quad \cos \epsilon - 1 + \frac{\epsilon^2}{2} = o(\epsilon^3), \quad e^{o(\epsilon)} = 1 + o(\epsilon). \quad (2.41) \]

The trouble with little Oh is that it hides too much information: if something tends to zero we usually want to know how it tends to zero. For example

\[ \ln(1 + 2e^{-x} + 3e^{-2x}) = o\left(e^{-x/2}\right), \quad \text{as} \quad x \to \infty, \quad (2.42) \]

is not as informative as

\[ \ln(1 + 2e^{-x} + 3e^{-2x}) = \text{ord}\left(e^{-x}\right), \quad \text{as} \quad x \to \infty. \quad (2.43) \]

**Asymptotic equivalence**

Finally “asymptotic equivalence” $\sim$ is useful. We say $f(\epsilon) \sim \phi(\epsilon)$ as $\epsilon \to 0$ if

\[ \lim_{\epsilon \to 0} \frac{f(\epsilon)}{\phi(\epsilon)} = 1. \quad (2.44) \]

Notice that

\[ f(\epsilon) \approx \phi(\epsilon), \quad \Leftrightarrow \quad f(\epsilon) = \phi(\epsilon) \left[1 + o(\epsilon)\right]. \quad (2.45) \]

Some $\epsilon \to 0$ examples are

\[ \epsilon + \frac{\sin \epsilon}{\ln(1/\epsilon)} \sim \epsilon, \quad \text{and} \quad \sqrt{1 + \epsilon} - 1 \sim \frac{\epsilon}{2}. \quad (2.46) \]

Some $x \to \infty$ examples are

\[ \sinh x \sim \frac{e^x}{2}, \quad \text{and} \quad \frac{x^3}{1 + x^2} + \sin x \sim x, \quad \text{and} \quad x + \ln (1 + e^{2x}) \sim 3x. \quad (2.47) \]

**Exercise:** Show by counterexample that $f(x) \approx g(x)$ as $x \to \infty$ does not imply that $\frac{df}{dx} \approx \frac{dg}{dx}$, and that $f(x) \approx g(x)$ as $x \to \infty$ does not imply that $e^f \approx e^g$. 

21
Gauge functions

The $\phi(\epsilon)$’s referred to above are gauge functions — simple functions that we use to compare a complicated $f(\epsilon)$ with. A sequence of gauge functions $\{\phi_0, \phi_1, \cdots\}$ is asymptotically ordered if

$$\phi_{n+1}(\epsilon) = o[\phi_n(\epsilon)], \quad \text{as } \epsilon \to 0. \quad (2.48)$$

In practice the $\phi$’s are combinations of powers and logarithms:

$$\epsilon^n, \quad \ln \epsilon, \quad \epsilon^m (\ln \epsilon)^p, \quad \ln \ln \epsilon \text{ etc.} \quad (2.49)$$

Exercise Suppose $\epsilon \to 0$. Arrange the following gauge functions in order, from the largest to the smallest:

$$\epsilon, \quad \ln \left(\ln \frac{1}{\epsilon}\right), \quad e^{-\ln \epsilon}, \quad e^{1/\sqrt{\epsilon}}, \quad e^0, \quad \ln \frac{1}{\epsilon} \quad (2.50)$$

$$e^{-1/\epsilon}, \quad \epsilon^{1/3}, \quad \epsilon^{1/\pi}, \quad \epsilon \ln \frac{1}{\epsilon}, \quad \frac{1}{\ln \epsilon}, \quad \epsilon^{\ln \epsilon}. \quad (2.51)$$

2.3 The definition of asymptoticity

Asymptotic power series

Consider a sum based on the simplest gauge functions $\epsilon^n$:

$$\sum_{n=0}^{\infty} a_n \epsilon^n. \quad (2.52)$$

This sum is an $\epsilon \to 0$ asymptotic approximation to a function $f(\epsilon)$ if

$$\lim_{\epsilon \to 0} \frac{f(\epsilon) - \sum_{n=0}^{N} a_n \epsilon^n}{\epsilon^N} = 0. \quad (2.53)$$

The numerator in the fraction above is the remainder after summing $N + 1$ terms, also known as $R_{N+1}(\epsilon)$. So the series in (2.52) is asymptotic to the function $f(\epsilon)$ if the remainder $R_{N+1}(\epsilon)$ goes to zero faster than the last retained gauge function $\epsilon^N$. We use the notation $\sim$ to denote an asymptotic approximation:

$$f(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n, \quad \text{as } \epsilon \to 0. \quad (2.54)$$

The right hand side of (2.54) is called an asymptotic power series or a Poincaré series, or an asymptotic representation of $f(\epsilon)$.

Our erf-example satisfies this definition with $\epsilon = x^{-1}$. If we retain only one term in the series (2.28) then the remainder is

$$R_1 = 2 \frac{\sqrt{\pi}}{\sqrt{x}} \int_x^{\infty} e^{-t^2} \frac{e^{-t^2}}{2t^2} \, dt. \quad (2.55)$$

In (2.11) we showed that

$$R_1 \leq \frac{1}{4x^2}. \quad (2.56)$$

Thus as $x \to \infty$ the remainder is much less than the last retained term. According to the definition above, this is the first step in justifying the asymptoticness of the series.

Exercise: Show from the definition of asymptoticity that

$$e^{-1/\epsilon} \sim 0 + 0 + 0 \epsilon^2 + 0 \epsilon^3 + \cdots \quad \text{as } \epsilon \downarrow 0. \quad (2.57)$$
A problem with applying the definition is that one has to be able to say something about the remainder in order to determine if a series is asymptotic. This is not the case with convergence. For example, one can establish the convergence of
\[ \sum_{n=0}^{\infty} \ln(n+2)x^n, \] (2.58)
without knowing the function to which this mysterious series converges. Convergence is an intrinsic property of the coefficients \( \ln(n+2) \). The ratio test shows that the series in (2.58) converges if \( |x| < 1 \) and we don’t have to know what (2.58) is converging to. On the other hand, asymptoticity depends on both the function and the terms in the asymptotic series.

**Example** The famous Stieltjes series
\[ S(x) \overset{\text{def}}{=} \sum_{n=0}^{\infty} (-)^n n! x^n \] (2.59)
does not converge unless \( x = 0 \). In fact, as it stands, \( S(x) \) does not define a function of \( x \). \( S(x) \) is a formal power series. And we can’t say that \( S(x) \) is an asymptotic series because we have to ask asymptotic to what?

The proof is integration by parts, which yields the identity
\[ F(x) \overset{\text{def}}{=} \int_{0}^{\infty} \frac{e^{-t}}{1 + xt} \, dt. \] (2.61)

Because of the dubious steps between (2.59) and (2.61), I’ve simply defined \( F(x) \) by the integral above. But now that we have a well defined function \( F(x) \), we’re entitled to ask is the sum \( S(x) \) asymptotic to \( F(x) \) as \( x \to 0 \)? The answer is yes.

**Exercise:** Find another function with the same \( x \to 0 \) asymptotic expansion as \( F(x) \) in (2.61).

**Example:** Dawson’s integral is
\[ D(x) \overset{\text{def}}{=} e^{-x^2} \int_{0}^{x} e^{t^2} \, dt. \] (2.65)
The integrand is strongly peaked near \( t = x \), where the integrand is equal to \( e^{x^2} \). The width of this peak is order \( x^{-1} \). Thus we expect that the answer is something like
\[ D(x) \sim e^{-x^2} \frac{\pi^{1/2}}{x} = \frac{\pi^{1/2}}{x}, \] (2.66)
where \( \pi \) is an unidentified number.

To more precisely estimate \( D(x) \) for \( x \gg 1 \) we try IP:
\[ \int_{0}^{x} e^{t^2} \, dt = \int_{0}^{x} \frac{d}{dt} e^{t^2} \, dt = \left[ e^{t^2} \right]_{0}^{x} + \int_{0}^{x} 2te^{t^2} \, dt. \] (2.67)

\[ = \left[ e^{t^2} \right]_{0}^{x} + \int_{0}^{x} 2te^{t^2} \, dt. \] (2.68)
The expression above is meaningless — we’ve taken a perfectly sensible integral and written it as the difference of two infinities.

A correct approach is to split the integral like this

\[ \int_0^x e^{t^2} dt = \int_0^1 e^{t^2} dt + \int_1^x \frac{e^{t^2}}{2t} dt , \]  
\[ = \int_0^1 e^{t^2} dt + \left[ e^{t^2} \right]_1^x + \int_1^x \frac{e^{t^2}}{2t} dt , \]  
\[ = \int_0^1 e^{t^2} dt - \frac{e^{x^2}}{2x} + \int_1^x \frac{e^{t^2}}{2t} dt , \]  
\[ \sim \frac{e^{x^2}}{2x} , \text{ as } x \to \infty. \]  

Thus

\[ D(x) \sim \frac{1}{2x} , \text{ as } x \to \infty. \]  

Back in (2.69) we split the range at \( t = 1 \) — this was an arbitrary choice. We could split at another arbitrary value such as \( t = \frac{32}{2345465} \). The point is that as \( x \to \infty \) all the terms on the right of (2.71) are much less than the single dominant term \( e^{x^2/2x} \). If we want the next term in (2.73), then that comes from performing another IP on the next biggest term on the right of (2.71), namely

\[ R(x) = \int_1^x \frac{e^{t^2}}{2t} dt . \]  

To show that (2.72) is a valid asymptotic approximation according to the definition of Poincaré — with \( \epsilon = x^{-1} \) and \( N = 1 \) in definition (2.53) — we should show that \( R(x) \) in (2.74) is very much less than the leading term, or in other words that

\[ \lim_{x \to \infty} \frac{\int_1^x e^{t^2/2t^2} dt}{e^{x^2/2x}} = 0. \]  

Exercise: Use l’Hôpital’s rule to verify the result above.

### 2.4 Manipulation of asymptotic series

Many — but not all— of the expansion expansions you’ll encounter have the form of an asymptotic power series. But in the previous lectures we saw examples with fractional powers of \( \epsilon \) and \( \ln \epsilon \) and \( \ln[\ln(1/\epsilon)] \). These expansions have the form

\[ f(\epsilon) \sim \sum_{n=0}^{\infty} a_n \phi_n(\epsilon) , \]  

where \( \{\phi_n\} \) is an asymptotically ordered set of gauge functions. The Poincaré definition of asymptoticity is generalized to say that the sum on the right of (2.76) is an asymptotic approximation to \( f(\epsilon) \) as \( \epsilon \to 0 \) if

\[ \lim_{\epsilon \to 0} \frac{f(\epsilon) - \sum_{n=0}^{N} a_n \phi_n(\epsilon)}{\phi_N(\epsilon)} = 0. \]  

In other words, once \( \epsilon \) is sufficiently small the remainder is less than the last term.

**Example:** Using the \( x \to \infty \) gauge functions \( \{x^n/12\} \), where \( n \) is an integer, we have the generalized asymptotic expansion

\[ \frac{x^{1/2} + x^{1/3}}{x^{1/2} + 1} = x^{5/12} - x^{-1/3} + 2x^{-1/4} - 2x^{-1/6} + 2x^{-1/12} - 2 + \text{ord}(x^{-1/12}) . \]  

**Example:** Another example produced by

\[ \text{Series}[x^{-2}(x - x^2), \{x, 0, 3\}] \]
in Mathematica is
\[ x^{x-x^2} \sim 1 + x \ln x + x^2 \left( \frac{1}{2} \ln^2 x - \ln x \right) + x^3 \left( \frac{1}{6} \ln^3 x - \ln^2 x \right) + O(x^4). \]  
(2.79)

Evidently in this example the \( x \to 0 \) gauge functions are \( x^p \ln^q x \) where \( p \) and \( q \) are non-negative integers.

**Uniqueness**

If a function has an asymptotic expansion in terms of a particular set of gauge function then that expansion is unique. For example, using the \( \theta \to 0 \) gauge functions \( \theta^n \), the function \( \sin 2\theta \) can be expanded as
\[ \sin 2\theta = 2\theta - \frac{4\theta^3}{3} + \text{ord} \left( \theta^5 \right), \]  
(2.80)
and that’s the only asymptotic expansion of \( \sin \theta \) using \( \theta^n \). In this sense asymptotic expansions are unique.

The converse is not true: if you’ve done the exercise in (2.76) then you know that two different functions might share an asymptotic expansion because they differ by a quantity that is asymptotically smaller than every gauge function. For example, as \( \theta \downarrow 0 \)
\[ \sin 2\theta + e^{-1/\theta} \sim \sum_{n=0}^{\infty} (-1)^n \frac{(2\theta)^{2n+1}}{(2n+1)!}. \]  
(2.81)

The right of (2.81) is also the asymptotic expansion of \( \sin 2\theta \) in terms of the gauge functions \( \theta^n \).

A given function can also have multiple asymptotic expansions in terms of different gauge functions. For example, consider the \( \theta \to 0 \) gauge functions \( \sin^n \theta \), for which
\[ \sin 2\theta = 2\sin^2 \theta - \sin^3 \theta + \text{ord} \left( \sin^5 \theta \right). \]  
(2.82)
Or gauge functions \( \tan^n \theta \), for which
\[ \sin 2\theta = 2\tan \theta - 2\tan^3 \theta + \text{ord} \left( \tan^5 \theta \right). \]  
(2.83)

**Manipulation of asymptotic expansions**

If we have two \( \epsilon \to 0 \) asymptotic power series
\[ f \sim \sum_{n=0}^{\infty} a_n \epsilon^n, \quad \text{and} \quad g \sim \sum_{n=0}^{\infty} a_n \epsilon^n. \]  
(2.84)
then we can do what comes naturally as far as adding, multiplying and dividing these expansions.

If \( f \) and \( g \) are represented by the generalized asymptotic series in (2.76) then we have a minor problem with multiplication: \( \phi_m \phi_n \) may not be a member of our set of gauge functions. In this case we can simply enlarge the set of gauge functions — provided that the expanded set can be ordered as \( \epsilon \to 0 \). (I can’t think of an example in which this is not possible.)

**Exercise:** Noting that
\[ \frac{1}{\epsilon(1+\epsilon)} \sim \frac{1}{\epsilon} \quad \text{as} \quad \epsilon \to 0, \]  
(2.85)
is
\[ \exp \left( \frac{1}{\epsilon(1+\epsilon)} \right) \sim e^{1/\epsilon} \]  
(2.86)
Asymptotic series can be integrated: if
\[ f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad \text{as } x \to x_0, \quad (2.87) \]
then
\[ \int_{x_0}^{x} f(t) \, dt \sim \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1}, \quad \text{as } x \to x_0. \quad (2.88) \]

Asymptotic series cannot in general be differentiated. Thus
\[ x + \sin x \sim x, \quad \text{as } x \to \infty, \quad (2.89) \]
but the derivative \( 1 + \cos x \) is not asymptotic to 1. Note however that BO section 3.8 discusses some useful special cases in which differentiation is permitted.

### 2.5 Stokes lines

### 2.6 Problems

#### Problem 2.1.
(i) Find a leading-order \( x \to \infty \) asymptotic approximation to
\[ A(x; p, q) \overset{\text{def}}{=} \int_{x}^{\infty} e^{-pt^q} \, dt. \quad (2.90) \]
Show that the remainder is asymptotically negligible as \( x \to \infty \). Above, \( p \) and \( q \) are both positive real numbers.

#### Problem 2.2.
Find two terms in the \( x \to \infty \) behaviour of
\[ F(x) = \int_{0}^{x} e^{-v/v^{1/3}} \, dv. \quad (2.91) \]

#### Problem 2.3.
(i) Use integration by parts to find the leading-order term in the \( x \to \infty \) asymptotic expansion of the exponential integral:
\[ E_1(x) \overset{\text{def}}{=} \int_{x}^{\infty} \frac{e^{-v}}{v} \, dv. \quad (2.92) \]
Show that this approximation is asymptotic i.e., prove that the remainder is asymptotically less than the leading term as \( x \to \infty \). (ii) With further integration by parts, find an expression for the \( n \)’th term, and the remainder after \( n \) terms. (iii) Show that the remainder after \( N \) terms is asymptotically less than the \( N \)’th terms as \( x \to \infty \).

#### Problem 2.4.
Consider the first-order differential equation:
\[ y' - y = -\frac{1}{x}, \quad \text{with the condition } \lim_{x \to \infty} y(x) = 0. \quad (2.93) \]
(i) Find a valid two-term dominant balance in the differential equation and thus deduce the leading-order asymptotic approximation to \( y(x) \) for large positive \( x \). (ii) Use an iterative procedure to deduce the full asymptotic expansion of \( y(x) \). (iii) Is the expansion convergent? (iv) Use the integrating function method to solve the differential equation exactly in terms of the exponential integral in (2.92). Use MATLAB (help expint) to compare the exact solution of (2.93) with asymptotic expansions of different order. Summarize your study as in Figure 2.5.
Figure 2.5: Solution of problem 2.4. Upper panel compares the exact solution with truncated asymptotic series. Lower panel shows the asymptotic approximation at $x = 5$ as a function of the truncation order $n$ i.e., $n = 1$ is the one-term approximation. The solid line is the exact answer.

Problem 2.5. The exponential integral of order $n$ is

$$E_n(x) \overset{\text{def}}{=} \int_x^\infty \frac{e^{-t}}{t^n} \, dt.$$  \hspace{1cm} (2.94)

Show that

$$E_{n+1}(x) = e^{-x} \frac{x^n}{n} - \frac{1}{n} E_n(x).$$  \hspace{1cm} (2.95)

Find the leading-order asymptotic approximation to $E_n(x)$ as $x \to \infty$.

Problem 2.6. (i) Solve the differential equation

$$y' - xy = -1, \quad \text{with} \quad \lim_{x \to \infty} y(x) = 0,$$  \hspace{1cm} (2.96)

in terms of erf and use the results from this lecture to find the full asymptotic expansion of the solution as $x \to \infty$. (ii) Find this expansion without using explicit solution: identify a two-term $x \to \infty$ balance in the differential equation, and then proceed to higher order via iteration or some other scheme.

Problem 2.7. Find an example of a infinitely differentiable function satisfying the inequalities

$$\max_{0 < x < 1} |f(x)| < 10^{-10}, \quad \text{and} \quad \max_{0 < x < 1} \left| \frac{df}{dx} \right| > 10^{10}. \hspace{1cm} (2.97)$$

This is why the differential operator $d/dx$ is “unbounded”: $d/dx$ can take a small function and turn it into a big function.

Problem 2.8. Prove that

$$\int_0^\infty \frac{e^{-t}}{1 + xt^2} \, dt \sim \sum_{n=0}^\infty (-1)^n (2n)! x^n, \quad x \to 0.$$  \hspace{1cm} (2.98)
Problem 2.9. True or false as \( x \to \infty \)

\[
(i) \ x + \frac{1}{x} \sim x, \quad (ii) \ x + \sqrt{x} \sim x, \quad (iii) \ \exp \left( x + \frac{1}{x} \right) \sim \exp(x), \\
(iv) \ \exp \left( x + \sqrt{x} \right) \sim \exp(x), \quad (v) \ \cos \left( x + \frac{1}{x} \right) \sim \cos x, \quad (v) \ \frac{1}{x} \sim 0? \quad (2.99)
\]

Problem 2.10. Let’s investigate the Stieltjes series \( S(x) \) in (2.59) and the function \( F(x) \) in (2.61)

\( (i) \) Compute the integral \( F(0.1) \) numerically.

\( (ii) \) With \( x = 0.1 \), compute partial sums of the divergent series (2.59) with \( N = 2, 3, 4, \cdots 20 \). Which \( N \) gives the best approximation to \( F(0.1) \)?

\( (iii) \) I think the best answer is obtained by truncating the series \( S(0.1) \) just before the smallest term. Is that correct?
Lecture 3

Integration by parts (IP)

Our earlier example

\[
\text{erfc}(z) \sim \frac{e^{-z^2}}{z\sqrt{\pi}} \left[ 1 - \frac{1}{2z^2} + \frac{1 \times 3}{(2z^2)^2} - \frac{1 \times 3 \times 5}{(2z^2)^3} + O(z^{-8}) \right], \quad \text{as } z \to \infty,
\]

illustrated the use of integration by parts (IP) to obtain an asymptotic series. In this lecture we discuss other integrals that also yield to IP

3.1 The Taylor series, with remainder

We can very quickly use integration by parts to prove that a function \( f(x) \) with \( n \) derivatives can be represented \textit{exactly} by \( n \) terms of a Taylor series, plus a remainder. The fundamental theorem of calculus is

\[
f(x) = f(a) + \int_a^x f'(\xi) \, d\xi,
\]

If we drop the final term, \( R_1(x) \), we have a one-term Taylor series for \( f(x) \) centered on \( x = a \). To generate one more terms we integrate by parts like this

\[
f(x) = f(a) + (x - a)f'(a) - \int_a^x f''(\xi)(\xi - x) \, d\xi,
\]

And again

\[
f(x) = f(a) + (x - a)f'(a) - \int_a^x f''(\xi)(\xi - x)^2 \, d\xi.
\]

If \( f(x) \) has \( n \)-derivatives we can keep going till we get

\[
f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x - a)^{n-1} + R_n(x),
\]

\( = n \) terms, let’s call this \( f_n(x) \)
where the remainder after \( n \)-terms is

\[
R_n(x) = \frac{1}{(n-1)!} \int_a^x f^{(n)}(\xi)(x-\xi)^{n-1} \, d\xi. \tag{3.8}
\]

Using the first mean value theorem, the remainder can be represented as

\[
R_n(x) = \frac{f^{(n)}(\bar{x})}{n!} (x-a)^n, \tag{3.9}
\]

where \( \bar{x} \) is some unknown point in the interval \([a, x]\). This is the form given in section 4.6 of \text{RHB}.

Some remarks about the result in (3.7) through (3.9) are:

1. \( f(x) \) need not have derivatives of all order at the point \( x \): the representation in (3.7) and (3.9) makes reference only to derivatives of order \( n \), and that is all that is required.

2. Using (3.9), we see that the ratio of \( R_n(x) \) to the last retained term in the series is proportional to \( x - a \) and therefore vanishes as \( x \to a \). Thus, according to our definition in (2.53), \( f_n(x) \) is an asymptotic expansion of \( f(x) \).

3. The convergence of the truncated series \( f_n(x) \) as \( n \to \infty \) is not assumed: (3.7) is exact. The remainder \( R_n(x) \) may decrease up to a certain point and then start increasing again.

4. Even if \( f_n(x) \) diverges with increasing \( n \), we may obtain a close approximation to \( f(x) \) — with a small remainder \( R_n \) — if we stop summing at a judicious value of \( n \).

5. The difference between the convergent case and the divergent case is that in the former instance the remainder can be made arbitrarily small by increasing \( n \), while in the latter case the remainder cannot be reduced below a certain minimum.

Above we are recapitulating many remarks we made previously regarding the asymptotic expansion of \( \text{erf} \) in (3.1).

**Example:** Taylor series, even when they diverge, are still asymptotic series. Let’s investigate this by revisiting problem 1.1:

\[
x(\epsilon)^2 = 9 + \epsilon. \tag{3.10}
\]

Notice that even before taking this class you could have solved this problem by arguing that

\[
x(\epsilon) = 3 \left(1 + \frac{\epsilon}{9}\right)^{1/2}, \tag{3.11}
\]

and then recollecting the standard Taylor series

\[
(1 + z)^{\alpha} = 1 + \alpha z + \frac{\alpha(\alpha - 1)}{2!} z^2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!} z^3 + \cdots \tag{3.12}
\]

The perturbation expansion you worked out in problem 1.1 is laboriously reproducing the special case \( \alpha = 1/2 \) and \( z = \epsilon/9 \).

You should recall from part B that the radius of convergence of (3.12) is limited by the nearest singularity to the origin in the complex \( z \)-plane. With \( \alpha = 1/2 \) the nearest singularity is the branch point at \( z = -1 \). So the series in problem 1.1 converges provided that \( \epsilon < 9 \). Let us ignore this red flag and use the Taylor series with \( \epsilon = 16 \) to estimate \( x(16) = \sqrt{25} = 5 \). We calculate a lot of terms with the mathematica command:

\text{Series[Sqrt[9 + u], \{u, 0, 8\}].}

This produces the series

\[
x(\epsilon) = 3 + \frac{\epsilon}{6} - \frac{\epsilon^2}{216} + \frac{\epsilon^3}{3888} - \frac{5\epsilon^4}{279936} + \frac{7\epsilon^5}{5038848} - \frac{7\epsilon^6}{60466176} + \frac{11\epsilon^7}{1088391168} - \frac{143\epsilon^8}{156728328192} + \text{ord} (\epsilon^9). \]

Thus

\[
x(16) \sim 3 + \frac{8}{3} - \frac{32}{27} + \frac{256}{243} - \frac{2560}{2187} + \frac{28672}{19683} - \frac{114688}{59049} + \cdots \tag{3.13}
\]
The fourth term is the smallest term. Stopping short of the smallest term, the sum of the first three terms is

\[ x(16) \approx \frac{121}{27} = 4.48148, \]  

(3.14)

which is a relative error of about 10%. If we include half of the smallest term then

\[ x(16) \approx \frac{1217}{243} = 5.00823, \]  

(3.15)

with relative error 0.00165. This is a good result when working with the “small” parameter 16/9.

### 3.2 Large-s behaviour of Laplace transforms

The \( s \to \infty \) behaviour of the Laplace transform

\[ \bar{f}(s) \overset{\text{def}}{=} \int_0^\infty e^{-st} f(t) \, dt \]  

(3.16)

provides a typical and important example of IP. But before turning to IP, we argue that as \( \Re s \to \infty \), the maximum of the integrand in (3.16) is determined by the rapidly decaying \( e^{-st} \) and is therefore at \( t = 0 \). In fact, \( e^{-st} \) is appreciably different from zero only in a peak at \( t = 0 \), and the width of this peak is \( s^{-1} \ll 1 \). Within this peak \( t = O(s^{-1}) \) the function \( f(t) \) is almost equal to \( f(0) \) (assuming that \( f(0) \) is non-zero) and thus

\[ \bar{f}(s) \approx f(0) \int_0^\infty e^{-st} \, dt = \frac{f(0)}{s}. \]  

(3.17)

This argument suggests that the large \( s \)-behaviour of the Laplace transform of any function \( f(t) \) with a Taylor series around \( t = 0 \) is given by

\[ \int_0^\infty e^{-st} f(t) \, dt = \int_0^\infty e^{-st} \left[ f(0) + tf'(0) + \frac{t^2}{2!} f''(0) + \cdots \right] e^{-st} \, dt, \]  

(3.18)

\[ \sim \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{f''(0)}{s^3} + \cdots \]  

(3.19)

This heuristic answer is in fact a valid asymptotic series. We obtain an improved version of (3.19) using successive integration by parts starting with (3.16):

\[ \bar{f}(s) = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{f''(0)}{s^2} + \cdots + \frac{f^{(n-1)}(0)}{s^n} + \frac{1}{s^n} \int_0^\infty e^{-st} f^{(n)}(t) \, dt. \]  

(3.20)

The improvement over (3.19) is that on the right of (3.20), IP has provided an explicit expression for the remainder \( R_n(s) \).

#### Example: A Laplace transform

Find the large-\( s \) behaviour of the Laplace transform

\[ \mathcal{L} \left[ \frac{1}{\sqrt{1+t^2}} \right] = \int_0^\infty \frac{e^{-st}}{\sqrt{1+t^2}} \, dt. \]  

(3.21)

When \( s \) is large the function \( e^{-st} \) is non-zero only in a peak located at \( t = 0 \). The width of this peak is \( s^{-1} \ll 1 \). In this region the function \( (1+t^2)^{-1/2} \) is almost equal to one. Hence heuristically

\[ \int_0^\infty \frac{e^{-st}}{\sqrt{1+t^2}} \, dt \approx \int_0^\infty e^{-st} \, dt = \frac{1}{s}. \]  

(3.22)

\(^1\)See the next section, Watson’s lemma, for justification.
This is the correct leading-order behaviour.

To make a more careful estimate we can use integration by parts:

\[
\mathcal{L}\left[\frac{1}{\sqrt{1+t^2}}\right] = -\frac{1}{s} \int_0^\infty \frac{1}{\sqrt{1+t^2}} \, dt = \frac{1}{s} \int_0^\infty \frac{e^{-st}}{\sqrt{1+t^2}} \, dt - \frac{1}{s} \int_0^\infty \frac{te^{-st}}{(1+t^2)^{3/2}} \, dt, \quad (3.23)
\]

As \(s \to \infty\) the remainder \(R_1(s)\) is negligible with respect to \(s^{-1}\) and the heuristic \((3.22)\) is confirmed. Why is \(R_1(s)\) much smaller than \(s^{-1}\) in the limit? Notice that in the integrand of \(R_1\)

\[
\frac{te^{-st}}{(1+t^2)^{3/2}} \leq te^{-st}, \quad \text{and therefore} \quad R(s) < \frac{1}{s} \int_0^\infty te^{-st} \, dt = \frac{1}{s^2}. \quad (3.26)
\]

The estimates between \((3.23)\) and \((3.26)\) are a recap of arguments we’ve been making in the previous lectures. The proof of Watson’s lemma below is just a slightly more general version of these same estimates.

To get more terms in the asymptotic expansion we invoke Watson’s lemma, so as \(s \to \infty\):

\[
\mathcal{L}\left[\frac{1}{\sqrt{1+t^2}}\right] = \int_0^\infty e^{-st} \left[1 - \frac{t^2}{2} + \frac{3t^4}{8} - \frac{5t^6}{16} + O(t^8)\right] \, dt, \quad (3.27)
\]

\[
\sim \frac{1}{s} - \frac{1}{s^3} + \frac{9}{s^5} - \frac{225}{s^7} + O\left(s^{-9}\right). \quad (3.28)
\]

Because of the rapid growth of the numerators this is clearly an asymptotic series. The Taylor series of \((1+t^2)^{-1/2}\) does not converge beyond \(t = 1\). The limited radius of convergence doesn’t matter: Watson’s lemma assures us that we get the right asymptotic expansion even if we integrate into the region where the Taylor series diverges. In fact, the expansion of the integral is asymptotic, rather than convergent, because we’ve integrated a Taylor series beyond its radius of convergence.

We obtain the entire asymptotic series by noting that

\[
\frac{1}{\sqrt{1-4x}} = 1 + 2x + 6x^2 + 20x^3 + 70x^4 + \ldots \quad (3.29)
\]

where the coefficient of \(x^n\) above is the “central binomial coefficient” \((2n)!/(n!)^2\). Thus, with \(x = -t^2/4\), we have

\[
\mathcal{L}\left[\frac{1}{\sqrt{1+t^2}}\right] \sim \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} (-1)^n \left(\frac{1}{2}\right)^{2n} \int_0^\infty t^{2n} e^{-st} \, dt, \quad (3.30)
\]

\[
= \frac{1}{s} \sum_{n=1}^{\infty} (-1)^n \left(\frac{(2n)!}{(n!)^2}\right)^2 \frac{1}{(2s)^{2n}}. \quad (3.31)
\]

Example: Another Laplace transform. Consider

\[
\mathcal{L}\left[\frac{H(t)}{\sqrt{1-t^2}}\right] = \int_0^1 \frac{e^{-st}}{\sqrt{1-t^2}} \, dt, \quad (3.32)
\]

\[
\sim \frac{1}{s} + \frac{1}{s^3} + \frac{9}{s^5} + \frac{225}{s^7} + O\left(s^{-9}\right). \quad (3.33)
\]

This is the same as \((3.28)\), except that all the signs are positive. The integrable singularity at \(t = 1\) makes only an exponentially small contribution as \(s \to \infty\).

Example: Yet another Laplace transform. Find the large-\(s\) behaviour of the Laplace transform

\[
\mathcal{L}\left[\sqrt{1+e^t}\right] = \int_0^\infty e^{-st} \sqrt{1+e^t} \, dt. \quad (3.34)
\]

In this case \(f(0) = \sqrt{2}\) and we expect that the leading order is

\[
\tilde{f} \sim \frac{\sqrt{2}}{s}. \quad (3.35)
\]
Let’s confirm this using IP:

\[ \tilde{f}(s) = \frac{\sqrt{2}}{s} - \frac{1}{s} \int_0^\infty e^{-st} \frac{e^t}{2\sqrt{1 + e^t}} \, dt. \]  

(3.36)

Notice that in this example \( f'(t) \sim e^{t/2} \) as \( t \to \infty \), and thus we cannot bound the remainder using \( \max_{t > 0} f'(t) \). Instead, we bound the remainder like this

\[ R_1 = \frac{1}{s} \int_0^\infty e^{-(s - \frac{1}{2})t} \frac{1}{2\sqrt{1 + e^{-t/2}}} \, dt < \frac{1}{s} \frac{1}{2s - 1}. \]  

(3.37)

This maneuver works in examples with \( f(t) \sim e^{\alpha t} \) as \( t \to \infty \).

**Example: A thinly disguised Laplace transform.** Consider

\[ S(x) \overset{\text{def}}{=} \int_0^1 e^{xt^2} \, dt \]  

(3.38)

as \( x \to \infty \). The integrand is strongly peaked near \( t = 1 \). Changing variable to \( v = 1 - t^2 \) we obtain

\[ S(x) = \frac{e^x}{t} \int_0^1 e^{-xv} \, dv \]  

(3.39)

\[ \sim \frac{e^x}{t} \int_0^\infty e^{-xv}(1 - \frac{6}{7}v + \cdots) \, dv \]  

(3.40)

etc.

**Example: An integral with a parameter.** Consider

\[ I(x, \nu) \overset{\text{def}}{=} \int_0^\infty t^\nu e^{-x \sinh t} \, dt. \]  

(3.41)

The minimum of \( \phi(t) = \sinh t \) is at \( t = 0 \), so

\[ I(x) \sim \int_0^\infty t^\nu e^{-xt} \, dt \sim \frac{\Gamma(\nu + 1)}{x^{\nu + 1}}, \quad \text{as } x \to \infty. \]  

(3.42)

To get the next term in the asymptotic series, keep one more term in the expansion of \( \sinh t \):

\[ e^{-x \sinh t} \approx e^{-xt} e^{xt^2/6 - xt^4/120 + \cdots} \approx e^{-xt} \left( 1 + \frac{xt^3}{6} + O(xt^5) \right). \]  

(3.43)

Thus

\[ I(x) \sim \int_0^\infty t^\nu e^{-xt} \left( 1 - \frac{xt^3}{6} + O(xt^5) \right) \, dt, \]  

(3.44)

\[ \sim \frac{\Gamma(\nu + 1)}{x^{\nu + 1}} - \frac{\Gamma(\nu + 4)}{6x^{\nu + 3}} + O(x^{-\nu - 5}). \]  

(3.45)

Notice we have to keep the dominant term \( xt \) up in the exponential.

If we desire more terms, and are obliged to justify the heuristic above, we should change variables with \( u = \sinh t \) in (3.41), and use Watson’s lemma. The transformed integral is a formidable Laplace transform:

\[ I(x, \nu) \overset{\text{def}}{=} \int_0^\infty e^{-xu} \ln^\nu \left( \sqrt{1 + u^2} + u \right) \frac{du}{\sqrt{1 + u^2}}. \]  

(3.46)

With mathematica

\[ \frac{\ln^\nu \left( \sqrt{1 + u^2} + u \right)}{\sqrt{1 + u^2}} = u^\nu \left[ 1 - \frac{3 + \nu}{6} u^2 + \frac{135 + 52\nu + 5\nu^2}{360} u^4 + O(u^6) \right]. \]  

(3.47)

The coefficient of \( u^{2n} \) in this expansion is a polynomial — let’s call it \((-)^n P_n(\nu)\) — of order \( n \). Substituting (3.47) into (3.46) and integrating term-by-term

\[ I(x, \nu) \sim \frac{1}{x^{\nu + 1}} \left[ \frac{\Gamma(\nu + 1) - P_2(\nu)}{x^2} \Gamma(\nu + 3) + \frac{P_4(\nu)}{x^4} \Gamma(\nu + 5) + O(x^{-6}) \right]. \]  

(3.48)
3.3 Watson’s Lemma

All the examples in the previous section are a special cases of Watson’s lemma. So let’s prove the lemma by considering a Laplace transform

\[ \tilde{f}(s) = \int_0^\infty e^{-st} t^\xi g(t) \, dt, \]

(3.49)

where the factor \( t^\xi \) includes whatever singularity exists at \( t = 0 \); the singularity must be integrable i.e., \( \xi > -1 \). We assume that the function \( g(t) \) has a Taylor series with remainder

\[ g(t) = g_0 + g_1 t + \cdots + g_n t^n + R_{n+1}(t). \]

(3.50)

This is a \( t \to 0 \) asymptotic expansion in the sense that there is some constant \( K \) such that

\[ |R_{n+1}| < K t^{n+1}. \]

(3.51)

Notice we are not assuming that the Taylor series converges.

Of course, we do assume convergence of the Taylor series \( 3.49 \) as \( t \to \infty \), which most simply requires that \( f(t) = t^\xi g(t) \) eventually grows no faster than \( e^{\gamma t} \) for some \( \gamma \). Notice that the possibility of a finite upper limit \( 3.49 \) is encompassed if \( f(t) \) is zero once \( t > T \).

With these modest constraints on \( t^\xi g(t) \):

\[ \tilde{f}(s) = \int_0^\infty e^{-st} t^\xi (g_0 + g_1 t + \cdots + g_n t^n + R_{n+1}(t)) \, dt. \]

(3.52)

The second integral in \( 3.52 \) is

\[ I_2 < K \int_0^\infty e^{-st} t^{n+1+\xi} \, dt = O \left( \frac{1}{s^{\xi+n+2}} \right). \]

(3.53)

Using

\[ \int_0^\infty e^{-st} t^{\xi+n} \, dt = \frac{\Gamma(n + \xi + 1)}{s^{n+\xi+1}}, \]

(3.54)

we integrate \( I_1 \) term-by-term and obtain Watson’s lemma:

\[ \tilde{f}(s) \sim g_0 + g_1 \frac{\Gamma(\xi + 1)}{s^{\xi+1}} + g_2 \frac{\Gamma(\xi + 2)}{s^{\xi+2}} + \cdots + g_n \frac{\Gamma(\xi + n + 1)}{s^{\xi+n+1}} + O \left( \frac{1}{s^{\xi+n+2}} \right). \]

(3.55)

Watson’s lemma justifies doing what comes naturally.

3.4 Problems

**Problem 3.1.** (i) Obtain the leading-order asymptotic approximation for the integral

\[ \int_{-1}^1 e^{xt^3} \, dt, \quad \text{as } x \to \infty. \]

(3.56)

(ii) Justify the asymptoticness of the expansion. (iii) Find the leading-order asymptotic approximation for \( x \to -\infty \).
**Problem 3.2.** In our evaluation of \( \text{Ai}(0) \) we encountered a special case, namely \( n = 3 \), of the integral
\[
Z(n, x) \overset{\text{def}}{=} \int_{0}^{\pi/(2n)} e^{-x \sin n\theta} \, d\theta.  
\] (3.57)

Convert \( Z(n, x) \) to a Laplace transform and use Watson’s lemma to obtain the first few terms of the \( x \to \infty \) asymptotic expansion.

**Problem 3.3.** In lecture 3 we obtained the full asymptotic series for \( \text{erfc}(z) \) via IP:
\[
\text{erfc}(x) \sim \frac{e^{-x^2}}{\sqrt{\pi x}} \sum_{n=0}^{\infty} \frac{(2n-1)!!}{n!} \left( -\frac{1}{2x^2} \right)^n.  
\] (3.58)

Obtain this result by making a change of variables that converts \( \text{erfc}(z) \) into a Laplace transform, and then use Watson’s lemma.

**Problem 3.4.** Use integration by parts to find \( x \to \infty \) asymptotic approximations of the integrals
\[
A(x) = \int_{0}^{x} e^{-t^4} \, dt,  
\]
\[
B(x) = \int_{0}^{x} e^{+t^4} \, dt,  
\]
\[
C(x) = \int_{0}^{\infty} e^{-xt} \ln(1 + t^2) \, dt,  
\]
\[
D(x) = \int_{0}^{\infty} e^{-xt} \frac{1}{t^a(1+t)} \, dt, \quad \text{with } a < 1;  
\]
\[
E(x) = \int_{1}^{\infty} e^{-xtp} \, dt, \quad \text{with } p > 0;  
\]
(3.59) (3.60) (3.61) (3.62) (3.63)

In each case obtain a two-term asymptotic approximation and exhibit the remainder as an integral. Explain why the remainder is smaller than the second term as \( x \to \infty \).

**Problem 3.5.** Using repeated IP, find the full \( x \to \infty \) asymptotic expansion of Dawson’s integral \( (2.65) \). Is this series convergent?

**Problem 3.6.** Consider \( f(x) = (1 + x)^{5/2} \), and the corresponding Taylor series \( f_n(x) \) centered on \( x = 0 \). (i) Show that for \( n \geq 3 \) and \( x > 0 \):
\[
R_n < \frac{f^{(n)}(0)}{n!} x^n,  
\]
i.e., the remainder is smaller than the first neglected term for all positive \( x \). (ii) The Taylor series converges only up to \( x = 1 \). But suppose we desire \( f(2) = 3^{5/2} \). How many terms of the series should be summed for best accuracy? Sum this optimally truncated series and compare with the exact answer. (iii) Argue from the remainder in (3.39) that the error can be reduced by adding half the first neglected term. Compare this corrected series with the exact answer.
Lecture 4

Laplace’s method

Laplace’s method applies to integrals in which the integrand is concentrated in the neighbourhood of a few (or one) isolated points. The value of the integral is determined by the dominant contribution from those points. This happens most often for integrals of the form

\[ L(x) = \int_{a}^{b} f(t)e^{-x\phi(t)} \, dt, \quad \text{as } x \to \infty. \]  

(4.1)

If \( \phi(t) \geq 0 \) for all \( t \) in the interval \( a \leq t \leq b \) then as \( x \to +\infty \) the integrand will be maximal where \( \phi(t) \) is smallest. This largest contribution becomes more and more dominant as \( x \) increases.

Look what happens if we apply IP to (4.1):

\[
L(x) = -\int_{a}^{b} \frac{f(t)}{x\phi'(t)} \, dt, \\
= -\left[ \frac{f(t)e^{-x\phi}}{x\phi'(t)} \right]_{a}^{b} + \int_{a}^{b} e^{-x\phi} \frac{df(t)}{dt} \frac{d}{dt} \left( \frac{f(t)}{x\phi'(t)} \right) \, dt.
\]  

(4.3)

There is a problem if \( \phi' \) has a zero anywhere in the closed interval \([a, b]\). However if \( \phi' \) is non-zero throughout \([a, b]\) then IP delivers the goods. For example, suppose

\[ \phi' > 0 \quad \text{for } a \leq t \leq b, \]  

(4.4)

then from (4.3)

\[ L(x) \sim \frac{f(a)}{x\phi'(a)} e^{-x\phi(a)}, \quad \text{as } x \to \infty. \]  

(4.5)

All of the examples in section 3.2 have this form.

In the case of (4.5), the integrand is concentrated near \( x = a \) and the asymptotic approximation in (4.3) depends only on \( f(a) \), \( \phi(a) \) and \( \phi'(a) \). We can quickly obtain (4.5) with the following approximations in (4.3):

\[ L(x) \sim \int_{a}^{\infty} f(a)e^{-x\phi(a) - x\phi'(a)t} \, dt. \]  

(4.6)

Exercise: Find the leading order asymptotic approximation to \( I(x) \) if \( \phi' < 0 \) for \( a \leq t \leq b \). Show that

\[ A(x) \overset{\text{def}}{=} \int_{a}^{\pi} e^{x\cosh t} \, dt \sim \frac{e^{x\cosh \pi}}{x\sinh \pi}, \quad \text{as } x \to \infty. \]  

(4.7)

To summarize, if \( \phi' \) is non-zero throughout \([a, b]\) then the integrand is concentrated at one of the end points, and IP quickly delivers the leading-order term. And, if necessary, one can write the integral as a Laplace transform by changing variables to

\[ v = \phi(t). \]  

(4.8)
in (4.1). Then Watson’s lemma delivers the full asymptotic expansion. We turn now to discussion of the case in which \( \phi'(t) \) has a zero somewhere in \([a, b]\).

### 4.1 An example — the Gaussian approximation

As an example of Laplace’s method with a zero of \( \phi' \) we study the function defined by

\[
U(x, y) \overset{\text{def}}{=} \int_0^y e^{-x \cosh t} \, dt, \quad (4.9)
\]

and ask for an asymptotic approximations as \( x \to +\infty \) with \( y \) fixed. In this example \( \phi' = \sinh t \) is zero at \( t = 0 \) and IP fails.

**Exercise:** Find a two-term approximation of \( U(x, 1) \) when \( |x| \ll 1 \).

With \( x \to \infty \), the main contribution to \( U(x, y) \) in (4.9) is from \( t \approx 0 \). Thus, according to Laplace, the leading-order behaviour is

\[
U(x, y) \sim \int_0^\infty e^{-x(1 + \frac{1}{2}t^2)} \, dt, \quad (4.10)
\]

\[
e^{-x} \sqrt{\frac{\pi}{2x}}, \quad \text{as} \quad x \to +\infty. \quad (4.11)
\]

The peak of the integrand is centered on \( t = 0 \) and has width \( x^{-1/2} \ll 1 \). All the approximations we’ve made above are good in the peak region. They’re lousy approximations outside the peak e.g., near \( t = 1/2 \). But both the integrand and our approximation to the integrand are tiny near \( t = 1/2 \) and thus those errors do not seriously disturb our estimate of the integral.

Notice that in (4.10) the range of integration is extended to \( t = \infty \) — we can then do the integral without getting tangled up in error functions. The point is that the leading-order behaviour of \( U(x, y) \) as \( x \to \infty \) is independent of the fixed upper limit \( y \). If you’ve understood the argument above regarding the peak width, then you’ll appreciate that if \( y = 1/10 \) then \( x \) will have to be roughly as big as 100 in order for (4.11) to be accurate.

Let’s bash out the second term in the \( x \to \infty \) asymptotic expansion. According to MATHEMATICA, the integrand is

\[
e^{-x \cosh t} = e^{-x - xt^2/2} e^{-xt^4/24! - xt^6/6! + \cdots} \approx e^{-x - xt^2/2} \left( 1 - \frac{xt^4}{24} - \frac{xt^6}{720} + O(x^2 t^8) \right). \quad (4.12)
\]

Notice the \( x^2 \) in the big Oh error estimate above — this \( x^2 \) will bite us below. We now substitute the expansion (4.12) into the integral (4.9) and integrate term-by-term using

\[
\int_0^\infty t^p e^{-at^2} \, dt = \frac{1}{2^p} a^{-p+1} \Gamma \left( \frac{p+1}{2} \right). \quad (4.13)
\]

Thus we have

\[
U(x, y) = e^{-x} \int_0^\infty e^{-\frac{1}{2}xt^2} \left[ \frac{1}{\sim x^{-1/2}} - \frac{1}{24} \frac{xt^4}{x^{-3/2}} - \frac{1}{720} \frac{xt^6}{x^{-5/2}} + O\left( \frac{x^2 t^8}{x^{-5/2}} \right) \right] \quad (4.14)
\]

The underbraces indicate the order of magnitude of each term after using (4.13) to evaluate the integral. Notice that a term of order \( x^2 t^6 \) is of order \( x^{-5/2} \) after integration. Thus, if we desire a systematic expansion, we should not keep the term \( xt^6 \) and drop \( x^2 t^8 \). After integration both these terms are order \( x^{-5/2} \), and we should keep them both, or drop them both.
Figure 4.1: Upper panel compares the two-term asymptotic expansion in (4.17) with evaluation of the integral by numerical quadrature using the MATLAB routine quad. The lower panel compares the three term expansion in (4.21) with quadrature.
Proceeding with the integration

\[ U(x, y) \sim e^{-x} \int_0^{2x} \int_0^\infty e^{-v^2} \left[ 1 - \frac{v^4}{6x} - \frac{8v^6}{720x^2} + O\left( v^8 x^{-2} \right) \right] \, dv , \]  
\[ = e^{-x} \int_0^{\frac{\pi}{2x}} e^{-xu} \left[ 1 - \frac{1}{6x} \times \frac{3}{4} - \frac{8}{720x^2} \times \frac{15}{8} + O\left( x^{-2} \right) \right] \, du , \]  
\[ \sim e^{-x} \sqrt{\frac{\pi}{2x}} \left[ 1 - \frac{1}{8x} + O\left( x^{-2} \right) \right] . \]  
\[ (4.15) \]

Discretion is the better part of valor, so I’ve dropped the inconsistent term and written \( O(x^{-2}) \) above.

Another way to generate more terms in the expansion is to convert \( U(x, y) \) into a Laplace transform via \( u = \cosh t - 1: \)

\[ U(x, y) \sim e^{-x} \int_0^\infty \frac{e^{-xu}}{\sqrt{2u + u^2}} \, du , \]  
\[ = e^{-x} \int_0^\infty e^{-xu} \frac{1}{\sqrt{2u}} \left[ 1 - \frac{u}{4} + \frac{3u^2}{32} - \frac{5u^3}{128} + O\left( u^4 \right) \right] \, du , \]  
\[ \sim e^{-x} \sqrt{\frac{\pi}{2x}} \left[ \Gamma\left( \frac{1}{2} \right) - \frac{1}{4x} \Gamma\left( \frac{3}{2} \right) + \frac{3}{32x^2} \Gamma\left( \frac{5}{2} \right) + O\left( x^{-3} \right) \right] , \]  
\[ = e^{-x} \sqrt{\frac{\pi}{2x}} \left[ 1 - \frac{1}{8x} + \frac{9}{128x^2} + O\left( x^{-3} \right) \right] . \]  
\[ (4.17) \]

The Laplace-transform approach is more systematic because the coefficients in the series expansion are not functions of \( x \), and the expansion is justified using Watson’s lemma. However the argument about the dominance of the peak provides insight and is all one needs to quickly obtain the leading-order asymptotic expansion.

### 4.2 Another Laplacian example

Consider

\[ I_n \overset{\text{def}}{=} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} (\cos t)^n \, dt . \]  
\[ (4.22) \]

With a little integration by parts one can show that

\[ I_n = \left( 1 - \frac{1}{n} \right) I_{n-2} . \]  
\[ (4.23) \]

Then, since \( I_0 = 1 \) and \( I_1 = 2/\pi \), it is easy to compute the exact integral at integer \( n \) recursively.

Let’s use Laplace’s method to find an \( n \to \infty \) asymptotic approximation. We write the integral as

\[ I_n = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} e^{n \ln \cos t} \, dt , \]  
\[ (4.24) \]

and then make the small \( t \)-approximation

\[ \ln \cos t = \ln \left( 1 - \frac{t^2}{2} \right) \approx -\frac{t^2}{2} . \]  
\[ (4.25) \]
Thus the leading order is obtained by evaluating a gaussian integral

\[
I_n \sim \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-nt^2/2} \, dt,
\]

\[
= \sqrt{\frac{2}{\pi n}}.
\]

Figure 4.2 compares this approximation to the exact integral. Suppose we’re disappointed with the performance of this approximation at \( n = 5 \), and want just one more term. The easiest way to bash out an extra term is

\[
\ln \cos t = \ln \left( 1 - \frac{t^2}{2} + \frac{t^4}{24} + \text{ord}(t^6) \right),
\]

\[
= \left( \frac{t^2}{2} - \frac{t^4}{24} + \text{ord}(t^6) \right) + \frac{1}{2} \left( \frac{t^2}{2} + \text{ord}(t^4) \right)^2 + \text{ord}(t^6),
\]

\[
= -\frac{t^2}{2} - \frac{t^4}{12} + \text{ord}(t^6),
\]

and then

\[
I_n \sim \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-nt^2/2} e^{-nt^4/12} \, dt,
\]

\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-nt^2/2} \left( 1 - \frac{nt^4}{12} \right) \, dt,
\]

\[
= \sqrt{\frac{2}{\pi n}} \left( 1 - \frac{1}{4n} \right).
\]

This works very well at \( n = 5 \). In the unlikely event that more terms are required, then it is probably best to be systematic: change variables with \( v = -\ln \cos t \) and use Watson’s lemma.
4.3 Laplace’s method with moving maximum

Large $s$ asymptotic expansion of a Laplace transform

Not all applications of Laplace’s method fall into the form (4.1). For example, consider the Laplace transform

$$L\left[ e^{-1/t} \right] = \int_0^\infty e^{-\frac{1}{t} - st} \, dt, \quad \text{as } s \to \infty. \quad (4.34)$$

Watson’s lemma is defeated by this example.

In the exponential in (4.34) have $\chi \overset{\text{def}}{=} t^{-1} + st$, and

$$\frac{d\chi}{dt} = 0, \quad \Rightarrow -\frac{1}{t^2} + s = 0. \quad (4.35)$$

Thus the integrand is biggest at $t_\ast = s^{-1/2}$ — the peak is approaching $t = 0$ as $s$ increases. Close to the peak

$$\chi = \chi(t_\ast) + \frac{1}{2} \chi''(t_\ast)(t - t_\ast)^2 + O(t - t_\ast)^3,$$

$$= 2s^{1/2} + s^{-3/2}(t - s^{-1/2})^2 + O(t - t_\ast)^3. \quad (4.37)$$

The width of the peak is $s^{-3/4} \ll s^{-1/2}$, so it helps to introduce a change of variables

$$v \overset{\text{def}}{=} s^{3/4}(t - s^{-1/2}) \quad (4.38)$$

In terms of the original variable $t$ the peak of the integrand is moving as $s$ increases. We make the change of variable in (4.38) so that the peak is stationary at $v = 0$. The factor $s^{3/4}$ on the right of (4.38) ensures that the width of the $v$-peak is not changing as $s \to \infty$.

Notice that $t = 0$ corresponds to $v = -s^{1/4} \to -\infty$. But the integrand has decayed to practically to zero once $v \gg 1$. Thus the lower limit can be taken to $v = -\infty$. The Laplace transform is therefore

$$L\left[ e^{-1/t} \right] \sim s^{-3/4}e^{-2s^{1/2}} \int_{-\infty}^\infty e^{-v^2} \, dv \Rightarrow \sqrt{\pi}\int_{-\infty}^\infty e^{-v^2} \, dv = \sqrt{\pi} \quad (4.39)$$

This Laplace transform is exponentially small as $s \to \infty$, and of course the original function was also exponentially small as $t \to 0$. I trust you’re starting to appreciate that there is an intimate connection between the small-$t$ behaviour of $f(t)$ and the large-$s$ behaviour of $f(s)$.

Remark: the Laplace transform of any function must vanish as $s \to \infty$. So, if you’re asked to find the inverse Laplace transform of $s$, the answer is there is no function with this transform.

Stirling’s approximation

A classic example of a moving maximum is provided by Stirling’s approximation to $n!$. Starting from

$$\Gamma(x + 1) = \int_0^\infty t^x e^{-t} \, dt, \quad (4.40)$$

let’s derive the fabulous result

$$\Gamma(x + 1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \left( 1 + \frac{1}{12x} + O\left( x^{-2} \right) \right), \quad \text{as } x \to \infty. \quad (4.41)$$
At \( x = 1 \), we have from the leading order 1 \( \approx \sqrt{2\pi/e} = 0.9221 \), which is not bad! And with the next term \( \sqrt{2\pi/e} \times (13/12) = 0.99898 \). It only gets better as \( x \) increases.

We begin by moving everything in (4.40) upstairs into the exponential:

\[
\Gamma(x + 1) = \int_0^\infty e^{-\chi} \, dt, \tag{4.42}
\]

where

\[
\chi \overset{\text{def}}{=} x \ln t - t. \tag{4.43}
\]

The maximum of \( \chi \) is at \( t_s = x \) — the maximum is moving as \( x \) increases. We can expand \( \chi \) around this moving maximum as

\[
\chi = x \ln x - x + \frac{(t - x)^2}{2x} + O(t - x)^3, \tag{4.44}
\]

\[
\chi = x \ln x - x - v^2, \tag{4.45}
\]

where \( v \overset{\text{def}}{=} (t - x)/\sqrt{2x} \) is the new variable of integration. With this Gaussian approximation we have

\[
\Gamma(x + 1) = e^{x \ln x - x} \sqrt{2x} \int_{-\infty}^{\infty} e^{-v^2} \, dv \bigg|_{0}^{\infty} = \sqrt{\pi}. \tag{4.46}
\]

This is the leading order term in (4.40).

Exercise: Obtain the next term, \( 1/12x \), in (4.40).

Example: Find the leading order approximation to

\[
\Lambda(x) = \int_0^\infty t^x e^{-t} \, dt \tag{4.47}
\]

It is necessary to move all functions upstairs into the exponential, and after some algebra I found

\[
\Lambda(x) \sim \sqrt{2\pi x} \left( \frac{x - 2}{e} \right)^{x - 2}, \quad \text{as } x \to \infty. \tag{4.48}
\]

I’m about 80% sure that this is correct.

### 4.4 Uniform approximations

Consider a function of two variables defined by:

\[
J(x, \alpha) \overset{\text{def}}{=} \int_0^\infty e^{-x (\sinh t - \alpha t)} \, dt, \quad \text{with } x \to \infty, \text{ and } \alpha \text{ fixed.} \tag{4.49}
\]

In this case

\[
\phi = \sinh t - \alpha t, \quad \text{and} \quad \frac{d\phi}{dt} = \cosh t - \alpha. \tag{4.50}
\]

The location of the minimum of \( \phi \) crucially depends on whether \( \alpha \) is greater or less than one.

\footnote{This function is related to the Anger function

\[
A_v(x) \overset{\text{def}}{=} \int_0^\infty \exp(-vt - x \sinh t) \, dt.
\]}

42
If $\alpha < 1$ then the minimum of $\phi$ is at $t = 0$ and
\[
J(x, \alpha < 1) \sim \int_0^\infty e^{-x(1-\alpha)t} \, dt, \quad (4.51)
\]
\[
= \frac{1}{(1-\alpha)x}, \quad \text{as } x \to \infty, \text{ and } \alpha < 1 \text{ fixed.} \quad (4.52)
\]

If $\alpha > 1$, the minimum of $\phi(t)$ moves away from $t = 0$ and enters the interior of the range of integration. Let's call the location of the minimum $t_\ast(\alpha)$:
\[
\cosh t_\ast(\alpha) = \alpha, \quad \text{and therefore} \quad t_\ast = \ln \left( \alpha + \sqrt{\alpha^2 - 1} \right). \quad (4.53)
\]

If $\alpha > 1$ then $t_\ast$ is real and positive. Notice that
\[
\phi(t_\ast) = \sinh t_\ast - \alpha t_\ast = \sqrt{\alpha^2 - 1} - \alpha t_\ast(\alpha), \quad (4.54)
\]
and
\[
\phi''(t_\ast) = \sinh t_\ast = \sqrt{\alpha^2 - 1}. \quad (4.55)
\]

Then we expand $\phi(t)$ in a Taylor series round $t_\ast$:
\[
\phi(t) = \phi(t_\ast) + \frac{1}{2} (t - t_\ast)^2 \phi''(t_\ast) + O(t - t_\ast)^3. \quad (4.56)
\]

To leading order
\[
J(x, \alpha > 1) \sim e^{-x\phi_{\ast}} \int_{-\infty}^\infty e^{-x\frac{1}{2}(t-t_\ast)^2}\phi''(t_\ast) \, dt, \quad (4.57)
\]
Notice we’ve extended the range of integration to $t = -\infty$ above. The error is small, and this enables us to evaluate the integral exactly
\[
J(x, \alpha > 1) \sim e^{-\phi_{\ast}(\alpha)} \sqrt{\frac{2\pi}{x\phi''_{\ast}(\alpha)}}, \quad \text{as } x \to \infty. \quad (4.58)
\]

If we use the expressions for $t_\ast(\alpha)$ and $\phi''_{\ast}(\alpha)$ above then we obtain an impressive function of the parameter $\alpha$:
\[
J(x, \alpha > 1) \sim \sqrt{\frac{2\pi}{x\sqrt{\alpha^2 - 1}}} \exp \left( -x\sqrt{\alpha^2 - 1} \right) (\alpha + \sqrt{\alpha^2 - 1})^{\alpha x}, \quad \text{as } x \to \infty. \quad (4.59)
\]

Comparing (4.52) with (4.59), we wonder what happens if $\alpha = 1$? And how does the asymptotic expansion change continuously from the simple form in (4.52) to the complicated expression in (4.59) as $\alpha$ passes continuously through 1?

Notice that as $x \to \infty$:
\[
J(x, 1) = \int_0^\infty e^{-x(\sinh t - t)} \, dt, \quad (4.60)
\]
\[
\sim \int_0^\infty e^{-xt^3/6} \, dt, \quad (4.61)
\]
\[
= 2^{1/3} 3^{-2/3} \Gamma \left( \frac{1}{3} \right) x^{-1/3}. \quad (4.62)
\]

So, despite the impression given by (4.52) and (4.59), $J(x, 1)$ is not singular.
We’re interested in the transition where \( \alpha \) is close to 1, so we write

\[
\alpha = 1 + \epsilon \tag{4.63}
\]

where \( \epsilon \) is small. Then

\[
J(x, \alpha) \sim \int_0^\infty e^{x \epsilon t - \frac{1}{3} x^3 t^3} \, dt = x^{-1/3} \int_0^\infty e^{\xi \tau - \frac{1}{3} \tau^3} \, d\tau, \tag{4.64}
\]

where \( \xi \) is a similarity variable:

\[
\xi \overset{\text{def}}{=} (\alpha - 1) x^{2/3}. \tag{4.65}
\]

The transition from (4.52) to (4.59) occurs when \( \alpha - 1 = O(x^{-2/3}) \), and \( \xi = O(1) \). The transition is described uniformly by a special function

\[
J(\xi) \overset{\text{def}}{=} \int_0^\infty e^{\xi \tau - \frac{1}{3} \tau^3} \, d\tau. \tag{4.66}
\]

Our earlier results in (4.52), (4.59) and (4.62) are obtained as special cases by taking \( \xi \to -\infty \), \( \xi \to +\infty \) and \( \xi = 0 \) in \( J(\xi) \).

### 4.5 Problems

**Problem 4.1.** Considering \( U(x, y) \) in (4.9), show that

\[
x^2 U_{xx} + x U_x - x^2 U = U_{yy}. \tag{4.67}
\]

Evaluate \( U(x, \infty) \) in terms of modified Bessel functions.

**Problem 4.2.** Consider

\[
V(x, k, p) \overset{\text{def}}{=} \int_0^{kx-p} e^{-x \cosh t} \, dt, \quad \text{as } x \to \infty. \tag{4.68}
\]

Find a leading-order approximation to (i) \( V(x, k, 1) \); (ii) \( V(x, k, 1/2) \) and (iii) \( V(x, k, 1/4) \). Hint: In one of the three cases you’ll need to use the error function.

**Problem 4.3.** Show that

\[
\int_0^1 e^t \left( \frac{t}{1 + t^2} \right)^n \, dt \sim \frac{\pi}{2n} \frac{e^{2n}}{2^n}, \quad \text{as } n \to \infty. \tag{4.69}
\]

**Problem 4.4.** Show that

\[
\int_0^\pi t^n \sin t \, dt \sim \frac{\pi^{n+2}}{n^2}, \quad \text{as } n \to \infty. \tag{4.70}
\]

**Problem 4.5.** The beta function is

\[
B(x, y) \overset{\text{def}}{=} \int_0^1 t^{x-1} (1 - t)^{y-1} \, dt. \tag{4.71}
\]

With a change of variables show that

\[
B(x, y) = \int_0^\infty e^{-xv} (1 - e^{-v})^{y-1} \, dv. \tag{4.72}
\]
Suppose that \( y \) is fixed and \( x \to \infty \). Use Laplace’s method to obtain the leading order approximation
\[
B(x, y) \sim \frac{\Gamma(y)}{xy}.
\]
(4.73)

Go to the Digital Library of Special Functions, chapter 5 and find the relation between the beta function and the gamma function. (You can probably also find this formula in RHB, or any text on special functions.) Use this relation to show that
\[
\frac{\Gamma(x)}{\Gamma(x + y)} \sim \frac{1}{xy}, \quad \text{as } x \to \infty.
\]
(4.74)

Remark: this result can also be deduced from Stirling’s approximation, but it’s a rather messy calculation.

**Problem 4.6.** Find an asymptotic approximation of
\[
\int_{0}^{\infty} \int_{0}^{\infty} e^{-n(x^2+y^2)} \frac{dx}{(1+x+y)^n} \quad \text{as } n \to \infty.
\]
(4.75)

**Problem 4.7.** Find the \( x \to \infty \) leading-order behaviour of the integrals
\[
\begin{align*}
A(x) &= \int_{-1}^{1} e^{-xt^3} \, dt, \\
B(x) &= \int_{-1}^{1} e^{+xt^3} \, dt, \\
C(x) &= \int_{-1}^{1} e^{-xt^4} \, dt, \\
D(x) &= \int_{-1}^{1} e^{+xt^4} \, dt, \\
E(x) &= \int_{0}^{\infty} e^{-xt-t^4/4} \, dt, \\
F(x) &= \int_{-\infty}^{\infty} e^{+xt-t^4/4} \, dt, \\
G(x) &= \int_{-\infty}^{\infty} e^{-t^2/(1+t^2)x} \, dt, \\
H(x) &= \int_{-\infty}^{\infty} e^{t^2/(1+t^2)x} \, dt, \\
I(x) &= \int_{0}^{\pi/2} e^{-x \sec^2 t} \, dt, \\
J(x) &= \int_{0}^{\pi/2} e^{-x \sin^2 t} \, dt, \\
K(x) &= \int_{-1}^{1} (1-t^2) e^{-x \cosh t} \, dt, \\
L(x) &= \int_{-1}^{1} (1-t^2) e^{x \cosh t} \, dt.
\end{align*}
\]
(4.76)-(4.81)

**Problem 4.8.** Find the leading order asymptotic expansion of
\[
M(x) := \int_{0}^{\infty} e^{xt-t^t} \, dt
\]
as \( x \to \infty \) and as \( x \to -\infty \).

**Problem 4.9.** Find the first two terms in the asymptotic expansion of
\[
N(x) := \int_{0}^{\infty} t^ne^{-t^2-rac{x}{t}} \, dt
\]
as \( x \to \infty \).

**Problem 4.10.** Show that
\[
\int_{0}^{\infty} e^{-x} \left( \frac{1}{1+e^{-x}} \right)^n \, dx \sim \sqrt{2\pi} \frac{(n-1)^{n-\frac{3}{2}}}{n^{n-\frac{1}{2}}} \quad \text{as } n \to \infty.
\]
(4.84)

(I am 80% sure this is correct.)
Problem 4.11. (i) Draw a careful graph of \( \phi(t) = (1 - 2t^2)^2 \) for \(-2 \leq t \leq 2\). (ii) Use Laplace’s method to show that as \( x \to \infty \)

\[
\int_0^{1/2} \sqrt{1 + t} e^{x\phi} \, dt \sim e^x \left( \frac{1}{4} \sqrt{\frac{\pi}{x}} + \frac{p}{x} + \frac{q}{x^{3/2}} + \cdots \right),
\]

and determine the constants \( p \) and \( q \). Find asymptotic expansion as \( x \to \infty \) of

\[
(ii) \quad \int_0^1 \sqrt{1 + t} e^{x\phi} \, dt, \quad (iii) \quad \int_{-1}^1 \sqrt{1 + t} e^{x\phi} \, dt.
\]

Calculate the expansion up to and including terms of order \( x^{-3/2} e^x \).
Problem 4.12. Consider the function
\[ F(x) \equiv \int_0^\infty \exp\left(-\frac{t^3}{3} + xt\right) \, dt. \]  
(4.87)

(i) \( F(x) \) satisfies a second-order linear inhomogeneous differential equation. Find the ODE and give the initial conditions \( F(0) \) and \( F'(0) \) in terms of the \( \Gamma \)-function. (ii) Perform a local analysis of this ODE round the irregular singular point at \( x = \infty \) and say what you can about the large \( x \) behaviour of \( F(x) \). (iii) Use Laplace’s method on (4.87) to obtain the complete \( x \to \infty \) leading-order approximation to \( F(x) \). (iv) Numerically evaluate (4.87) and make a graphical comparison with Laplace’s approximation on the interval \( 0 \leq x \leq 3 \) (see figure 4.3).

% MATLAB script for Laplace’s method.
% You’ll have to supply the ??’s and code \{\tt myfun\}.
clear
xx = [0:0.05:3];
nloop = length(xx);
FF = zeros(1,nloop); % Store function values in FF
uplim = 10; %10=\infty for the upper limit of quad?
lowlim = realmin; % avoid a divide-by-zero error
for n=1:nloop
    F = quad(@(t)myfun(t,xx(n)),lowlim,uplim);
    FF(n) = F;
end
plot(xx,FF)
hold on
approx = sqrt(??)*xx.^(-??).*exp(2*xx.^(??)/3);
plot(xx,approx,'--')
hold off
xlabel('x')
ylabel('F(x)')

Figure 4.3: A comparison of \( F(x) \) computed from (4.87) using MATLAB (solid curve) with the asymptotic approximation (dashed curve).
Figure 4.4: Upper panel is the exact integrand in (4.88) (the solid curve) and the Gaussian approximation (dashed). Lower panel compares the $F(x)$ obtained by numerical quadrature (solid) with the asymptotic approximation. The comparison is not great — problem 4.13 asks you to calculate the next term in the asymptotic expansion and add that to the figure.

**Problem 4.13.** Find the first few terms in the $x \to \infty$ asymptotic expansion of

$$F(x) \overset{\text{def}}{=} \int_{0}^{1} \exp \left( -\frac{xt^2}{1+t} \right) \, dt.$$ (4.88)

Improve figure 4.4 by adding the higher-order approximations to the lower panel.

**Problem 4.14.** Find the first two terms in the $x \to \infty$ expansion of

$$Y(x) \overset{\text{def}}{=} \int_{0}^{e^x} e^{-xt^2/(1+t^2)} \, dt.$$ (4.89)

**Problem 4.15.** Show that as $x \to \infty$

$$\int_{x}^{\infty} \frac{e^{-t}}{t^x} \, dt \sim e^{-x} \left[ \frac{1}{2x} + \frac{1}{8x^2} + \text{ord} \left( x^{-3} \right) \right].$$ (4.90)
Lecture 5

Fourier Integrals and Stationary phase

5.1 Fourier Series

Recall that we can represent almost any function \( f(x) \) defined on the fundamental interval \(-\pi < x < \pi\) as a Fourier series

\[
f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots
\]  
(5.1)

(see chapter 12 of RHB). Determining the coefficients in the series above devolves to evaluating the integrals:

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx, \quad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos kx \cdot f(x) \, dx, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin kx \cdot f(x) \, dx.
\]  
(5.2)

(Notice the irritating factors of 2 in \( a_0 \) versus \( a_k \).) We’re interested in how fast these Fourier coefficients decay as \( k \to \infty \): the series is most useful if the coefficients decay rapidly.

A classic example is the discontinuous square wave function

\[
sqr(x) \overset{\text{def}}{=} \text{sgn} \left[ \sin(x) \right].
\]  
(5.4)

Applying the recipe above to \( \text{sqr}(x) \), we begin by observing that because \( \text{sqr}(x) \) is an odd function, all the \( a_k \)'s are zero. To evaluate \( b_k \) notice that the integrand of (5.3) is even so that we need only integrate from 0 to \( \pi \)

\[
b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin kx \cdot f(x) \, dx.
\]  
(5.3)

\[
(b_k = 2\pi \int_0^\pi \sin kx \, dx = - \left[ \frac{2}{\pi k} \cos kx \right]_0^\pi = \left[ 1 - (-1)^k \right] \frac{2}{\pi k}.)
\]  
(5.5)

The even \( b_k \)'s are also zero — this is clear from the anti-symmetry of the integrand above about \( x = \pi/2 \). A sensitive awareness of symmetry is often a great help in evaluating Fourier coefficients. Thus we have

\[
sqr(x) = \frac{4}{\pi} \left[ \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \frac{1}{7} \sin 7x + \cdots \right].
\]  
(5.6)

The wiggly convergence of (5.6) is illustrated in figure 5.1. (Perhaps we’ll have time to say more about the wiggles later.) The point of this square-wave example is that Fourier series is converging very slowly: the coefficients decrease only as \( k^{-1} \), and the series is certainly not absolutely convergent.
Figure 5.1: Convergence of the Fourier series of $\text{sqr}(t)$. The left panel shows the partial sum with 1, 4 and 16 terms. The right panel is an expanded view of the Gibbs oscillations round the discontinuity at $x = 0$. Notice that the overshoot near $x = 0$ does not get smaller if $n$ is increased from 16 to 256. FourSer4.eps

**Exercise:** Deduce the **Gregory-Leibniz series**

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$  \hspace{1cm} (5.7)

from (5.6).

Now let’s go to the other extreme and consider very rapidly convergent Fourier series, such as

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x.$$  \hspace{1cm} (5.8)

Another example of a rapidly convergent Fourier series is

$$\frac{1 - r^2}{1 + r^2 - 2r \cos x} = 1 + 2r \cos x + 2r^2 \cos 2x + 2r^3 \cos 3x + \cdots$$  \hspace{1cm} (5.9)

If $|r| < 1$ then the coefficients decrease as $r^k = e^{k \ln r}$, which is faster than any power of $k$. In examples like this we get a great approximation with only a few terms.

We can use IP to prove that if $f(x)$ and its first $p-1$ derivatives are continuous and differentiable in the *closed* interval $-\pi \leq x \leq \pi$, and if the $p$th derivative exists apart from jump discontinuities at some points, then the Fourier coefficients are $O(n^{-p-1})$ as $n \to \infty$. Functions such as $\text{sqr}(x)$, with jump discontinuities, correspond to $p = 0$ — the coefficients decay slowly as $n^{-1}$. Very smooth functions such as (5.8) and (5.9) correspond to $p = \infty$.

The asymptotic estimate in the previous paragraph is obtained by evaluating the Fourier coefficient

$$f_n = \int_{-\pi}^{\pi} f(x)e^{inx} \, dx,$$  \hspace{1cm} (5.10)

using integration by parts. Suppose we can break the fundamental interval up into sub-intervals so that $f(x)$ is smooth (i.e., infinitely differentiable) in each subinterval. Non-smooth behavior, such a jump in some derivative, occurs only at the ends of the interval. Then the contribution of the
sub-interval \((a, b)\) to \(f_n\) is

\[
I_n = \int_a^b f(x)e^{inx} \, dx ,
\]

\[
= \frac{1}{in} \int_a^b f(x) \frac{de^{inx}}{dx} \, dx ,
\]

\[
= \frac{1}{in} \left[ f(x)e^{inx} \right]_a^b - \frac{1}{in} \int_a^b f'(x)e^{inx} \, dx .
\]

\(\equiv J_n\) (5.11)

Since \(f(x)\) is smooth, we can apply integration by parts to \(J_n\) to obtain

\[
I_n = \frac{1}{in} \left[ f(x)e^{inx} \right]_a^b + \frac{1}{n^2} \left[ f'(x)e^{inx} \right]_a^b - \frac{1}{n^2} \int_a^b f''(x)e^{inx} \, dx .
\]

\(\equiv K_n\) (5.12)

Obviously we can keep going and develop a series in powers of \(n^{-1}\). Thus we can express \(I_n\) in terms of the values of \(f\) and its derivatives at the end-points.

It is sporting to show that we actually generate an asymptotic series with this approach. For instance, looking at (5.12), we should show that the ratio of the remainder, \(n^{-2}K_n\), to the previous term limits to zero as \(n\) increases. Assuming that \(f'\) is not zero at both end points, this requires that

\[
\lim_{n \to \infty} \frac{\int_a^b f''(x)e^{inx} \, dx}{\int_a^b f'(x)e^{inx} \, dx} = 0.
\]

(5.13)

We can bound the integral easily

\[
\left| \int_a^b f''(x)e^{inx} \, dx \right| \leq \int_a^b |f''(x)||e^{inx}| \, dx \leq \int_a^b |f''(x)| \, dx .
\]

(5.14)

But this doesn’t do the job.

Instead, we can invoke the Riemann-Lebesgue lemma\(^1\): If \(\int_a^b |F(t)| \, dt\) exists then

\[
\lim_{a \to \infty} \int_a^b e^{iat} F(t) \, dt = 0.
\]

(5.15)

Riemann-Lebesgue does not tell us how fast the integral vanishes. So, by itself, Riemann-Lebesgue is not an asymptotic estimate. But RL does assure us that the remainder in (5.12) is vanishing faster than the previous term as \(n \to 0\) i.e., dropping the remainder we obtain an \(n \to \infty\) asymptotic approximation.

An alternative to Riemann-Lebesgue is to change our perspective and think of (5.12) like this:

\[
I_n = \frac{1}{in} \left[ f(x)e^{inx} \right]_a^b + \frac{1}{n^2} \left[ f'(x)e^{inx} \right]_a^b - \frac{1}{n^2} \int_a^b f''(x)e^{inx} \, dx .
\]

(5.16)

The bound in (5.14) then shows that the new remainder is asymptotically less than the first term on the right as \(n \to \infty\). We can then continue to integrate by parts and prove asymptoticity by using the last two terms as the remainder.

\(^1\)The statement above is not the most general and useful form of RL — see section 3.4 of *Asymptotics and Special Functions* by F.W.J. Olver — particularly for cases with \(a = -\infty\) or \(b = +\infty\).
Some examples

Suppose, for example, we have a function such as those in (5.8) and (5.9). These examples are smooth throughout the fundamental interval. In this case we take $a = -\pi$ and $b = \pi$ and use the result above. Since $f(x)$ and all its derivatives have no jumps, even at $x = \pm \pi$, all the end-point terms vanish. Thus in this case $f_n$ decreases faster than any power of $n$ e.g., perhaps something like $e^{-n}$, or $e^{-\sqrt{n}}$. In this case integration-by-parts does not provide the asymptotic rate of decay of the Fourier coefficients — we must deploy a more potent method such as steepest descent.

Example: An interesting example of a Fourier series is provided by a square in the $(x,y)$-plane defined by the four vertices $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$. The square can be represented in polar coordinates as $r = R(\theta)$. In the first quadrant of the $(x,y)$-plane, the edge of the square is line $x + y = 1$, or

$$R(\theta) = \cos \theta + \sin \theta, \quad 0 \leq \theta \leq \pi/2. \quad (5.17)$$

With some huffing and puffing we could write down $R(\theta)$ in the other three quadrants. But instead we simplify matters using the obvious symmetries of the square.

Because $R(\theta) = R(-\theta)$ we only need the cosines in the Fourier series. But we also have $R(\theta) = R(\theta + \pi/2)$, and this symmetry implies that

$$R(\theta) = a_0 + a_4 \cos 4\theta + a_8 \cos 8\theta + \cdots \quad (5.18)$$

We can save some work by leaving out $\cos \theta$, $\cos 2\theta$, $\cos 3\theta$ etc because these terms reverse sign if $\theta \to \theta + \pi/2$. Thus the $a_k$’s corresponding to these harmonics will turn out to be zero.

The first term in the Fourier series is therefore

$$a_0 = \frac{1}{2\pi} \int R(\theta) \, d\theta, \quad (5.19)$$

$$= \frac{2}{\pi} \int_0^{\pi/2} d\theta \cos \theta + \sin \theta \quad (5.20)$$

We’ve used symmetry to reduce the integral from $-\pi$ to $\pi$ to four times the integral over the side in the first quadrant. The mathematica command

\text{Integrate}[1/(\sin(x) + \cos(x)), \{x, 0, \pi/2\}] \]

tells us that

$$a_0 = \frac{2\sqrt{2}}{\pi} \tanh^{-1} \left( \frac{1}{\sqrt{2}} \right) \approx 0.7935. \quad (5.21)$$
The higher terms in the series are

\[ a_{4k} = \frac{1}{\pi} \oint \cos(4k\theta) R(\theta) \, d\theta = \frac{4}{\pi} \int_0^{\pi/2} \cos(4k\theta) \frac{d\theta}{\cos \theta + \sin \theta}. \tag{5.22} \]

With mathematica, we find

\[ a_4 = \frac{4}{\pi} \left[ \frac{4}{3} - \tau \right] = 0.1106, \quad a_8 = -\frac{4}{\pi} \left[ \frac{128}{105} - \tau \right] = 0.0349, \tag{5.23} \]
\[ a_{12} = \frac{4}{\pi} \left[ \frac{4364}{3465} - \tau \right] = 0.0166, \quad a_{16} = \frac{4}{\pi} \left[ \frac{55808}{45045} - \tau \right] = 0.0096, \tag{5.24} \]

where

\[ \tau \overset{\text{def}}{=} \sqrt{2\tanh^{-1} \left( \frac{1}{\sqrt{2}} \right)} = 1.24645. \tag{5.25} \]

Figure 5.2 shows that the first three terms of the Fourier series can be used to draw a pretty good square. We might have anticipated this because the coefficients above decrease quickly. In fact, we now show that \( a_{4k} = O(k^{-2}) \) as \( k \to \infty \).

After this preamble, we consider the problem of estimating the Fourier integral

\[ S(N) = \int_0^{\pi/2} \frac{\cos N\theta}{\sin \theta + \cos \theta} \, d\theta, \tag{5.26} \]

as \( N \to \infty \). (I’ve changed notation: \( N \) is a continuously varying quantity i.e., not necessarily integers 4, 8 etc.)

The Riemann-Lebesgue (RL) lemma assures us that

\[ \lim_{N \to \infty} S(N) = 0. \tag{5.27} \]

But in asymptotics we’re not content with this — we want to know how \( S(N) \) approaches zero. Let’s try IP

\[ S(N) = \frac{1}{N} \left[ \sin N\theta \right]_0^{\pi/2} \frac{\cos N\theta}{\sin \theta + \cos \theta} \, d\theta, \tag{5.28} \]

\[ = \frac{1}{N} \left[ \sin N\theta \left( \frac{\cos \theta - \sin \theta}{(\sin \theta + \cos \theta)^2} \right) \right]_0^{\pi/2} \sin N\theta \, d\theta + \frac{1}{N} \int_0^{\pi/2} \frac{\cos \theta - \sin \theta}{(\sin \theta + \cos \theta)^2} \, d\theta. \]

We’ve invoked the Riemann-Lebesgue lemma above. Thus, provided that \( \sin(N\pi/2) \neq 0 \), the leading order term is

\[ S(N) \sim \frac{\sin(N\pi/2)}{N}. \tag{5.29} \]

If \( N \) is an even integer (and in the problem that originated this example, \( N = 4k \) is an even integer) then to find a non-zero result we have to integrate by parts again:

\[ S(N) = \frac{\sin(N\pi/2)}{N} - \frac{1}{N^2} \left[ \cos \theta - \sin \theta \right]_0^{\pi/2} \frac{\cos N\theta}{(\sin \theta + \cos \theta)^2} \, d\theta, \]
\[ = \frac{\sin(N\pi/2)}{N} - \frac{1}{N^2} \left[ \cos \theta - \sin \theta \right]_0^{\pi/2} \cos N\theta \, d\theta + \frac{1}{N^2} \int_0^{\pi/2} \left[ \cos \theta - \sin \theta \right]_0^{\pi/2} \frac{\cos N\theta}{(\sin \theta + \cos \theta)^2} \, d\theta, \]
\[ \sim \frac{\sin(N\pi/2)}{N} + \frac{1 + \cos(N\pi/2)}{N^2} + o \left( N^{-2} \right). \tag{5.30} \]

We’ve used RL to justify the \( o(N^{-2}) \). We can keep integrating by parts and develop an asymptotic series is powers of \( N^{-1} \).

With \( N = 4 \) and 8 we find from (5.30)

\[ a_4 \approx \frac{1}{2\pi} = 0.1592, \quad a_8 \approx \frac{1}{8\pi} = 0.0398, \tag{5.31} \]
\[ a_{12} \approx \frac{1}{18\pi} = 0.0177, \quad a_{16} \approx \frac{1}{32\pi} = 0.0099. \tag{5.32} \]

Comparing these asymptotic estimates with (5.24), we see that the errors are 44%, 14%, 6% and 4% respectively.
Example: partial failure of IP  Previously we evaluated the Fourier transform
\[ \pi e^{-|k|} = \int_{-\infty}^{\infty} e^{-ikx} \, dx. \]  
(5.33)

Can we find a \( k \to \infty \) asymptotic expansion using IP? Let’s try:
\[ f(k) = \int_{-\infty}^{\infty} \frac{1}{1+x^2} \frac{e^{ikx}}{ik} \, dx, \]  
(5.34)
\[ = O \left( k^{-1} \right), \quad \text{(use RL)}. \]  
(5.35)

We could IP again, but again the terms that fall outside the integral are zero. In retrospect, this can’t work — after \( n \) integrations we’ll find
\[ f(k) = O \left( k^{-n} \right). \]  
(5.36)

This is true: using the exact answer in (5.33)
\[ \lim_{k \to \infty} k^n e^{-k} = 0, \quad \text{for all } n. \]  
(5.37)

IP will never recover an exponentially small integral. I call this a partial failure, because at least integration by parts correctly tells us that the Fourier transform is smaller than any inverse power of \( k \). This is the case for any infinitely differentiable function: just keep integrating by parts.

5.2 Generalized Fourier Integrals

Generalized Fourier integrals are the imaginary analog of the Laplace integrals in (4.1). The generalized Fourier integral is
\[ J(x) = \int_{a}^{b} f(t) e^{ix\psi(t)} \, dt, \]  
(5.38)
where \( \psi(t) \) is a real phase function. As \( x \to \infty \) the integrand is very oscillatory. We previously considered the special case \( \psi(t) = t \) and obtained the asymptotic expansion with IP. Let’s try IP again:
\[ J = -\frac{i}{x} \int_{a}^{b} f(t) \frac{d}{dt} e^{ix\psi(t)} \, dt, \]  
(5.39)
\[ = -\frac{i}{x} \left[ f(t) e^{ix\psi} \right]_{a}^{b} + \frac{i}{x} \int_{a}^{b} e^{ix\psi(t)} \frac{d}{dt} \frac{f(t)}{\psi'(t)} \, dt. \]  
(5.40)

The asymptotic expansion of \( J(x) \) can be obtained via IP provided that \( f(t)/\psi'(t) \) is non-zero at either \( a \) or \( b \) and provided that \( f(t)/\psi'(t) \) is not singular for \( a \leq t \leq b \). If \( f(t)/\psi'(t) \) is singular then IP fails. Instead we need the method of stationary phase.

Example: Let’s find the leading term in the asymptotic expansion of
\[ A(x) = \int_{0}^{\pi/2} e^{-r^2} \sin x t \, dt, \quad \text{as } x \to \infty. \]  
(5.41)

This is an ordinary Fourier integral and integration by parts is the method of choice:
\[ A(x) = -\frac{1}{x} \int_{0}^{\pi/2} e^{-r^2} \frac{d}{dt} \cos x t \, dt, \]  
(5.42)
\[ = \frac{1}{x} \left( 1 - e^{-\pi^2/4} \cos \frac{x\pi}{2} \right) - \frac{2}{x} \int_{0}^{\pi/2} e^{-t^2} \cos x t \, dt. \]  
(5.43)

The RL lemma shows that the final term is \( o(x^{-1}) \). We can keep integrating by parts to generate the full asymptotic series in inverse powers of \( x \).
Example: Find the asymptotic expansion of the generalized Fourier integral
\[ B(x) = \int_0^{\pi/2} e^{-i t^2} \sin(x \cos t) \, dt, \quad \text{as } x \to \infty. \tag{5.44} \]

We set up the integral for IP
\[ B(x) = \frac{1}{x} \int_0^{\pi/2} e^{-i t^2} \frac{d}{dt} \cos(x \cos t) \, dt. \tag{5.45} \]

But then we see that in this case the integration by parts will fail because \( e^{-i t^2} / \sin t \) is singular at \( t = 0 \). This happens because the phase, \( \cos t \), has a stationary point at \( t = 0 \). We’ll use stationary phase on this integral.

Example: Find the asymptotic expansion of
\[ C(x) = \int_0^{\pi/2} t \sin(x \cos t) \, dt, \quad \text{as } x \to \infty. \tag{5.46} \]

In this case we get away with integration by parts only once:
\[ C(x) = \frac{1}{x} \int_0^{\pi/2} \frac{t}{\sin t} \, d \cos(x \cos t) \, dt, \tag{5.47} \]
\[ = \frac{1}{x} \left[ \frac{\pi}{2} - \cos x \right] - \frac{1}{x} \int_0^{\pi/2} \cos(x \cos t) \frac{\sin t - t \cos t}{\sin^2 t} \, dt, \tag{5.48} \]
\[ = \frac{1}{x} \left[ \frac{\pi}{2} - \cos x \right] + o \left( x^{-1} \right). \tag{5.49} \]

We can’t integrate by parts again — we encounter the same \( t = 0 \) singularity that stopped us in the example \( B(x) \) above.

Example: Find the asymptotic expansion of
\[ D(x) \overset{\text{def}}{=} \int_0^\infty e^{i x t^2} \, dt, \quad \text{as } x \to \infty. \tag{5.50} \]

This example is important for a complete understanding of stationary phase: we’ll obtain the full asymptotic expansion of \( D(x) \).

Write the integral as
\[ D(x) = \int_0^\infty e^{i x t^2} \, dt - \int_a^\infty e^{i x t^2} \, dt, \tag{5.51} \]
\[ = \frac{1}{\sqrt{x}} \int_0^\infty e^{i x t^2} \, dt + \frac{i}{2x} \int_a^\infty \frac{1}{t} \, dt e^{i x t^2} \, dt. \tag{5.52} \]

The first term on the right is the Fresnel integral, and the second can be integrated by parts to obtain
\[ D(x) = \frac{1}{\sqrt{x}} \int_0^{\infty} e^{i x t^2} \, dt + \frac{1}{2x} \int_a^\infty e^{i x t^2} \, dt. \tag{5.53} \]

The final integral in (5.54) is easily bounded
\[ \left| \int_a^\infty e^{i x t^2} \, dt \right| \leq \int_a^\infty \frac{dt}{t^2} = \frac{1}{a}. \tag{5.54} \]

Thus the final two terms in (5.53) are \( O(x^{-1}) \), showing that
\[ D \sim \frac{1}{\sqrt{x}} \text{e}^{i\pi x}, \quad \text{as } x \to \infty. \tag{5.55} \]

Successive integration by parts in (5.53) will generate the full asymptotic expansion of \( D(x) \). In fact, we can reduce this problem to the function
\[ \mathcal{F}(z, p) \overset{\text{def}}{=} \int_a^\infty e^{i x t^2} \, dt \tag{5.56} \]
studied in problem (5.31) making the change of variables \( v = x t^2 \), the final integral in (5.54) is
\[ \int_a^\infty e^{i x t^2} \, dt = \frac{1}{2\sqrt{x}} \int_a^\infty e^{i v} \, \frac{dv}{2v}, \tag{5.57} \]
\[ = \mathcal{F} \left( a^2 x, \frac{1}{2} \right), \tag{5.58} \]
\[ \sim \frac{i e^{i a^2 x}}{2 a \sqrt{x}} \sum_{n=0}^\infty \frac{\Gamma(n + \frac{1}{2})}{(i a^2 x)^n}. \tag{5.59} \]

55
Fresnel Integral Factoids

The Fresnel integrals are

\[ \int_0^\infty \cos a^2 t^2 \, dt = \int_0^\infty \sin a^2 t^2 \, dt = \frac{1}{2a} \sqrt{\frac{\pi}{2}}. \]

Alternatively, if \( \beta \) is a real number

\[ \int_0^\infty e^{\pm i\beta v^2} \, dv = \frac{1}{2} \sqrt{\frac{\pi}{|\beta|}} \exp \left[ \pm \frac{i\pi}{4} \text{sgn}(\beta) \right]. \]

The Fresnel integrals are special cases of

\[ \int_0^\infty e^{it^2} \, dt = \Gamma \left( 1 + \frac{1}{n} \right) e^{i\pi/2n}, \quad \text{provided } n > 1. \]

The integral above is used in the case of higher-order stationary points.
The stationary phase approximation

Let’s reconsider the integral that defeated IP:

\[ B(x) = \int_{\pi/2}^{0} e^{-t^2} \sin(x \cos t) \, dt, \quad \text{as } x \to \infty. \]  

(5.60)

The phase \( \psi = \cos t \) is stationary at \( t = 0 \) and thus we suspect that the dominant contribution to \( B(x) \) comes from the neighbourhood of \( t = 0 \). Looking at figure 5.3 and proceeding heuristically

\[ B(x) \sim \int_{0}^{\infty} \sin \left( x \left( 1 - \frac{t^2}{2} \right) \right) \, dt, \]  

(5.61)

\[ = \sin x \int_{0}^{\infty} \cos \frac{x t^2}{2} \, dt - \cos x \int_{0}^{\infty} \sin \frac{x t^2}{2} \, dt, \]  

(5.62)

\[ \sim \frac{1}{2} \sqrt{\frac{\pi}{x}} (\sin x - \cos x), \quad \text{as } x \to \infty. \]  

(5.63)

We justify the heuristic above by dividing the range of integration into two regions:

\[ B = \underbrace{\int_{0}^{\delta} e^{-t^2} \sin(x \cos t) \, dt}_{B_1} + \int_{\delta}^{\infty} e^{-t^2} \sin(x \cos t) \, dt}_{B_2}, \]  

(5.64)

where \( \delta \ll 1 \). In integral \( B_1 \), the variable \( t \) is always much less than one and therefore \( \exp(-t^2) \approx 1 \). Thus \( B_1 \) is asymptotically determined following the argument in example \( D(x) \): we obtain (5.61). In integral \( B_2 \) there are no stationary points in the range of integration and therefore with IP \( B_2 \sim x^{-1} \), which is negligible relative to \( B_1 \) as \( x \to \infty \).
Example: Find the leading term in the $x \to \infty$ asymptotic expansion of

$$E(x) \overset{\text{def}}{=} \int_0^\infty \exp \left[ ix(t - \sinh t) \right] \, dt. \quad (5.65)$$

The phase is

$$\psi = t - \sinh t = -\frac{t^3}{6} + \text{ord}(t^5). \quad (5.66)$$

This is a higher order stationary point because both $\psi'$ and $\psi''$ vanish at the stationary point. The leading term is

$$E(x) \sim \int_0^\infty \exp \left( -\frac{ix^3}{6} \right) \, dt = \left( \frac{6}{x} \right)^{1/3} \Gamma \left( \frac{4}{3} \right) e^{-ix/6}. \quad (5.67)$$

Example: Find the leading term in the $x \to \infty$ asymptotic expansion of

$$F(x,p) \overset{\text{def}}{=} \int_0^\infty t - p \cos \left( x \sqrt{\sinh t - t} \right) \, dt, \quad \text{where } 0 \leq p < 1. \quad (5.68)$$

### 5.3 The Airy function

An important example leading to asymptotic analysis of a Fourier integral is the $x \to -\infty$ asymptotic expansion of the Airy function

$$\text{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos \left( kx + \frac{k^3}{3} \right) \, dk. \quad (5.69)$$

We now use the method of stationary phase to determine the leading order asymptotic expansion of $\text{Ai}(x)$ as $x \to -\infty$.

We begin by calculating the derivative of the phase function

$$\frac{d}{dk} \left( kx + \frac{k^3}{3} \right) = x + k^2. \quad (5.70)$$

If $x < 0$ we see that there are stationary points at $k = \pm|x|^{1/2}$ (and only $k = +|x|^{1/2}$ is in the range of integration). This observation motivates the change of variable

$$t = \frac{k}{|x|^{1/2}}, \quad (5.71)$$

so that

$$\text{Ai}(x) = \frac{|x|^{1/2}}{\pi} \int_0^\infty \cos \left( X \psi(t) \right) \, dt, \quad (5.72)$$

where

$$X \overset{\text{def}}{=} |x|^{3/2}, \quad \text{and} \quad \psi(t) \overset{\text{def}}{=} t - \frac{t^3}{3}. \quad (5.73)$$

The change of variables in (5.71) puts (5.69) into the form of a generalized Fourier integral in (5.72).

When $X$ is large the phase of the cosine is changing rapidly and integrand is violently oscillatory. Successive positive and negative lobes almost cancel. The main contribution to the integral comes from the neighbourhood of the point $t = t_*$ where the oscillations are slowest. This is the point of “stationary phase”, defined by

$$\frac{d\psi}{dt} = 0, \quad \Rightarrow \quad t_* = 1. \quad (5.74)$$

We get a leading-order approximation to the integral by expanding the phase function round $t_*$:

$$\psi = \psi_* + \frac{1}{2} (t - 1)^2 \psi_*'' + \text{ord}(t - 1)^3, \quad (5.75)$$
Figure 5.4: The solid blue curve is $\text{Ai}(x)$ and the red dashed curve is the $x \to -\infty$ asymptotic approximation in (5.78).

where $\psi_* \overset{\text{def}}{=} \psi(t_*)$ and $\psi''_* \overset{\text{def}}{=} \psi''(t_*)$. Thus as $x \to -\infty$:

$$\text{Ai}(x) \sim \frac{|x|^{1/2}}{\pi} \int_{-\infty}^{\infty} \cos \left[ X \left( \frac{2}{3} - (t - 1)^2 \right) \right] \, dt, \quad (5.76)$$

$$\sim \frac{|x|^{1/2}}{\pi X^{1/2}} \Re e^{2iX/3} \int_{-\infty}^{\infty} e^{-iu^2} \, dv. \quad (5.77)$$

To ease the integration we extend the range to $-\infty$, and now invoke the Fresnel integral to obtain

$$\text{Ai}(x) \sim \frac{1}{\sqrt{\pi} |x|^{1/4}} \cos \left( \frac{2|x|^{3/2}}{3} - \frac{\pi}{4} \right), \quad \text{as } x \to -\infty. \quad (5.78)$$

This asymptotic approximation is compared with $\text{Ai}$ in Figure 5.4

### 5.4 Problems

**Problem 5.1.** (i) Use IP to obtain the leading-order asymptotic approximation for the integral

$$\int_{-\infty}^{\infty} \frac{e^{it}}{t} \, dt, \quad \text{as } x \to \infty. \quad (5.79)$$

(ii) Justify the asymptoticness of the expansion. *Hint:* see the discussion surrounding (5.16).

**Problem 5.2.** Using integration by parts to find $x \to \infty$ asymptotic approximations of the integrals

$$A(x) = \int_{1}^{2} \frac{\cos xt}{t} \, dt, \quad \text{and} \quad B(x) = \int_{0}^{1} \cos t^2 e^{ixt} \, dt. \quad (5.80)$$

**Problem 5.3.** Consider

$$f(x) = \int_{0}^{\frac{\pi}{4}} \cos(xt^2) \left( 1 - e^{-t^2} \right) \, dt, \quad \text{as } x \to \infty. \quad (5.81)$$

Show that IP can be used to compute the leading-order term, but not the second term. Compute the second term using stationary phase.
Problem 5.4. Consider the Fresnel-type function

\[ F(x, p) \overset{\text{def}}{=} \int_x^\infty \frac{e^{it}}{tp} \, dt, \quad (5.82) \]

which converges for all positive \( x \) if \( p > 0 \). Use integration by parts to show that

\[ F(x, p) = \frac{ie^{ix}}{xp} - ipF(x, p + 1). \quad (5.83) \]

Prove that the full \( x \to \infty \) asymptotic expansion of \( F(x, p) \) is

\[ F(x, p) \sim \frac{ie^{ix}}{xp} \sum_{n=0}^{\infty} \frac{\Gamma(p + n)}{\Gamma(p)(ix)^n}. \quad (5.84) \]

Be sure to explain carefully why the remainder after \( N \) terms is asymptotically less than the absolute value of the \((N+1)\)st term.

Problem 5.5. Find two terms in the \( x \to 0 \) and \( x \to \infty \) expansion of the Fresnel integrals

\[ C(x) = \int_x^\infty \cos t^2 \, dt, \quad \text{and} \quad S(x) = \int_x^\infty \sin t^2 \, dt. \quad (5.85) \]

Problem 5.6. The Bessel function of order zero is defined by

\[ J_0(x) \overset{\text{def}}{=} \frac{2}{\pi} \int_0^{\pi/2} \cos (x \cos t) \, dt. \quad (5.86) \]

(i) Show that \( J_0'' + x^{-1}J_0' + J_0 = 0 \). (ii) Plot the integrand of (5.86) as a function of \( t \) at \( x = 100 \). (iii) Show that

\[ J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\pi}{4} \right). \quad (5.87) \]

(iv) Use MATLAB (help besselj) to compare the leading order approximation with the exact Bessel function on the interval \( 0 < x < 5\pi \). The comparison is splendid: see Figure 5.5

Problem 5.7. The Bessel function of integer order \( n \) has an integral representation

\[ J_n(x) = \frac{1}{\pi} \int_0^\pi \cos (x \sin t - nt) \, dt. \quad (5.88) \]

Show that

\[ J_n(n) \sim \frac{\Gamma \left( \frac{1}{3} \right)}{\pi^{2/3} 3^{1/3} \sqrt{n}} \quad \text{as} \quad n \to \infty. \quad (5.89) \]
Problem 5.8. (i) Using the integral representation for $J_n(x)$ in the previous problem show that

$$J_n(ny) \sim \sqrt{\frac{\pi}{2n}} (y^2 - 1)^{-1/4} \cos \left[ n \left( \sqrt{y^2 - 1} - \cos^{-1} \left( \frac{y}{n} \right) - \frac{\pi}{4} \right) \right], \quad (5.90)$$

where $y > 1$ is fixed as $n \to \infty$. (ii) Now instead of fixing $y$, write $y = 1 + n^{-p}v$ with $p > 0$ and $v$ fixed. Thus as $n \to \infty$, $y \to 1$. Determine the scaling exponent $p$ that results in an interesting distinguished limit. (iii) Find a uniformly valid approximation to $J_n(ny)$ for large $n$ and $y$ close to one. You should recover (5.89) if $y = 1$ (equivalently $v = 0$).

Problem 5.9. Find a leading order $x \to \infty$ asymptotic approximation to

$$B(x) = \int_0^\pi e^{ix(t+\cos t)} \, dt. \quad (5.91)$$

Problem 5.10. Find leading-order, $x \to \infty$ asymptotic expansion of

\begin{align*}
(i) \quad & \int_0^1 \sqrt{1 + t} e^{ix(1-2t)^2} \, dt, \\
(ii) \quad & \int_1^\infty \sqrt{1 + t} e^{ix(1-2t)^2} \, dt. \quad (5.92)
\end{align*}

Problem 5.11. According to article 238 in Lamb, the surface elevation produced by a two-dimensional splash is given by

$$\eta(x,t) = \frac{1}{\pi} \int_0^\infty \cos \left( kx - \sqrt{g} k t \right) \, dk. \quad (5.93)$$

Show that as $t \to \infty$

$$\eta(x,t) \sim \frac{\sqrt{gt}}{2\pi^{3/2} \sqrt{x}} \left( \cos \frac{gt^2}{4x} + \sin \frac{gt^2}{4x} \right). \quad (5.94)$$

Verify that the frequency and wavenumber in (5.94) are connected by the water-wave dispersion relation $\omega = \sqrt{gk}$. 

61
Lecture 6

Dispersive wave equations

6.1 Group velocity

An important application of stationary phase is estimating the Fourier integrals that arise when we solve dispersive wave problems using the Fourier transform. Some first-order dispersive wave equations, and their Fourier transforms ($\partial_x \rightarrow ik$), are

\begin{align*}
iA_t + A_{xx} &= 0, & \tilde{A}_t + ik^2 \tilde{A} &= 0, \\
A_{xt} - A &= 0, & \tilde{A}_t + ik^{-1} \tilde{A} &= 0, \\
A_t - A_{xxx} &= 0, & \tilde{A} + ik^3 \tilde{A} &= 0.
\end{align*}

(6.1)

In each case the solution is

\begin{equation}
A(x,t) = \int_{-\infty}^{\infty} e^{i(kx - t\omega(k))} \tilde{A}_0(k) \frac{dk}{2\pi},
\end{equation}

(6.4)

where $\tilde{A}_0(k)$ is the Fourier transform of the initial condition and the function $\omega(k)$ is the dispersion relation. In the three cases above

\begin{equation}
\omega(k) = k^2, \quad \omega(k) = \frac{1}{k}, \quad \omega(k) = k^3.
\end{equation}

(6.5)

We’ll see later that very similar results apply to second-order wave equations.

The inverse Fourier transform can be written in the form of a generalized Fourier integral

\begin{equation}
A(x,t) = \int_{-\infty}^{\infty} e^{i\psi(k,u)} \tilde{A}_0(k) \frac{dk}{2\pi},
\end{equation}

(6.6)

where the phase function is

\begin{equation}
\psi(k,u) \overset{\text{def}}{=} uk - \omega(k), \quad \text{with} \quad u \overset{\text{def}}{=} \frac{x}{t}.
\end{equation}

(6.7)

In the limit $t \to \infty$, with $u = x/t$ fixed, the stationary phase condition that $\partial_k \psi = 0$ leads to

\begin{equation}
\frac{d\omega}{dk} \overset{\text{group velocity}}{=} u.
\end{equation}

(6.8)

To apply the method of stationary phase to the generalized Fourier integral in (6.6), we have to find all solutions of (6.8), call them $k_1(u)$, $k_2(u)$, $\cdots$ and $k_n(u)$. Usually $n$ is a modest number: $n = 1$ in the following example. Then we expand the phase functions around each $k_m(u)$ as

\begin{equation}
\psi(k,u) = \psi_m - \frac{1}{2}(k - k_m)^2 \omega_m'' + O(k - k_m)^3;
\end{equation}

(6.9)
Figure 6.1: Upper panel shows the initial condition in (6.27) with $\ell = 12$ and $p = 1$, and the solution at $t = 40$ computed with MATLAB FFT. The lower panel compares the MATLAB solution with the stationary phase approximation in (6.19).

where $m = 1, 2 \cdots n$. Also in (6.9)

$$
\psi_m \overset{\text{def}}{=} \psi(k_m), \quad \text{and} \quad \omega''_m \overset{\text{def}}{=} \partial_k^2 \omega(k_m). \tag{6.10}
$$

Thus the integral in (6.6) becomes

$$
A(x, t) \sim \sum_{m=1}^{n} e^{i[k_m x - \omega(k_m)t]} \tilde{A}_0(k_m) \int_{-\infty}^{\infty} e^{-\frac{1}{2} (k-k_m)^2 \omega''_m} \frac{dk}{2\pi}. \tag{6.11}
$$

Evaluating the Fresnel integrals we find

$$
A(x, t) \sim \sum_{m=1}^{n} \frac{\tilde{A}_0(k_m)}{2\pi t|\omega''_m|} \exp \left[ i \left( k_m x - \omega(k_m)t - \operatorname{sgn}(\omega''_m) \frac{\pi}{4} \right) \right]. \tag{6.12}
$$

Note the fiddly $|\omega''_m|$ and $\operatorname{sgn}(\omega''_m)$ in (6.12). More important, note that $k_m$ in (6.12) is a function of $x/t$ which is obtained by solving the group velocity equation (6.8).

Remark: if $\omega''_m = 0$ the formula in (6.12) is invalid. This higher-order stationary point requires separate analysis.

**Example:** Use stationary phase to approximate the $t \to \infty$ solution of

$$
A_t - A_{xxx} = 0, \quad \text{with IC} \quad A(x, 0) = \frac{e^{-x^2/2\ell^2}}{\ell \sqrt{2\pi}} \cos px. \tag{6.13}
$$

Using the Fourier transform, the initial condition is

$$
\tilde{A}_0(k) = \frac{1}{2} \left( e^{-\ell^2(k-p)^2/2} + e^{-\ell^2(k+p)^2/2} \right). \tag{6.14}
$$
and the solution is then
\[ A(x,t) = \int_{-\infty}^{\infty} e^{i(kx - k^3 t)} \tilde{A}_0(k) \frac{dk}{2\pi}, \]
(6.15)
\[ = 2 \int_0^{\infty} \cos(kx - k^3 t) \tilde{A}_0(k) \frac{dk}{2\pi}. \]
(6.16)

In (6.15), because \( \tilde{A}_0(k) = A_0(-k) \), we can restrict attention to \( k > 0 \). The stationary phase equation is
\[ 3k^2 = \frac{x}{t}, \quad \text{or} \quad k_\pm(x,t) = \pm \sqrt{\frac{x}{3t}}. \]
(6.17)

There are two stationary phase points, but only \( k_+(x,t) \) lies in the range of integration in (6.16). The expansion of the phase function \( ku - k^3 \) around \( k_+ \) is:
\[ \psi = 2k_+^3 - 3k_+(k - k_+)^2 + \text{ord}(k - k_+)^3. \]
(6.18)

The stationary-phase approximation to the Fourier integral in (6.16) is then
\[ A \sim \frac{\tilde{A}_0(k_+)}{\pi} \left\{ \cos(2k_+^3 t) \int_{-\infty}^{\infty} \cos \left[ \frac{4}{3}k^2(k - k_+)^2 t \right] dk + \sin(2k_+^3 t) \int_{-\infty}^{\infty} \sin \left[ \frac{4}{3}k^2(k - k_+)^2 t \right] dk \right\}, \]
(6.19)

Figure 6.1 compares this approximation to a numerical solution.
6.2 The 1D KG equation

Let’s use stationary phase to analyze the Klein-Gordon equation:

$$A_{tt} - a^2 A_{xx} + \sigma^2 A = 0,$$

with the initial condition such as

$$A(x, 0) = e^{-x^2/2\ell^2} \cos px, \quad \text{and} \quad A_t(x, 0) = 0.$$

We introduce non-dimensional variables

$$\bar{x} \stackrel{\text{def}}{=} \frac{\sigma x}{a}, \quad \text{and} \quad \bar{t} = \sigma t, \quad \text{and} \quad \bar{A} = \frac{a}{\sigma} A.$$

With this change in notation, the non-dimensional problem is

$$\bar{A}_{\bar{t} \bar{t}} - \bar{A}_{\bar{x} \bar{x}} + \bar{A} = 0,$$

with the initial condition

$$\bar{A}(\bar{x}, 0) = e^{-\bar{x}^2/2\eta^2} \cos \nu \bar{x}, \quad \text{and} \quad \bar{A}_t(x, 0) = 0.$$

The IC contains the non-dimensional parameters

$$\nu \stackrel{\text{def}}{=} \frac{pa}{\sigma}, \quad \text{and} \quad \eta \defeq \frac{\ell \sigma}{a}.$$

We proceed dropping all the bars decorating the non-dimensional variables.

Remark: in the limit $\eta \to 0$, the initial condition is $A(x, 0) \to \delta(x)$. By taking $\eta \to 0$ we recover the Green’s function of the Klein-Gordon equation.

The Fourier transform of $\psi$ is

$$\tilde{A}(k, t) \defeq \int_{-\infty}^{\infty} e^{-ikx} A(x, t) \, dx,$$

and with the operational rule $\partial_x \to ik$ we find the transformed equation Klein-Gordon equation

$$\tilde{A}_{tt} + (k^2 + 1) \tilde{A} = 0.$$

The solution that satisfies the transformed initial condition is

$$\tilde{A} = \cos \omega t \tilde{A}_0(k) = \frac{1}{2} \left( e^{i\omega t} + e^{-i\omega t} \right) \tilde{A}_0(k).$$

Above, the Klein-Gordon dispersion relation is

$$\omega(k) = \sqrt{k^2 + 1}.$$
From (6.28), the Fourier Integral Theorem now delivers the solution in the form

\[ A(x,t) = \int_{-\infty}^{\infty} e^{ikx} \cos(\omega t) \tilde{A}_0(k) \frac{dk}{2\pi}, \]  
\[ = \frac{1}{2} \int_{-\infty}^{\infty} e^{ikx-\omega t} \tilde{A}_0(k) \frac{dk}{2\pi} + \frac{1}{2} \int_{-\infty}^{\infty} e^{ikx+\omega t} \tilde{A}_0(k) \frac{dk}{2\pi}. \]  
\[ \overset{\text{def} = B}{=} B^\ast. \]  

(6.31)

(6.32)

To show that the two integrals on the right of (6.32) are complex conjugates of each other we use the reality condition \( \tilde{A}_0^\ast(k) = \tilde{A}_0(-k) \).

Should simplify to an integral over \( x > 0 \) by restrict attention to even functions of \( x \) so that \( \tilde{A}_0(-k) = \tilde{A}_0(k) \).

The Klein-Gordon equation is a second-order wave equation and has two modes, one with dispersion relation \( \sqrt{k^2+1} \) and the other with \( -\sqrt{k^2+1} \). One wave goes to the left and the other to the right. Our initial condition excites both waves. These are the two terms \( B(x,t) \) and \( B^\ast(x,t) \) above.

Let’s consider \( B(x,t) \), written as:

\[ B(x,t) = \frac{1}{2} \int_{-\infty}^{\infty} e^{it\psi} \tilde{A}_0(k) \frac{dk}{2\pi}, \]  
\[ \psi(k,u) \overset{\text{def}}{=} uk - \sqrt{k^2+1}. \]  

(6.33)

(6.34)

The stationary-phase condition for \( \psi \) is that

\[ u = \frac{k_\ast}{\sqrt{k_\ast^2+1}}. \]  

(6.35)

Solving (6.35) for \( k_\ast \) as a function of \( u = x/t \) we have

\[ k_\ast(u) = \frac{u}{\sqrt{1-u^2}}, \quad \text{and} \quad 1+k_\ast^2 = \frac{1}{1-u^2}, \]  

(6.36)

provided that \(-1 < u < 1\). The restriction on \( u \) is because an observer moving with speed \( |u| > 1 \) is out-running the waves — we return to this point later.

Close to the stationary point \( k_\ast(u) \), the phase \( \psi \) is therefore

\[ \psi(k) = -\sqrt{1-u^2} - \frac{1}{2} (1-u^2)^{3/2} (k-k_\ast)^2 + O(k-k_\ast)^3. \]  

(6.37)

Notice that \( \omega''(k_\ast) = (1-u^2)^{3/2} \) is positive.

The Fourier integral is therefore

\[ B \sim \frac{1}{2} e^{-it\sqrt{1-u^2}} \int_{-\infty}^{\infty} e^{-i\frac{1}{2}(1-u^2)^{3/2}(k-k_\ast)^2} \tilde{A}_0(k) \frac{dk}{2\pi}. \]  

(6.38)

Invoking the Fresnel integral, we obtain

\[ B \sim \frac{1}{2} \cos \left( \frac{t}{\sqrt{1-u^2}} + \frac{\pi}{4} \right) \tilde{A}_0 \left( \frac{u}{\sqrt{1-u^2}} \right), \]  

(6.39)
provided that $0 < u \overset{\text{def}}{=} x/t < 1$. The total solution is $A = B + B^*$, or

$$A(x,t) \sim \cos \left( \frac{t \sqrt{1 - u^2} + \frac{x}{4}}{\sqrt{2\pi t(1 - u^2)^{3/2}}} \right) \Re \tilde{A}_0 \left( \frac{u}{\sqrt{1 - u^2}} \right) .$$

(6.40)

Figure 6.2 compares the stationary phase approximation (6.40) with a numerical solution of the Klein-Gordon equation.

Should do $\nu \neq 0$ first.

**Visualization of the stationary phase solution**

Let’s visualize this asymptotic solution using the Gaussian initial condition with the Fourier transform $\tilde{A}_0$ in (??).

First consider $\eta \to 0$, so that $A(x,0) \to \delta(x)$. In this case the Klein-Gordon equation has an exact solution

$$A(x,t) = -\frac{J_1 \left( t \sqrt{1 - u^2} \right)}{2 \sqrt{1 - u^2}} + \frac{1}{2} \left[ \delta(x-t) + \delta(x+t) \right] ,$$

(6.41)

where $J_1(z)$ is the first order Bessel function. Figure ?? compares the stationary phase solution in (6.39) (with $\tilde{A}_0 = 1$) to the exact solution in (6.41). The approximation is OK provided we don’t get too close to $x = t$. The stationary phase approximations says nothing about the $\delta$-pulses that herald the arrival of the signal.

Figure ?? shows the stationary phase approximation (6.39) with $\eta \neq 0$. The initial disturbance now has finite width and therefore contains no high-wavenumbers. Thus the rapid oscillations near the from at $x = t$ have been removed. This is good: it probably means that the stationary phase approximation is accurately reproducing the waveform. But on the other hand, because of the $\delta(x \pm t)$ in (6.41), there should be terms like

$$\frac{e^{-(x\pm t)^2/2\eta^2}}{\sqrt{2\pi\eta}}$$

(6.42)

in the exact solution. The stationary phase approximation fails when $u = x/t$ is close to $\pm 1$, and therefore the approximation gives no hint of these pulses.

**Caustics: $u \approx 1$**

To analyze the solution (6.32) close to front at $x = t$, we write $x = t + x'$ where $x' \ll 1$. The stationary-phase approximation, analogous to (6.39), is therefore

$$\psi_2(k) = (t + x')k - \sqrt{1 + k^2} t ,$$

$$\approx |k|(1 + \frac{1}{2}k^2)$$

$$\approx x'k - \frac{t}{2k} , \quad \text{provided } k > 0 .$$

(6.43)

The approximation above is valid close to the stationary wavenumber, $k = \infty$. The inverse transform is therefore

$$A_2(x,t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{i(x'k - \frac{t}{2k})} \tilde{A}_0(k) \, dk ,$$

$$= \frac{1}{4\pi} \sqrt{\frac{t}{2x'}} \int_{-\infty}^{\infty} e^{i\xi(k^{-1} - \kappa)} \tilde{A}_0 \left( \sqrt{\frac{t}{2x'}} \kappa \right) \, d\kappa ,$$

(6.44)
Figure 6.2: Comparison of a numerical solution of the Klein-Gordon equation (solid black curve) with the asymptotic approximation (red dashed curve) in (6.40) at $t = 400$. In this illustration $\nu = 0$. The MATLAB code is given in the following verbatim box.

where

$$\xi \overset{\text{def}}{=} \sqrt{\frac{1}{2}tx'}.$$  

(6.45)
%% Comparison of stat phase approx with FFT solution of KG
Lx = 1e3; nx = 2e5; dx = Lx/nx;  \%grid
xx = 0:dx:Lx-dx; xx=xx-Lx/2;
k1 = 2*pi/Lx; kk = k1*[0:nx/2 (-nx/2+1):-1];  \%wavenumbers
t = 400;  \%final time
nLoop=0;
for eta =[0.5 1 2]
    nLoop= nLoop+1;
    \% IC in physical space, and in Fourier space
    A0 = ( eta*sqrt(2*pi) )^(-1) * exp(-xx.^2/(2*eta^2));
    A0h = fft(A0);
    \% solution to Klein-Gordon
    Ah = A0h .* cos( sqrt(1+kk.^2)*t );
    A = real(ifft(Ah));  \% physical space solution
    subplot(3,1,nLoop)
    plot(xx,A,'k-')
    axis([-0.2*t , 1.1*t , -1.1*max(A), 1.1*max(A)])
    hold on
    \% compare final solution to stationary phase
    uu = xx/t; sq = sqrt(1-uu.^2); ut = uu ./ sq;
    At = exp( -(eta*ut).^2 );
    Asp = cos( t*sq + pi/4 ) ./ sqrt( 2*pi*t*sq.^3 ) .* At;
    subplot(3,1,nLoop), hold on
    plot(xx,real(Asp),'r--')
    xlabel('$x$','interpreter','latex','fontsize',16)
    ylabel('$A(x,t)$','interpreter','latex','fontsize',16)
    ss = ['$\eta =$',num2str(eta), '$$'];
    text(-50,0.75*max(A),ss,'interpreter','latex','fontsize',12)
end
6.3 Problems

Problem 6.1. Use a Fourier transform to solve
\[ Q_t = \frac{i}{2} Q_{xx} - \frac{1}{3} Q_{xxx}, \quad \text{with IC } Q(x,0) = \delta(x). \] (6.46)

Use the method of stationary phase to estimate
\[ \lim_{t \to \infty} Q(ut,t) \]
with (a) \( u = 0 \); (b) \( u = 1/4 \); (c) all values of \( u \).

Problem 6.2. Here are some one-dimensional dispersive wave equations
\[ A_t - A_{xxx} = 0, \quad F_{xt} - F = 0, \quad B_t + B_{xxx} = 0, \quad G_{xt} + G = 0. \] (6.48)

Figure 6.3 shows the solution of one of these equations with the initial condition
\[ \frac{e^{-x^2/200}}{\sqrt{2\pi} 10} \cos x. \] (6.49)

(i) Which equation has been solved to produce the figure?  (ii) Construct the solution of the equation from part (i) using the Fourier Integral Theorem.  (iii) The width of the wave packet increases like \( t^\alpha \), while its amplitude decreases like \( t^{-\beta} \). Determine the exponents \( \alpha \) and \( \beta \).
Lecture 7

Constant-phase (a.k.a. steepest-descent) contours

The method of steepest descents, and its big brother the saddle-point method, is concerned with complex integrals of the form

$$I(\lambda) = \int_{C} h(z)e^{\lambda f(z)} \, dz,$$

where $C$ is some contour in the complex ($z = x + iy$) plane. The basic idea is to convert $I(\lambda)$ to

$$I(\lambda) = \int_{D} h(z)e^{\lambda f(z)} \, dz,$$

where $D$ is a contour on which $\Re f$ is constant. Thus if

$$f = \phi + i\psi,$$

then on the contour $D$

$$I(\lambda) = e^{\lambda i\psi_D} \int_{D} h(z)e^{\lambda \phi} \, dt.$$

Above $\psi_D$ is the constant imaginary part of $f(z)$. We refer $\psi$ as the phase function and $D$ is therefore a constant-phase contour.

Contour $D$ is orthogonal to $\nabla \psi$ and, because

$$\nabla \phi \cdot \nabla \psi = 0,$$

$D$ is also tangent to $\nabla \phi$. Thus as one moves along $D$ one is always ascending or descending along the steepest direction of the surface formed by $\phi(x, y)$ above the $(x, y)$-plane. The main advantage to integrating along $D$ is that the integral will be dominated by the neighbourhood of the point on $D$ where $\phi$ is largest.

Because $\phi$ is changing most rapidly as one moves along $D$, $D$ is also a contour of steepest descent (or of steepest ascent).

Exercise: Prove (7.5) and the surrounding statements.

The method and its advantages are best explained via well chosen examples. In fact we’ve already used the method to calculate $\text{Ai}(0)$ back in section 1.5 — you should review that example.
7.1 Asymptotic evaluation of an integral using a constant-phase contour

An example of the constant phase method is provided by

\[ B(\lambda) \overset{\text{def}}{=} \int_{0}^{1} e^{i\lambda x^2} \, dx. \]  

(7.6)

If \( \lambda \gg 1 \) we quickly obtain a leading order approximation to \( B \) using stationary phase. We improve on this leading-order approximation by considering

\[ f(z) = iz^2 = -2xy + i(x^2 - y^2). \]  

(7.7)

The end-points of (7.6) define two constant-phase contours — see Figure 7.1. The end-point at \((x, y) = (0, 0)\) has \( \psi = 0 \), and the other end-point at \((x, y) = (1, 0)\) has \( \psi = 1 \). The constant-phase contours through these endpoints are

\[ D_0 : z = re^{i\pi/4}, \quad \text{and} \quad D_1 : z = \sqrt{1 + y^2} + iy. \]  

(7.8)

(See the figure.) There are no singularities in the region enclosed by \( C, D_0, D_1 \) and \( \mathcal{E} \). Thus, as \( \mathcal{E} \) recedes to infinity, we have

\[ B(\lambda) = \int_{D_0} e^{i\lambda z^2} \, dz + \int_{D_1} e^{i\lambda z^2} \, dz. \]  

(7.9)

The first integral along \( D_0 \) in (7.9) is the standard Fresnel integral. In the second integral along \( D_1 \), the exponential function is

\[ e^{i\lambda z^2} = e^{i\lambda} e^{-2\lambda y \sqrt{1+y^2}}. \]  

(7.10)

This verifies that on \( D_1 \) the integrand decreases monotonically away from the maximum at \( z = 1 \). On \( D_1 \) we strive towards a Laplace integral in (7.9) by making the change of variable

\[ iz^2 = i - v, \quad \text{or} \quad z = \sqrt{1 + iv}. \]  

(7.11)

Thus

\[ \int_{D_1} e^{i\lambda z^2} \, dz = -\frac{i}{2} e^{i\lambda} \int_{0}^{\infty} \frac{e^{-\lambda v}}{\sqrt{1 + iv}} \, dv. \]  

(7.12)

The minus sign is because we integrate along \( D_1 \) starting at \( v = \infty \).

Assembling the results above we integrate along \( D_1 \) starting at \( v = \infty \).

Watson’s lemma delivers the full asymptotic expansion of the final integral in (7.13).

There is an alternative derivation that using the contour \( \mathcal{F} \) in Figure ??, \( \mathcal{F} \) is tangent to \( D_1 \) at \( z = 1 \), so that

\[ \int_{D_1} e^{i\lambda z^2} \, dz \sim \int_{\mathcal{F}} e^{i\lambda z^2} \, dz \quad \text{as} \quad \lambda \to \infty. \]  

(7.14)

Now in the neighbourhood of \( z = 1 \):

\[ z = 1 + iy \]  

(7.15)

Thus

\[ \int_{\mathcal{F}} e^{i\lambda z^2} \, dz \sim -e^{i\lambda} \int_{0}^{\infty} e^{-2\lambda y} e^{i\lambda y^2} \, dy. \]  

(7.16)
7.2 Problems

Problem 7.1. Show that if $p > 1$ and $0 < a$:

$$\int_0^\infty e^{\pm i a v^p} dv = \frac{1}{a} \Gamma \left( 1 + \frac{1}{p} \right) e^{\pm i \pi/2p}. \quad (7.17)$$

The case $p = 2$ is the Fresnel integral and $p = 3$ is the example in (1.42).

Problem 7.2. Consider the integral

$$P(a,b) \overset{\text{def}}{=} \int_0^\infty e^{-ax^2} \cos bx \, dx = \frac{1}{2} \int_C e^{-ax^2+ibx} \, dx, \quad (7.18)$$

where $C$ is the real axis, $-\infty < x < \infty$. Complete the square in the exponential and show that the line $y = b/2a$ is a contour $\mathcal{D}$ on which the integrand is not oscillatory. Evaluate $P$ exactly by integration along $\mathcal{D}$.

Problem 7.3. Use a constant phase contour to asymptotically evaluate

$$Q(\lambda) \overset{\text{def}}{=} \int_0^\pi e^{i \lambda x} \ln x \, dx \quad (7.19)$$

as $\lambda \to \infty$. 

Figure 7.1: The contours $\mathcal{D}_0$ and $\mathcal{D}_1$. On $\mathcal{D}_0$, $x = y$ and the phase is constant.
Lecture 8

The saddle-point method

The saddle-point method is a combination of the method of steepest descents and Laplace’s method in the complex plane.

In the previous lecture we considered integrals along constant-phase contours $D$. In those earlier examples the maximum of $\phi$ on $D$ was at the endpoints of the contour. But it is possible that the maximum of $\phi$ is in the middle of the contour. If $s$ is the arclength along the contour then

$$\frac{d\phi}{ds} = \nabla \phi \cdot \hat{\tau}_D$$

where $\hat{\tau}_D$ is the unit tangent to $D$. In fact, from the Cauchy-Riemann equations,

$$\hat{\tau}_D = \frac{\nabla \phi}{|\nabla \phi|},$$

so that

$$\frac{d\phi}{ds} = |\nabla \phi|.$$  \hspace{1cm} (8.2)

Thus if there is an interior maximum on $D$ at that point $z_*$ then

$$\left. \frac{d\phi}{ds} \right|_{z_*} = 0,$$

which requires

$$|\nabla \phi| = 0 \text{ at } z_*.$$  \hspace{1cm} (8.3)

However from the Cauchy-Riemann equations

$$|\nabla \phi| = |\nabla \psi| = \left| \frac{df}{dz} \right|.$$  \hspace{1cm} (8.4)

So at the point $z_*$, $|\nabla \phi|$, $|\nabla \psi|$ and $|df/dz|$ are all zero. In short, we can locate the saddle points by solving

$$\frac{df}{dz} = 0.$$  \hspace{1cm} (8.5)

8.1 The Airy function as $x \to \infty$

Let’s recall our Fourier transform representation of the Airy function:

$$\text{Ai}(x) = \int_{-\infty}^{\infty} e^{i(kx+k^3/3)} \frac{dk}{2\pi}$$

In an earlier lecture we used stationary phase to investigate the $x \to -\infty$ behaviour of $\text{Ai}$. Now consider what happens if $x \to \infty$. In this case the stationary point is in the complex $k$-plane, at
$k = \pm \sqrt{x_1}$. We’ll use this to illustrate the saddle-point method. But first we make the change of variable

$$ k = \sqrt{x}\kappa $$

(8.7)

so that

$$ \text{Ai}(x) = \frac{\sqrt{x}}{2\pi} \int_C e^{xf(\kappa)} d\kappa, $$

(8.8)

where $C$ is the real axis and

$$ X \overset{\text{def}}{=} x^{3/2}, \quad \text{and} \quad f(\kappa) \overset{\text{def}}{=} i \left( \kappa + \frac{\kappa^3}{3} \right). $$

(8.9)

If $\kappa = p + iq$ then in this problem we can write $f(\kappa)$ explicitly without too much work:

$$ f(\kappa) = -q - p^2q + \frac{q^3}{3} + i \left( p + \frac{p^3}{3} - pq^2 \right). $$

(8.10)

Contours of constant $\psi$ are shown in Figure 8.1.

The saddle points at $\kappa = \kappa_*$ are located by

$$ \frac{df}{dk} = 0, \quad \text{or} \quad \kappa_* = \pm i. $$

(8.11)

Notice that

$$ f(i) = -\frac{2}{3}, $$

(8.12)

and thus the constant-phase contours passing through $\kappa = i$ are determined by $\Im f = 0$, implying that

$$ p = 0, \quad \text{and} \quad \frac{p^3}{3} - q^2 + 1 = 0. $$

(8.13)
Our attention is drawn to the saddle at \( +i \), and we deform \( C \) onto the contour \( D_+ \) passing through \( \kappa = +i \). We can parameterize the integral on \( D_+ \) via
\[
p = \sqrt{3} \sinh t, \quad \text{and} \quad q = \cosh t.
\] (8.14)

Notice then that
\[
\phi = \frac{2}{3} (3 \cosh t - 4 \cosh^3 t) = -\frac{2}{3} \cosh 3t,
\] (8.15)
and
\[
\frac{d\kappa}{dt} = \sqrt{3} \cosh t + i \sinh t.
\] (8.16)

Thus we have a new integral representation for the Airy function
\[
\text{Ai}(x) = \frac{\sqrt{3x}}{2\pi} \int_{-\infty}^{\infty} \cosh t e^{-\frac{2}{3} t \cosh 3t} dt.
\] (8.17)

Laplace’s method applied to this integral quickly gives the leading-order result
\[
\text{Ai}(x) \sim e^{-\frac{2}{3} x^{3/2}} \frac{1}{2\sqrt{\pi} x^{1/4}}.
\] (8.18)

Notice that if we had to numerically evaluate \( \text{Ai}(x) \) for positive \( x \) the representation in (8.14) would be very useful.

We were distracted by the exciting parameterization (8.14). A better way to obtain the asymptotic approximation of \( \text{Ai}(x) \) is to use a path that is tangent to the true steepest-descent path at \( \kappa = i \) i.e., the path
\[
\kappa = i + p, \quad \text{on which} \quad f = -\frac{2}{3} - p^2 + i \frac{p^3}{3}.
\] (8.19)

This is not a path of constant phase (notice the \( ip^3/3 \)) but that doesn’t matter. We then have
\[
\text{Ai}(x) \sim \frac{\sqrt{x}}{2\pi} e^{\frac{2}{3} X} \int_{-\infty}^{\infty} e^{-Xp^2} e^{Xip^3} dp.
\] (8.20)

Watson’s lemma now delivers the full asymptotic expansion of the Airy function:
\[
\text{Ai}(x) \sim \frac{e^{-\frac{2x^{3/2}}{3}}}{2\sqrt{\pi} x^{1/4}} \sum_{n=0}^{\infty} \frac{\Gamma(3n + \frac{1}{2})}{(2n)!} \left( -\frac{1}{9X} \right)^n, \quad \text{as} \quad x \to \infty.
\] (8.21)

### 8.2 The Laplace transform of a rapidly oscillatory function

As an another example, consider the large-\( s \) behaviour of the Laplace transform
\[
\mathcal{L} \left[ \sin \frac{1}{t} \right] = \int_0^\infty e^{-st} \sin \left( \frac{1}{t} \right) dt.
\] (8.22)

Although this is a real integral, we make an excursion into the complex plane by writing
\[
\mathcal{L} \left[ \sin \frac{1}{t} \right] = \Im \int_0^\infty e^{-st + it} dt.
\] (8.23)

Our previous experience with Laplace’s method suggests considering the function
\[
\phi \overset{\text{def}}{=} \frac{1}{t} - st,
\] (8.24)

real(φ) and imag(φ) = √s

Figure 8.2: The green curves show ℜ(iz⁻¹ − z), and the black curve is ℑ(iz⁻¹ − z) = ±√2s. The saddle point S is at z = e⁻ⁱπ/₄, and the curve of steepest descent is OSR.

in the complex t-plane. Our attention focusses on saddle points, which are located by
\[ \frac{dφ}{dt} = 0, \quad ⇒ \quad t_* = ±\frac{e^{-iπ/4}}{√s}. \quad (8.25) \]

Studying \( t_* \), we’re motivated to introduce a complex variable \( z = x + iy \) defined by
\[ z = √st \quad \text{and therefore} \quad \phi = √s \left( \frac{i}{z} - z \right). \quad (8.26) \]

In terms of \( z \), the Laplace transform in (8.23) is therefore
\[ \mathcal{L} \left[ \sin \frac{1}{t} \right] = ℑ\left( √s \int_0^∞ e^{√sφ} \, dz \right). \quad (8.27) \]

Back in (8.26), the \( √s \) in the definition of \( z \) ensures that the saddle points are at \( z = ω \) and \( z = -ω \), where
\[ ω = e^{-iπ/4} = √{-i} = \frac{1 - i}{√2}. \quad (8.28) \]

That is, as \( s → ∞ \) the saddle points don’t move in the \( z \)-plane. ℜ\( \dot{φ} \) is shown in the \( z \)-plane as green contours in Figure 8.2. The black curves in Figure 8.2 are ℑ\( \dot{φ} = ±√2 \) and the saddle point at \( z = ω \) is indicated by S. The curve of steepest descent, which goes through S, is OSR.

Invoking Cauchy’s theorem on the closed curve OURSO:
\[ \frac{1}{√s} \int_{OURSO} e^{√s\dot{φ}} \, dz = 0, \quad (8.29) \]
and therefore, taking the points R and U to infinity,
\[ \mathcal{L} \left[ \sin \frac{1}{t} \right] = ℑ\left( √s \int_{OSR} e^{√s\dot{φ}} \, dz \right). \quad (8.30) \]

When \( s ≫ 1 \) we can evaluate the integral on the right of (8.30) using
\[ \dot{φ}(z) = -2ω + ω^3(z - ω)^2 + O(z - ω)^3. \quad (8.31) \]
To go over the saddle point on OSR

\[ z - \omega = \omega^{1/2} \zeta \quad \text{so that} \quad \omega^3 (z - \omega)^2 = -\zeta^2. \] 

(8.32)

Thus

\[ \mathcal{L} \left[ \sin \frac{1}{t} \right] \sim \Im \left( \frac{\sqrt{\pi}}{s^{3/4}} \omega^{1/2} e^{-2\omega \sqrt{s}} \right) = \frac{\sqrt{\pi}}{s^{3/4}} e^{-\sqrt{2s}} \cos \left( \sqrt{2s} + \frac{\pi}{8} \right). \] 

(8.33)

Must check!

The saddle point method requires careful scrutiny of the real and imaginary parts of analytic functions. It is comforting to contour these functions with MATLAB. The script that generates Figure 8.2 is in the box, and it can be adapted to other examples we encounter.
Note the use of \texttt{1i} for $\sqrt{-1}$ in MATLAB, and the use of \texttt{meshgrid} to set up the complex matrix $z = xx + 1i*yy$. To get a good plot you have to supply some analytic information to \texttt{contour} e.g., we make MATLAB plot the saddle point contours $\Im \phi = \pm \sqrt{2s}$ in the second call to \texttt{contour}. And you can’t trust MATLAB to pick sensible contour levels because the pole at $z = 0$ means that there are some very large values in the domain — so the vector $V$ determines the contour levels using the convenient scale $\sqrt{2s}$. Since the real and imaginary parts of the analytic function $\phi$ intersect at right angles (recall $\nabla \phi_r \cdot \nabla \phi_i = 0$), we preserve angles with the command \texttt{axis equal}. 

```matlab
%% Saddle point plot
%% phi = 1i/t - s t with t = z/sqrt(s)
%% phi = sqrt(2s)(1i/z - z)
close all
clc
clear
s = 10; a=sqrt(2*s);
x = linspace(-1,8,200); y = linspace(-2,1,200);
[xx,yy] = meshgrid(x,y);
z = xx + 1i*yy;
phi = sqrt(s)*( 1i./z - z);
phImag = imag(phi);
phReal = real(phi);
% contour levels for real(phi):
V=[-6 :0.4: 2]*a;
subplot(2,1,1)
contour(xx,yy,phReal, V,'g' )
hold on
% plot the saddle point contour
contour( xx , yy , phImag, [-a a],'k' )
axis equal %best for analytic functions
hold on
xlabel('$x$','interpreter','latex')
ylabel('$y$','interpreter','latex')
axis([min(x) max(x) min(y) max(y) ] )
% plot the x-axis:
plot([min(x) max(x)],[0 0],’k’,'linewidth',1.0)
% plot the y-axis:
plot([0 0] , [min(y) max(y)],’k’,'linewidth’,1.0)
title('real($\phi$) and imag($\phi$)=\sqrt{2 s}','interpreter','latex')
%label the steepest descent contour
text(-0.05 , 0 , ’0’)
b = 1/sqrt(2);
text(1*b, -1.2*b , ’S’)
text(8.1 , -1.3 , ’R’)
```
8.3 Inversion of a Laplace transform

We can further illustrate the saddle-point method by considering the problem of inverting a Laplace transforms using the Bromwich method. Recall again that the Laplace transform of a function \( f(t) \) is

\[
\tilde{f}(s) \overset{\text{def}}{=} \int_0^\infty f(t)e^{-st} \, dt.
\]  

(8.34)

We refuse to Laplace transform functions with nonintegrable singularities, such as \( f(t) = (t - 1)^{-2} \) and \( 1/\ln t \).

The inversion theorem says that

\[
f(t) = \frac{1}{2\pi i} \int_B e^{st} \tilde{f}(s) \, ds,
\]

(8.35)

where the “Bromwich contour” \( B \) is a straight line parallel to the \( s \)-axis and to the right of all singularities of \( \tilde{f}(s) \). The theorem requires that \( f(t) \) is absolutely integrable over all finite integrals i.e., that

\[
\int_0^T |f(t)| \, dt < \infty, \quad \text{for } 0 \leq T < \infty.
\]

(8.36)

Absolute integrability over the infinite interval is not required: \( e^{-st} \) takes care of that.

Now consider the Laplace transform

\[
\tilde{f}(x,s) = \frac{e^{x\sqrt{s}}}{s}.
\]

(8.37)

According to the inversion theorem

\[
f(x,t) = \frac{1}{2\pi i} \int_B e^{st-x\sqrt{s}} \frac{ds}{s},
\]

(8.38)

The Bromwich contour must be in the RHP, to the right of the pole at \( s = 0 \).

Suppose we don’t have enough initiative to look up this inverse transform in a book, or to evaluate it exactly by deformation of \( B \) to a branch-line integral (see problems). Suppose further that we don’t notice that \( f(x,t) \) is a similarity solution of the highly regarded diffusion problem

\[
\frac{f_t}{f_{xx}} = 1, \quad f(0,t) = 1, \quad f(x,0) = 0.
\]

(8.39)

We’re dim-witted, and thus we try to obtain the \( t \to \infty \) asymptotic expansion of \( f(x,t) \).
Lecture 9

Evaluating integrals by matching

9.1 Singularity subtraction

Considering

\[ F(\epsilon) = \int_0^\pi \frac{\cos x}{\sqrt{x^2 + \epsilon^2}} \, dx , \quad \text{as } \epsilon \to 0, \quad (9.1) \]

we cannot set \( \epsilon = 0 \) because the resulting integral is logarithmically divergent at \( x = 0 \). An easy way to make sense of this limit is to write

\[ F(\epsilon) = -\int_0^\pi \frac{1 - \cos x}{x} \, dx + \int_0^\pi \frac{dx}{\sqrt{x^2 + \epsilon^2}} , \quad (9.2) \]

\[ \sim -\int_0^\pi \frac{1 - \cos x}{x} \, dx + \ln \left( \pi + \sqrt{\pi^2 + \epsilon^2} \right) - \ln \epsilon , \quad (9.3) \]

\[ \sim \ln \frac{1}{\epsilon} - \int_0^\pi \frac{1 - \cos x}{x} \, dx + \ln 2\pi , \quad (9.4) \]

with errors probably \( \text{ord}(\epsilon) \). This worked nicely because we could exactly evaluate the elementary integral above. This method is called singularity subtraction — to evaluate a complicated nearly-singular integral one finds an elementary integral with the same nearly-singular structure and subtracts the elementary integral from the complicated integral. To apply this method one needs a repertoire of elementary nearly singular integrals.

Exercise: generalize the example above to

\[ F(\epsilon) = \int_0^a \frac{f(x)}{\sqrt{x^2 + \epsilon^2}} \, dx . \quad (9.5) \]

Example: find the small \( x \) behaviour of the exponential integral

\[ E(x) = \int_x^\infty \frac{e^{-t}}{t} \, dt . \quad (9.6) \]

Notice that

\[ \frac{dE}{dx} = -\frac{e^{-x}}{x} = -\frac{1}{x} + 1 - \frac{x}{2} + \cdots \quad (9.7) \]

If we integrate this series we have

\[ E(x) = -\ln x + C + x - \frac{x^2}{4} + \text{ord}(x^3) . \quad (9.8) \]

The problem has devolved to determining the constant of integration \( C \). We do this by subtracting the singularity. We use an elementary nearly-singular-as \( x \to 0 \) integral:

\[ \ln x = -\int_x^1 \frac{dt}{t} . \quad (9.9) \]
We use this elementary integral to subtract the logarithmic singularity from (9.6):

\[ E(x) + \ln x = - \int_x^1 \frac{1 - e^{-t}}{t} \, dt + \int_t^\infty \frac{e^{-t}}{t} \, dt. \] (9.10)

Now we take the limit \( x \to 0 \) and encounter only convergent integrals:

\[ C = \lim_{x \to 0} [E(x) + \ln x] , \]
\[ = - \int_0^1 \frac{1 - e^{-t}}{t} \, dt + \int_t^\infty \frac{e^{-t}}{t} \, dt , \] (9.12)
\[ = -\gamma_E. \] (9.13)

Above, we’ve used the result from problem 1.7 to recognize Euler’s constant \( \gamma_E \approx 0.57721 \). To summarize, as \( x \to 0 \)

\[ E(x) \sim -\ln x - \gamma_E + x - \frac{x^2}{4} + \text{ord}(x^3). \] (9.14)

### 9.2 Local and global contributions

Consider

\[ A(\epsilon) \overset{\text{def}}{=} \int_0^1 \frac{e^\epsilon \, dx}{\sqrt{\epsilon + x}}. \] (9.15)

The integrand is shown in Figure 9.1. How does the function \( A(\epsilon) \) behave as \( \epsilon \to 0 \)? The leading order behaviour is perfectly pleasant:

\[ A(0) = \int_0^1 \frac{e^\epsilon \, dx}{\sqrt{x}}. \] (9.16)

This integral is well behaved and we can just evaluate it, for example as

\[ A(0) = \int_0^1 \frac{e^\epsilon \, dx}{\sqrt{x}} \approx 2 + 2 + 1 + \frac{2}{21}, \]
\[ = 2.91429. \] (9.17)

Alternatively, with the Mathematica command \texttt{NIntegrate}, we find \( A(0) = 2.9253. \)

To get the first dependence of \( A \) on \( \epsilon \), we try taking the derivative:

\[ \frac{dA}{d\epsilon} = -\frac{1}{2} \int_0^1 \frac{e^\epsilon \, dx}{(\epsilon + x)^{3/2}}. \] (9.18)

But now setting \( \epsilon = 0 \) we encounter a divergent integral. We’ve just learnt that the function \( A(\epsilon) \) is not differentiable at \( \epsilon = 0 \). Why is this?

Referring to Figure 9.1 we can argue that the peak contribution to the integral in (9.15) is

\[ \text{peak width, ord}(\epsilon) \times \text{peak height, ord}(\epsilon^{-1/2}) = \text{ord}(\epsilon^{1/2}). \] (9.19)

Therefore the total integral is

\[ A(\epsilon) = \text{an ord}(1) \text{ global contribution} \]
\[ + \text{an ord}(\epsilon^{1/2}) \text{ contribution from the peak} \]
\[ + \text{higher-order terms} — \text{probably a series in } \sqrt{\epsilon}. \] (9.20)

The \( \text{ord}(\epsilon^{1/2}) \) is not differentiable at \( \epsilon = 0 \) — this is why the integral on the right of (9.18) is divergent. This argument suggests that

\[ A(\epsilon) = 2.9253 + c \sqrt{\epsilon} + \text{higher-order terms}. \] (9.21)

How can we obtain the constant \( c \) above?
Figure 9.1: The integrand in (9.15) with $\epsilon = 0.01$. There is a peak with height $\epsilon^{-1/2} \gg 1$ and width $\epsilon \ll 1$ at $x = 0$. The peak area scales as $\epsilon^{1/2}$, while the outer region makes an $O(1)$ contribution to the integral.

**Method 1: subtraction**

We have

$$A(\epsilon) - A(0) = \int_0^1 e^x \left( \frac{1}{\sqrt{\epsilon + x}} - \frac{1}{\sqrt{x}} \right) \, dx,$$

$$\approx \int_0^\infty \frac{1}{\sqrt{\epsilon + x}} - \frac{1}{\sqrt{x}} \, dx,$$

$$= \sqrt{\epsilon} \int_0^\infty \frac{1}{\sqrt{1 + t}} - \frac{1}{\sqrt{t}} \, dt,$$

$$= -2\sqrt{\epsilon}.$$

**Exercise:** Explain the transition from (9.22) to (9.23).

Although this worked very nicely, it is difficult to get further terms in the series with subtraction.

**Method 2: range splitting and asymptotic matching**

We split the range at $x = \delta$, where

$$\epsilon \ll \delta \ll 1,$$

and write the integral as

$$A(\epsilon) \overset{\text{def}}{=} \int_0^\delta \frac{e^x \, dx}{\sqrt{\epsilon + x}} + \int_\delta^1 \frac{e^x \, dx}{\sqrt{\epsilon + x}}.$$

We can simplify $A_1(\epsilon, \delta)$ and $A_2(\epsilon, \delta)$ and add the results together to recover $A(\epsilon)$. Of course, the artificial parameter $\delta$ must disappear from the final answer. This cancellation provides a good check on the consistency of our argument and the correctness of algebra.

To simplify $A_1$ note

$$A_1 = \int_0^\delta \frac{1 + x + \text{ord}(x^2)}{\sqrt{\epsilon + x}} \, dx.$$

This is a splendid approximation because $x$ is small everywhere in the range of integration.
integrals are elementary, and we obtain

\[ A_1 = 2\sqrt{\epsilon + \delta} - 2\sqrt{\epsilon} + \frac{1}{3} \epsilon^{3/2} + \frac{2}{3} \delta \sqrt{\epsilon} + \epsilon - \frac{1}{3} \epsilon \sqrt{\delta} + \epsilon + \text{ord}(\delta^{5/2}) , \quad (9.29) \]

\[ = 2\sqrt{\delta} + \frac{\epsilon}{\sqrt{\delta}} - 2\sqrt{\epsilon} + \frac{1}{3} \epsilon^{3/2} + \frac{2}{3} \delta^{3/2} + \text{ord}\left(\delta^{5/2}, \frac{\epsilon^2}{\delta^{3/2}}, \delta^{1/2}\epsilon\right) . \quad (9.30) \]

To be consistent about which terms are discarded we gear by saying that \( \delta = \text{ord}(\epsilon^{1/2}) \). Then all the terms in ord garbage heap in \( (9.30) \) are of order \( \epsilon^{5/4} \).

To simplify \( A_2 \) we use the approximation

\[ A_2 = \int_{\delta}^{1} e^x \left( \frac{1}{\sqrt{x}} - \frac{\epsilon}{2x^{3/2}} + \text{ord}(\epsilon^2 x^{-5/2}) \right) \mathrm{d}x . \quad (9.31) \]

This approximation is good because \( x \geq \delta \gg \epsilon \) everywhere in the range of integration. Now we can evaluate some elementary integrals:

\[ A_2 = \int_0^1 \frac{e^x}{\sqrt{x}} \mathrm{d}x - \int_0^{\delta} \frac{e^x}{\sqrt{x}} \mathrm{d}x + \epsilon \int_{\delta}^{1} e^x \frac{\mathrm{d}x}{x^{1/2}} + \text{ord}\left(\frac{\epsilon^2}{\delta^{3/2}}\right) , \quad (9.32) \]

\[ = A(0) - \int_0^{\delta} x^{-1/2} + x^{1/2} \mathrm{d}x + \epsilon \int_0^{1} x^{-1/2} e^{x} \mathrm{d}x + \text{ord}\left(\frac{\epsilon^2}{\delta^{3/2}}, \delta^{5/2}\right) , \quad (9.33) \]

\[ = A(0) - 2\sqrt{\delta} - \frac{2}{3} \delta^{3/2} + \epsilon e - \frac{\epsilon}{\sqrt{\delta}} - \epsilon A(0) + \text{ord}\left(\frac{\epsilon^2}{\delta^{3/2}}, \delta^{5/2}, \epsilon^{1/2}\right) . \quad (9.34) \]

The proof of the pudding is when we sum \( (9.30) \) and \( (9.34) \) and three terms containing the arbitrary parameter \( \delta \), namely

\[ 2\sqrt{\delta}, \quad \frac{2}{3} \delta^{3/2}, \quad \text{and} \quad \frac{\epsilon}{\sqrt{\delta}} , \quad (9.35) \]

all cancel. We are left with

\[ A(\epsilon) = A(0) - 2\epsilon^{1/2} + [e - A(0)] \epsilon + \frac{1}{3} \epsilon^{3/2} + \text{ord}(\epsilon^2) . \quad (9.36) \]

The terms of order \( \epsilon^0 \) and \( \epsilon^1 \) come from the outer region, while the terms of order \( \epsilon^{1/2} \) and \( \epsilon^{3/2} \) came from the inner region (the peak).

**Another example of matching**

Let us complete problem 1.5 by finding a few terms in the \( t \to 0 \) asymptotic expansion of

\[ \dot{x}(t) = \int_0^\infty \frac{ve^{-vt}}{1 + v^2} \mathrm{d}v . \quad (9.37) \]

If we simply set \( t = 0 \) then the integral diverges logarithmically. We suspect \( \dot{x} \sim \ln t \). Let’s calculate \( \dot{x}(t) \) at small \( t \) precisely by splitting the range at \( v = a \), where

\[ 1 \ll a \ll \frac{1}{t} . \quad (9.38) \]

For instance, we could take \( a = \text{ord}(t^{-1/2}) \). Then we have

\[ \dot{x} = \int_0^a \frac{v - v^2 t + \cdots}{1 + v^2} \mathrm{d}v + \int_a^\infty e^{-vt} \left( \frac{1}{v} - \frac{1}{v^3} + \cdots \right) \mathrm{d}v \quad (9.39) \]
Now we have a variety of integrals that can be evaluated by elementary means, and by recognizing
the exponential integral:
\[ \dot{x} \sim \frac{1}{2} \ln(1 + a^2) - at + t \tan^{-1} a + E(at) + \cdots \] (9.40)
\[ \sim \ln a - at + \frac{\pi t}{2} - \ln(at) - \gamma + \text{ord} \left( \frac{a^2 t^2, t, a^{-2}}{at} \right), \] (9.41)
\[ = \ln \frac{1}{t} - \gamma + \frac{\pi t}{2} + \text{ord}(t^2). \] (9.42)

9.3 An electrostatic problem — H section 3.5

Here is a crash course in the electrostatics of conductors:

\[ \nabla \cdot e = \rho, \quad \text{and} \quad \nabla \times e = 0. \] (9.43)

Above \( e(x) \) is the electric field at point \( x \) and \( \rho(x) \) is the density of charges (electrons per cubic meter). Both equations can be satisfied at once by introducing the electrostatic potential \( \phi \):

\[ e = -\nabla \phi, \quad \text{and therefore} \quad \nabla^2 \phi = -\rho. \] (9.44)

To obtain the electrostatic potential \( \phi \) we must solve Poisson’s equation above.

This is accomplished using the Green’s function

\[ \nabla^2 g = -\delta(x), \quad \Rightarrow \quad g = \frac{1}{4\pi r}, \] (9.45)

where \( r \overset{\text{def}}{=} |x| \) is the distance from the singularity (the point charge). Hence if there are no boundaries

\[ \phi(x) = \frac{1}{4\pi} \int \frac{\rho(x')}{|x - x'|} \, dx'. \] (9.46)

So far, so good: in free space, given \( \rho(x) \), we must evaluate the three dimensional integral above. The charged rod at the end of this section is a non-trivial example.

If there are boundaries then we need to worry about about boundary conditions e.g., on the surface of a charged conductor (think of a silver spoon) the potential is constant, else charges would flow along the surface. In terms of the electric field \( e \), the boundary condition on the surface of a conducting body \( B \) is that

\[ e \cdot t_B = 0, \quad \text{and} \quad e \cdot n_B = \sigma \] (9.47)

where \( t_B \) is any tangent to the surface of \( B \), \( n_B \) is the unit normal, pointing out of \( B \), and \( \sigma \) is the charge density (electrons per square meter) sitting on the surface of \( B \).

Example: a sphere. The simplest example is sphere of radius \( a \) carrying a total charge \( q \), with surface charge density

\[ \sigma = \frac{q}{4\pi a^2}. \] (9.48)

Outside the sphere \( \rho = 0 \) and the potential is

\[ \phi = \frac{q}{4\pi r}, \quad \text{so that} \quad e = \frac{qr}{4\pi r^2}. \] (9.49)

where \( r \) is a unit vector pointing in the radial direction (i.e., our notation is \( x = r r \)). The solution above is the same as if all the charge is moved to the center of the sphere.
For a non-spherical conducting body \( B \) things aren’t so simple. We must solve \( \nabla^2 \phi = 0 \) outside the body with \( \phi = \phi_B \) on the surface \( B \) of the body, where \( \phi_B \) is an unknown constant. (We are considering an isolated body sitting in free space so that \( \phi(x) \to 0 \) as \( r \to \infty \).)

We don’t know the surface charge density \( \sigma(x) \), but only the total charge \( q \), which is the surface integral of \( \sigma(x) \):

\[
q = \int_B \sigma \, dS = \int_B e \cdot n_B \, dS.
\]  

(9.50)

This is a linear problem, so the solution \( \phi(x) \) will be proportional to the total charge \( q \). We define the capacity \( C_B \) of the body as

\[
q = C_B \phi_B.
\]  

(9.51)

The capacity is an important electrical property of \( B \).

**Example** The electrostatic energy is defined via the volume integral

\[
E \overset{\text{def}}{=} \frac{1}{2} \int \abs{\epsilon}^2 dV,
\]

(9.52)

where the integral is over the region outside of \( B \). Show that

\[
E = \frac{1}{2} C_B \phi_B.
\]  

(9.53)

If you have a sign error, consider that the outward normal to body, \( n_B \), is the inward normal to free space.

**Example: a charged rod.** Find the potential due to a line distribution of charge with density \( \eta \) (electrons per meter) along \( -a < z < a \).

In this example the charge density is

\[
\rho = \eta \frac{\delta(s) \chi(z)}{2\pi s},
\]

(9.54)

where \( s = \sqrt{x^2 + y^2} \) is the cylindrical radius. The signature function \( \chi(z) \) is one if \( -a < z < +a \), and zero otherwise.

We now evaluate the integral in (9.46) using cylindrical coordinates, \((\theta, s, z)\) i.e., \( dV = d\theta ds dz \). The \( s \) and \( \theta \) integrals are trivial, and the potential is therefore

\[
\phi(s, z) = \frac{\eta}{4\pi} \int_{-a}^{a} \frac{d\xi}{\sqrt{(z - \xi)^2 + s^2}},
\]

(9.55)

\[
= \frac{\eta}{4\pi} \int_{-(a+z)/s}^{+(a-z)/s} \frac{dt}{\sqrt{1 + t^2}},
\]

(9.56)

\[
= \frac{\eta}{4\pi} \ln \left( t + \sqrt{1 + t^2} \right) \bigg|_{-(a+z)/s}^{+(a-z)/s},
\]

(9.57)

\[
= \frac{\eta}{4\pi} \ln \left( \frac{r_+ - z + a}{r_- - z - a} \right),
\]

(9.58)

where

\[
r_\pm \equiv \sqrt{s^2 + (a \mp z)^2}.
\]  

(9.59)

\( r_\pm \) is the distance between \( x \) and the end of the rod at \( z = \pm a \).

Using

\[
z = \frac{r_+^2 - r_-^2}{4a},
\]

(9.60)

the expression in (9.58) can alternatively be written as

\[
\phi = \frac{\eta}{4\pi} \ln \left( \frac{r_+ + r_- + 2a}{r_+ + r_- - 2a} \right).
\]  

(9.61)

If you dutifully perform this algebra you’ll be rewarded by some remarkable cancellations. The expression in (9.61) shows that the equipotential surfaces are confocal ellipsoids — the foci are at \( z = \pm a \). The solution is shown in Figure 9.2.
A slender body

In section 5.3, \( H \) considers an axisymmetric body \( B \) defined in cylindrical coordinates by

\[
\sqrt{x^2 + y^2} = \epsilon B(z) . \tag{9.62}
\]

(I’m using different notation from \( H \): above \( s \) is the cylindrical radius.) The integral equation in \( H \) is then

\[
1 = \frac{1}{4\pi} \int_{-1}^{1} \frac{f(\xi; \epsilon) \, d\xi}{\sqrt{(z - \xi)^2 + \epsilon^2 B(z)^2}} . \tag{9.63}
\]

I think it is easiest to attack this integral equation by first asymptotically estimating the integral

\[
\phi(s, z) = \frac{1}{4\pi} \int_{-1}^{1} \frac{f(\xi; \epsilon) \, d\xi}{\sqrt{(z - \xi)^2 + s^2}} , \quad \text{as } s \to 0. \tag{9.64}
\]

Notice that we can’t simply set \( s = 0 \) in (9.64) because the “simplified” integral,

\[
\frac{1}{4\pi} \int_{-1}^{1} \frac{f(\xi; \epsilon) \, d\xi}{|z - \xi|} ,
\]

is divergent. Instead, using the example below, we can show that

\[
\phi(s, z) = \frac{f(z)}{2\pi} \ln \left( \frac{2\sqrt{1 - z^2}}{s} \right) + \frac{1}{4\pi} \int_{-1}^{1} \frac{f(\xi) - f(z)}{|\xi - z|} \, d\xi + O(s) . \tag{9.65}
\]

Thus the integral equation (9.63) is approximated by

\[
1 \approx \frac{f(z; \epsilon)}{2\pi} \ln \left( \frac{2\sqrt{1 - z^2}}{\epsilon B(z)} \right) + \frac{1}{4\pi} \int_{-1}^{1} \frac{f(\xi; \epsilon) - f(z; \epsilon)}{|\xi - z|} \, d\xi . \tag{9.66}
\]
As $\epsilon \to 0$ there is a dominant balance between the left hand side and the first term on the right, leading to

$$f(z; \epsilon) \approx \frac{2\pi}{\ln \left( \frac{2\sqrt{1-z^2}}{\epsilon B(z)} \right)},$$  \hspace{1cm} (9.67)

where $L \overset{\text{def}}{=} \ln \frac{1}{\epsilon} \gg 1$. Thus expanding the denominator in (9.68) we have

$$f(z; \epsilon) \approx \frac{2\pi}{L} + \frac{2\pi}{L^2} \ln \left( \frac{B(z)}{2\sqrt{1-z^2}} \right) + \text{ord} \left( L^{-3} \right).$$  \hspace{1cm} (9.69)

This is the solution given by $H$. For many purposes we might as well stop at (9.67), which provides the sum to infinite order in $L^{-n}$. However if we need a nice explicit result for the capacity,

$$C(\epsilon) = \int_{-1}^{1} f(z; \epsilon) \, dz,$$  \hspace{1cm} (9.70)

then the series in (9.69) is our best hope.

**Example:** obtain the approximation (9.66). This is a good example of singularity subtraction. We subtract the nearly singular part from (9.64):

$$\phi(s, z) = \frac{1}{4\pi} \int_{-1}^{1} \frac{f(\xi) - f(z)}{\sqrt{(z - \xi)^2 + s^2}} \, d\xi + \frac{f(z)}{4\pi} \int_{-1}^{1} \frac{d\xi}{\sqrt{(z - \xi)^2 + s^2}},$$  \hspace{1cm} (9.71)

In the first integral on the right of (9.71) we can set $s = 0$ without creating a divergent integral: this move produces the final term in (9.66), with the denominator $|z - \xi|$. The final term in (9.71) is the potential of a uniform line density of charges on the segment $-1 < z < 1$ i.e., the potential of a charged rod back in (9.58) (but now with $a = 1$). We don’t need (9.58) in its full glory — we’re taking $s \to 0$ with $-1 < z < 1$. In this limit (9.61) simplifies to

$$\frac{1}{4\pi} \int_{-1}^{1} \frac{d\xi}{\sqrt{(z - \xi)^2 + s^2}} \approx \frac{1}{2\pi} \ln \left( \frac{2\sqrt{1-z^2}}{s} \right).$$  \hspace{1cm} (9.72)

Thus we have

$$\phi(s, z) = \frac{1}{4\pi} \int_{-1}^{1} \frac{f(\xi) - f(z)}{|z - \xi|} \, d\xi + \frac{f(z)}{2\pi} \ln \left( \frac{2\sqrt{1-z^2}}{s} \right).$$  \hspace{1cm} (9.73)

### 9.4 Problems

**Problem 9.1.** Find the leading-order behavior of

$$H(\epsilon) = \int_{0}^{\pi} \frac{\cos x}{x^2 + \epsilon^2} \, dx,$$  \hspace{1cm} as $\epsilon \to 0$.  \hspace{1cm} (9.74)

**Problem 9.2.** Consider the integral

$$I(x) \equiv \int_{x}^{\infty} \frac{W(z)}{z} \, dz,$$  \hspace{1cm} (9.75)

where $W(x)$ is a smooth function that decays at $x = \infty$ and has a Taylor series expansion about $x = 0$:

$$W(x) = W_0 + x W'_0 + \frac{1}{2} x^2 W''_0 + \cdots$$  \hspace{1cm} (9.76)
Some examples are: $W(z) = \exp(-z)$, $W(z) = \sech(z)$, $W(z) = (1 + z^2)^{-1}$ etc. (i) Show that the integral in (4) has an expansion about $x = 0$ of the form

$$I(x) = W_0 \ln \left( \frac{1}{x} \right) + C - W_0' x - \frac{1}{4} W_0'' x^2 + O(x^3),$$  

where the constant $C$ is

$$C = \int_0^1 \left[ W(z) + W(1/z) - W_0 \right] \frac{dx}{z}. \hspace{1cm} (9.77)$$

(ii) Evaluate $C$ if $W(z) = (1 + z^2)^{-1}$. (iii) Evaluate the integral exactly with $W(z) = (1 + z^2)^{-1}$, and show that the expansion of the exact solution agrees with the formula above.

**Problem 9.3.** Find useful approximations to

$$F(x) \overset{\text{def}}{=} \int_0^\infty \frac{du}{\sqrt{x^2 + u^2 + u^4}} \hspace{1cm} (9.79)$$

as (i) $x \to 0$; (ii) $x \to \infty$.

**Problem 9.4.** Find the first two terms in the $\epsilon \to 0$ asymptotic expansion of

$$F(\epsilon) \overset{\text{def}}{=} \int_0^\infty \frac{dy}{(1 + y)^{1/2}(\epsilon^2 + y)}. \hspace{1cm} (9.80)$$

**Problem 9.5.** Consider

$$H(r) \overset{\text{def}}{=} \int_0^\infty \frac{x \, dx}{(r^2 + x)^{3/2}(1 + x)}. \hspace{1cm} (9.81)$$

(i) First, with $r \to 0$, find the first two non-zero terms in the expansion of $H$. (ii) With $r \to \infty$, find the first two non-zero terms, counting constants and $\ln r$ as the same order.

**Problem 9.6.** Find two terms in the expansion the elliptic integral

$$K(m) \overset{\text{def}}{=} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m^2 \sin^2 \theta}}, \hspace{1cm} (9.82)$$

as $m \uparrow 1$.

**Problem 9.7.** Find three terms (counting $\epsilon^n$ and $\epsilon^n \ln \epsilon$ as different orders) in the expansion of the elliptic integral

$$J(m) \overset{\text{def}}{=} \int_0^{\pi/2} \sqrt{1 - m \cos^2 \theta} \, d\theta. \hspace{1cm} (9.83)$$

as $m \uparrow 1$.

**Problem 9.8.** This is H exercise 3.8. Consider the integral equation

$$x = \int_{-1}^1 \frac{f(t; \epsilon) \, dt}{\epsilon^2 + (t - x)^2}, \hspace{1cm} (9.84)$$

posed in the interval $-1 \leq x \leq 1$. Assuming that $f(x; \epsilon)$ is $O(\epsilon)$ in the end regions where $1 - |t| = \text{ord}(\epsilon)$, obtain the first two terms in an asymptotic expansion of $f(x; \epsilon)$ as $\epsilon \to 0$.

**Problem 9.9.** Show that as $\epsilon \to 0$:

$$\int_0^1 \frac{\ln x}{\epsilon + x} \, dx = -\frac{1}{2} \ln^2 \left( \frac{1}{\epsilon} \right) - \frac{\pi^2}{6} + \epsilon \left( \frac{1}{4} - \frac{\epsilon^2}{9} - \frac{\epsilon^3}{16} + \cdots \right). \hspace{1cm} (9.85)$$