SIO203C/MAE294C, Spring 2019, Final

3:00pm to 6:00pm

Problem 1

Solve $\zeta_{tt} = \zeta_{xx} + \mathrm{e}^x$ with initial conditions $\zeta(x,0) = \zeta_t(x,0) = 0$.

Problem 2

(i) Find two real solutions of Laplace's equation in the (x,y)-plane by fiddling around with the complex function $1/z$, where $z = x + \mathrm{i}y$. *(ii)* Find a solution, $U(x,y)$, of Laplace's equation that is non-singular in the upper half-plane $y > 0$, with the boundary condition

$$U(x,0) = \frac{1}{a^2 + x^2}\,. \tag{1}$$

Problem 3

Consider an age-stratified population, with histogram $h(a,t)$ satisfying

$$h_t + h_a = -(1+t)h\,. \tag{2}$$

The initial condition on the half-line $a > 0$ is $h(a,0) = N\mathrm{e}^{-a}$ and the birth rate $h(0,t)$ is adjusted so that the population,

$$N \stackrel{\text{def}}{=} \int_0^\infty h(a,t)\,\mathrm{d}a\,, \tag{3}$$

is constant in time. Solve the PDE and exhibit $h(a,t)$. Check your answer by showing that $h(a,a) = N\exp\left(-a - \tfrac{1}{2}a^2\right)$.

Problem 4

The evolution of $\eta(x,t)$ is governed almost everywhere by the conservation law

$$\eta_t + \left(\tfrac{1}{2}\eta^2\right)_x = 0\,. \tag{4}$$

The shock-tracking condition is used to prevent multi-valuedness of η, and ensure integral conservation. The initial condition is

$$\eta(x,0) = \begin{cases} 1\,, & 0 < x < 1\,, \\ 0\,, & \text{otherwise.} \end{cases} \tag{5}$$

Find $\eta(x,t)$ and sketch snapshots of η as a function of x at $t = 1$ and at $t = 3$.

Problem 5

Note:

$$\text{sech}x = \frac{1}{\cosh x}$$

Consider the wave equation

$$\cosh x \, U_{tt} - \left(\text{sech}x \, U_x\right)_x = 0 \,. \tag{6}$$

(i) Show that (6) has an energy conservation law,

$$E_t + J_x = 0 \,, \tag{7}$$

and find expressions for the energy density E and flux J in terms of U_t, U_x, $\cosh x$ etc. *(ii)* Find the general solution of (6) in terms of two arbitrary functions. *(iii)* Solve (6) with the initial condition

$$U(x,0) = \text{sech}x \,, \qquad U_t(x,0) = 0 \,. \tag{8}$$

To check your answer, show that $U(0,t) = 1/\sqrt{1+t^2}$.

Problem 6

Consider heat conduction in a 2D, uniformly heated sheet of metal. The temperature at the boundary of the sheet is fixed at $T = 0$. Poisson's equation for the steady-state temperature distribution, $T(x, y)$, is

$$0 = \kappa \left(T_{xx} + T_{yy}\right) + h \,,$$

where the constant $h > 0$ is the uniform heating. Define the "average temperature" at a point $\boldsymbol{x}$ as

$$\bar{T}(\boldsymbol{x}, r) \stackrel{\text{def}}{=} \oint T(\boldsymbol{x} + \boldsymbol{r}) \, \frac{\mathrm{d}\theta}{2\pi} \,,$$

where $\boldsymbol{r} \stackrel{\text{def}}{=} r(\cos\theta, \sin\theta)$. Show that provided the circle $\boldsymbol{x} + \boldsymbol{r}$ lies within the sheet

$$\bar{T}(\boldsymbol{x}, r) = T(\boldsymbol{x}) - \frac{hr^2}{4\kappa} \,.$$