SIO203C/MAE2904C: PDE Notes A

W.R. Young [1]

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[1]Scripps Institution of Oceanography, University of California at San Diego, La Jolla, CA 92093–0230, USA.

Contents

Lecture 1

Mainly first-order partial differential equations

1.1 The simplest partial differential equation

Here is the very simplest example of a first-order partial differential equation (PDE)

$$h_x = 0\,, \tag{1.1}$$

Notation: $h_x = \dfrac{\partial h}{\partial x}$

where $h(x, y)$ is a function of two variables x and y. For example, h might be topographic height of a landscape above the (x, y)-plane. It is fundamental that the solution of (1.1) is a two-dimensional *solution surface* living in a three-dimensional space. The solution of the PDE (1.1) is that

$$h(x, y) = a(y)\,. \tag{1.2}$$

where a is an *arbitrary* (even discontinuous) function of y. With all due ceremony, consider the example

$$a(y) = \begin{cases} 1\,, & \text{if } y \text{ is rational;} \\ 0\,, & \text{if } y \text{ is irrational.} \end{cases} \tag{1.3}$$

This solution of (1.2) is discontinuous everywhere, but differentiable along the lines of constant y, which is all that (1.1) requires.

This reasoning might remind you that the general solution of an ODE involves arbitrary constants, which are determined by initial or boundary conditions. The general solution of a PDE involves *arbitrary functions*, which are determined from boundary or initial conditions.

The lines of constant y are called *characteristic curves*, or simply *characteristics* of the PDE (1.1), and a solution such as (1.2) that contains an arbitrary function of the characteristic variable is called a *general solution* of the PDE. It is remarkable that the solution on a particular characteristic curve is independent of the solution on other characteristics — even infinitesimally close characteristics.

Although the PDE (1.1) is trivial, it can be used to make several general points arising in the determination of $a(y)$ from boundary information.

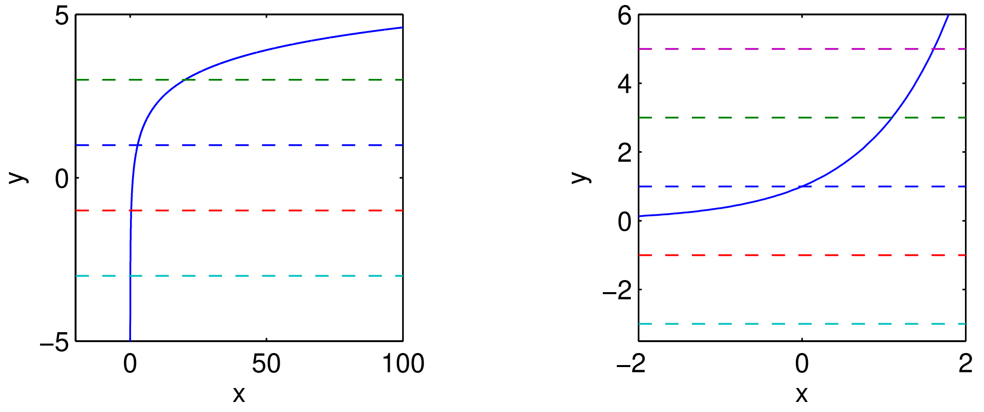


Figure 1.1: Left panel: the solid curve is the data curve $y = \ln x$ (with $x > 0$). The characteristics are the dashed lines $y =$ constant. Right panel: the solid curve is the data curve $y = \exp(x)$ (with $-\infty < x < \infty$). The characteristics are the dashed lines $y =$ constant. `simpfig1.eps`

Example: Solve the PDE $h_x = 0$ subject to the boundary condition that on the curve $y = \ln x$ (with $x > 0$), $h = \sin x$.

$h(x, \ln x) = \sin x$

The left panel of figure 1.1 shows the *data curve* $y = \ln x$ on which $h(x, y)$ is specified as being equal to $\sin x$. The dashed straight lines are the *characteristics*, on which $h(x, y)$ is constant. Each characteristic intersects the data curve only once, and at this intersection the arbitrary function $a(y)$ is determined. To determine the arbitrary function $a(y)$, on the data curve we have $\sin x = a(\ln x)$, or $a = \sin\exp$ and therefore $h(x, y) = \sin\exp y$. Notice that the solution is defined everywhere in the (x, y)-plane by the PDE and specification of h on the data curve.

Example: Solve the PDE $h_x = 0$ subject to the boundary condition that on the curve $y = \mathrm{e}^x$ (with $-\infty < x < \infty$), $h = \sin x$.

$h(x, \mathrm{e}^x) = \sin x$

In the right panel of Figure 1.1 we show the data curve $y = \exp(x)$ and the dashed straight lines which are the characteristics of the PDE $h_x = 0$. In this case, only the characteristics in the upper half plane intersect the data curve. Consequently the data determines the arbitrary function $a(y)$ only in the upper half plane. In the lower half plane the function $a(y)$ is still arbitrary: the characteristics in the lower half plane do not find the data curve. Thus in this example we have incomplete determination of $h(x, y)$.

Focussing then on the upper-half plane, at the intersection of the characteristics with the data curve we have $\sin x = a(\mathrm{e}^x)$, so in the region $y > 0$ we have $h(x, y) = \sin\ln y$. We say that the *domain of definition* of the solution is the upper half plane $y > 0$.

Example: Solve PDE $h_x = 0$ subject to the boundary condition that on $y = 0$, $h = \sin x$.

$h(x, 0) = \sin x$

In this case there is no solution: the PDE says that h is constant on every line of constant y, and the boundary condition contradicts this on the particular line $y = 0$. If the boundary condition is changed to $h = \pi$ on $y = 0$ then there is no longer an inconsistency — in this case $h(x, 0) = \pi$. But $h(y)$ remains undetermined for $y \neq 0$.

Let's dwell on the concept of an arbitrary function by doing an inverse problem. Suppose we have an axisymmetric topography, such as volcano. Axisymmetry means that the height above the (x, y)-plane is

$$h(x, y) = a\big(\sqrt{x^2 + y^2}\big)\,, \tag{1.4}$$

where a is an arbitrary function. Does h in (1.4) satisfy a first-order PDE? We calculate the first derivatives

$$h_x = \frac{x}{\sqrt{x^2+y^2}}\, a'\,, \qquad \text{and} \qquad h_y = \frac{y}{\sqrt{x^2+y^2}}\, a'\,. \tag{1.5}$$

We then quickly see that

$$y h_x - x h_y = 0\,. \tag{1.6}$$

Exercise: Suppose $m(x,y) = a(y/x)$ where a is an arbitrary function. Find the first-order PDE satisfied by m. Repeat this exercise for $n(x,y) = a(y^2 - x^2)$ and $p(x,y) = a(xy)$ and $q(x,y) = a(y - x^2)$.

1.2 The constant coefficient transport equation

The transport equation for the unknown function $h(x,y)$ is

$$u h_x + v h_y = 0\,, \qquad \text{where } u \text{ and } v \text{ are constants.} \tag{1.7}$$

The PDE in (1.1) is the special case $u = 1$ and $v = 0$. The transport equation is a simple PDE that occurs frequently in applications. By inspection, the general solution of the transport equation (1.7) is

$$h(x,y) = a(vx - uy)\,, \tag{1.8}$$

where a is an arbitrary function. The characteristics are straight lines $vx - uy =$ constant.

We can write the transport equation as

$$\boldsymbol{v} \cdot \boldsymbol{\nabla} h = 0\,, \tag{1.9}$$

where $\boldsymbol{v} \stackrel{\text{def}}{=} (u,v)$. In this form we interpret (1.7) geometrically as saying that the directional derivative of $h(x,y)$ along the vector $\boldsymbol{v}$ is zero. We could say that h is transported along the vector $\boldsymbol{v}$. Then the variation of h must be in the direction orthogonal to $\boldsymbol{v}$ i.e.,

$$h(x,y) = a(\boldsymbol{p} \cdot \boldsymbol{x})\,, \tag{1.10}$$

where $\boldsymbol{p}$ is any vector orthogonal to $\boldsymbol{v}$: In writing the general solution (1.8) we happened to use $\boldsymbol{p} = \boldsymbol{v} \times \hat{\boldsymbol{z}} = (v, -u)$. It is educational to verify by substitution that the PDE is satisfied. $\boldsymbol{p} \cdot \boldsymbol{v} = 0$

This observation is the key to solving the constant-coefficient transport equation in higher dimensions e.g., in three dimensions we can interpret (1.9) as saying that the directional derivative of $h(x,y,z)$ along the vector $\boldsymbol{v} \stackrel{\text{def}}{=} (u,v,w)$ is zero. If $\boldsymbol{v}$ is a constant vector then the general solution of the three-dimensional transport equation is

$\boldsymbol{\nabla} a(\boldsymbol{p} \cdot \boldsymbol{x}) = \boldsymbol{p}\, a'(\boldsymbol{p} \cdot \boldsymbol{x})$

$$h = a(\boldsymbol{p} \cdot \boldsymbol{x}, \boldsymbol{q} \cdot \boldsymbol{x})\,, \tag{1.11}$$

where $\boldsymbol{p}$ and $\boldsymbol{q}$ are any two linearly independent vectors orthogonal to $\boldsymbol{v}$ and a is an arbitrary function with two arguments. It is educational to verify by substitution that the PDE is satisfied.

Exercise: find the general solution of $h_x + h_y + h_z = 0$.

Exercise: find the general solution of the system

$$u_1\psi_x + v_1\psi_y + w_1\psi_z = 0\,, \qquad \text{and} \qquad u_2\psi_x + v_2\psi_y + w_2\psi_z = 0\,. \tag{1.12}$$

The coefficients, u_1, v_2 etc. are all constant and $\psi(x, y, z)$ satisfies *both* of the PDEs above.

Linear superposition and undetermined coefficient

Many techniques for solving ODEs also work on PDEs. For example, consider the transport equation with a non-zero right hand side:

$$h_x + h_y = x\,. \tag{1.13}$$

This is a linear problem so one way to solve it is to spot a particular solution and then add a homogenous solution. In this case finding a particular solution is easy and the general solution is therefore

$$h = \tfrac{1}{2}x^2 + a(x - y)\,, \tag{1.14}$$

or equivalently

$$h = xy - \tfrac{1}{2}y^2 + b(x - y)\,. \tag{1.15}$$

Above a and b are related arbitrary functions.

Exercise: Find the relation between $a(s)$ and $b(s)$.

Example: Find a general solution of

$$h_x + 2h_y = xy\,. \tag{1.16}$$

We know that the homogeneous solution is $a(2x - y)$. But it is not entirely easy to guess a particular solution. Trying undetermined coefficients

$$\text{particular solution} \stackrel{?}{=} \alpha x^3 + \beta x^2 y\,, \tag{1.17}$$

we quickly find that the general solution is

$$h = \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 y + a(2x - y)\,. \tag{1.18}$$

A systematic approach (no guessing) is provided by the *method of characteristics*.

Easy transport equations with non-constant coefficients

A more general form of the transport equation is obtained by considering that u and v back in (1.7) are functions of x and y. In frequently occurring cases the transport equation with non-constant coefficients can be transformed into a constant-coefficient transport equation.

Example: Find a general solution of

$$xh_x - yh_y = 0\,. \tag{1.19}$$

If we change variables to $\xi = \ln x$ and $\eta = \ln y$ then the equation is transformed to the transport equation

$\xi = \ln x \;\Rightarrow\; \frac{\partial}{\partial \xi} = x\frac{\partial}{\partial x}$

$$h_\xi - h_\eta = 0\,. \tag{1.20}$$

The general solution is that

$$h = a(\xi + \eta) = a\left(\ln(xy)\right) = b(xy)\,, \tag{1.21}$$

where a and b are arbitrary functions.

Example: Find a general solution of

$\eta = \frac{1}{2}y^2 \;\Rightarrow \partial_\eta = y^{-1}\partial_y$.

$$x h_x + y^{-1} h_y + h_z = 0\,. \tag{1.22}$$

If we change variables to $\xi = \ln x$ and $\eta = \frac{1}{2}y^2$ then the equation is transformed to the constant-coefficient transport equation

$$h_\xi + h_\eta + h_z = 0\,. \tag{1.23}$$

To use the solution in (1.11), we need to find two vectors perpendicular to $(1, 1, 1)$. For example, $(1, 0, -1)$ and $(0, 1, -1)$. Then general solution is that

$$h(x, y, z) = a(\xi - z, \eta - z) = a\big(\ln x - z, \tfrac{1}{2}y^2 - z\big)\,, \tag{1.24}$$

where a is an arbitrary function with two arguments. An alternative way to write the general solution is

$$h(x, y, z) = b\big(x\mathrm{e}^{-z}, \tfrac{1}{2}y^2 - z\big)\,, \tag{1.25}$$

where b is another arbitrary function. As always, it is educational to verify these solutions by substitution.

But if the coefficients are sufficiently complicated we won't be able to reduce the general case to constant coefficients. For example, consider

$$h_x + (x + y)^2 h_y = 0\,. \tag{1.26}$$

I don't see anyway to reduce this to a constant-coefficient transport equation. This is a job for the *method of characteristics*.

1.3 Some second-order partial differential equations

Arbitrary functions are also involved in the solution of some second-order equations. For example, the general solution of

$$h_{xy} = 0 \tag{1.27}$$

is, by inspection,

$$h = a(x) + b(y)\,, \tag{1.28}$$

where a and b are arbitrary functions.

How about

$$7f_{xx} + 20f_{xy} - 3f_{yy} = 0\,? \tag{1.29}$$

Look for a solution $f = a(y - \lambda x)$, where a is arbitrary. Substituting into (1.29)

$$\big(7\lambda^2 - 20\lambda - 3\big)a'' = 0\,. \tag{1.30}$$

Solving the quadratic equation $7\lambda^2 - 20\lambda - 3 = 0$ we find

$$\lambda = 3\,, \qquad \text{or} \qquad -\frac{1}{7}\,. \tag{1.31}$$

Thus the general solution is

$$f = a(y - 3x) + b(y + \tfrac{1}{7}x)\,, \tag{1.32}$$

where a and b are arbitrary.

The 2D Laplace equation,

$$\phi_{xx} + \phi_{yy} = 0\,, \tag{1.33}$$

can also be solved with this trick. Look for a solution of the form

$$\phi = a(x + \tau y)\,. \tag{1.34}$$

Substitution gives

$$1 + \tau^2 = 0\,, \qquad \Rightarrow \qquad \tau = \pm\mathrm{i}\,. \tag{1.35}$$

Hence the general solution is

$$\phi = a(x + \mathrm{i}y) + b(x - \mathrm{i}y)\,, \tag{1.36}$$

where a and b are arbitrary. If you insist on a real solution, then you should take the real or imaginary part of the above. For example, if $a(z) = b(z) = z^p$ then

$$\phi = \tfrac{1}{2}(x + \mathrm{i}y)^p + \tfrac{1}{2}(x - \mathrm{i}y)^p \tag{1.37}$$

is a real solution of Laplace's equation for any power p. You might write out the cases $p = \pm1$, ±2 and ±3 and see if you recognize some solutions you've previously seen.

Exercise: Find a real solution of Laplace's equation if $a(s) = \mathrm{e}^s$ and if $a(s) = \mathrm{e}^{\mathrm{i}s}$.

1.4 The Jacobian

If

$$f = x^2 - y^2\,, \qquad \text{and} \qquad g = \cos x^2 \cos y^2 + \sin x^2 \sin y^2 \tag{1.38}$$

then

$$g = \cos f\,. \tag{1.39}$$

This is an example of a functional relation. The contours, or level sets, of f and g in the (x, y)-plane coincide.

If $f(x, y)$ and $g(x, y)$ are differentiable functions of x and y then a necessary and sufficient condition for the existence of a functional relation,

$$\phi(f, g) = 0\,, \qquad \text{or perhaps} \qquad g = A(f)\,, \tag{1.40}$$

is that the *Jacobian*

$$\frac{\partial(f, g)}{\partial(x, y)} \stackrel{\text{def}}{=} f_x g_y - f_y g_x \tag{1.41}$$

is zero.

Let's prove this algebraically by differentiating $\phi(f, g) = 0$ with respect to x and y:

$$f_x\phi_f + g_x\phi_g = 0\,, \tag{1.42}$$

$$f_y\phi_f + g_y\phi_g = 0\,. \tag{1.43}$$

This is a 2×2 linear homogeneous system for ϕ_f and ϕ_g. The condition for a non-trivial solution is that the determinant is zero, or

$$\frac{\partial(f,g)}{\partial(x,y)} = 0\,. \tag{1.44}$$

Therefore vanishing of the Jacobian is necessary for the existence of a functional relation between $f(x,y)$ and $g(x,y)$. The condition is also sufficient.

The geometric interpretation of the condition (1.44) is that $f(x,y)$ and $g(x,y)$ are functionally related if their contours coincide, which is equivalent to saying that $\boldsymbol{\nabla} f$ and $\boldsymbol{\nabla} g$ are parallel at every point. Now recall that

$$\hat{\mathbf{z}}\cdot(\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g) = |\boldsymbol{\nabla} f||\boldsymbol{\nabla} g| \sin\theta \tag{1.45}$$

where θ is the angle between $\boldsymbol{\nabla} f$ and $\boldsymbol{\nabla} g$. So $\boldsymbol{\nabla} f$ and $\boldsymbol{\nabla} g$ are parallel at a point if and only if $|\boldsymbol{\nabla} f| \neq 0$, $|\boldsymbol{\nabla} g| \neq 0$ and $\theta = 0$. But it is easy to verify that

$$\hat{\mathbf{z}}\cdot(\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g) = \frac{\partial(f,g)}{\partial(x,y)}\,. \tag{1.46}$$

Solving the 2D transport equation with Jacobians

One way of solving a PDE such as

$$(x \sin y) h_x + (\cos y) h_y = 0 \tag{1.47}$$

is by recognizing a thinly disguised Jacobian:

$$(-x\cos y)_y h_x - (-x \cos y)_x h_y = 0\,. \tag{1.48}$$

Thus the general solution is that h is an arbitrary function of $x \cos y$.

But unfortunately not all PDEs of the form

$$u h_x + v h_y = 0 \tag{1.49}$$

are disguised Jacobians: for the LHS to be a Jacobian there must be a function ψ such that

$$u = -\psi_y\,, \qquad \text{and} \qquad v = \psi_x\,. \tag{1.50}$$

For this to be true one must have

$$\underbrace{u_x}_{-\psi_{xy}} + \underbrace{v_y}_{\psi_{xy}} = 0\,. \tag{1.51}$$

Example: Find the general solution of

$$(x + 2y) h_x - (6x + y) h_y = 0 \tag{1.52}$$

by showing that the LHS is a Jacobian.

Note that $u_x + v_y = 0$ so this PDE is indeed a Jacobian. In more words, if the LHS is a Jacobian then there is a function $\psi(x,y)$ such that

$$\psi_y = x + 2y\,, \qquad \Rightarrow \qquad \psi_{yx} = 1\,. \tag{1.53}$$

But we also require

$$\psi_x = 6x + y\,, \qquad \Rightarrow \qquad \psi_{xy} = 1\,. \tag{1.54}$$

The equation passes the test so we know that the object of our desire, $\psi(x, y)$, exists. Now we proceed to uncover ψ by integration

$$\begin{aligned} \psi_x &= 6x + y\,, && \Rightarrow && \psi = 3x^2 + xy + A(y)\,, \\ \psi_y &= x + 2y\,, && \Rightarrow && \psi = xy + y^2 + B(x)\,. \end{aligned} \tag{1.55}$$

Subtracting our different expressions for ψ we soon see that

$$\psi = 3x^2 + xy + y^2 + \text{constant}\,. \tag{1.56}$$

Thus the solution is that h is an arbitrary function of $\psi = 3x^2 + xy + y^2$.

Exercise Is the PDE

$$(x + 2y)h_x + (6x + y)h_y = 0 \tag{1.57}$$

a Jacobian?

Exercise: Consider the PDE

$$(ax + by)g_x + (cx + dy)g_y = 0\,, \tag{1.58}$$

where a, b, c and d are all constants. Find the condition ensuring that this PDE is a Jacobian, and then find the general solution.

2D incompressible fluid mechanics

The condition (1.51) might remind you of two-dimensional (2D) incompressible fluid mechanics. The velocity field of a 2D incompressible fluid, denoted $\boldsymbol{v} = u\hat{\boldsymbol{x}} + v\hat{\boldsymbol{y}}$, is incompressible if

$$\begin{aligned} \boldsymbol{\nabla\cdot v} &= u_x + v_y\,, \qquad &(1.59)\\ &= 0\,. &(1.60) \end{aligned}$$

In this case one can introduce a streamfunction ψ as in (1.50). Now the advection equation for a passive scalar (also known as "tracer") is

(1.50): $u = -\psi_y, \quad v = \psi_x$

$$c_t + \underbrace{uc_x + vc_y}_{=\frac{\partial(\psi,c)}{\partial(x,y)}} = 0\,. \tag{1.61}$$

If the system is steady — that is $c_t = 0$ — then the general solution of (1.61) is that c is an arbitrary function of ψ. In other words, in a steady 2D incompressible flow tracer is transported along streamlines.

Integrating factors

Some simple PDEs fail the Jacobian test but can still be solved using an *integrating factor*. For example, the PDE

$$h_x - \frac{y}{x}h_y = 0 \tag{1.62}$$

is not equivalent to a Jacobian. However if we multiply by the integrating factor x we have

$$xh_x - yh_y = 0\,. \tag{1.63}$$

This now passes the Jacobian test and the solution is that h is an arbitrary function of xy.

Perhaps we can always find an integrating factor? Suppose we multiply (1.49) by $\mu(x,y)$ to obtain

$$\mu u h_x + \mu v h_y = 0\,. \tag{1.64}$$

Applying the test to this new equation we require μ to satisfy

$$(\mu u)_x + (\mu v)_y = 0\,, \tag{1.65}$$

or

$$u\,(\ln\mu)_x + v\,(\ln\mu)_y + (u_x + v_y) = 0\,. \tag{1.66}$$

Sometimes it is easy to guess a solution of (1.66) e.g., try μ's that depend only on x or only on y, or on x and y in some special combination.

Example: Solve

$$h_x + (1 - xy)h_y = 0\,. \tag{1.67}$$

This equation does not pass the Jacobian test:

$$u_x + v_y = -x \neq 0\,. \tag{1.68}$$

The integrating factor equation (1.66) is

$$(\ln\mu)_x + (1-xy)\,(\ln\mu)_y - x = 0\,. \tag{1.69}$$

We can find a solution of (1.69) by assuming that $\mu_y = 0$, resulting in

$$\mu = \mathrm{e}^{x^2/2}\,. \tag{1.70}$$

Thus

$$\underbrace{\mathrm{e}^{x^2/2}}_{-\psi_y}\,h_x + \underbrace{\mathrm{e}^{x^2/2}(1-xy)}_{\psi_x}\,h_y = 0 \tag{1.71}$$

must be a Jacobian. We quickly find

$$\psi = \int_0^x \mathrm{e}^{t^2/2}\,\mathrm{d}t - y\mathrm{e}^{x^2/2}\,, \tag{1.72}$$

and the general solution of (1.67) is that h is an arbitrary function of ψ.

1.5 Change of coordinates

1.6 The method of characteristics

Consider some function $h(x,y)$ which you can visualize as the height of a surface above the (x,y) plane. If you move around on the (x,y)-plane following a curve $y = y(x)$ then you will observe changes in h both because x changes and also y changes. In fact, the *total derivative* of $h(x,y)$ following this moving point is

$$\frac{\mathrm{d}}{\mathrm{d}x}h(x,y) = h_x + \frac{\mathrm{d}y}{\mathrm{d}x}h_y\,. \tag{1.73}$$

We will use this result again and again and again.

Suppose we are confronted with the inhomogeneous 2D transport equation

$$uh_x + vh_y = e\,, \tag{1.74}$$

where u, v and e are specified functions of x and y. We can rewrite this PDE as

$$h_x + \frac{v}{u}h_y = \frac{e}{u}\,. \tag{1.75}$$

In this form we can interpret the problem as saying that on curves determined by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v}{u}\,, \tag{1.76}$$

the variation of h is determined by integration of

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{e}{u}\,. \tag{1.77}$$

Let's see some examples.

Example 1

We start by finding the general solution of

$$\alpha_x + 2x\alpha_y = xy\,. \tag{1.78}$$

On curves defined by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x\,, \tag{1.79}$$

the function $\alpha(x, y)$ varies as

$$\frac{\mathrm{d}\alpha}{\mathrm{d}x} = xy\,. \tag{1.80}$$

The integration of (1.79) produces

$$y = x^2 + \eta\,. \tag{1.81}$$

Each value of η corresponds to a particular *characteristic curve* of the PDE i.e. η is the name of a characteristic. Some of these characteristics are show in figure 1.5.

Now we turn to (1.80). Don't make the dumb mistake of integrating (1.80) treating y as a constant — y is not constant as we move along a characteristic. Instead it is η that is constant on characteristics. So must eliminate y in favor of η using (1.81):

$$\frac{\mathrm{d}\alpha}{\mathrm{d}x} = x(x^2 + \eta)\,. \tag{1.82}$$

Now we integrate (1.82) to obtain

$$\alpha = a(\eta) + \frac{x^4}{4} + \eta x^2\,, \tag{1.83}$$

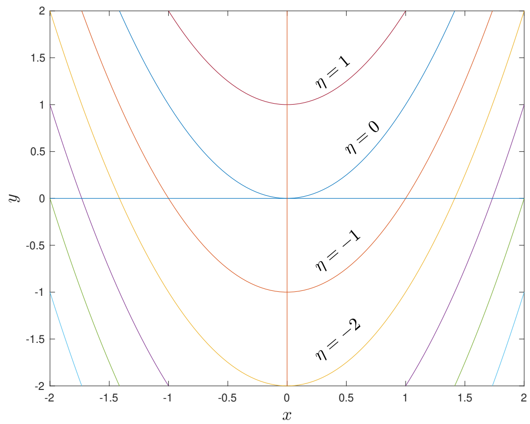


Figure 1.2: The characteristics in (1.81).

where a is an arbitrary function. Now we can use (1.81) again to eliminate η from (1.83)

$$\alpha = a(y-x^2) + \frac{x^4}{4} + (y-x^2)x^2 \,, \tag{1.84}$$

$$= a(y-x^2) - \tfrac{1}{4}x^4 + x^2 y \,, \tag{1.85}$$

This is the general solution — there is an undetermined arbitrary function a.

Now suppose we are supplied with boundary data along the x-axis

$$\alpha(x,0) = 0 \,. \tag{1.86}$$

We apply this condition to (1.85) to obtain

Boundary data

$\alpha(x,0) = 0$

$$0 = a(-x^2) - \tfrac{1}{4}x^4 \,. \tag{1.87}$$

To solve this equation, write $s = -x^2$. Thus

$$a(s) = \tfrac{1}{4}s^2 \,, \tag{1.88}$$

and therefore (1.85) is

$$\alpha = \tfrac{1}{4}(y-x^2)^2 - \tfrac{1}{4}x^4 + x^2 y \,. \tag{1.89}$$

But looking at the geometry of characteristics in figure 1.5 we see that we have only determined a on characteristics that pass through the x-axis — these are the characteristics with $\eta < 0$. The *domain of definition* of the solution in (1.89) is the region

$$y \le x^2 \,. \tag{1.90}$$

In the domain $y > x^2$ the arbitrary function a in (1.85) remains undetermined.

Example: Discuss the solution of (1.78) with boundary data $\alpha(x,0) = x$.

(1.85):

$\alpha = a(y-x^2) - \frac{1}{4}x^4 + x^2 y$

You might think it is just a matter of substituting $\alpha(x,0) = x$ into the general solution (1.85). This results in

$$x \stackrel{?}{=} a\big(-x^2\big) - \tfrac{1}{4}x^4 \,. \tag{1.91}$$

But this equation for a has no solution: evaluating it at $x = -1$ and then at $x = +1$ leads to contradictory determinations of $a(-1)$. This is the case for all values of x. The issue is clear geometrically: any characteristic with $\eta < 0$ in figure 1.5 intersects the x-axis twice, at $x = +\sqrt{|\eta|}$ and again at $x = -\sqrt{|\eta|}$. These intersections enforce contradictory information onto the characteristic.

One way to get a well defined problem is to prescribe boundary data over only half of the x-axis. For example, boundary data $a(x \geq 0, 0) = x$ leads to

$$x \geq 0: \qquad x = a\big(-x^2\big) - \tfrac{1}{4}x^4 \tag{1.92}$$

with solution

$$a(s) = \tfrac{1}{4}s^2 + \sqrt{-s}\,, \qquad \text{for } s \leq 0. \tag{1.93}$$

The domain of definition of this solution is $y \leq x^2$.

If one insists on prescribing boundary data over the entire x-axis, say $a(-\infty < x < \infty, 0) = f(x)$, then for a well-posed problem f must be even function. In this case the arbitrary function is determined as

$$a(s) = f(\pm\sqrt{-s}) + \tfrac{1}{4}s^2\,, \qquad \text{for } s < 0. \tag{1.94}$$

The domain of definition of this solution is again $y \leq x^2$.

Example 2

The homogeneous transport

$$\beta_x + (x+y)^2\beta_y = 0\,, \tag{1.95}$$

is not a disguised Jacobian and one can't guess the solution. We see, however, that if we move in the (x, y)-plane along any path $\mathcal{P}$ satisfying

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x+y)^2 \tag{1.96}$$

then on $\mathcal{P}$ the PDE (1.95) implies that

$$\frac{\mathrm{d}\beta}{\mathrm{d}x} = 0\,. \tag{1.97}$$

This means that $\beta(x, y)$ is a constant on $\mathcal{P}$ i.e., $\mathcal{P}$ is a contour or level-set of the function $\beta(x, y)$.

Now we must calculate $\mathcal{P}$ by integration of (1.96): let $v \stackrel{\text{def}}{=} x + y$ so (1.96) becomes

$$\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v^2\,. \tag{1.98}$$

Separation of variables gives

$$\frac{\mathrm{d}v}{1+v^2} = \mathrm{d}x\,, \qquad \Rightarrow \qquad \tan^{-1}(v) = x + \xi\,, \tag{1.99}$$

where ξ is a constant of integration. Now let's change our perspective and take the expression above as the *definition* of a function:

$$\xi(x, y) \stackrel{\text{def}}{=} \tan^{-1}(x+y) - x\,. \tag{1.100}$$

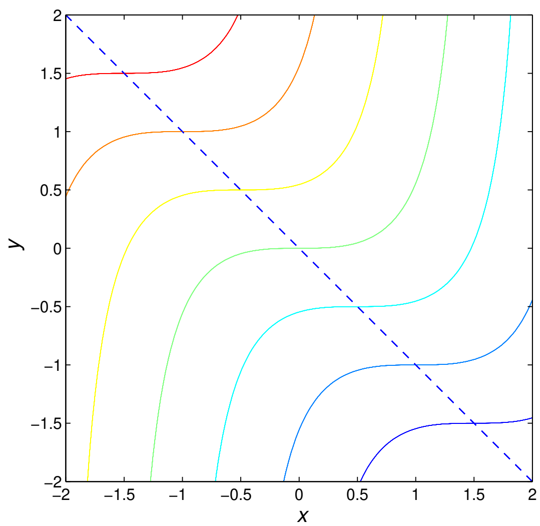


Figure 1.3: The solid curves are the characteristics in (1.100). The dashed line $y + x = 0$ intersects each characteristic once. `atanchar.eps`

Curves of constant ξ — also known as the "level sets" of ξ — are shown in Figure 1.3. Think of ξ as the name of characteristic and (1.100) answers the following question: if I am sitting at (x, y), which characteristic is passing through my position?

Each curve of constant ξ is a possible path $\mathcal{P}$ on which the PDE collapses to (1.97), and so on each $\mathcal{P}$, β is a constant. Thus the solution of (1.95) is

$$\beta(x, y) = a\big(\tan^{-1}(x + y) - x\big) , \tag{1.101}$$

where a is an arbitrary function. You should test your powers of differentiation by substituting this putative solution into the PDE and verifying that it works.

To determine the arbitrary function in (1.101) we need additional information provided by *boundary data*. Suppose that we are told that $\beta(x, -x) = -x$. In this case the boundary data is supplied on the dashed line $x + y = 0$ shown in figure 1.3. The characteristics passing through this line cover the whole of the (x, y) plane and so the solution is defined in the whole plane. Applying the boundary data to the general solution in (1.101) we have $\beta(x, -x) = -x$

$$-x = a(-x) , \qquad \Rightarrow \qquad a(s) = s . \tag{1.102}$$

Thus in this case the solution is

$$\beta = \tan^{-1}(x + y) - x . \tag{1.103}$$

The domain of definition is the whole (x, y)-plane.

The collection of special paths $\mathcal{P}$ along which a PDE collapses to an ODE such as (1.97) are *characteristics*. We'll give a more careful definition in the next few lectures. The first example below is important because it shows that *characteristics are not level sets of the solution.*

Example: Find the solution of the inhomogeneous transport equation

$$f_x + (x+y)^2 f_y = 1\,, \tag{1.104}$$

subject to $f(x,-x) = 0$.
On the the characteristics defined by (1.96) the PDE is

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 1\,, \qquad \text{or} \qquad f = x + a\left(\tan^{-1}(x+y) - x\right)\,, \tag{1.105}$$

with a an arbitrary function. This is the general solution. The arbitrary function a is determined by the boundary condition that $f = 0$ on the curve $y + x = 0$:

$$0 = x + a(-x)\,. \tag{1.106}$$

This implies that $a(s) = s$ and therefore

$$f = \tan^{-1}(x+y)\,. \tag{1.107}$$

Figure 1.3 shows that the characteristics passing through the data curve fill the whole (x, y) plane. Therefore in this problem the domain of definition is the entire (x, y)-plane. *Warning:* the solution f is *not* constant on characteristics: the defining quality of characteristics is that these are special paths in the (x, y)-plane along which the variation of f is determined by solving an ODE. Characteristics are not the level sets of the solution (unless the problem is homogenous).

Exercise Find a solution satisfying the boundary data $\beta(0, y) = y$. What's the domain of definition of this solution?

Example: Find a solution satisfying the boundary data $\beta(x, 0) = x$.
Evaluating the general solution (1.101) at $y = 0$ we have

$$x = a\big(\tan^{-1}(x) - x\big)\,. \tag{1.108}$$

We can't determine $a(s)$ analytically. But we can draw a graph of a by specifying x and then calculating $s = \tan^{-1}(x) - x$: see figure 1.4 and the following MATLAB code.

```
%% plot a no longer arbitrary function
x = linspace(-4,4); s = x - atan(x);
plot(s,x)
xlabel('$s$','interpreter','latex','fontsize',16)
ylabel('$a(s)$','interpreter','latex','fontsize',16)
```

This is all very pleasant. But if we want to use the solution (1.101) to produce a contour plot of $\beta(x, y)$ then we must determine the arbitrary function by numerical inversion of (1.108). Again the solution is defined in the whole (x, y)-plane.

$\beta(x, 0) = x$

Example 3

Consider the inhomogeneous transport equation

$$y\phi_x + x\phi_y = 1\,, \tag{1.109}$$

with the boundary data

$$\phi(x, 0) = 0\,. \tag{1.110}$$

Can you spot a particular solution? Probably not. But don't fret — the method of characteristics will produce it.

In this example it is convenient to divide (1.109) by x and write it as

$$\phi_y + \frac{y}{x}\phi_x = \frac{1}{x}\,. \tag{1.111}$$

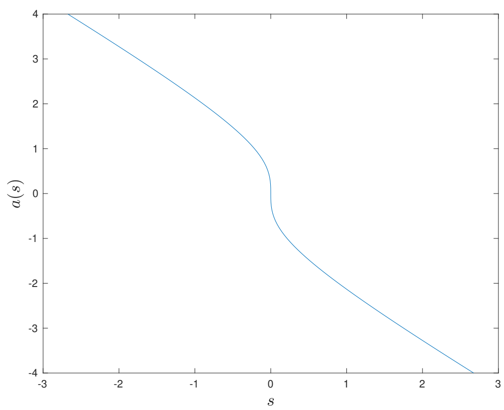


Figure 1.4: The function $a(s)$ defined by (1.108). There is a vertical tangent at $s = 0$. Local analysis of (1.108) shows that if $|s| \ll 1$, then $a(s) \approx -\operatorname{sgn}(s)(3|s|)^{1/3}$.

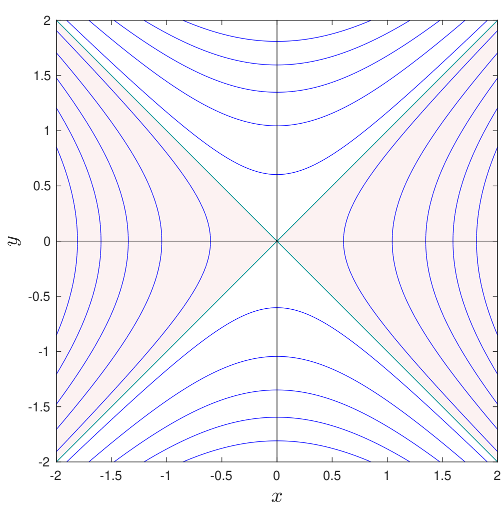


Figure 1.5: The characteristics in (1.116). The shaded region is where $x^2 - y^2 > 0$; this is the domain of definition of the solution in (1.121).

We use y, rather than x, as the independent variable and interpret (1.111) as

$$\frac{\mathrm{d}\phi}{\mathrm{d}y} = \frac{1}{x}\,, \tag{1.112}$$

on paths (characteristics) determined by

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{y}{x}\,. \tag{1.113}$$

If you don't like this, then I encourage you to use x as the independent variable and see the resulting mess.

Separating variables in (1.113) and integrating, we see that the characteristics are curves are defined by

$$x^2 - y^2 = \text{constant}\,. \tag{1.114}$$

In this example the boundary data is supplied along the x-axis and the domain of definition of the solution is therefore the region defined by

$$x^2 - y^2 > 0\,. \tag{1.115}$$

Thus we can write (1.113) as

$$x^2 - y^2 = \xi^2\,, \tag{1.116}$$

where ξ is the value of x at which the characteristic named ξ intersects the x-axis — by definition, $\xi < 0$ for characteristics in the left half plane.

Note: $\xi = \mathrm{sgn}(x)\sqrt{x^2 - y^2}$

Determine ϕ by writing (1.112) as

$$\frac{\mathrm{d}\phi}{\mathrm{d}y} = \frac{\mathrm{sgn}(x)}{\sqrt{\xi^2 + y^2}}\,, \tag{1.117}$$

and then integrating from $y' = 0$ to y

$$\phi = \mathrm{sgn}(x)\int_0^y \frac{\mathrm{d}y'}{\sqrt{\xi^2 + y'^2}}\,, \tag{1.118}$$

$$= \mathrm{sgn}(x)\Big[\ln\left(y' + \sqrt{\xi^2 + y'^2}\right)\Big]_0^y\,, \tag{1.119}$$

$$= \mathrm{sgn}(x)\ln\left[\frac{(y + \sqrt{\xi^2 + y^2})}{|\xi|}\right]\,. \tag{1.120}$$

The solution in (1.120) satisfies the boundary data $\phi(x, 0) = 0$. Now we use (1.116) to eliminate ξ and write ϕ as a function of x and y:

$$\phi = \mathrm{sgn}(x)\ln\left[\frac{(y + |x|)}{\sqrt{x^2 - y^2}}\right]\,. \tag{1.121}$$

The domain of definition of this solution is the region $x^2 > y^2$. The solution is undefined in the remainder of the (x, y)-plane.

1.7 Some books

Four good general texts on PDEs written by applied mathematicians are:

CP *Partial Differential Equations: Theory and Technique* by G.F. Carrier & C.E. Pearson.

Z *Partial Differential Equations of Applied Mathematics* by E. Zauderer.

K *Partial Differential Equations: Analytic Solution Techniques* by J. Kevorkian.

Sn *Elements of Partial differential Equations* by I.N. Sneddon.

CP and **Sn** have the advantage of brevity. **Z** is a leisurely intoduction, with lots of problems. The first two chapters of **Sn** are a good introduction to the geometric theory of first-order PDE's.

There are several 'math methods' books written by physicists for physicists which contain a lot of material on PDE's. Four good ones are:

So *Partial Differential Equation* by A. Sommerfeld.

MW *Mathematical Methods of Physics* by J. Matthews & R.L. Walker.

W *Mathematical Analysis of Physical Problems* by P.R. Wallace.

MF *Methods of Theoretical Physics; Part I and II* by P.M. Morse & H. Feshbach.

MF is useful as a compendium of formulas and methods. **So** is a very readable classic.

Two other classic textbooks on wave dynamics, characteristics and much more are:

W *Linear and Nonlinear Waves* by G.B. Whitham.

ZR *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena* by Ya. B. Zeldovich & Yu. P. Raizer.

W is a tough book, but well worth the effort. The first chapter of **ZR** is a good introduction to gas dynamics and the wave equation.

General texts on mathematical methods are:

BO *Advanced Mathematical Methods for Scientists and Engineers* by C. Bender & S.A. Orszag.

JJ *Methods of Mathematical Physics* by H. Jeffreys & B.S. Jeffreys.

Useful handbooks are:

OLBC *NIST Handbook of Mathematical Functions* Edited by F.W.J. Olver, D.W. Lozier, R.F. Boisvert & C.W. Clark.

GR *Tables of Integrals, Series and Products* by I.S. Gradshteyn & I.M. Rhyzik. (The recent editions are edited by A. Jeffrey & D. Zwillinger).

1.8 Problems

Problem 1.1. Find the general solution $h(x, y)$ of $h_x = yh$. Determine the arbitrary function in the general solution by applying the boundary condition $h(x, x) = \cos x$.

Problem 1.2. Reduce the following PDE's to the constant-coefficient transport equation and construct the general solution:

$$y\alpha_x + x\alpha_y = 0\,, \qquad x\beta_x + y\beta_y = 0\,, \qquad \gamma_y + e^x\gamma_x = \gamma\,, \qquad \eta_y + (x\eta)_x = 0\,. \tag{1.122}$$

Problem 1.3. Find the general solution of $h_x - xh_y = \cos x$.

Problem 1.4. Find the general solution of the PDEs

$$2\alpha_{xx} + 5\alpha_{xy} - 3\alpha_{yy} = 0\,, \tag{1.123}$$

$$\beta_{xx} + 2\beta_{xy} + \beta_{yy} = 0\,. \tag{1.124}$$

Problem 1.5. *(i)* Find the general solution of the PDE $xyh_x + h_y = 0$ and sketch the characteristic curves. *(ii)* Find the solution satisfying the boundary data $h(x, 0) = x$. What is the domain of definition of this solution?

Problem 1.6. Solve the PDE $h_x + h_y = 0$ with the data $h(x, cx) = x$. For one value of the constant c there is no solution. What's going on?

Problem 1.7. *(i)* Find the solution and *the domain of definition*, of the PDE $h_x - xh_y = 0$, subject to the boundary condition that: *(i)* $h(0 < x < \infty, x^2/2) = x$; *(ii)* $h(-\infty < x < 0, x^2/2) = x$; *(iii)* $h(-\infty < x < \infty, x^2/2) = x^2$; *(iv)* $h(-\infty < x < \infty, -x^2/2) = x^2$. (In one the four cases there is no solution.)

Problem 1.8. Find an example of a function $f(x, y)$ which is discontinuous at $(x, y) = (0, 0)$, even though all directional derivatives exist at $(x, y) = (0, 0)$.

Problem 1.9. Consider the PDE

$$h_x + (x + y)^2 h_y = 0 \tag{1.125}$$

discussed in the lecture. Give an example of boundary data for which this PDE has no solution.

Problem 1.10. What do the following functions

$$f_1(x, y) = x^4 + 4(x^2y + y^2 + 1)\,,$$

$$f_2(x, y) = \sin x^2 \cos 2y + \cos x^2 \sin 2y\,,$$

$$f_3(x, y) = \frac{x^2 + 2y + 2}{3x^2 + 6y + 5}\,, \tag{1.126}$$

have in common? Evaluate the derivative

$$(\partial_x - x\partial_y) \ln\left(\frac{f_1 f_3}{1 - \sin^{1/3}(f_2)}\right) \cos(x - y)\,, \tag{1.127}$$

where the functions $f_n(x, y)$ with $n = 1$, 2 and 3 are defined in (1.126)

Problem 1.11. Is the LHS of

$$e^x \cos y\, h_x - e^x \sin y\, h_y = 0\,, \tag{1.128}$$

a Jacobian? How about

$$(\cos x \cosh y - \sin x \sinh y)h_x - (\sin x \sinh y + \cos x \cosh y)h_y = 0\,? \tag{1.129}$$

Problem 1.12. Find the general solution of $h_x - x h_y = f'(x)g(x^2 + 2y)$. Here f and g are given functions and f' is the derivative of f.

Problem 1.13. Find a solution of the PDE

$$u_{xx}u_{yy} - u_{xy}^2 = 0\,, \quad u(x,0) = \tfrac{1}{2}x^2\,, \quad u_y(x,0) = x\,. \tag{1.130}$$

(Hint: look for a Jacobian.)

Problem 1.14. Let

$$f(x,y) \stackrel{\text{def}}{=} \int_0^\infty \frac{t \exp[-y\sqrt{1+t^2}]\sin xt}{\sqrt{1+t^2}}\,\mathrm{d}t\,. \tag{1.131}$$

Use integration by parts to show that

$$f(x,y) = \frac{x}{y}\int_0^\infty \exp[-y\sqrt{1+t^2}]\cos xt\,\mathrm{d}t\,. \tag{1.132}$$

Next, differentiate with respect to x and y and show that

$$f_y = [(y/x)f]_x\,. \tag{1.133}$$

Show that the solution of this PDE is $f = xa(x^2 + y^2)$ where a is an arbitrary function. Find a simple form for a. (From **CP**.)

Problem 1.15. Solve

$$2f_x + f_y = xy\,, \qquad \text{with boundary data} \quad f(x,0) = x\,. \tag{1.134}$$

Problem 1.16. *(i)* Solve

$$f_x + (x+y)f_y = y\,, \tag{1.135}$$

with the boundary data $f(x,-x) = 0$. *(ii)* Plot the characteristics and determine the domain of definition of your solution. (For the plotting chore, I'd use MATLAB with commands such as `meshgrid` — that's how I drew figure 1.3.)

Problem 1.17. Find the general solution of the PDE

$$g_x + (x+y)g_y = 0\,. \tag{1.136}$$

Check your answer by substitution. Draw three or four characteristics in the (x,y)-plane. Now solve the PDE with the boundary condition $g(0,y) = \exp(-y)$.

Problem 1.18. Try to solve the PDE $h_x = 0$ for $h(x,y)$, subject to the boundary condition that $h = x$ on the circle $x^2 + y^2 = 1$. What is the domain of definition of putative solutions? What assumptions do you have to make in order to determine the arbitrary function?

Problem 1.19. Consider the PDE $h_x + \beta h_y = 0$ with the boundary condition $h(x, x-1) = x$. Find the solution and state any condition which must be imposed on the constant β in order for the solution to exist.

Problem 1.20. Consider the three functions

$$\phi \stackrel{\text{def}}{=} x + y + z\,, \quad \theta = x^2 + y^2 + z^2\,, \quad \chi \stackrel{\text{def}}{=} xy + yz + zx\,. \tag{1.137}$$

(i) Show that one of these can be written as a function of the other two. *(ii)* Find a PDE of the form

$$u\psi_x + v\psi_y + w\psi_z = 0\,, \qquad \text{with } u,\ v \text{ and } w \text{ functions of } (x,y,z)\,, \tag{1.138}$$

whose general solution is that ψ is an arbitrary function of ϕ and θ.

Problem 1.21. Suppose you are given three function of (x,y,z) (e.g., as in problem 1.20). Find a three-dimensional generalization of the Jacobian criterion which tests if such a functional relation exists between the three. Give both an algebraic proof and a geometric interpretation.

Hint: Denote the three given functions by $u(x,y,z)$, $v(x,y,z)$ and $w(x,y,z)$. Suppose that there is some unknown functional relation

$$\Phi(u,v,w) = 0\,, \tag{1.139}$$

connecting u, v and w. Differentiating with respect to x, y and z we find

$$u_x\Phi_u + v_x\Phi_v + w_x\Phi_w = 0\,, \tag{1.140}$$
$$u_y\Phi_u + v_y\Phi_v + w_y\Phi_w = 0\,, \tag{1.141}$$
$$u_z\Phi_u + v_z\Phi_v + w_z\Phi_w = 0\,. \tag{1.142}$$

This is 3×3 set of linear equations for the three unknowns (Φ_u, Φ_v, Φ_w). If the only solution is $(\Phi_u, \Phi_v, \Phi_w) = (0,0,0)$ then there is no nontrivial functional relation.

Problem 1.22. Find the general solution of the PDEs

$$u(x)\alpha_x + v(y)\alpha_y = 0\,, \tag{1.143}$$
$$u(y)\beta_x + v(x)\beta_y = 0\,, \tag{1.144}$$
$$u(x)\gamma_x + v(y)\gamma_y + w(z)\gamma_z = 0\,. \tag{1.145}$$

Lecture 2

Applications of first-order partial differential equations

2.1 1D conservation laws

First-order PDE's arise in applications frequently as a result of conservation laws. The fundamental idea here is simple: count the amount of stuff in some control interval, $a < x < b$, and then:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\text{Amount of Stuff}] = [\text{ Stuff entering}] - [\text{Stuff leaving}]\,. \tag{2.1}$$

Consider traffic on the northbound lanes of a highway as an example. We use the coordinate x to denote distance along the road. The *density*, $\rho(x,t)$, is the number of cars per length. In the interval $a < x < b$ the total number of cars is

$$\text{Number of cars in the interval } a < x < b = \int_a^b \rho(x,t)\,\mathrm{d}x. \tag{2.2}$$

We have made the *continuum approximation* by assuming that there is a well defined density which is a smooth function of position x. If a car is 5 meters long it makes no sense in (2.2) to pick an interval of length 1 meter. The introduction of the density $\rho(x,t)$ requires a separation in length scales between the distance over which $\rho(x,t)$ changes appreciably (e.g. one kilometer) and the distance between cars (e.g. a few meters).

We apply the principle in (2.1) by picking an interval of highway $a < x < b$, and counting the number of cars which pass $x = a$ in a time $\mathrm{d}t$. Thus we define the *flux* (cars per second passing $x = a$) as $f(a,t)$. We can do the same at the other end of the control length and so determine $f(b,t)$. We are ignoring off-ramps and on-ramps which might introduce cars into the middle of (a,b). Then (2.1) tells us that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,[\text{Number of cars in the interval } a < x < b] = f(a,t) - f(b,t). \tag{2.3}$$

In other words

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \rho(x,t)\,\mathrm{d}x + f(b,t) - f(a,t) = 0\,. \tag{2.4}$$

Letting $a \to b$ in (2.4), with x sandwiched in the middle, we obtain the differential statement of the conservation law:

Assumption: ρ is continuous and f is differentiable.

$$\rho_t + f_x = 0\,. \tag{2.5}$$

If we can find a relation between the flux $f(x,t)$ and the density $\rho(x,t)$ then the conservation law in (2.5) is a PDE for the density of traffic.

Example: Consider a semi-infinite pipe with cross-sectional area $\alpha(x)$ (square meters), lying along the positive x-axis. Seawater is pumped into the pipe at a rate $Q(t)$ (kilograms per second) through the inlet at $x = 0$. The salinity at the inlet is $S_0(t)$ (grams of salt per kilogram of seawater). The flow is safely subsonic so the density of seawater, ρ (kilograms per cubic meter), is constant. Write down the PDE for salinity $S(x,t)$. Is this a statement of conservation of salt?

There are two conservation laws, one for seawater and the other for salt. The conservation law for pure water, H_2O, can be deduced from these two. First, consider seawater. The sectionally averaged seawater velocity through the pipe is $U(x,t) = Q(t)/\rho\alpha(x)$. This ensures that the flux of incompressible seawater is $Q(t)$ at all x.

The total amount of salt between x and $x + \mathrm{d}x$ is

$$\rho S(x,t) \times \underbrace{\alpha \mathrm{d}x}_{\mathrm{d}V} \text{ (grams of salt)} \tag{2.6}$$

where $\mathrm{d}V$ is a control volume. The salt flux through any section is

$$Q(t)S(x,t) \text{ (grams of salt per second)}\,. \tag{2.7}$$

Hence the salt conservation law is

$$\partial_t\left(\rho\alpha S(x,t)\right) + \partial_x\left(QS\right) = 0\,. \tag{2.8}$$

To get a PDE for $S(x,t)$ we divide (2.8) $\rho\alpha$ to obtain

$$S_t + US_x = 0\,. \tag{2.9}$$

$$U = \frac{Q(t)}{\rho\alpha(x)}$$

The PDE for S in (2.9) does not have the form of a conservation law in (2.5). But if we use $\mathcal{S} \stackrel{\text{def}}{=} \alpha S$ as a new variable then (2.8) becomes the conservational law

$$\partial_t \mathcal{S} + \partial_x\left(U\mathcal{S}\right) = 0\,. \tag{2.10}$$

The variable $\mathcal{S}$ is grams per meter of pipe.

Exercise: Referring to the example above, and starting from $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u} = 0$, prove that "The sectionally averaged seawater velocity through the pipe is $U(x,t) = Q(t)/\alpha(x)$." Note $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u} = 0$ is an approximation to the exact equation for mass conservation $\rho_t + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho\boldsymbol{u}) = 0$.

The transport equation

Suppose, for example, that $\rho(x,t)$ is the density of photons and c is the speed of light. In this case

$$f = c\rho\,, \tag{2.11}$$

and (2.5) becomes a simple and important example of a first order PDE:

$$\boxed{\rho_t + c\rho_x = 0\,.} \tag{2.12}$$

The boxed equation is known variously as the *transport equation* or the *1D advection equation* or the *one-way wave equation* or the *chiral wave equation*. It has lots of names because it's a central model in many scientific fields.

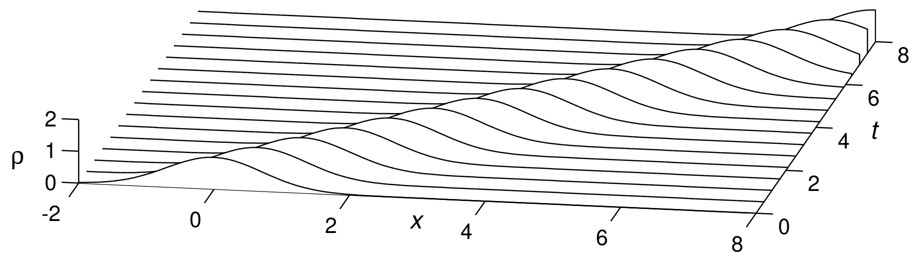


Figure 2.1: The solution (2.14) of the advection equation. The location of the Gaussian pulse moves with constant velocity c. `gaussWave.eps`

If the speed c is constant then, by inspection, the *general solution* of (2.12) is

$$\rho(x,t) = a(x-ct)\,, \tag{2.13}$$

where a is an arbitrary function. To determine a we need a combination of initial and boundary conditions. The simplest case is the initial value problem in which a is determined by specifying an initial condition which applies for all x. For example, if the initial condition is

$$\rho(x,0) = \exp(-x^2)\,, \tag{2.14}$$

then the solution of the transport equation is

$$\rho(x,t) = \exp\left[-(x-ct)^2\right]\,. \tag{2.15}$$

The pulse preserves its shape and moves with constant velocity c (see the top panel of figure 2.1). This is a *traveling wave solution* of the PDE in (2.12).

Exercise: draw the characteristic curves in the (x,t) plane for the constant-c PDE in (2.12).

Exercise: Solve the damped transport equation $\rho_t + c\rho_x = -\alpha\rho$ with $\rho(x,0) = \rho_0(x)$; c and α are constants.

Now consider a more interesting case in which the speed c depends on x. Using the highway metaphor, this might be because of bad road conditions resulting in a reduction of c in some section of the highway. The traffic density obeys the conservation equation

$$\rho_t + (c\rho)_x = 0\,. \tag{2.16}$$

Notice that with $\mathrm{d}c/\mathrm{d}x \neq 0$ and with this change the conservation equation (2.16) is not the same as the transport equation

$$\eta_t + c\eta_x = 0\,. \tag{2.17}$$

It is easy to find the *steady* solutions of (2.16) and (2.17) i.e., solutions with $\rho_t = 0$ and $\eta_t = 0$. You'll readily appreciate that the steady solution of (2.16) is consistent with some features of your driving experience.

Below we obtain the general solutions of (2.16) and (2.17) using the method of characteristics.

Fick's law and diffusion

Stuff moves along the x-axis because of advection, but also because of *diffusion*. Think of red dye in a stationary fluid. Suppose the initial condition is red colored fluid to the left ($x < 0$) of an impermeable membrane and clear fluid to the right ($x > 0$). If the membrane is removed then soon the initially sharp boundary between the two water masses is a blurred pink i.e., the initial discontinuity evolves into a smooth transition. Let $n(x, t)$ (red molecules per cubic meter) denote the dye concentration. Then it is an empirical fact — known as Fick's law — that the flux-gradient relation is

$$f = -\kappa n_x \,. \tag{2.18}$$

Notice that if the dye concentration is uniform, and the water is not flowing, then there is no flux. This seems sensible. The diffusive Fickian flux acts to equalize concentration by transporting molecules of dye from regions of high concentration to regions of low concentration. In (2.18) the diffusivity κ is positive and the minus sign on the right means that the flux f has the opposite sign to the dye gradient n_x: we say that the flux is *down-gradient*.

Using Fick's flux-gradient relation, the conservation law (2.5) produces a second-order linear PDE

$$\boxed{n_t = \kappa n_{xx} \,.} \tag{2.19}$$

This is the diffusion equation. If for some reason κ is a function of x then

$$n_t = (\kappa n_x)_x \,. \tag{2.20}$$

Note that $n(x, t) =$ constant is a *steady* solution even if κ varies with x.

Example: sedimentation

We can, of course, have advection and diffusion operating together. An important example is small macroscopic particles that settle in a fluid at the Stokes velocity. Let's use the vertical coordinate z pointing upwards, opposite gravity, and suppose that $n(z, t)$ is the density of particles i.e., particles per volume. The bottom of the container is at $z = 0$, the top is at $z = h$ and the Stokes settling speed is s. If the particles are spheres with radius a then $s = mg'/6\pi\mu a$ where g' is the gravity reduced by the buoyancy force acting on the spheres, and m is the mass of a particle.

(In using the Stokes law for a single particle we are assuming that the concentration is dilute so that hydrodynamic interactions between the particles are negligible. In the problem of *hindered settling* these interactions are not negligible and so the settling speed depends on the local concentration of the particles. The settling speed simplifies to the Stokes law as the concentration approaches zero. See Kynch (1951) and the problems at the end of this lecture.)

The sediment conservation equation is

$$n_t + f_z = 0 \,, \tag{2.21}$$

where the flux is

$$f = -sn - \kappa n_z\,. \tag{2.22}$$

The advective part of the flux, $-sn$, is negative because the particles settle towards the bottom the container i.e. the setting velocity is negative. Substituting (2.21) into (2.22) and rearranging we obtain the PDE

$$n_t - sn_z = \kappa n_{zz}\,. \tag{2.23}$$

This is the advection-diffusion equation. The boundary conditions applied to (2.23) are:

$$\text{at } z = 0 \text{ and } z = h: \qquad sn + \kappa n_z = 0\,? \tag{2.24}$$

Let's find a very simple solution of (2.23) with the boundary conditions in (2.24). We look for a *steady* solution i.e., a solution with $n_t = 0$. We quickly find

$$n = a + b\mathrm{e}^{-sz/\kappa}\,, \tag{2.25}$$

where a and b are constants of integration. The flux is

$$f = -sn - \kappa n_z \tag{2.26}$$

$$= -as\,. \tag{2.27}$$

The boundary conditions in (2.24) tell us that $f = 0$ at $z = 0$ and $z = h$ and therefore $a = 0$. In the steady state the sediment flux f is therefore zero at all levels. The steady solution above is set up starting from all initial conditions. We can relate the remaining constant of integration b to properties of the initial condition by demanding that the load

$$\int_0^h n(z,t)\,\mathrm{d}z \tag{2.28}$$

is constant. Thus the load in the steady state is equal to the initial load:

$$b\frac{\kappa}{s}\left(1 - \mathrm{e}^{-sh/\kappa}\right) = \int_0^h n_0(z)\,\mathrm{d}z\,, \tag{2.29}$$

where $n_0(z)$ is the initial condition.

Exercise: Prove (2.29) by integrating the conservation equation (2.21) from $z = 0$ to $z = h$ and using the boundary conditions (2.24).

Let's assume that the vessel is much deeper than a scale height so that $\mathrm{e}^{-sh/\kappa}$ is negligible. With this simplification we have the final $t = \infty$ steady sediment distribution

$$n(z) = \frac{s}{\kappa}\int_0^\infty n_0(z)\,\mathrm{d}z\,\mathrm{e}^{-sz/\kappa}\,. \tag{2.30}$$

The sediment concentration varies exponentially with a *scale height* κ/s.

Example: Paint draining down a wall

Using lubrication theory, a stripe of paint draining down a vertical wall satisfies the 1D conservation equation

$$h_t + \frac{g}{3\nu}\left(h^3\right)_x = 0\,. \tag{2.31}$$

Here gravity is directed along the positive x-axis and $z = h(x,t)$ is the thickness of the paint layer. For realistically thin layers of paint the worst approximation made in (2.31) is probably neglect of surface tension. Similar non-Newtonian models are used to describe sliding glaciers.

Example: chromotography

In chromotography one can spatially separate the components of a mixture by pumping it through an immobile bed of absorbing particles. Suppose we have a dissolved chemical with concentration (kilograms per cubic meter) $\rho(x,t)$ in the liquid phase and $\mu(x,t)$ in the solid phase. Then the conservation law is

$$\rho_t + \mu_t + c\rho_x = \kappa\rho_{xx}\,, \tag{2.32}$$

where c (assumed constant) is the bulk velocity of the liquid and κ is the diffusivity of the chemical. The exchange of the chemical between the solid and liquid phases is determined by an equation of the form

$$\mu_t = E(\mu,\rho)\,. \tag{2.33}$$

In equilibrium, the exchange vanishes:

$$E(\mu,\rho) = 0 \tag{2.34}$$

which can be solved to obtain $\mu = \mu_{eq}(\rho)$. Assuming that absorption is a rapid process we can substitute $\mu(x,t) \approx \mu_{eq}(\rho(x,t))$ into (2.32) to obtain

$$\rho_t + \frac{c\,\rho_x}{1+\mu_{eq}'(\rho)} = \kappa\rho_{xx}\,. \tag{2.35}$$

According to **W**, a specific model for the exchange E is

$$E = \alpha(\mu_* - \mu)\rho - \beta\mu(\rho_* - \rho)\,. \tag{2.36}$$

Here α, β, μ_* and ρ_* are all constants. The term $\alpha(\mu_* - \mu)\rho$ is deposition from the liquid onto to the solid at a rate proportional to the concentration in the liquid, but limited by the amount already deposited, up to a maximum μ_*. The term $-\beta\mu(\rho_* - \rho)$ is reverse transfer from the solid to the liquid. The equilibrium condition, $E = 0$, implies that

$$\mu = \frac{\alpha\mu_*\rho}{(\alpha-\beta)\rho + \beta\rho_*}\,. \tag{2.37}$$

2.2 Evolution equations

Consider the transport equation

$$\eta_t + c\eta_x = 0\,, \tag{2.38}$$

with an initial condition

$$\eta(x,0) = \eta_0(x)\,. \tag{2.39}$$

The speed c is specified as some function of x.

Consider the solution $\eta(x,t)$ as the height of a surface above the (x,t)-plane. If you move around on the (x,t)-plane following a curve $x = x(t)$ and measuring $\eta(x,t)$, then you will observe changes in η because x changes and also because t changes. In fact, the *total derivative* of $\eta(x,t)$ following this moving point is

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(x,t) = \eta_t + \frac{\mathrm{d}x}{\mathrm{d}t}\eta_x\,. \tag{2.40}$$

Comparing (2.40) with (2.38) we see that if

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c(x)\,, \qquad \text{then} \qquad \frac{\mathrm{d}\eta}{\mathrm{d}t} = 0\,. \tag{2.41}$$

Let's solve a simple PDE using this trick. Consider

$$\eta_t + x\eta_x = 0\,, \qquad \text{with initial condition } \eta(x,0) = \mathrm{e}^{-\frac{1}{2}x^2}. \tag{2.42}$$

Then on curves in the (x,t)-plane defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x\,, \tag{2.43}$$

the PDE in (2.42) is equivalent to

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = 0\,. \tag{2.44}$$

Solving (2.43) we find that these *characteristic curves* are

$$x = \xi\mathrm{e}^t\,, \tag{2.45}$$

where ξ is the value of x at $t = 0$. We can turn (2.45) around and use it as the definition of the *characteristic coordinate*:

$$\xi(x,t) = x\mathrm{e}^{-t}\,. \tag{2.46}$$

The solution of (2.44) is that $\eta(x,t)$ is constant on curves of constant ξ i.e. η is an arbitrary function of ξ in (2.46). We determine the arbitrary function by applying the initial condition: at $t = 0$, $\xi = x$ and $\rho(x,0) = \mathrm{e}^{-\frac{1}{2}x^2}$. Thus the solution is

$$\eta = \mathrm{e}^{-\frac{1}{2}\xi^2} = \mathrm{e}^{-\frac{1}{2}\mathrm{e}^{-2t}x^2}\,. \tag{2.47}$$

Exercise: Show that

$$J(t) \overset{\text{def}}{=} \int_{-\infty}^{\infty} \eta(x,t)\,\mathrm{d}x\,, \tag{2.48}$$

$$= \sqrt{2\pi}\mathrm{e}^{-t}\,. \tag{2.49}$$

You can do this by directly computing the integral from (2.47) or by integrating the PDE (2.42) from $x = -\infty$ to ∞. Following the sneaky second route you'll find that $J(t)$ satisfies a very simple ODE.

Now consider the analogous conservation equation

$$\rho_t + (x\rho)_x = 0\,, \tag{2.50}$$

with initial condition

$$\rho(x,0) = \mathrm{e}^{-\frac{1}{2}x^2}\,. \tag{2.51}$$

We rewrite (2.50) as

$$\rho_t + x\rho_x = -\rho\,. \tag{2.52}$$

In we move along the curves of constant ξ in (2.45) then the PDE in (2.52) says that

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\,, \tag{2.53}$$

with solution

$$\rho = \mathrm{e}^{-t-\frac{1}{2}\xi^2} = \exp\left[t - \tfrac{1}{2}\mathrm{e}^{-2t}x^2\right]\,. \tag{2.54}$$

Notice that the total amount of ρ-stuff is independent of time:

$$\int_{-\infty}^{\infty} \mathrm{e}^{-t-\frac{1}{2}\mathrm{e}^{-2t}x^2}\,\mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}\xi^2}\,\mathrm{d}\xi = \sqrt{2\pi}\,. \tag{2.55}$$

This is expected because $\rho(x,t)$ is the solution of the PDE (2.50), which is a conservation equation for $\rho(x,t)$.

Example: Solve the initial value problem

$$\rho_t + (x\rho)_x = \cos x\,, \tag{2.56}$$

with the initial condition

$$\rho(x,0) = 0\,. \tag{2.57}$$

Write the PDE in the form

$$\rho_t + x\rho_x = -\rho + \cos x\,. \tag{2.58}$$

Then the characteristic equations are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x\,, \qquad \text{and} \qquad \frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho + \cos x\,. \tag{2.59}$$

The solution of the first equation in (2.59) is (2.45). We determine ρ by solving the second ODE for ρ in (2.59). Notice, however, that on a characteristic — that is a curve of constant $\xi(x,t)$ — x is a function of t. So it's a dumb mistake to integrate (2.59) treating x as a constant. But if we replace x by ξ we have

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho + \cos\left(\mathrm{e}^t\xi\right)\,. \tag{2.60}$$

Equation (2.45) is

$$x = \xi\mathrm{e}^t\,,$$

and therefore

$$\xi(x,t) = \mathrm{e}^{-t}x\,.$$

Now we can solve (2.60) as an ODE, treating ξ as a constant:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^t\rho\right) = \mathrm{e}^t\cos(\mathrm{e}^t\xi) \qquad \Rightarrow \qquad \rho = \mathrm{e}^{-t}\,\frac{\sin\left(\mathrm{e}^t\xi\right) - \sin\xi}{\xi}\,. \tag{2.61}$$

Finally, eliminate ξ in favor of x and t, with the result

$$\rho = \frac{\sin x - \sin\left(\mathrm{e}^{-t}x\right)}{x}\,. \tag{2.62}$$

Notice that $\rho(0,t) = 1 - \mathrm{e}^{-t}$.

Example: Solve

$$G_t + \mu(z-1)G_z = \lambda(z-1)G\,, \qquad \text{with IC} \qquad G(x,0) = G_0(x)\,. \tag{2.63}$$

The ODE for the characteristics is

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \mu(z-1)\,, \qquad \text{with initial condition} \qquad z(0) = \xi\,. \tag{2.64}$$

The solution is

$$z = (\xi-1)\mathrm{e}^{\mu t} + 1\,, \qquad \text{or} \qquad \xi = \mathrm{e}^{-\mu t}(z-1) + 1\,. \tag{2.65}$$

As we move along the characteristic curves in (2.65). the function G varies as

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \lambda(z-1)G = (\xi-1)\mathrm{e}^{\mu t}G\,, \tag{2.66}$$

with initial condition

$$G(\xi,0) = G_0(\xi)\,. \tag{2.67}$$

Integrating (2.66) from the initial condition in (2.67) to time t:

$$G(z,t) = G_0(\xi)\exp\left[(\xi-1)\tfrac{\lambda}{\mu}(\mathrm{e}^{\mu t}-1)\right]\,. \tag{2.68}$$

Eliminating $\xi(z,t)$ using (2.65) we obtain the solution

$$G(z,t) = G_0\left[\mathrm{e}^{-\mu t}(z-1)+1\right]\exp\left[(z-1)\tfrac{\lambda}{\mu}(1-\mathrm{e}^{-\mu t})\right]\,. \tag{2.69}$$

This example is used in the discussion of birth-death processes in section 2.4.

Example: Solve

$$\eta_t + \frac{1+x^2}{5+x^2}\eta_x = 0\,, \qquad \text{with the IC} \qquad \eta(x,0) = \exp\left[-9\,(x+2)^2\right]\,. \tag{2.70}$$

The speed $c(x)$ is a minimum at $x=0$, $c(0) = 1/5$, and asymptotes to a maximum as $x \to \pm\infty$, $c(x) \to 1$. The characteristic coordinate ξ is determined by solving

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1+x^2}{5+x^2}\,, \qquad \Rightarrow \qquad \int_\xi^x \frac{5+x'^2}{1+x'^2}\,\mathrm{d}x' = t\,. \tag{2.71}$$

Evaluating the integral:

$$(x + 4\arctan x) - (\xi + 4\arctan\xi) = t\,. \tag{2.72}$$

We can plot the characteristic diagram by specifying a particular value of ξ and then calculating t as a function of x (see figure 2.2). But in this example we cannot analytically solve (2.72) to exhibit $\xi(x,t)$. The best we can do is say that the solution of the initial value problem is

$$\eta(x,t) = \exp\left[-9\,(\xi(x,t)+2)^2\right]\,, \tag{2.73}$$

with $\xi(x,t)$ defined implicitly by (2.72). Then with MATLAB, for instance, we solve (2.72) for $\xi(x,t)$ with the command

```
xi = fzero(@(y) y + 4*atan(y) - x - 4*atan(x) + t,x)
```

Thus if we specify t we can plot a snapshot of the solution $\eta(x,t)$: see figure 2.3. It is clear from this figure that η is not a conserved density: the area under the curve is changing with time.

Exercise: Write a recipe for solving the PDE

$$\rho_t + c(x,t)\rho_x = s(x,t,\rho)\,, \qquad \text{with initial condition} \qquad \rho(x,0) = \rho_0(x)\,. \tag{2.74}$$

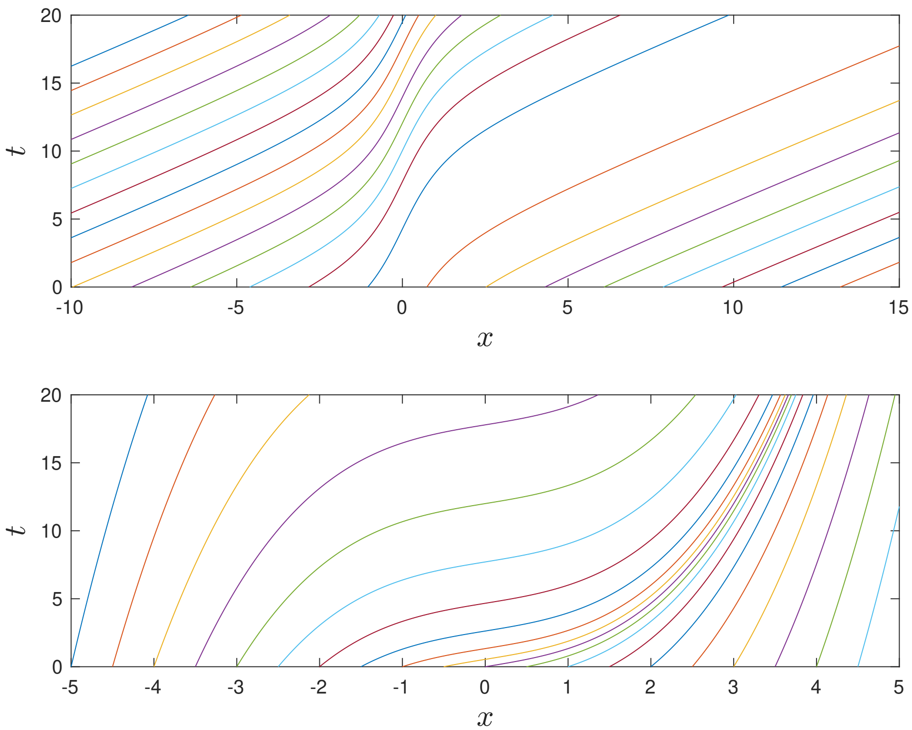


Figure 2.2: Upper panel shows the characteristics defined by (2.72) and the lower panel the characteristics for problem 2.13. `char2019.eps`

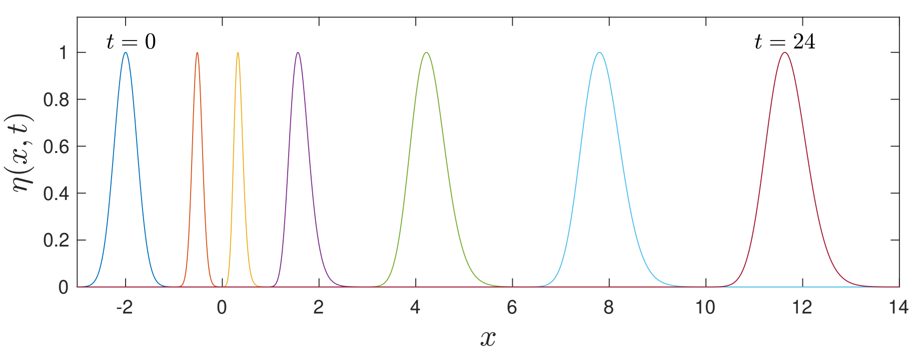


Figure 2.3: Snapshots of the solution in (2.73) at $t = [0\,,4\,,8\,,12\,,16\,,20\,,24]$. `pulse2019.eps`

General solution of the linear transport equation

Return to the 1D transport equation (2.38) with specified speed $c(x)$. Following the reasoning used in the earlier example (2.38), the *characteristic coordinate* $\xi(x,t)$, is defined via the solution of the ordinary differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c(x)\,, \qquad \text{with the initial condition} \qquad x(\xi,0) = x\,. \tag{2.75}$$

$\xi(x,t)$ is the initial ($t=0$) location of the particle that is passing through x at time t.

Separating variables in (2.75) we have

$$\int_{\xi}^{x} \frac{\mathrm{d}x'}{c(x')} = t\,, \tag{2.76}$$

or

$$\tau(x) - \tau(\xi) = t\,, \tag{2.77}$$

where the "transit time" is

$$\tau(x) \stackrel{\text{def}}{=} \int^{x} \frac{\mathrm{d}x'}{c(x')}\,. \tag{2.78}$$

If q is the inverse function to τ,

$$q(\tau(x)) = x\,, \tag{2.79}$$

then the characteristic coordinate is

$$\xi(x,t) = q\left(\tau(x) - t\right)\,. \tag{2.80}$$

Hence the solution of the transport equation (2.75) is

$$\eta = \eta_0\left[q\left(\tau(x) - t\right)\right]\,. \tag{2.81}$$

It is educational to check this by substitution.

See problem 2.5 for the general solution of the analogous conservation equation.

2.3 Entry on the half-line

Now we turn to problems defined on the half-line $x \geq 0$. Think of $x=0$ as the beginning of a highway and suppose we count the number of cars entering at $x=0$. If this entry rate increases then there is a bulge in the traffic density, ρ, which propagates down the roadway. This bulge is a signal indicating changes in the upstream conditions. In our simplest model, with constant speed c, the signal from $x=0$ reaches x at time x/c. Another example might be one-dimensional radiative transfer: $\rho(x,t)$ is the density of photons moving along the positive half of the x-axis with the speed of light c. The source at $x=0$ is a laser emitting new photons.

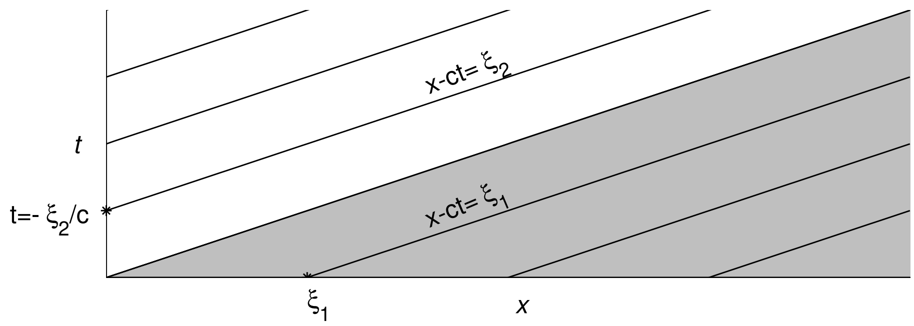


Figure 2.4: The (x,t)-diagram, or *characteristic diagram*, for the signaling problem with $c > 0$. The shaded region is $x - ct = \xi > 0$, in which the solution is determined by information from $t = 0$. The unshaded region is $x - ct = \xi < 0$, in which the solution is determined by information from $x = 0$. `signal.eps`

Note carefully we're discussing *entry* problems. That is

$$c > 0 \tag{2.82}$$

so that cars or photons are entering at $x = 0$. We'll come back at the end of this section and discuss the more subtle exit problem in which $c < 0$.

To formulate the problem mathematically, denote the rate at which cars enter at $x = 0$ by $R(t)$ (cars per second). The conservation equation is our old friend the linear advection equation

$$\rho_t + c\rho_x = 0\,. \tag{2.83}$$

For a PDE model we specify both an initial condition

$$\rho(x, 0) = F(x)\,, \qquad \text{for } x > 0\,, \tag{2.84}$$

and a boundary condition

$$c\rho(0, t) = R(t)\,, \qquad \text{for } t > 0\,. \tag{2.85}$$

Notice the dimensions — R is cars per second, ρ is cars per meter and c is meters per second.

To solve this problem it is essential to draw the (x,t)-plane and the family of curves on which $\rho(x,t)$ is constant. These *characteristics* are the lines

$$x - ct = \xi\,, \tag{2.86}$$

where ξ is the intercept of the characteristic with the x-axis (see figure 2.4). You can think of ξ as the name of each member of the family of lines in figure 2.4. Figure 2.4 is the *characteristic diagram* of the PDE— also known as the space-time diagram.

The solution of (2.83) is

$$\begin{aligned}\rho &= q(x - ct)\\ &= q(\xi)\,.\end{aligned}$$

The solution of (2.83) can also be written as $\rho = p(t - x/c)$.

The characteristics in the shaded region have $\xi > 0$ and they intersect the x-axis where (2.84) applies. Thus in the shaded region $\rho = F(x - ct)$: the signal from $x = 0$ has not had time to arrive, and the solution is identical to that of the initial value problem. The characteristics in the unshaded region intercept the t-axis where (2.85) applies. In this region the signal from $x = 0$ has finally arrived and we determine $q(x - ct)$ from

$$x = 0 : \quad cq(-ct) = R(t) \quad \Rightarrow \quad q(s) = c^{-1}R(-s/c) . \tag{2.87}$$

Thinking carefully about this argument, you'll see that characteristics are spacetime curves along which signals propagate. To determine the arbitrary function, q, we follow each characteristic till we hit the boundary of the space-time domain.

Summarizing, the solution of this "signaling" problem is

$$\rho(x,t) = \begin{cases} F(x - ct) , & \text{if } x - ct > 0, \\ c^{-1}R(t - x/c) , & \text{if } x - ct < 0. \end{cases} \tag{2.88}$$

In general $F(0) \neq R(0)$ so the solution might be discontinuous on the line $x = ct$. For instance, suppose $\rho(x, 0) = 0$ and R is a constant. Then the solution is $\rho(x,t) = c^{-1}RH(ct - x)$ where H is the Heaviside step function.

2.3.1 The exit problem, $c < 0$

If you redraw Figure 2.4 with $c < 0$ then you immediately see that the characteristics intersect the boundary *twice*: the lines of constant $x - ct$ intersect both the positive x-axis and the positive t-axis. Consequently if one arbitrarily specifies both $F(x)$ and $R(t)$ then usually the problem has no solution.

In discussing this non-existence one must be sensitively aware of the distinction between the mathematical formulation and the physical situation. The former is an approximation or idealization of the latter. If one is not aware of the physical motivation then one declares that the PDE plus boundary/initial condition has no solution, and that's the end of the matter. But thinking of the original motivation for this model in terms of traffic flow or photons it seems clear, to me at least, that one *can* prescribe the initial condition on the positive x-axis, and one *cannot* prescribe the exit condition on the positive t-axis. The exit at $x = 0$ takes what it gets and doesn't complain. If the exit does complain (e.g., if some fraction of the incident photons are reflected back into the domain) then one must reformulate the physical problem to account for this complication.

Example: Formulate a half-line ($x > 0$) model in which photons move in both directions with speed $\pm c$. Assume that all photons with speed $-c$ incident on $x = 0$ are reflected back with speed $+c$. Photons with speed $+c$ are also radiated at $x = 0$ at a rate $R(t)$. We need two densities, $\rho^+(x,t)$ for photons with $+c$ and $\rho^-(x,t)$ for photons with speed $-c < 0$. The PDEs are

$$\rho^+_t + c\rho^+_x = 0 , \qquad \text{and} \qquad \rho^-_t - c\rho^-_x = 0 . \tag{2.89}$$

The boundary condition at $x = 0$ is

$$c\rho^+(0,t) = R(t) + c\rho^-(0,t)\,. \tag{2.90}$$

We must also supply initial conditions for ρ^+ and ρ^- on the half-line. As a sanity check, you should show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \rho^+(x,t) + \rho^-(x,t)\,\mathrm{d}x = R(t)\,. \tag{2.91}$$

2.3.2 Age-Stratified Populations

Let us give another example of a context in which the half-line linear advection equation, (2.12) arises. To characterize the age structure of the population of San Diego at $t = 0$ we use a histogram:

$$h_0(a)\mathrm{d}a = \text{the number of people with age } a \in (a, a + \mathrm{d}a) \text{ at } t = 0. \tag{2.92}$$

The age-coordinate, a, is strictly positive so that the histogram lives on the half-line $a > 0$.

Suppose for a moment that this population is closed — we ignore births, deaths and immigration. Then it is obvious that at $t > 0$ the histogram of ages is $h(a,t) = h_0(a - t)$. Therefore the evolution of the age structure of this population is described by the PDE

$$h_t + h_a = 0\,. \tag{2.93}$$

In this example everyone strictly observes the speed limit by moving along the age axis at a rate one year per year — there is no doubt that the connection between flux and density is correct.

Now we face the facts of life by admitting that some people die, just as new people (babies) replace them. We can make the model more realistic by incorporating birth and death. The probability of death depends on age and also, perhaps, on time. This fact is incorporated by adding a loss term to the right hand side of (2.93):

$$h_t + h_a = -\mu(a,t)h\,, \tag{2.94}$$

where $\mu(a,t)$ is the probability that in the interval $(t, t + \mathrm{d}t)$ an individual of age a will die. Births correspond to the $a = 0$ boundary condition. If the birth rate is $b(t)$ (babies per second) then

$$h(0,t) = b(t)\,. \tag{2.95}$$

An increase in $b(t)$ produces a disturbance which will move through the histogram (a baby boom).

We can convince ourselves that this formulation is sensible by noticing that the population size is

$$N(t) = \int_0^\infty h(a,t)\,\mathrm{d}a\,. \tag{2.96}$$

If we integrate (2.94) from $a = 0$ to $a = \infty$, and use the boundary condition in (2.95), we get

$$\frac{\mathrm{d}N}{\mathrm{d}t} = b(t) - \int_0^\infty \mu(a,t)h(a,t)\,\mathrm{d}a\,. \tag{2.97}$$

This is the obvious conclusion that the total size of the population changes if there is an imbalance between the birth-rate and the death-rate.

Attrition of a cohort

Consider a population whose death rate depends only on age, a. A cohort of N babies that leaves the maternity ward: all members of the cohort start with with $a = 0$. Or a box of light bulbs leaves the factory. Or a bunch of N ^{14}C atoms are suddenly created in the upper atmosphere by a cosmic ray shower. The fate of this cohort is determined by solving

$$h_t + h_a = -\mu(a)h\,, \qquad h(a,0) = \delta(a)\,. \tag{2.98}$$

The quasi-obvious solution of this PDE is

$$h(a,t) = NS(a)\delta(a-t)\,, \tag{2.99}$$

where N is the initial size of the cohort and

$$S(a) \stackrel{\text{def}}{=} \exp\left(-\int_0^a \mu\left(a'\right)\mathrm{d}a'\right) \tag{2.100}$$

is the *survival function*. We have the interpretation

$$S(a) = \text{fraction of the cohort still alive at age } a.$$

Sadly the survival function S is a non-increasing function of a from its maximum $S(0) = 1$ to eventually zero at sufficiently large a.

Example: Solve the PDE in (2.98).

The familiar equations are

$$\frac{\mathrm{d}a}{\mathrm{d}t} = 1\,, \qquad \text{and} \qquad \frac{\mathrm{d}h}{\mathrm{d}t} = -\mu h\,. \tag{2.101}$$

The solution for the characteristic curves is $a = \alpha + t$ and then the evolution of h along a characteristic curve is

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\mu(\alpha + t)h\,, \tag{2.102}$$

which integrates to

$$h(a,t) = h_0(\alpha)\exp\Big[-\int_0^t \mu(\alpha + t')\mathrm{d}t'\Big]\,. \tag{2.103}$$

Applying the initial condition $h_0 = \delta$, and replacing α by $a - t$, we have

$$h(a,t) = \delta(a-t)\exp\Big[-\int_0^t \mu(a - t + t')\mathrm{d}t'\Big]\,, \tag{2.104}$$

$$= \delta(a-t)\underbrace{\exp\Big[-\int_0^a \mu(t')\mathrm{d}t'\Big]}_{S(a)}\,, \tag{2.105}$$

where a standard property of the δ-function has been used in the transition above.

Now suppose that $P(a)$ denotes the probability density function of lifespans e.g., if a is the age at death

$$\text{probability that } a_1 < a < a_2 = \int_{a_1}^{a_2} P(a')\,\mathrm{d}a' . \tag{2.106}$$

Let's find the relation between $S(a)$ and $P(a)$ i.e, given $S(a)$ how do we calculate $P(a)$? After some head scratching we see that

In other words $S(a)$ is the probability that a lifespan is longer than a.

$$S(a) = \int_a^\infty P\left(a'\right)\,\mathrm{d}a' . \tag{2.107}$$

In other words, $S(a)$ is the fraction of lifetimes longer than a. As a sanity check notice that $a = 0$ in (2.107) results in

Or

$$P = -\frac{\mathrm{d}S}{\mathrm{d}a}$$

$$1 = \int_0^\infty P(a)\,\mathrm{d}a . \tag{2.108}$$

That's the correct normalization for a probability density function.

Exercise: Show that

$$\tau \stackrel{\text{def}}{=} \int_0^\infty aP(a)\,\mathrm{d}a = \int_0^\infty S(a)\,\mathrm{d}a . \tag{2.109}$$

τ is the average lifespan.

Populations in equilibrium

Consider an age-stratified population whose death-rate is depends only on age, $\mu = \mu(a)$. Suppose further that the birth rate b is constant. Then we expect that eventually the age-structure of the population will stop changing — the population will be in equilibrium. We can determine this equilibrium by setting $h_t = 0$ and looking for a *steady solution* of the PDE (2.94) with the boundary condition in (2.95). This turns out to be very easy because now we deal only ODE's:

$$h_a = -\mu h , \tag{2.110}$$

with solution

$$h_{\mathrm{eq}}(a) = bS(a) . \tag{2.111}$$

Above $S(a)$ is the survival function in (2.100). The total population is

$$N = b\int_0^\infty S(a)\,\mathrm{d}a . \tag{2.112}$$

The expression for the mean lifespan τ in (2.109) confirms our intuitive expectation that for a population in equilibrium

Equation (2.109):

$$\tau = \int_0^\infty S(a)\,\mathrm{d}a$$

$$N = b \times \text{average life span} . \tag{2.113}$$

On the other hand, the average age of the equilibrium population is

$$\bar{a} = N^{-1}\int_0^\infty a h_{\mathrm{eq}}(a)\,\mathrm{d}a , \tag{2.114}$$

and using the results above

$$\bar{a} = \int_0^\infty aS(a)\,\mathrm{d}a \Big/ \int_0^\infty S(a)\,\mathrm{d}a\,. \tag{2.115}$$

There is not an obvious relation between the average lifespan τ and the average age $\bar{a}$ of the population.

Exercise: Show that if μ is constant, then the mean lifespan τ is equal to the average age $\bar{a}$ of the equilibrium population.

Exercise: Suppose that the survival function is

$$S(a) = \begin{cases} 1\,, & \text{if } 0 < a < a_*; \\ 0\,, & \text{if } a_* < a. \end{cases} \tag{2.116}$$

Design a death rate, $\mu(a)$, that produces an approximation to this $S(a)$. Show that $\tau = a_*$ and $\bar{a} = a_*/2$.

Example: a baby boom

Let us go back to our earlier model of an age-stratified population. Recall that the PDE formulation is

$$h_t + h_a = -\mu(a)h\,, \qquad h(0,t) = b(t)\,, \tag{2.117}$$

where $b(t)$ is flux of people into the histogram (babies per second). This is naturally a signaling problem on the axis $0 < a < \infty$.

Let us adopt a simple model of the death-rate: assume that μ is constant. Suppose that the birth rate is also a constant, b_0. If we wait for a very long time then the age structure of the population comes into *equilibrium*. In other words, the solution of (2.94) is *steady* and we can very quickly find:

$$h_{\mathrm{eq}}(a) = b_0 e^{-\mu a}\,. \tag{2.118}$$

The steady-state population is $N_0 = b_0/\mu$.

We suppose now that the population is in steady state when $t < 0$ but then at $t = 0$ a baby boom starts. We can model this situation by solving the PDE

$$h_t + h_a = -\mu h\,, \quad h(a,0) = b_0 e^{-\mu a}\,, \tag{2.119}$$

with the birth rate

$$h(0,t) = b_0(1 + \mathrm{e}^{-\alpha t})\,. \tag{2.120}$$

At $t = 0$ the birth rate $b(t)$ suddenly jumps to $2b_0$ and then $b(t)$ falls back again to b_0.

We solve this problem by noticing that the PDE in (2.119) can be written as

$$(\mathrm{e}^{\mu a}h)_t + (\mathrm{e}^{\mu a}h)_a = 0\,. \tag{2.121}$$

Thus $\tilde{h} \overset{\text{def}}{=} \exp(\mu a)h$ satisfies the linear advection equation and the general solution of the PDE in (2.119) is therefore

$$\tilde{h}(a,t) = q(a-t)\,, \quad \Rightarrow \quad h(a,t) = \mathrm{e}^{-\mu a}q(a-t)\,. \tag{2.122}$$

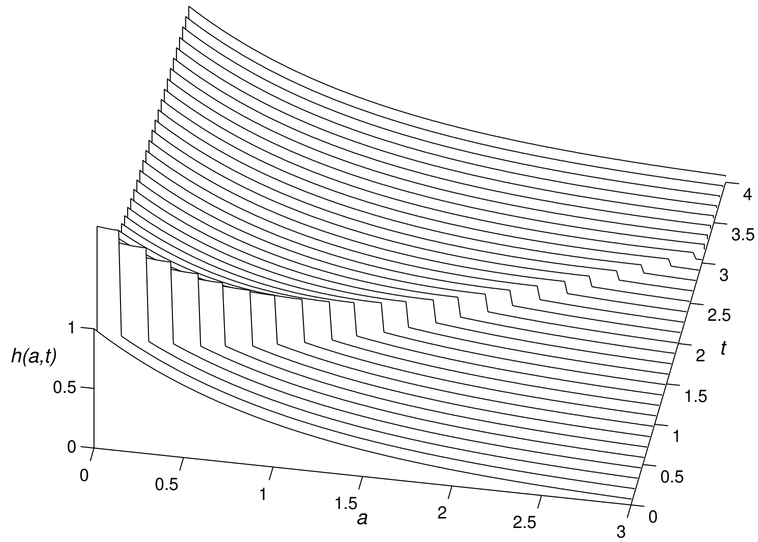


Figure 2.5: The baby boom solution in (2.123). `babyBoom.eps`

Now we have to determine the function q using the boundary condition at $a = 0$ and the initial condition at $t = 0$. This is exactly the signaling problem discussed earlier in this lecture. Here is the answer

$$h(a,t) = \begin{cases} b_0 \mathrm{e}^{-\mu a}\,, & \text{if } a - t > 0, \\ b_0 \mathrm{e}^{-\mu a}\left[1 + \mathrm{e}^{\alpha(a-t)}\right]\,, & \text{if } a - t < 0. \end{cases} \tag{2.123}$$

To visualize the solution we try a MATLAB `waterfall` plot (see figure 2.5). Is it intuitively obvious to you that the solution in the (a,t)-diagram does not change from the initial condition till $t > a$?

A final remark is that we might want to determine the total size of the population, $N(t)$ in (2.96). Of course we can do this by direct integration of the explicit solution (2.123) over $0 < a < \infty$. But there is an slicker approach which you can use to obtain $N(t)$ even before the PDE is solved. Do you see how to do this?

2.4 Probability generating functions

Probability theory provides interesting and nontrivial applications of the method of characteristics. An historical example is provided by the engineers of the Ericsson phone company, who are interested in how many phone lines are busy at time t in Stockholm. This is a random process, and so they define

$$p_n(t) = \text{probability that at } t \text{ there are } n \text{ lines busy}\,. \tag{2.124}$$

We assume that phone conversations are independent and that in an interval $\mathrm{d}t$ people terminate a call with a probability $\mu\mathrm{d}t$. Of course, if there are n

conversations the probability of a termination is $n\mu \mathrm{d}t$. We assume that a new call is initiated with a probability $\lambda \mathrm{d}t$.

Notice that the initiation process is independent of how many lines happen to be busy: we are assuming that the population of Stockholm is much greater than the number of busy lines. In this case the initiation of a new call in $(t, t + \mathrm{d}t)$ happens with probability $\lambda \mathrm{d}t$ no matter whether 10 people or 1000 people are chatting on the phone. This might is plausible if the population of Stockholm is 10^6, but not if the population is 10^4. We also neglect the probability that a call finishes and starts on the same line because that's proportional to $(\mathrm{d}t)^2$.

These assumptions leads to an infinite system of coupled ordinary differential equations

$$\begin{aligned}
\dot{p}_0 &= -\lambda p_0 + \mu p_1 \,, \\
\dot{p}_1 &= \lambda p_0 - (\lambda + \mu) p_1 + 2\mu p_2 \,, \\
\dot{p}_2 &= \lambda p_1 - (\lambda + 2\mu) p_2 + 3\mu p_3 \,, \\
\dot{p}_3 &= \lambda p_2 - (\lambda + 3\mu) p_3 + 4\mu p_4 \,,
\end{aligned} \tag{2.125}$$

and so on.

To explain how we arrive at (2.125) consider the gain and loss of probability of the realizations with no lines busy:

$$\begin{aligned}
\dot{p}_0(t) = &- \{\text{loss if a call is initiated}\} \\
&+ \{\text{gain if a realization with one busy line “hangs-up”}\} \,.
\end{aligned} \tag{2.126}$$

The loss occurs at a rate $\lambda p_0(t)$ and the gain at a rate $\mu p_1(t)$. If we considering realizations with 3 busy lines we would write

$$\dot{p}_3(t) = + \{\text{gain from state 2 due to a new call }\} \tag{2.127}$$

$$\begin{aligned}
&- \{\text{loss from state 3 to state 4 due to a new call}\} \\
&- \{\text{loss from state 3 to state 2 due to a hang-up}\} \\
&+ \{\text{gain from state 4 due to a hang-up}\} \,.
\end{aligned} \tag{2.128}$$

The four terms on the right hand side above correspond to the four terms on the right hand side of (2.125). Do you see why probabilists call this a "one-step" process — this a Markov chain in which state n gains and loses probability only from its immediate neighbours at $n \pm 1$.

An essential check on our argument is that if we sum the system (2.125) we find

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{n=0}^{\infty} p_n(t) = 0 \,, \qquad \Rightarrow \qquad \sum_{n=0}^{\infty} p_n(t) = 1 \,. \tag{2.129}$$

Thus probability is conserved.

The mean number of lines in action is

$$\bar{n}(t) = \sum_{n=0}^{\infty} n p_n(t) \,. \tag{2.130}$$

Either by intuitive arguments, or from (2.125) it follows that

$$\frac{\mathrm{d}\bar{n}}{\mathrm{d}t} = \lambda - \mu\bar{n}\,. \tag{2.131}$$

This is the mean field description of the process: as $t \to \infty$ the *expected* number of busy lines is λ/μ. However we do not know how large the *fluctuations* about this mean state are likely to be.

This is all very interesting, but what does it have to with PDE's? One way of obtaining the complete solution of (2.125) is to introduce the *probability generating function*:

$$G(z,t) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} p_n(t) z^n\,. \tag{2.132}$$

Fiddling around with (2.125) we can show that $G(z,t)$ satisfies

$$G_t + \mu(z-1)G_z = \lambda(z-1)G\,. \tag{2.133}$$

Once we specify an initial condition we can solve this first-order PDE and figure out the Taylor series expansion of $G(z,t)$. The coefficents in this expansion are the probabilities $p_n(t)$.

The PDE (2.133) has already been solved in an earlier example — see (2.69). Now suppose that at $t = 0$ there are no lines in action. Then $p_0(0) = 1$ and all the other p_n's are zero. In this case the initial condition is

$$G_0(z) = 1\,, \tag{2.134}$$

and the solution in (2.69) is

$$G(z,t) = \exp\left[(z-1)\frac{\lambda}{\mu}(1-\mathrm{e}^{-\mu t})\right]\,. \tag{2.135}$$

As a quick check on our work, we notice from the definition in (2.132) that we must have

$$G(1,t) = \sum_{n=0}^{\infty} p_n(t) = 1\,. \tag{2.136}$$

The expression in (2.135) passes this basic test.

We also see from (2.132) that

$$G_z(1,t) = p_1 + 2p_2(t) + 3p_3(t) + \cdots = \bar{n}(t)\,. \tag{2.137}$$

Taking the z-derivative of (2.69) and setting $z = 1$ we quickly find

$$\bar{n}(t) = \frac{\lambda}{\mu}\left(1 - \mathrm{e}^{-\mu t}\right)\,. \tag{2.138}$$

But we can also calculate $\bar{n}(t)$ by solving (2.131) with the initial condition $\bar{n}(0) = 0$. This approach also gives (2.138) — we are starting to have some confidence in (2.69)! Of course we should really check (2.69) by substituting it back into the PDE...

After checking (2.69) we are finished with the PDE, but not with generatingfunctionology. We can rewrite the probability generating function in (2.69) as

$$G(z,t) = \mathrm{e}^{(z-1)\bar{n}}\,, \tag{2.139}$$

where $\bar{n}(t)$ is given in (2.138). Recalling the Taylor series expansion

$$\mathrm{e}^{a} = \sum_{k=0}^{\infty} \frac{a^k}{k!}\,, \tag{2.140}$$

we then find from (2.139) that the probability that k lines are busy is

$$p_k(t) = \mathrm{e}^{-\bar{n}} \frac{\bar{n}^k}{k!}\,. \tag{2.141}$$

This is a Poisson distribution.

The birth-death processes

Consider a population of bugs reproducing and dying in your refrigerator. $p_k(t)$ is the probability that there are k bugs at time t. We suppose that the probability per time that a single bug divides is $\lambda \mathrm{d}t$. Thus, if there are n bugs the probability of a birth is $n\lambda \mathrm{d}t$. Likewise $n\mu \mathrm{d}t$ is probability of a death in $\mathrm{d}t$. The mean field model for the expected number of bugs is just

$$\frac{\mathrm{d}\bar{n}}{\mathrm{d}t} = \gamma \bar{n}\,, \tag{2.142}$$

where $\gamma \stackrel{\text{def}}{=} \lambda - \mu$ is the *growth rate*. This tells us that *on average* the bug population changes exponentially with time. Of course, even if $\gamma > 0$, bad luck might extinguish a particular bug culture. This is an example of a population *fluctuation*. To quantitatively understand this we turn to the *birth-death equations*:

$$\begin{aligned}
\dot{p}_0 &= \mu p_1\,,\\
\dot{p}_1 &= -(\lambda+\mu)p_1 + 2\mu p_2\,,\\
\dot{p}_2 &= \lambda p_1 - 2(\lambda+\mu)p_2 + 3\mu p_3\,,
\end{aligned} \tag{2.143}$$

and so on. The k'th equation is

$$\dot{p}_k = \lambda(k-1)p_{k-1} - (\lambda+\mu)kp_k + \mu(k+1)p_{k+1}\,. \tag{2.144}$$

As a sanity check you can verify that probability is conserved. This is like the trunking problem except that the probability of a new bug appearing in $\mathrm{d}t$ is proportional to the number of bugs. Notice also that extinction is an absorbing state — bugs don't appear out of thin air. (But telephone calls do.)

Once again, the generating function is defined by

$$G(z,t) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} z^k p_k(t)\,. \tag{2.145}$$

From (2.143), it follows that the generating function satisfies

$$G_t + (1-z)(\lambda z - \mu) G_z = 0 . \tag{2.146}$$

If each realization has n_0 organisms at $t=0$ then $p_k(0) = \delta_{k-n_0}$ and $G(z,0) = z^{n_0}$.

Equation (2.146) can be solved with method of characteristics. The characteristic coordinate, $\xi(z,t)$, is the solution of

$$\frac{\mathrm{d}z}{\mathrm{d}t} = (1-z)(\lambda z - \mu) , \qquad \text{with IC} \qquad z(0) = \xi . \tag{2.147}$$

We integrate this ODE by separating variables and using the partial fraction

$$\frac{1}{(1-z)(\lambda z - \mu)} = \frac{1}{\lambda - \mu}\left[\frac{1}{1-z} + \frac{\lambda}{\lambda z - \mu}\right] . \tag{2.148}$$

The result is

$$\ln\left[\frac{\lambda z - \mu}{1-z}\,\frac{1-\xi}{\lambda\xi - \mu}\right] = \gamma t . \tag{2.149}$$

Solving (2.149) for ξ we exhibit the characteristic coordinate

$$\xi(z,t) = \frac{\mu(1-e^{\gamma t}) + (\mu e^{\gamma t} - \lambda) z}{(\mu - \lambda e^{\gamma t}) + \lambda(e^{\gamma t} - 1) z} . \tag{2.150}$$

The general solution of (2.146) is therefore $G(z,t) = \mathcal{G}(\xi)$, where $\mathcal{G}$ is an arbitrary function. If there are n_0 organisms at $t=0$ then $G(z,0) = z^{n_0}$, and consequently $\mathcal{G}(\xi) = \xi^{n_0}$. Thus, one finds that

$$G(z,t) = \left[\frac{\mu(1-e^{\gamma t}) + (\mu e^{\gamma t} - \lambda) z}{(\mu - \lambda e^{\gamma t}) + \lambda(e^{\gamma t} - 1) z}\right]^{n_0} . \tag{2.151}$$

The probability of extinction is $p_0(t) = G(0,t)$ and so from (2.151):

$$p_0(t) = \left[\frac{\mu - \mu e^{(\lambda-\mu)t}}{\mu - \lambda e^{(\lambda-\mu)t}}\right]^{n_0} . \tag{2.152}$$

If $\lambda - \mu > 0$ then, as $t \to \infty$, the probability of extinction is $p_0 \to (\mu/\lambda)^{n_0}$. What happens if $\lambda - \mu < 0$?

In general it is difficult to find the coefficient of z^k in the Taylor series expansion of (2.151). But in the special case $n_0 = 1$ one can expand $G(z,t)$ in (2.151) and obtain

$$p_0(t) = \mu\frac{e^{(\lambda-\mu)t} - 1}{\lambda e^{(\lambda-\mu)t} - \mu}, \qquad p_k(t) = (1-p_0)\left(1 - \frac{\lambda}{\mu}p_0\right)\left(\frac{\lambda}{\mu}p_0\right)^{k-1} \quad \text{if } k \geq 1 . \tag{2.153}$$

Notice that even if the growth rate, $\gamma = \lambda - \mu$, is positive the probability of ultimate extinction is nonzero i.e., $p_0(\infty) = \mu/\lambda$.

References specialized to pgfs

Chapter XVII of the classic:

Fe *An Introduction to Probability Theory and its Applications* by W. Feller.

is my source. (This chapter is self-contained and can be understood without reading the preceding 16 chapters.) Another useful reference is chapter VI of

vK *Stochastic Processes in Physics and Chemistry* by N.G. van Kampen.

For lots more on generating functions see

Wi *Generatingfunctionology* by Herbert S. Wilf

2.5 Problems

Problem 2.1. Read the discussion of sedimentation in section 2.1 of the notes. To model *hindered sedimentation* suppose that the sedimentation speed is

$$s = s_0 \left(1 - \frac{n}{n_*}\right) , \tag{2.154}$$

where s_0 is the Stokes velocity of a single particle and n_* is the close-packing density. *(i)* How must (2.16) be modified to account for hindering? State the boundary conditions at $z = 0$ and h that apply to your model of hindered settling *(ii)* Verify that sediment is conserved i.e. show that

$$\int_0^h n(z,t)\, \mathrm{d}z \tag{2.155}$$

is independent of time. *(iii)* Assume that the vessel is very tall so the top boundary condition at $z = h$ can be replaced by $n(z) \to 0$ as $z \to \infty$. With this simplification, find the steady solution, analogous to (2.23) in the notes. *(iv)* Check that you recover (2.23) when there is a small amount of sediment in the vessel. You'll need to identify a non-dimensional parameter that quantifies "a small amount of sediment".

Problem 2.2. Solve $\theta_t - \alpha x \theta_x = 0$ with the initial condition $\theta(x, 0) = \cos kx$.

Problem 2.3. Solve the advection-reaction equations

$$\theta_t + \theta_x = -\theta\phi , \qquad \phi_t + \phi_x = \theta\phi ,$$

with the initial condition $\theta(x, 0) = \phi(x, 0) = \frac{1}{2} f(x)$.

Problem 2.4. Find an integral expression for the solution of the forced linear advection problem (with constant $c > 0$):

$$\rho_t + c\rho_x = s(x,t) , \tag{2.156}$$

in the domain $-\infty < x < \infty$ and $t > 0$ with the initial condition $\rho(x, 0) = 0$. Make sure you check your result by substitution.

Problem 2.5. Show that solution of the PDE

$$\rho_t + [c(x)\rho]_x = 0\,, \qquad \rho(x,0) = \rho_0(x)\,, \tag{2.157}$$

is

$$\rho(x,t) = \frac{\partial \xi}{\partial x}\rho_0(\xi) = \frac{c(\xi)}{c(x)}\rho_0(\xi)\,, \tag{2.158}$$

where $\xi(x,t)$ is the characteristic coordinate defined in (2.75).

Problem 2.6. Find an integral expression for the solution of the forced linear advection problem (with constant $c > 0$):

$$\rho_t + c\rho_x = s(x,t)\,, \tag{2.159}$$

in the domain $0 < x < \infty$ and $t > 0$ with the initial and boundary conditions

$$\rho(x,0) = 0\,, \qquad \text{and} \qquad \rho(0,t) = 0\,. \tag{2.160}$$

Problem 2.7. Consider a semi-infinite pipe with cross-sectional area α square meters, lying along the positive x-axis. A solution of water and a chemical A is pumped into the pipe at a constant rate q kilograms of solution per second though the inlet at $x = 0$. The chemical A spontaneously transforms into a product B at a rate β (inverse seconds):

$$A \xrightarrow{\beta} B\,. \tag{2.161}$$

The density of the solution, ρ kilograms per cubic meter, is not affected by this transformation and is constant.

Assume that both A and B are well mixed over the cross section of the pipe. Denote the sectionally integrated concentrations of A and B by $a(x,t)$ and $b(x,t)$, both with dimensions kilograms per meter of pipe. At the inlet at $x = 0$, A enters the pipe at a rate $r(t)Q$ kilograms of A per second; there is no B entering the pipe. At $t = 0$ the pipe contains pure water. Write down the coupled PDE conservation laws for a and b and state boundary and initial conditions. Suppose $r(t) = r_0 H(t)$ where $H(t)$ is the Heaviside step function. Find the steady state ($t \to \infty$) concentrations $a(x)$ and $b(x)$. What is the steady-state sectionally integrated concentration of water?

Problem 2.8. Solve the PDE:

$$\rho_t + (\mathrm{e}^{-x}\rho)_x = 0\,, \qquad \text{with IC } \rho(x,0) = \mathrm{e}^{-\mu^2 x^2}\,. \tag{2.162}$$

Use MATLAB to visualize the solution (see Figure 2.6 for the answer).

Problem 2.9. As a simple model of radiative relaxation to a uniform thermal gradient, Γ, consider

$$\theta_t + a \sin \omega t\, \theta_x = \beta(\Gamma x - \theta)\,. \tag{2.163}$$

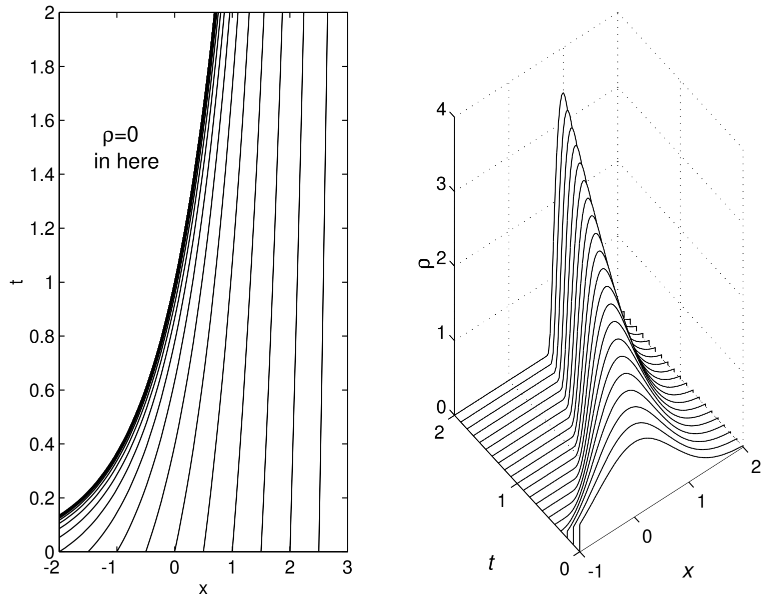


Figure 2.6: The left panel shows the characteristics of (2.162). The right panel is the solution. `pileUpFig.eps`

$\theta(x,t)$ is temperature and the distance a is the amplitude of an oscillatory excursion due to a wave with frequency ω. Solve this PDE with the initial condition $\theta(x,0)=0$. Show that in the long time limit, $1 \ll \beta t$, the time-averaged flux of θ-stuff is

$$f = a\omega\,\overline{\sin\omega t\,\theta}\,, \tag{2.164}$$

$$= -D\Gamma\,, \tag{2.165}$$

where the effective diffusivity is

$$D = \tfrac{1}{2}\omega a^2 \frac{\beta\omega}{\beta^2+\omega^2}\,. \tag{2.166}$$

Note there be a net flux of θ even though air parcels have no net displacement.

Problem 2.10. *(i)* With constant c, solve the PDE

$$\theta_t + c\theta_x = \frac{1}{1+x^2}\,, \qquad \text{with IC } \rho(x,0)=0. \tag{2.167}$$

(ii) Evaluate

$$m(t) \overset{\text{def}}{=} \int_{-\infty}^{\infty} \theta(x,t)\,\mathrm{d}x\,, \tag{2.168}$$

the easy way. *(iii)* Visualize the solution using MATLAB.

Problem 2.11. Solve the PDE

$$\rho_t - (\tanh x\,\rho)_x = 0\,, \qquad \rho(x,0)=\rho_0(x)\,. \tag{2.169}$$

Exhibit the characteristic coordinate $\xi(x,t)$ and use MATLAB to draw the characteristic curves in the (x,t)-plane. Show directly from the PDE that $\rho(0,t) = \mathrm{e}^t \rho_0(0)$ and make sure your answer agrees with this.

Problem 2.12. *(i)* Solve

$$\rho_t + (c\rho)_x = 0 \tag{2.170}$$

with speed

$$c(x) = \frac{\mathrm{e}^x + 2}{2\mathrm{e}^x + 2}\,, \tag{2.171}$$

and the initial condition $\rho(x,0) = \exp(-(x+5)^2)$. *(ii)* Use MATLAB to visualize ρ as a function of x at selected times. Show that at large times the peak value of $\rho(x,t)$ is 2.

Problem 2.13. Solve

$$(1+x^2)\eta_t + \eta_x = 0\,, \qquad \text{with the IC} \qquad \eta(x,0) = \exp\left[-9\,(x+2)^2\right]\,. \tag{2.172}$$

Check your answer by reproducing the characteristic diagram in the lower panel of figure 2.2. Use MATLAB to draw a figure analogous to 2.3 showing snapshots of the solution at selected times.

Problem 2.14. Consider the trunking problem and suppose that at $t = 0$ exactly one line is busy. *(i)* Find the probability generating function by solving (**??**) with the appropriate initial condition. *(ii)* Use your solution to obtain $p_0(t)$ and $p_1(t)$.

Problem 2.15. Consider the system

$$\dot{p}_1 = -p_1\,, \quad \dot{p}_n = (n-2)p_{n-2} - np_n\,, \tag{2.173}$$

with the initial condition $p_n(0) = \delta_{1,n}$. Show that the generating function is

$$G(z,t) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} p_n(t) z^n\,, \tag{2.174}$$

$$= \frac{\mathrm{e}^{-t} z}{\sqrt{1 - (1 - \mathrm{e}^{-2t})z^2}}\,. \tag{2.175}$$

Using the binomial theorem, or MATHEMATICA, find an expression fore $p_n(t)$.

Problem 2.16. Consider the birth-death process in (2.143). Define

$$\alpha_0(t) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} p_k\,, \qquad \alpha_1 \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} kp_k\,, \qquad \alpha_2 \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} k(k-1)p_k\,. \tag{2.176}$$

Show from (2.143) that

$$\dot{\alpha}_0 = 0\,, \qquad \dot{\alpha}_1 = \gamma\alpha_1\,, \qquad \dot{\alpha}_2 = 2\lambda\alpha_1 + 2\gamma\alpha_2\,, \tag{2.177}$$

where $\gamma \stackrel{\text{def}}{=} \lambda - \mu$ is the growth rate. Interpret these results probabilistically. In particular show that if $\gamma \le 0$ then eventually fluctuations in the size of the population must become comparable to the average population.

Problem 2.17. Consider the special case in which the birth rate is equal to the death rate, $\lambda = \mu$, in (2.146) of the notes:

$$G_t + \mu(1 - z^2)G = 0\,. \tag{2.178}$$

(i) Start from scratch and solve (2.178) using the initial condition which is appropriate if $p_1(0) = 1$ and all other $p_k(0)$'s are zero i.e., there is exactly one bug at $t = 0$. *(ii)* Show that

$$p_0(t) = \frac{\mu t}{1 + \mu t}\,, \qquad \text{and} \qquad p_k(t) = \frac{(\mu t)^{k-1}}{(1 + \mu t)^{k+1}} \quad \text{if } k \geq 1. \tag{2.179}$$

What is the probability of extinction as a function of time? What is the average size of the population as a function of time?

Problem 2.18. A power plant supplies m machines which use the current intermittently. A machine switches off with a probability μ per unit time and switches on with a probability λ per unit time. All the machines operate independently of each other. *(i)* Construct the coupled ODE's describing the evolution of

$$p_k(t) = \text{Probability of } k \text{ machines drawing power at } t\,, \tag{2.180}$$

where $k = 0,\, 1,\, \cdots\, m$. *(ii)* Find the PDE satisfied by

$$\mathcal{G}(z, t) = \sum_{k=0}^{m} p_k(t) z^k\,. \tag{2.181}$$

(iii) Find the solution of this PDE with the initial condition $p_0(0) = 1$. *(iv)* Find the steady asymptotic ($t \to \infty$) solution.

Problem 2.19. Consider the trunking problem and suppose that the probability of initiation of a new call is

$$\lambda(t) = \mathrm{e}^{-\mu t}\lambda_0\,. \tag{2.182}$$

The probability of termination is still the constant μ. Supposing that $p_0(0) = 1$, find $p_n(t)$.

Problem 2.20. Solve the PDE:

$$\theta_t + c\theta_x = \alpha(1 - \theta)\,, \quad \theta(x, 0) = 0\,, \quad \theta(0, t) = 0\,, \tag{2.183}$$

in the domain $x > 0$ and $t > 0$. $\theta(x, t)$ might be the temperature of fluid which is being pumped with speed c through a semi-infinite pipe ($x > 0$). The pipe is heated uniformly, and fluid enters at $x = 0$ with $\theta = 0$. Discuss both $c > 0$ and $c < 0$. Sketch the solution at $\alpha t = 1$.

Problem 2.21. Consider a highway, $x > 0$, on which the speed is $c = c_0 \exp(-\alpha x)$. Suppose there are no cars on the highway at $t = 0$ and then cars enter the highway at $x = 0$ at a constant rate, R. *(i)* Formulate a PDE description of this problem. *(ii)* Check your formulation by showing that the number of cars on the highway at time $t > 0$ is Rt. *(iii)* Find the position of the car which leaves $x = 0$ at $t = \tau$. *(iv)* Solve the PDE and determine $\rho(x,t)$.

Problem 2.22. Consider a population in which the death rate is a decreasing function of time. As an illustrative model consider:

$$h_t + h_a = -\frac{\mu h}{1 + \alpha t} \,. \tag{2.184}$$

Suppose the age structure is in equilibrium at $t = 0$: $h(a, 0) = b_0 \exp(-\mu a)$. Notice that both μ and α have dimensions time^{-1}, so $\nu \stackrel{\text{def}}{=} \mu/\alpha$ is dimensionless. The constant ν appears prominently in the solution.
(i) How must the birth-rate, $b(t)$, decrease so that the total size of the population remains constant? *(ii)* Calculate the average age, $\bar{a}(t)$, of the population in this situation. Hint: you should answer *(i)* and *(ii)* without solving a PDE. *(iii)* Solve the PDE and obtain the age structure of the population.

Problem 2.23. Consider a steady-state population stratified by age a. Because of efficient medical care the death rate is

$$\mu(a) = \begin{cases} 0 \,, & \text{if } 0 < a < a_*; \\ \mu_* \,, & \text{if } a_* < a. \end{cases} \tag{2.185}$$

The birth rate, $b = h(0)$, is adjusted so that the population, $N = \int_0^\infty h(a)\,\mathrm{d}a$, is constant. What is the average lifespan τ of an individual? What is the average age of the population, $\bar{a} = \int_0^\infty a h(a) \mathrm{d}a/N$?

Problem 2.24. Consider a solution of photosensitive molecules occupying the region $x > 0$, and denote the density of molecules (i.e., molecules per cubic meter) by $n(x,t)$. The solution is irradiated by a beam of photons, with density $\rho(x,t)$ photons per cubic meter. A photon is absorbed by a molecule, which then decomposes so that the solution becomes more transparent to light. The model is

$$\rho_t + c\rho_x = -\alpha n \rho \,, \tag{2.186}$$

$$n_t = -\alpha n \rho \,, \tag{2.187}$$

where c is the speed of light. The initial conditions are

$$n(x,0) = n_0 \qquad \rho(x,0) = 0 \,, \tag{2.188}$$

and boundary condition is $\rho(0,t) = \rho_0$ i.e., starting at $t = 0$, $I_0 \stackrel{\text{def}}{=} c\rho_0$ photons enter the solution at $x = 0$. Determine the non-dimensional parameters in this problem and identify a limit in which the term ρ_t can be neglected i.e., the radiation is "quasi-static". Solve the problem in this limit by finding analytic expressions for $\rho(x,t)$ and $n(x,t)$.

Lecture 3

δ-functions and Green's functions

Can I assume that you understand the concept of a Green's function, and δ-functions, at least as far as ODEs are concerned? The material in this lecture is in section 1.5 of **BO**, but here is a little review in case you need it. In this lecture we'll only discuss ordinary differential equations, and we'll begin with evolution problems in which the independent variable is t = time. In this context the function $\delta(t)$ embodies the concept of impulsive action. For example, when a hammer pounds a nail into a plank it must exert a force on the head of the nail. If you can draw a graph of this force as function of time then you have a good idea of what $\delta(t)$ means.

Notation:

$$\dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

3.1 Review of "patching"

Before we discuss δ-functions we review the solution of ordinary differential equations with discontinuous coefficients and forcing functions. This is the method of "patching". We illustrate patching by solving a few simple examples from mechanics.

A prime example of a discontinuous functions is the Heaviside step function

$$\mathrm{H}(t) = \begin{cases} 0\,, & \text{if } t < 0; \\ 1\,, & \text{if } t > 0. \end{cases} \tag{3.1}$$

Notice we've left $\mathrm{H}(0)$ undefined. As we solve the following differential equations it will become clear that the value of $\mathrm{H}(0)$ is irrelevant. If it makes you feel better you can define it as $\mathrm{H}(0) = 1/2$, but it makes no difference.

As an example we consider the forced oscillator problem

$$\ddot{\theta} + \sigma^2\theta = \mathrm{H}(t)\,, \tag{3.2}$$

with the condition that

$$\theta(t) = 0\,, \qquad \text{when } t < 0. \tag{3.3}$$

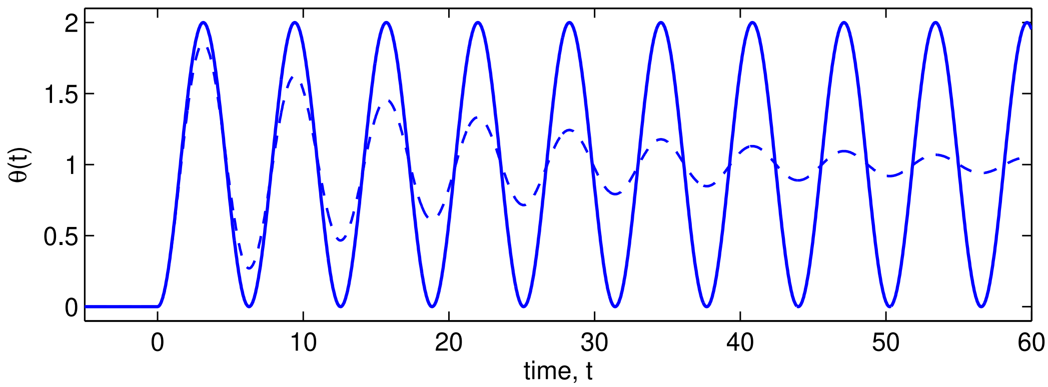


Figure 3.1: The solid curve is the undamped-oscillator solution in (3.7). With no damping, the oscillator rings around the equilibrium solution $\theta = 1$. The dashed curve is the solution for a lightly damped oscillator in problem 3.5 ($\mu/\sigma = 0.1$ in this illustration). `oscillator.eps`

When $t < 0$ the oscillator is at rest minding its own business. Then suddenly at $t = 0$ a constant force is switched on. How does the oscillator respond to this discontinuous force?

When $t > 0$ the general solution of (3.2) is

$$\theta(t) = \underbrace{\sigma^{-2}}_{=\text{particular solution}} + \underbrace{A\cos\sigma t + B\sin\sigma t}_{=\text{homogeneous solution}} . \tag{3.4}$$

The constants A and B are determined so that the solution is smooth at $t = 0$. Smooth means that both θ and $\dot\theta$ are continuous at $t = 0$. That is[1]

$$\theta(0^+) = 0 , \qquad \text{and} \qquad \dot\theta(0^+) = 0 . \tag{3.5}$$

The second derivative $\ddot\theta$ is discontinuous: in (3.2) the discontinuous function $H(t)$ on the right is balanced by a discontinuous function $\ddot\theta$ on the left:

$$\ddot\theta(0^-) = 0 , \qquad \text{and} \qquad \ddot\theta(0^+) = 1 . \tag{3.6}$$

Applying the 0^+ initial conditions in (3.5) to (3.4) we find the solution

$$\theta(t) = \sigma^{-2}\,(1 - \cos\sigma t)\ \mathrm{H}(t) , \tag{3.7}$$

shown as the solid curve in Figure 3.1. The $\mathrm{H}(t)$ is inserted in (3.7) so that the solution is valid on the whole time axis $-\infty < t < \infty$.

A third-order example

Let's consider another example of a differential equation involving $\mathrm{H}(t)$:

$$\dddot y - 7\dot y - 6y = \mathrm{H}(t) . \tag{3.8}$$

[1]Notation: $\theta(0^+)$ means $\theta(t)$ evaluated at t plus a little bit.

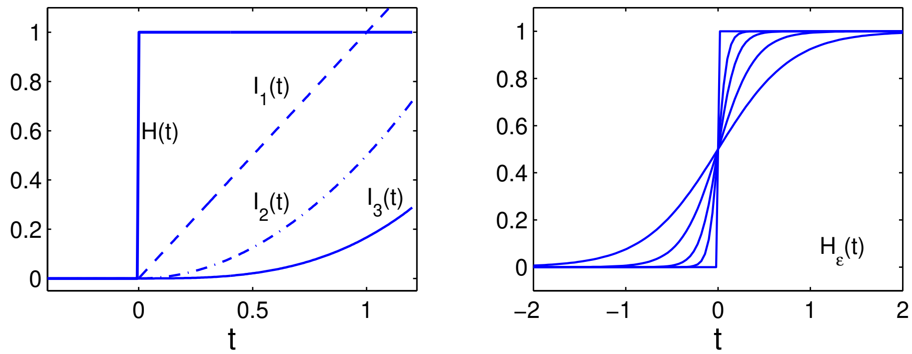


Figure 3.2: Left panel: The step function $H(t)$, and three successive integrals using the definition in (3.11). The singular behavior at $t = 0$ is smoothed out by integration. Right panel: the smoothed step function $H_\epsilon(t)$ defined in (3.13). In this illustration $\epsilon = [0.4, 0.2, 0.1, 0.05, 0.025]$. `jumpy.eps`

Something on the left hand side of (3.8) must balance the discontinuous forcing function on the right. Suspicion falls heavily on the term with the most derivatives i.e., $\dddot{y}$. This means that at $t = 0$ the third derivative $\dddot{y}$ is discontinuous:

$$\dddot{y}(0^+) - \dddot{y}(0^-) = 1 \,. \tag{3.9}$$

The terms with fewer derivatives are better behaved i.e., $\ddot{y}(t)$, $\dot{y}(t)$ and $y(t)$ are all continuous at $t = 0$. This is a general principle: *Taking derivatives makes a singularity worse.* The converse is: *Integrating makes a singularity better.*

Integrating makes a singularity better.

Let's beat this point to death by considering the Heaviside step $\mathrm{H}(t)$, and the functions obtained by successive integration

$$I_1(t) = \int_{-\infty}^{t} \mathrm{H}(t')\,\mathrm{d}t' \,, \tag{3.10}$$

$$I_2(t) = \int_{-\infty}^{t} I_1(t')\,\mathrm{d}t' \,, \tag{3.11}$$

and so on. It is easy to see that

$$I_n(t) = \mathrm{H}(t)\frac{t^n}{n!} \,. \tag{3.12}$$

The left panel of figure 3.2 shows that the sudden behaviour at $t = 0$ becomes less prominent as we successively integrate.

Another way of looking at this is to view $\mathrm{H}(t)$ as the limit of a sequence of very smooth functions. This limiting process is illustrated in the right panel of Figure 3.2 using

$$\mathrm{H}(t) = \lim_{\epsilon \to 0} \underbrace{\frac{1}{1 + \exp(-t/\epsilon)}}_{\stackrel{\mathrm{def}}{=} H_\epsilon(t)} \,. \tag{3.13}$$

We can then view the solution of (3.2) as the $\epsilon \to 0$ limit of the family of problems

$$\ddot{\theta}_\epsilon + \sigma^2 \theta_\epsilon = \mathrm{H}_\epsilon \,. \tag{3.14}$$

As long as ϵ is not zero $H_\epsilon(t)$ and $\theta_\epsilon(t)$ are infinitely differentible functions of t. But the $\epsilon = 0$ limiting functions have singularities at $t = 0$.

3.2 The oscillator Green's function

Let's follow the historical development of the subject and sneak up on the construction of the Green's function without using δ-functions. We begin by considering another example of an oscillator with discontinuous forcing

$$\ddot{\phi} + \sigma^2 \phi = \underbrace{\mathrm{H}(t) - \mathrm{H}(t-\tau)}_{\stackrel{\text{def}}{=} b(t,\tau)}, \tag{3.15}$$

with $\phi(t) = 0$ when $t < 0$. On the right of (3.15), $b(t,\tau)$ is the "block" function.

Let's solve (3.15) the easy way. Using linearity and temporal translation invariance, the solution of (3.15) is expressed in terms of the solution of the θ-problem in (3.7) as

$$\begin{aligned} \phi(t) &= \theta(t) - \theta(t-\tau)\,, \\ &= \frac{1-\cos\sigma t}{\sigma^2}\,\mathrm{H}(t) - \frac{1-\cos\sigma(t-\tau)}{\sigma^2}\,\mathrm{H}(t-\tau)\,. \end{aligned} \tag{3.16}$$

One absolutely *must* include the $\mathrm{H}(t)$ in (3.7) if this beautiful trick is to work. Back in (3.7) the $\mathrm{H}(t)$ seems like a fussy little device. But in this example the $\mathrm{H}(t)$ ensures that the solution $\phi(0 < t < \tau)$ does not anticipate that the forcing on the right of (3.15) is going to switch off at the future time $t = \tau$.

When $t > \tau$ the solution in (3.16) is

$$\phi(t > \tau) = \frac{\cos\sigma(t-\tau) - \cos\sigma t}{\sigma^2}\,. \tag{3.17}$$

We can pick τ so that the $\phi(t)$ above is equal to zero e.g., take $\sigma\tau = 2\pi$. Alternatively if $\sigma\tau = \pi$ then the oscillation has maximum amplitude viz., $\phi(t > \tau) = -2\cos\sigma t$. So the ultimate amplitude of the oscillation can be controlled by timing when the force is switched off.

Example: Solve (3.15) the hard way — by patching at both $t = 0$ and $t = \tau$.

Next consider an oscillator driven by a sum of blocks:

$$\ddot{\psi} + \sigma^2 \psi = \sum_{n=0}^{\infty} f_n \, b\,(t - t_n, \tau)\ , \tag{3.18}$$

with $\psi(t) = 0$ when $t < 0$ and

$$t_n \stackrel{\text{def}}{=} n\tau\,. \tag{3.19}$$

Someone should draw a figure showing this blocky forcing function. Using linearity, the solution of (3.18) is therefore

$$\psi = \sum_{n=0}^{\infty} f_n \phi\left(t - t_n\right) \,, \tag{3.20}$$

where $\phi(t)$ is the solution produced by a single block in (3.16).

Finally consider

$$\ddot{\chi} + \sigma^2 \chi = f(t) \,, \tag{3.21}$$

with both $\chi(t)$ and $f(t)$ zero when $t < 0$. In (3.21) the forcing $f(t)$ is some smooth function. In the sense of a Riemann integral we can approximate $f(t)$ arbitrarily closely as a sum of blocks

$$f(t) = \lim_{\tau \to 0} \sum_{n=0}^{\infty} f(t_n)\, b\left(t - t_n\right) \,. \tag{3.22}$$

This is the same as the forcing on the right of (3.18), so the solution of (3.21) is

$$\chi = \lim_{\tau \to 0} \sum_{n=0}^{\infty} f(t_n) \phi(t - t_n) \,. \tag{3.23}$$

To take the $\tau \to 0$ limit, we simplify ϕ in (3.17) using $\sigma\tau \ll 1$:

$$\phi(t) = \mathrm{H}(t) \frac{\sin \sigma t}{\sigma} \tau + \mathrm{ord}(\tau^2) \,. \tag{3.24}$$

Discarding the τ^2 terms, the solution in (3.23) is

$$\begin{aligned} \chi &= \lim_{\tau \to 0} \sum_{n=0}^{\infty} f(t_n)\, \mathrm{H}(t - t_n) \frac{\sin \sigma(t - t_n)}{\sigma} \tau \,, \\ &= \int_0^{\infty} f(t')\, \underbrace{\mathrm{H}(t - t') \frac{\sin \sigma(t - t')}{\sigma}}_{\stackrel{\text{def}}{=} g(t - t')} \underbrace{\mathrm{d}t'}_{=\tau} \,. \end{aligned} \tag{3.25}$$

The function g is the Green's function of the oscillator problem. Because of the step function in (3.25) the integrand is zero once t' is greater than t and so we can write the solution of the oscillator problem (3.21) as

$$\chi(t) = \int_0^t f(t')\, \frac{\sin \sigma(t - t')}{\sigma}\, \mathrm{d}t' \,. \tag{3.26}$$

With (3.26) we have a general expression for the solution of the forced oscillator problem in (3.21).

Exercise: Verify by substitution that (3.26) is a solution of (3.21).

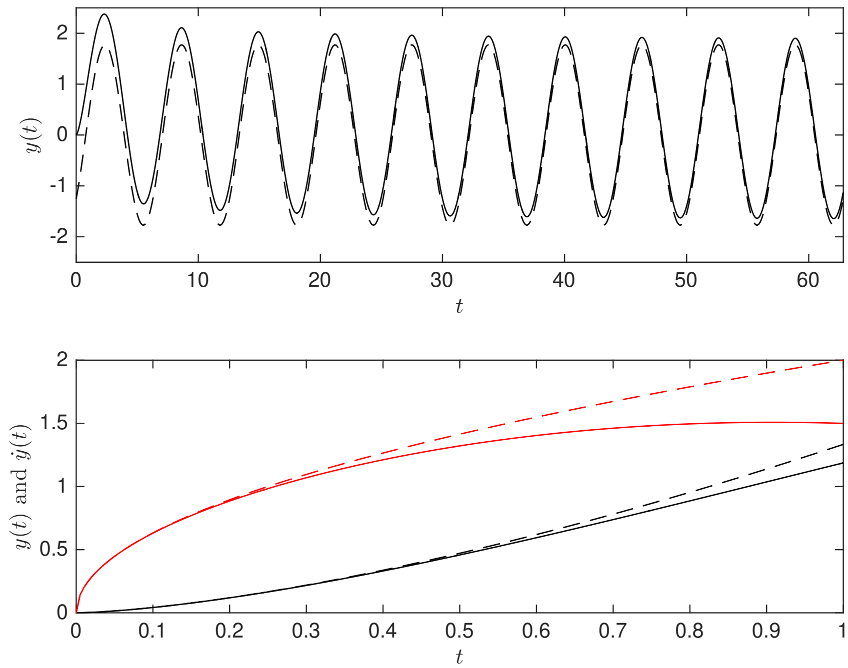


Figure 3.3: The solid curve in the the upper panel shows the solution in (3.36); the dashed curve is the large time approximation in (3.38). The solid curves in the lower panel are y and $\dot{y}$ from (3.36) and the dashed curves are the small-time approximations in (3.33). `fresnelOscillator.eps`

Example: Solve the forced oscillator problem

$$\ddot{x} + \omega^2 x = \mathrm{e}^{-\gamma t} H(t)\,, \tag{3.27}$$

where $x(t)$ is zero when $t < 0$. We can write the solution as an integral

$$x(t) = \int_0^t \mathrm{e}^{-\gamma t'}\, \frac{\sin\omega(t-t')}{\omega}\, \mathrm{d}t'\,. \tag{3.28}$$

But don't be a doofus — evaluating the integral above is complicated. I believe it is much easier to use undermined coefficients

$$x = A\mathrm{e}^{-\gamma t} + P\cos\omega t + Q\sin\omega t\,, \tag{3.29}$$

leading quickly to

$$x = \frac{\mathrm{H}(t)}{\gamma^2+\omega^2}\left[\mathrm{e}^{-\gamma t} - \cos\omega t + \frac{\gamma}{\omega}\sin\omega t\right]. \tag{3.30}$$

Notice that x and $\dot{x}$ are continuous at $t = 0$.

Example: Solve the forced oscillator problem

$$\ddot{y} + y = \frac{\mathrm{H}(t)}{\sqrt{t}}\,, \tag{3.31}$$

with $y(t)$ zero if $t < 0$. Before solving (3.31), notice that it easy to anticipate the structure of the solution when $0 < t \ll 1$: at small times, when y is still small, there is a two-term dominant balance in (3.31)

$$\ddot{y} \approx \frac{1}{\sqrt{t}}\,. \tag{3.32}$$

Integrating the simple equation above one finds

$$\dot{y} \approx 2\sqrt{t}\,, \qquad \text{and} \qquad y \approx \frac{4}{3}t^{3/2}\,, \tag{3.33}$$

where we've applied the initial conditions that $y(0^+) = \dot{y}(0^+) = 0$ i.e., both y and $\dot{y}$ are continuous at $t = 0$. Of course $\ddot{y}$ is worse than discontinuous at $t = 0$.

Turning now to the exact solution of (3.31) we don't get very far with undetermined coefficients — it is impossible to guess the form of a particular integral. Instead, the Green's function solution is

$$\begin{aligned} y &= \int_0^t \frac{\sin(t-t')}{\sqrt{t'}}\,\mathrm{d}t'\,, \\ &= \sin t \int_0^t \frac{\cos t'}{\sqrt{t'}}\,\mathrm{d}t' - \cos t \int_0^t \frac{\sin t'}{\sqrt{t'}}\,\mathrm{d}t'\,. \end{aligned} \tag{3.34}$$

With the change of variables

$$v \stackrel{\text{def}}{=} \sqrt{\frac{2t'}{\pi}}\,, \qquad \text{and} \qquad \frac{\mathrm{d}t'}{\sqrt{t'}} = \sqrt{2\pi}\mathrm{d}v\,, \tag{3.35}$$

we express the integrals in (3.34) in terms of the *Fresnel integrals*:

$$y = \sqrt{2\pi}\,\sin t \underbrace{\int_0^{\sqrt{2t/\pi}} \cos\left(\frac{\pi v^2}{2}\right)\mathrm{d}v}_{C(\sqrt{2t/\pi})} - \sqrt{2\pi}\,\cos t \underbrace{\int_0^{\sqrt{2t/\pi}} \sin\left(\frac{\pi v^2}{2}\right)\mathrm{d}v}_{S(\sqrt{2t/\pi})} \tag{3.36}$$

The Fresnel integrals $C(z)$ and $S(z)$ are `fresnels` and `fresnelc` in MATLAB and their main properties are summarized in books on special functions and on the **DLMF**. So the result in (3.36) is not completely useless. For instance, the **DLMF** tells us that

$$\lim_{z\to\infty} C(z) = \lim_{z\to\infty} S(z) = \frac{1}{2}\,. \tag{3.37}$$

Therefore at large times the solution in (3.36) becomes

$$y \approx \sqrt{\frac{\pi}{2}}\,\sin t - \sqrt{\frac{\pi}{2}}\,\cos t\,. \tag{3.38}$$

The solution is summarized in figure 3.3.

3.3 δ-sequences

The δ–function is not a real function; instead, the δ–function is interpreted as the limit ($\epsilon \to 0$) of a sequence of functions with two properties:

$$\begin{aligned} &\text{(a):} \quad \int_{-\infty}^{\infty} \delta_\epsilon(t)\mathrm{d}t = 1; \\ &\text{(b):} \quad \delta_\epsilon(t) \to 0 \text{ as } \epsilon \to 0 \text{ with } t \neq 0 \text{ fixed.} \end{aligned} \tag{3.39}$$

Because of property (a), the integral of each member of the sequence is independent of the parameter ϵ. Property (b) is satisfied because the functions become very sharply peaked as $\epsilon \to 0$. We will denote the δ–sequence by $\delta_\epsilon(t)$ but for brevity we denote the limit (when $\epsilon = 0$) simply by $\delta(t)$.

There is a simple recipe for producing δ–sequences. Start with some 'hump-

There are other recipes for obtaining a δ–sequence; this one is the most elementary.

like' function, $h(t)$, which is normalized so that the integral of h is equal to one. Here are some examples:

$$h_1(t) = \frac{e^{-t^2}}{\sqrt{\pi}}, \qquad h_2(t) = \frac{1}{\pi}\frac{1}{1+t^2}, \qquad h_3(t) = \frac{\sin t}{\pi t}. \tag{3.40}$$

To obtain a δ-sequence, take

$$\delta_\epsilon(t) = \frac{1}{\epsilon} h\left(\frac{t}{\epsilon}\right). \tag{3.41}$$

Make sure you understand why the construction above satisfies the properties in (3.39).

Exercise: Make a δ–sequence with the function $\operatorname{sech}^2(x)$.

The limit of a δ-sequence is only defined when the sequence is inside an integral. For example, if $f(t)$ is a smooth function then

$$\int_{-\infty}^{\infty} f(t_1)\delta(t-t_1)\,\mathrm{d}t_1 \stackrel{\text{def}}{=} \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(t_1)\delta_\epsilon(t-t_1)\,\mathrm{d}t_1. \tag{3.42}$$

Because of the integrand on the right of (3.42) is nonzero only in the immediate neighbourhood of $t_1 = t$, we can pull the slowly varying function $f(t_1)$ outside the integral so that:

$$\boxed{\int_{-\infty}^{\infty} f(t_1)\delta(t-t_1)\,\mathrm{d}t_1 = f(t)\,.} \tag{3.43}$$

This is the 'sifting–property' of a δ–function.

Here is a list of some of the more useful properties of the δ-function:

$$f(t)\delta(t-t_1) = f(t_1)\delta(t-t_1) \tag{3.44}$$

$$t\delta(t) = 0 \tag{3.45}$$

$$\delta(-t) = \delta(t) \tag{3.46}$$

$$\delta(at) = |a|^{-1}\delta(t)\,. \tag{3.47}$$

In addition, there is the crucial sifting property in (3.43). Integration by parts shows that the derivative of the δ-function has the property

$$\int_{-\infty}^{\infty} f(t)\delta'(t)\,\mathrm{d}t = -f'(0). \tag{3.48}$$

Exercise: Explain why the dimension of $\delta(t)$, with t=time, is $(\text{time})^{-1}$. In general δ-functions are dimensional quantities (unlike *bona fide* functions such as $\cos\omega t$).

Exercise: Show that $t\delta'(t) = -\delta(t)$.

It is easy to see that the t-derivative of the sequence $\mathrm{H}_\epsilon(t)$ defines a perfectly good δ-sequence:

$$\delta_\epsilon(t) = \frac{\mathrm{d}\,\mathrm{H}_\epsilon}{\mathrm{d}t}\,. \tag{3.49}$$

Thus blithely saying 'the derivative of the limit is the limit of the derivative', we get

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}(t) = \delta(t)\,.} \tag{3.50}$$

This is an extremely useful result, and it reinforces our earlier point that differentiating a singular function makes the singularity worse — $\delta(t)$ is more singular than $H(t)$.

Example: Suppose we build a δ-function using the recipe in (3.41) and the asymmetric hump function:

$$h_4(t) = \frac{\mathrm{e}^{-t^2}}{\sqrt{\pi}}(1+\alpha t)\,.$$

Is (3.46) satisfied, even though $h_4(t) \neq h_4(-t)$?

Yes: (3.46) means that if $f(t)$ is any function which is continuous at $t=0$ then

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(t)\frac{1}{\epsilon}h_4\left(\frac{t}{\epsilon}\right)\,\mathrm{d}t = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(t)\frac{1}{\epsilon}h_4\left(\frac{-t}{\epsilon}\right)\,\mathrm{d}t\,.$$

The result above is correct because the symmetry-breaking term proportional to α in $h_4(t)$ gives a vanishing contribution in the limit.

If a function is discontinuous at a point, its derivative at that point is a δ-function with an amplitude that is proportional to the size of the discontinuity. Equation (3.50) is one example, and another is the function $\mathrm{sgn}(t)$ which is equal to ± 1 according to whether x is positive of negative. Thus

$\mathrm{sgn}(t) = 2\,\mathrm{H}(t) - 1$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{sgn}(t) = 2\delta(t)\,. \tag{3.51}$$

This is not mysterious. If $a < 0 < b$ then, on one hand, a pedestrian application of the definition of $\mathrm{sgn}(t)$ gives:

$$\int_a^b \mathrm{sgn}(t)\frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t = \int_0^b \frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t - \int_a^0 \frac{df}{\mathrm{d}t}\,\mathrm{d}t \tag{3.52}$$

$$= f(b) - 2f(0) + f(a)\,. \tag{3.53}$$

On the other hand, we can use (3.51) to integrate by parts:

$$\int_a^b \mathrm{sgn}(t)\frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t = [\mathrm{sgn}(t)f(t)]_a^b - \int_a^b 2\delta(t)f(t)\,\mathrm{d}t\,,$$

$$= f(b) - 2f(0) + f(a)\,. \tag{3.54}$$

Although we've differentiated a discontinuous function, everything is hunky dory.

3.4 The oscillator Green's function again

Let's return to our oscillator example

$$\ddot{\theta} + \sigma^2\theta = f(t)\,, \tag{3.55}$$

with f and θ both zero when $t<0$. Suppose we somehow solve the equation

$$\ddot{g}+\sigma^2 g=\delta(t)\,, \tag{3.56}$$

with $g=0$ when $t<0$. Then the solution of (3.55) is

$$\theta(t)=\int_{-\infty}^{\infty} g(t-t')f(t')\,\mathrm{d}t'\,. \tag{3.57}$$

Now that we understand δ-functions the proof is substitution

$$\begin{aligned}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\sigma^2\right)\theta(t)&=\int_{-\infty}^{\infty}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\sigma^2\right)g(t-t')f(t')\,\mathrm{d}t'\,,\\ &=\int_{-\infty}^{\infty}\delta(t-t')f(t')\,\mathrm{d}t'\,,\\ &=f(t)\,. \end{aligned}\tag{3.58}$$

The function $g(t)$ is called the *Green's function*, or sometimes the fundamental solution, or other times the propagator, of (3.55). The Green's function is the oscillation induced by a δ-function kick.

Noting that $g(t)$ and $f(t)$ are both zero when $t<0$, we can re-write (3.57) as

$$\theta(t)=\int_0^t g(t-t')f(t')\,\mathrm{d}t'\,. \tag{3.59}$$

We've used *causality*: $g(t)$ is the response of an oscillator to an impulsive kick. The oscillator is at rest before its kicked i.e., $g(t<0)=0$.

This might seem simpler than (3.57). However the limits $\pm\infty$ in (3.57) were convenient in the calculation leading to (3.58).

This is all very well, but we still have to solve (3.56). We begin by noting that the forcing term on the right hand side is zero except in the neighborhood of $t=0$. Thus the solution when $t<0$ is simply $g(t)=0$. Then, at $t=0$, the oscillator gets an impulsive kick from the δ-function. The effect of the kick is calculated by integrating (3.56) over a small time interval that surrounds $t=0$. Because $\dot{g}(0^-)=g(0^-)=0$ the result of this integration is

$$\dot{g}(0^+)+\sigma^2\int_{0^-}^{0^+} g(t)\,\mathrm{d}t=1\,. \tag{3.60}$$

As the interval surrounding $t=0$ shrinks to zero one of the terms on the left hand side must balance the 1 on the right hand side. Some thought shows that it must be $\dot{g}(0^+)$ since the other term vanishes in this limit. This means there is a jump in $\dot{g}$ at $t=0$: $g(t)$ is continuous, $\dot{g}(t)$ is discontinuous and $\ddot{g}(t)$ has a δ-function component.

Notation: $0^+$is 0 plus a little bit, and 0^- is 0 minus a little bit.

The solution of (3.56) that satisfies these patching conditions at $t=0$ is

$$g(t)=\frac{\sin\sigma t}{\sigma}H(t)\,. \tag{3.61}$$

Notice that the bare mathematical problem

$$\ddot{g}+\sigma^2 g=\delta(t)\,, \tag{3.62}$$

has the solution in (3.61), but also at least two other solutions

$$g_2 = -\frac{\sin\sigma t}{\sigma}H(-t)\,, \qquad \text{and} \qquad g_3 = \tfrac{1}{2}\frac{\sin\sigma|t|}{\sigma}\,. \tag{3.63}$$

The g's above satisfy the patching conditions

$$g(0-) = g(0^+)\,, \qquad \text{and} \qquad \dot g(0^+) - \dot g(0^-) = 1\,. \tag{3.64}$$

We have made specific use of *causality* to identify (3.61) as the relevant Green's function. The solutions above might (and in fact do) appear in other physical problems.

Example: Find an integral representation of the causal solution of

$$\dddot y - 7\dot y - 6y = f(t)\,.$$

The forcing $f(t)$ is zero when $t < 0$, and so is $y(t)$. Find a condition on $f(t)$ which ensures that $\lim_{t\to\infty} y(t) < \infty$.

The causal Green's function satisfies

$$\dddot g - 7\dot g - 6g = \delta(t)\,, \qquad g(t<0) = 0\,.$$

"Causal" means nothing happens before the kick arrives at $t = 0$ i.e., $g(t < 0) = 0$. When $t > 0$ the general solution of the homogeneous equation is obtained with $g = \mathrm{e}^{\lambda t}$, leading to

$$\lambda^3 - 7\lambda - 6 = 0 \qquad \text{or } \lambda = -1,\ -2 \text{ and } +3.$$

The cubic polynomial for λ above is solved by inspired guessing, or with the MATHEMATICA command

```
Roots[x^3 - 7 x + 6 == 0, x]
```

Thus the general solution when $t > 0$ is

$$g = a\mathrm{e}^{-t} + b\mathrm{e}^{-2t} + c\mathrm{e}^{3t}\,. \tag{3.65}$$

We have to determine the constants a, b and c by applying initial condition at $t = 0^+$, just after the impulse. At $t = 0$ the δ-kick is balanced by $\dddot y$, so integrating from $t = -\epsilon$ to $t = +\epsilon$ we have

$$\ddot g(\epsilon) - \underbrace{\ddot g(-\epsilon)}_{=0} = \underbrace{\int_{-\epsilon}^{\epsilon} \delta(t)\,\mathrm{d}t}_{=1}$$

In other words, the second derivative $\ddot g$ jumps discontinuously from zero to one as a result of the kick. The function $g(t)$ and the first derivative $\dot g(t)$ are continuous at $t = 0$, implying that $g(0)$ and $\dot g(0)$ are zero. Thus we have three effective initial conditions

$$g(0^+) = 0\,, \qquad \dot g(0^+) = 0\,, \qquad \ddot g(0^+) = 1 \tag{3.66}$$

to determine the three constant a, b and c in (3.65). The conditions above result in the 3×3 linear system

$$\begin{aligned} a + b + c &= 0\,, \\ a + 2b - 3c &= 0\,, \\ a + 4b + 9c &= 1\,. \end{aligned}$$

Solving the equations above we find

$$g = \left[-\tfrac{1}{4}\mathrm{e}^{-t} + \tfrac{1}{5}\mathrm{e}^{-2t} + \tfrac{1}{20}\mathrm{e}^{3t}\right]\mathrm{H}(t)\,, \tag{3.67}$$

where the H(t) is inserted for causality. The general causal solution is therefore

$$y(t) = \int_0^t g(t-t')f(t')\,\mathrm{d}t'\,.$$

The long-time behaviour of $y(t)$ will be dominated by the term in $g(t)$ proportional to e^{3t}:

$$y(t) = \frac{\mathrm{e}^{3t}}{20}\int_0^t \mathrm{e}^{-3t'} f(t')\,\mathrm{d}t' + \text{decaying terms } \mathrm{e}^{-t} \text{ and } \mathrm{e}^{-2t}.$$

To completely destroy the growing term as $t \to \infty$ one requires that the forcing satisfy

$$\lim_{t\to\infty} \mathrm{e}^{3t}\int_0^\infty \mathrm{e}^{-3t'} f(t')\,\mathrm{d}t' = 0\,.$$

The advantage of the Green's function is that one can obtain general results determining important qualitative properties of the solution, such as the condition above. If one wants to solve the equation with a particular $f(t)$ then it may be best to avoid the Green's function integral representation and use simpler techniques such as undetermined coefficients.

Example: Find an integral representation of the solution of

$$\dddot{h} - 7\dot{h} - 6h = \delta(t)\,.$$

satisfying $\lim_{t\to\pm\infty} h(t) = 0$.

This problem is different from the previous one because we don't seek a *causal* solution i.e., we might have $h(t<0) \neq 0$. Instead we're after a solution on the whole t-axis, $-\infty < t < \infty$, which is zero as $t \to -\infty$ and as $t \to +\infty$.

Lets work out the details. The homogeneous problem is the same as the previous case, and the conditions at $\pm\infty$ demand

$$h(t) = \begin{cases} p\mathrm{e}^{3t}\,, & \text{for } t<0; \\ q\mathrm{e}^{-t} + r\mathrm{e}^{-2t}\,, & \text{for } t>0. \end{cases} \tag{3.68}$$

The construction above secures the vanishing of $h(t)$ as $t \to \pm\infty$. Now the three constants of integration are determined by patching at $t=0$ i.e., by requiring that h and $\dot{h}$ are continuous at $t=0$, and that $\ddot{h}(0^+) - \ddot{h}(0^-) = 1$. Thus

$$\begin{aligned} p &= q + r\,, \\ 3p &= -q - 2r\,, \\ q + 4r - 9p &= 1\,. \end{aligned}$$

Solving the equations above, we find we can write the solution as

$$h(t) = g(t) - \tfrac{1}{20}\mathrm{e}^{3t}\,,$$

where $g(t)$ is the causal solution back in (3.67). Duh —with hindsight this is obvious: the difference $g(t) - h(t)$ satisfies the homogeneous equation, and we construct $h(t)$ from $g(t)$ simply by subtracting the $t \to +\infty$ growing component e^{3t}.

3.5 The black box

I have introduced the concept of the Green's function as the impulse-response of the oscillator equation in (3.56). However the idea is much more general and important than this example might suggest. Suppose we have a device (the legendary 'black box') that accepts an input $U(t)$ and produces an output $V(t)$. We don't enquire about the internal workings of the black box. Obviously this is a very general concept e.g., one might view ones colleagues as black boxes.

Some black boxes are simpler than others. The very simplest examples are:

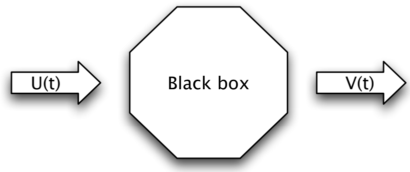


Figure 3.4: The black box with input $U(t)$ and output $V(t)$. `BlackBox.eps`

1 Causal; the black box can't anticipate future events.

2 Time translationally invariant; if the input $U(t)$ produces $V(t)$, then $U(t-t_1)$ produces $V(t-t_1)$.

3 Linear; the response to $\alpha U_1(t)+\alpha_2 U_2(t)$ is $\alpha V_1(t)+\alpha_2 V_2(t)$.

An example of this type of simple black box is a simple thermometer. The input to a thermometer might be the "true temperature" of the air surrounding the bulb; the output is the "measured temperature". If the true temperature is changing rapidly (for instance, if the thermometer is mounted on the wing of a plane) then the measured temperature will be some smoothed or averaged or "filtered" version version of the true temperature.

Another example is the measured deflection $V(t)$ of a structure at some point in response to a varying load $U(t)$ at some other point. If the load is not too great then we expect that the response varies linearly with the load. Buildings don't start shaking before the arrival of seismic waves from an earthquake, so the system is causal. And if the material in the structure isn't aging then the response to a standard load now and six months from now will be the same. These are the three requirements above.

The three assumptions are sufficient to *completely characterize* the operation of the black box if we know the response produced by the special input $\delta(t)$. This is the impulse response, or the Green's function of the black box, denoted by $G(t)$. Given $G(t)$, the response to an arbitrary input $U(t)$ is

$$V(t)=\int_{-\infty}^{t} G(t-t')U(t')\,\mathrm{d}t'. \tag{3.69}$$

Thus, if we measure or calculate the response of the black box to a single input, $\delta(t)$, then we know how the system responds to *any* input by evaluating the integral in (3.69).

Because the response $V(t)$ cannot depend on the future behaviour of the input $U(t)$, the upper limit of integration in (3.69) is $t'=t$. (Anyway, because $G(t<0)=0$ you can write $+\infty$ as the upper limit if you like — it won't make any difference.)

To prove (3.69), partition up the t-axis into small intervals of length $\mathrm{d}t$ and then approximate the input $U(t)$ as a 'staircase', with a constant value in each $\mathrm{d}t$-interval. The response to each 'pulse' in the staircase is then given by a

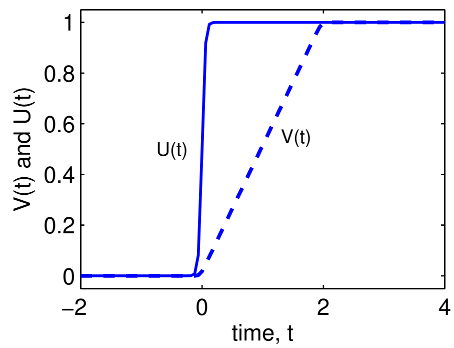


Figure 3.5: The input is $H(t)$ and output $V(t)$ ramps up over a time τ to the final level ($\tau = 2$ in this illustration). `ramp.eps`

single Green's function (multiplied by $\mathrm{d}t$) and (3.69) is the result of linearly superposing these responses.

Example: Suppose that the input temperature is $H(t)$ and the observed response to this step function input is

$$V(t) = \begin{cases} 0\,, & \text{if } t < 0; \\ t/\tau\,, & \text{if } 0 < t < \tau; \\ 1\,, & \text{if } \tau < t. \end{cases}$$

This ramp is shown in Figure 3.5. Find the Green's function. The constant τ is a response time of the thermometer.

The Green's function of the thermometer is obtained by using (3.50) i.e., by taking the derivative of the step–response $V(t)$ in Figure 3.5:

$$G(t) = \tau^{-1}\left[H(t) - H(t-\tau)\right]. \tag{3.70}$$

$\frac{\mathrm{d}}{\mathrm{d}t}H(t) = \delta(t)$

We've used linearity to say that if the input $U(t)$ elicits the response $V(t)$, then the response to $\mathrm{d}U/\mathrm{d}t$ is $\mathrm{d}V/\mathrm{d}t$.

Now using (3.69) and (3.70) we can write down a simple expression for the response of this hypothetical thermometer to any input:

$$V(t) = \frac{1}{\tau}\int_{t-\tau}^{t} U(t')\,\mathrm{d}t'. \tag{3.71}$$

We see that the thermometer is averaging the arriving signal $U(t)$ over the most recent times. The averaging is done 'on–the–fly', so that the measurement at t depends on the past temperature back to $t-\tau$.

Example: Find the response of the thermometer to a signal $u(t) = e^{\mathrm{i}\omega t}$. Discuss the two limits $\omega\tau \gg 1$ and $\omega\tau \ll 1$.

Example: Suppose two of these thermometers are connected in series so that the input of the second is the output is the first. Find the response of this composite system to the inputs $\delta(t)$ and $H(t)$.

3.6 Problems

Problem 3.1. Use patching to obtain non-trivial solutions of the differential equations

$$y' + \mathrm{sgn}(x)\,y = 0\,, \qquad y(\pm\infty) = 0\,, \tag{3.72}$$

$$z'' + \mathrm{sgn}(x)\,z = 0\,, \qquad z(-\infty) = 0\,. \tag{3.73}$$

Problem 3.2. Find the general solution to

$$y'' + \mathrm{sgn}(x)\, y = \mathrm{e}^{-|x|}\,. \tag{3.74}$$

Problem 3.3. Solve

$$y''' + y = \mathrm{H}(x)\,, \qquad y(-\infty) = 0\,, \quad y(\infty) = 1\,. \tag{3.75}$$

Problem 3.4. Consider the oscillator problem

$$\ddot{\chi} + \sigma^2 \chi = \underbrace{\frac{t}{\tau}\,\mathrm{H}(t) - \left(\tfrac{t}{\tau} - 1\right)\mathrm{H}(t-\tau)}_{\stackrel{\mathrm{def}}{=} r(t)}\,, \tag{3.76}$$

with $\chi(t<0) = 0$. Draw a graph of the right hand side $r(t)$ as a function of t. Solve the initial value problem and discuss how the amplitude of the ultimate oscillation depends on "ramp-up time" τ.

Problem 3.5. *(i)* Solve the forced-damped oscillator problem

$$\ddot{\theta} + \mu\dot{\theta} + \sigma^2\theta = \mathrm{H}(t)\,, \qquad \theta(t<0) = 0\,. \tag{3.77}$$

Assume that the oscillator is "underdamped" so that

$$\omega \stackrel{\mathrm{def}}{=} \sqrt{\sigma^2 - \mu^2/4} \tag{3.78}$$

is real. Check your answer by comparing it with the dashed curve in Figure 3.1. *(ii)* Show that as $t \to \infty$ half the work done by the force $H(t)$ is used to increase the potential energy of the oscillator and the other half is dissipated by drag force $-\mu\dot{\theta}$.

Problem 3.6. Solve

$$\ddot{\theta} + \sigma^2\theta = 0\,, \qquad \theta(t<0) = a\cos(\sigma_1 t + \alpha)\,, \tag{3.79}$$

where

$$\sigma^2(t) = \begin{cases} \sigma_1^2\,, & \text{if } t<0; \\ \sigma_2^2\,, & \text{if } t>0. \end{cases} \tag{3.80}$$

Problem 3.7. Solve the forced oscillator equation

$$\ddot{\theta} + \theta = \mathrm{e}^{-\alpha|t|}\,, \qquad \lim_{t\to-\infty} \theta(t) = 0\,. \tag{3.81}$$

Check your answer by showing that the amplitude of the oscillation at $t = +\infty$ is maximized by taking $\alpha = 1$.

Problem 3.8. Reconsider the forced oscillator problem in the example surrounding (3.15). Obtain the Green's function in (3.61) by taking the limit $\tau \to 0$ of the solution in (3.17).

Problem 3.9. *(i)* Show that

$$f(n, z) = \sqrt{n} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{[1 + (x - z)^2]^n} \tag{3.82}$$

is independent of z and becomes independent of n in the limit $n \to \infty$. (A heuristic argument based on the size of the interval in which the integrand is significantly different from zero is all that is required.) *(ii)* Use the well known result

$$\left(1 + \frac{y}{n}\right)^n \to e^y \qquad \text{as} \qquad n \to \infty \tag{3.83}$$

to evaluate $f(\infty, z)$. *(iii)* Approximately evaluate the integral

$$I \stackrel{\text{def}}{=} \int_0^{\infty} \frac{\cos x^2}{(1 + x^2)^4} \, \mathrm{d}x \, . \tag{3.84}$$

(The numerical value is 0.736237.)

Problem 3.10. Evaluate

$$I \stackrel{\text{def}}{=} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \sin\left(\mathrm{e}^x\right) \frac{\epsilon}{\epsilon^2 + x^2} \, \mathrm{d}x \, . \tag{3.85}$$

and

$$J = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \sin\left(\mathrm{e}^x\right) \frac{\mathrm{d}}{\mathrm{d}x} \frac{\epsilon}{\epsilon^2 + x^2} \, \mathrm{d}x \, . \tag{3.86}$$

Problem 3.11. Use integration by parts to evaluate $\int_{-1}^{1} |t| f''(t) \, \mathrm{d}t$. (Don't be afraid to differentiate $|t|$ twice.)

Problem 3.12. *(i)* Show that if $f(x)$ is a function which vanishes at x_n then

$$\delta[f(x)] = \sum_n \frac{\delta(x - x_n)}{|f'(x_n)|} . \tag{3.87}$$

(ii) Calculate

$$\int_{-\infty}^{\infty} \delta(\sin x) \exp(-|x|) \, \mathrm{d}x \, . \tag{3.88}$$

(iii) Show that

$$\delta\left(t^2 - a^2\right) = \left[\delta(t - a) + \delta(t + a)\right] / 2|a| \, . \tag{3.89}$$

Problem 3.13. If $(x, y) = r(\cos\theta, \sin\theta)$, show that

$$\oint \delta(a - x\cos\phi - y\sin\phi) \, \mathrm{d}\phi = \frac{2\,\mathrm{H}(r - a)}{\sqrt{r^2 - a^2}} \, . \tag{3.90}$$

Problem 3.14. Verify by substitution that

$$\theta = \int_0^t f(t') \, \frac{\sin\sigma(t - t')}{\sigma} \, \mathrm{d}t' \tag{3.91}$$

satisfies $\ddot{\theta} + \sigma^2\theta = f$ with initial conditions $\theta(0) = \dot{\theta}(0) = 0$.

Problem 3.15. Consider the pendulum in the strongly over-damped limit. After suitable scaling the equation of motion is

$$\mu\ddot{\theta} + \dot{\theta} + \theta = f(t)\,, \tag{3.92}$$

where $\mu \ll 1$. Obtain the Green's function $G(t)$ for this problem. Since $\mu \ll 1$ one might be tempted to neglect the term $\mu\ddot{\theta}$. Make this approximation and obtain the Green's function of the approximate equation. Use MATLAB to make a graphical comparison between the two Green's functions $\mu = 0.01$ and 0.1.

Problem 3.16. Find a particular solution of

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - y = \delta(x) \tag{3.93}$$

with $\lim_{x\to\pm\infty} y = 0$.

Problem 3.17. Find a particular solution of

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} - y = \delta(x) \tag{3.94}$$

which is (a) bounded as $x \to \pm\infty$ and (b) an even function of x. Check your answer against mine with $y(0) = 1/2$.

Problem 3.18. Find a particular solution of

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + y = \delta(x) \tag{3.95}$$

which is (a) bounded as $x \to \pm\infty$ and (b) an even function of x.

Problem 3.19. Suppose that the output of our hypothetical thermometer is observed to be:

$$V(t) = \begin{cases} 0\,, & \text{if } t < 0; \\ = 1 - \exp(-\alpha t)\,, & \text{if } t > 0. \end{cases} \tag{3.96}$$

Reconstruct the input, $U(t)$. (This one might be tough.)

Problem 3.20. Find the impulse response of a causal linear system in which the output $V(t)$ is obtained from from the input $U(t)$ by solving

$$\ddot{V} + 2\dot{V} + V = \dot{U}\,. \tag{3.97}$$

How does the system respond to the input $U = H(t)\mathrm{e}^{-\alpha t}$?

Problem 3.21. Find the Green's function of the equation

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} = f(t)\,, \quad \text{with} \qquad \lim_{t\to-\infty} y(t) = 0\,. \tag{3.98}$$

(Assume that $f(t)$ vanishes at least exponentially fast as $t \to -\infty$.) Consider the function $Y(t)$ defined by

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{17} Y = \mathrm{e}^{-t^2}\cos mt\,, \quad \text{with} \qquad \lim_{t\to-\infty} Y(t) = 0\,. \tag{3.99}$$

Show that when $t \gg 1$

$$Y(t) \sim A(m)t^{16} + O\left(t^{15}\right)\,, \tag{3.100}$$

and find $A(m)$.

Lecture 4

The 1D diffusion equation $\theta_t = \kappa\theta_{xx}$

4.1 Origin of the diffusion equation

I've already mentioned Fick's law. Let's briefly discuss two other contexts in which the diffusion equation arises.

Diffusion of heat in solids

The equation governing heat diffusion in a solid is

$$\varrho c u_t = \boldsymbol{\nabla} \cdot (k \boldsymbol{\nabla} u) \tag{4.1}$$

where $u(\boldsymbol{x}, t)$ is the temperature. Also in (4.1), where ϱ is the density of the solid (kilograms per cubic meter), c the heat capacity (Joules per kilogram per Kelvin) and k the conductivity (find the units yourself). In a solid c_p and c_v are almost equal, so it doesn't matter which heat capacity is used above. Equation (4.1) is in the form of a conservation law

$$\partial_t \,[\text{energy density}] + \boldsymbol{\nabla} \cdot [\text{energy flux}\ ] = 0\,. \tag{4.2}$$

The flux is $\boldsymbol{f} = -k\boldsymbol{\nabla} u$, so if the temperature is uniform then there is no flux of energy[1]In the 17th century, people thought of heat as "caloric fluid" that flows down the temperature gradient, from hot regions to colder regions. This caloric theory has been superseded by thermodynamics, but I think it still provides some useful intuition.

Notice that if we integrate (4.2) over a closed volume V, with surface ∂V, then with Gauss's theorem we have

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t} \int_V \varrho c u \,\mathrm{d}v}_{\text{energy in } V} = \underbrace{\int_{\partial V} k \boldsymbol{\nabla} u \cdot \boldsymbol{n} \,\mathrm{d}s}_{\text{energy flow through the surface } \partial V}\,, \tag{4.3}$$

[1] See Landau & Lifshitz, *Theory of Elasticity*, chapter V for a complete discussion, including the coupling of thermal conduction to elastic deformation.

where $\boldsymbol{n}$ is the unit outward normal to the surface of ∂V. This is the multi-dimensional analog of an integral conservation law.

If all three of the material properties ρ, c and k are constant we can bundle them into one combination,

$$\kappa \stackrel{\text{def}}{=} \frac{k}{\varrho c}\,, \tag{4.4}$$

called the diffusivity. Then the simplest form of the diffusion equation is

$$u_t = \kappa \triangle u\,. \tag{4.5}$$

Random walkers

Another context in which the diffusion equation arises is the random walk. In the one-dimensional case, $u(x,t)$ is the density (walkers per meter) and $f = -\kappa u_x$ is the flux (walkers per second). In this example

$$\kappa = \frac{(\text{mean square step length})^2}{2 \times \text{average time between steps}}\,. \tag{4.6}$$

Let's sketch the random walk model and explain how it produces the diffusion equation.

We begin with the classical model in which the random walk is discrete in both space and time. The walker (traditionally a drunk) takes a step at times $t = \tau,\ 2\tau, \cdots$. The steps are $\pm a$ with probability 1/2. We consider an ensemble of walkers and let $\rho(x,t)$ be the number of walkers at site x at time t. It is easy to see that the evolution of the ensemble of walkers is given by

$$\rho(t+\tau, x) = \tfrac{1}{2}\rho(t, x-a) + \tfrac{1}{2}\rho(t, x+a)\,. \tag{4.7}$$

i.e. half of the walkers at site $x-a$ hop to the right and land at x and so on. Equation (4.7) is a *partial difference equation* i.e., a discrete system with two independent variables. With an initial condition, such as

$$\rho(x,0) = \delta(x)\,, \tag{4.8}$$

it is easy to iterate a few times and find $\rho(x,\tau)$, $\rho(x,2\tau)$ etc. You might like to do this and see if you can guess the general result (hint: Pascal's triangle).

We can obtain the diffusion equation (4.5) as an approximation to (4.7) when $\rho(x,t)$ is slowly changing i.e., when the spatial variation of ρ is on a scale much larger than the hopping length a, so that in one iteration from t to $t+\tau$ there is only a small change in $\rho(x,t)$. This "macroscopic evolution equation" follows easily from (4.7) if we rewrite it as

$$\rho(t+\tau,x) - \rho(t,x) = \tfrac{1}{2}\left[\rho(t,x-a) - 2\rho(t,x) + \rho(t,x+a)\right]\,. \tag{4.9}$$

Now we make the slowly varying assumption: we expand everything in sight in the small parameters a and τ: Small relative to what?

$$\rho(t+\tau,x) = \rho(t,x) + \tau\rho_t(t,x) + O(\tau^2)\,, \tag{4.10}$$

and

Take $\tau = O(a^2)$ and neglect $O(\tau^2)$

$$\rho(t, x \pm a) = \rho(t, x) \pm a\rho_x(t, x) + \tfrac{1}{2}a^2\rho_{xx}(t, x) + O(a^3) . \tag{4.11}$$

Putting all this into (4.9) and isolating the surviving term on each side we get

$$\rho_t = \kappa\rho_{xx} , \tag{4.12}$$

where

$$\kappa \stackrel{\text{def}}{=} \frac{a^2}{2\tau} . \tag{4.13}$$

Why do we keep the $O(a^2)$ terms in (4.11), and neglect the $O(\tau^2)$ terms in (4.10)? This seems inconsistent. Neglect of the $O(\tau^2)$ terms can be justified by taking a *distinguished limit* in which $a \to 0$ and $\tau \to 0$ with ratio κ in (4.13) fixed; with this procedure τ and a^2 have the same order. In taking this limit we suppose that instead of (4.8) we have a smooth initial condition containing a length scale ℓ:

$$\rho(x, 0) = f\left(\frac{x}{\ell}\right) , \tag{4.14}$$

and that ℓ is fixed as a and τ go to zero with κ fixed. In this limit, the term $\tau\rho_{tt}$ — and all the other higher-order terms such as $a^4\rho_{xxxx}$ — disappear.

4.2 Diffusive smoothing of a discontinuity

Considering the problem

$$u_t = \kappa u_{xx} , \qquad u(x, 0) = A\,\mathrm{sgn}(x) , \tag{4.15}$$

we quickly see that the only dimensionally consistent form for the solution is

$$u(x, t) = A \times \text{some function of } x/\sqrt{\kappa t} . \tag{4.16}$$

Thus we are inspired to try the guess

$\xi_t = -\xi/2t$

$$u(x, t) = AU(\xi) , \qquad \xi \stackrel{\text{def}}{=} x/\sqrt{\kappa t} . \tag{4.17}$$

We start calculating

$U' \stackrel{\text{def}}{=} \dfrac{\mathrm{d}U}{\mathrm{d}\xi}$

$$u_x = \frac{A}{\sqrt{\kappa t}}U' , \qquad u_{xx} = \frac{A}{\kappa t}U'' , \qquad u_t = -\frac{A}{2t}\xi U' \tag{4.18}$$

and substitute into the PDE:

$$-\frac{A}{2t}\xi U' = \kappa\frac{A}{\kappa t}U'' \qquad \Rightarrow \qquad -\xi U' = 2U'' . \tag{4.19}$$

Our efforts are crowned with success when all the t's cancel and we are left with the boxed ODE for $U(\xi)$.

Now we solve that ODE using separation of variables. Integrating twice

$$U' = C_1\mathrm{e}^{-\xi^2/4} , \qquad U(\xi) = C_1\int_0^{\xi}\mathrm{e}^{-v^2/4}\,\mathrm{d}v + C_2 . \tag{4.20}$$

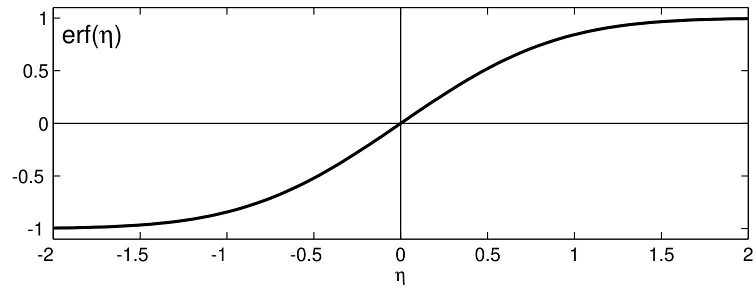


Figure 4.1: The function erf(η). `erfPic.eps`

The constants C_1 and C_2 are determined by insisting that

$$\lim_{\xi\to\pm\infty} U(\xi) = \pm 1\,, \tag{4.21}$$

or:

$$-1 = -\sqrt{\pi}C_1 + C_2\,, \qquad \text{and} \qquad 1 = \sqrt{\pi}C_1 + C_2\,, \tag{4.22}$$

$$\int_0^\infty e^{-v^2/4}\,dv = \sqrt{\pi}$$

with solution

$$C_1 = \frac{1}{\sqrt{\pi}}\,, \qquad \text{and} \qquad C_2 = 0\,. \tag{4.23}$$

Thus we now have

$$U(\xi) = \frac{1}{\sqrt{\pi}}\int_0^\xi e^{-v^2/4}\,dv\,. \tag{4.24}$$

Adjusting our notation, we rewrite the solution in (4.24) as

$$u(x,t) = A\,\mathrm{erf}(\eta)\,, \tag{4.25}$$

where the similarity variable is now

$$\mathrm{erf}(z) \overset{\text{def}}{=} \frac{2}{\sqrt{\pi}}\int_0^z e^{-u^2}du$$

$$\eta \overset{\text{def}}{=} \frac{x}{2\sqrt{\kappa t}}\,. \tag{4.26}$$

The erf function is shown in figure 4.1.

4.2.1 Discussion of the similarity method

This *similarity method* is a powerful technique for solving or simplifying both linear and nonlinear PDE's. In section 6.2 I have also demonstrated the powerful educational technique of glossing quickly over the profound part of the method — the guess in (4.17) — and dwelling at length on algebraic technicalities.

To explain the similarity method in more detail, suppose that Tom and Jerry are both struggling with (4.15). Tom is measuring length in meters and seconds, but Jerry is an engineer and prefers to use furlongs and fortnights. Tom uses x and t to denote space and time and Jerry uses x' and t'. The numerical values of these coordinates can be converted one to the other by

$$x' = \lambda x\,, \qquad t' = \mu t\,. \tag{4.27}$$

It follows that Tom and Jerry's numerical values for the diffusivity are related by $\kappa' = \lambda^2\kappa/\mu$.

Even before Tom & Jerry solve the PDE they realize that the solution must satisfy

$$u = AF(x,t,\kappa) = AF(\underbrace{\lambda x}_{x'}, \underbrace{\mu t}_{t'}, \underbrace{\lambda^2\kappa/\mu}_{\kappa'}) . \tag{4.28}$$

All that we have done is assume the existence of a formula which gives u once either set of variables is specified. The essential point is that this is the *same formula* no matter which set of units is used. Then as we freely vary both λ and μ the value of u at the same point in space-time is unchanged i.e., (4.28) is always true for all values of μ and λ. This implies that x, t and κ appear in F only in the combination $x^2/(\kappa t) = x'^2/(\kappa' t')$ — that's where we started back in (4.17).

One difficulty with the argument above is that the initial condition can never be totally discontinuous. Suppose the initial condition is really something like

$$u(x,0) = A\tanh(x/\ell) = A\tanh(x'/\ell') . \tag{4.29}$$

i.e., some smooth function which changes from $-A$ to $+A$ over a distance $\ell' = \mu\ell$. As soon as Tom and Jerry admit the existence of this length they must also amend (4.28) to

$$u = AF(x,t,\kappa,\ell) = AF(\lambda x, \mu t, \mu\kappa/\lambda^2, \lambda\ell) . \tag{4.30}$$

Now as one freely varies λ and μ the most that one can deduce is that

$$u = AF(x/\ell, \kappa t/\ell^2) = AF(x'/\ell', \kappa' t'/\ell'^2) . \tag{4.31}$$

This is a useful result because it tells us that we can economically present the solution using a function of two (rather than four) variables. But there is no longer a similarity solution.

On the other hand, it seems reasonable that if we consider a sequence of problems with smaller and smaller ℓ's then in the limit we should approach the $\mathrm{sgn}(x)$ initial condition and the erf similarity solution. Alternatively, with fixed ℓ, once the diffusively spreading front becomes much thicker than ℓ, then ℓ is an "irrelevant" parameter. In this case we get the erf solution as the *asymptotic* description of a diffusing front. This possibility is consistent with (4.30) if $t \gg \ell^2/\kappa$

$$(p,q) \to \infty \qquad \Rightarrow \qquad F(p,q) \to \tilde{F}(p/\sqrt{q}) . \tag{4.32}$$

4.3 The Green's function for diffusion on the line

To solve the initial value problem

$$u_t = \kappa u_{xx} , \qquad u(x,0) = f(x) , \tag{4.33}$$

we need the Green's function, which is defined by

$$g_t = \kappa g_{xx} , \qquad g(x,0) = \delta(x) . \tag{4.34}$$

Once we possess $g(x,t)$ it is easy to see by substitution that

$$u(x,t) = \int_{-\infty}^{\infty} f(x')g(x-x',t)\,\mathrm{d}x' \tag{4.35}$$

is the solution of (4.33). To find $g(x,t)$ we use the erf-solution as the thin end of the wedge: since

$$\delta(x) = \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{sgn}(x)\,, \tag{4.36}$$

the Green's function of the diffusion equation is

$$\begin{aligned} g(x,t) &= \tfrac{1}{2}\frac{\partial}{\partial x}\,\mathrm{erf}(\eta)\,, \\ &= \frac{\mathrm{e}^{-x^2/4\kappa t}}{\sqrt{4\pi\kappa t}}\,. \end{aligned} \tag{4.37}$$

4.3.1 Another derivation of the diffusion Green's function

An alternative route to (4.37) is to argue on dimensional grounds that

$$g(x,t) = \frac{1}{\sqrt{\kappa t}} G\left(\frac{x}{\sqrt{\kappa t}}\right)\,. \tag{4.38}$$

As an exercise you should substitute this guess into the diffusion equation and recover (4.37) by solving the resulting ODE.

Where does the factor $1/\sqrt{\kappa t}$ on the RHS of (4.38) come from? Because of the initial condition in (4.34), the dimensions of g are the same as the dimensions of $\delta(x)$. Now δ-functions have the peculiar property that

$$\dim\left[\delta(\theta)\right] = \frac{1}{\dim(\theta)}\,. \tag{4.39}$$

That is, δ-functions have the inverse dimension of their argument. (Other functions, such as $\cos x$ or $\ln t$, are dimensionless.) In the problem (4.34), the argument of the δ-function is x, with dimensions of "length", and therefore g has dimensions $(\text{length})^{-1}$.

Another way to understand the factor $1/\sqrt{\kappa t}$ is to observe that if we integrate the diffusion equation from $x=-\infty$ to $x=\infty$ we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} g(x,t)\,\mathrm{d}x = \left[\kappa g_x(x,t)\right]_{x=-\infty}^{x=\infty} = 0\,. \tag{4.40}$$

(We assume the solution decays quickly as $x\to\pm\infty$ so that there is no flux of g-stuff from $\pm\infty$.) Using the initial condition we deduce that

$$\int_{-\infty}^{\infty} g(x,t)\,\mathrm{d}x = 1\,. \tag{4.41}$$

Now the factor $1/\sqrt{\kappa t}$ is just what the similarity guess in (4.38) needs in order to the satisfy the conservation law (4.41):

$$1 = \int_{-\infty}^{\infty} G\left(\frac{x}{\sqrt{\kappa t}}\right)\;\frac{\mathrm{d}x}{\sqrt{\kappa t}} = \int_{-\infty}^{\infty} G\left(\xi\right)\;\mathrm{d}\xi\,. \tag{4.42}$$

This little argument based on conservation laws also tells us how to determine a constant of integration which arises as we solve the ODE which results from stuffing (4.38) into the diffusion equation.

Exercise: Substitute the *ansatz* in (4.38) into the diffusion equation and show that you obtain an ODE for $G(\eta)$. Solve this ODE and recover the Green's function in (4.37).

Convolution

In (4.35) we have the solution of the diffusion equation as the *convolution* of the initial condition $f(x)$ with the Green's function $g(x,t)$. Let us discuss the convolution of two functions in more detail. Alternative terms used to describe

$$c(x) = \underbrace{\int_{-\infty}^{\infty} a(x')b(x-x')\,\mathrm{d}x'}_{\text{the convolution of } a \text{ with } b} = \underbrace{\int_{-\infty}^{\infty} a(x-x')b(x')\,\mathrm{d}x'}_{\text{convolution is commutative}}, \tag{4.43}$$

are *faltung*, superposition integral, running mean, smoothing, scanning and blurring.

To appreciate the origin of these terms, consider the case when $b(x)$ in (4.43) is a "box-car filter"

$$b(x) = \begin{cases} 1\,, & \text{if } |x| < \frac{1}{2}, \\ 0\,, & \text{otherwise.} \end{cases} \tag{4.44}$$

Then at every point x, $c(x)$ is an averaged, or smoothed or low-pass filtered version of version of $a(x)$. Explaining this requires drawing some examples, such as visually computing the convolution

$$c(x) = \int_{-\infty}^{\infty} \mathrm{sgn}(x')b(x-x')\,\mathrm{d}x' = \begin{cases} -1\,, & \text{if } x < -\frac{1}{2}, \\ 2x\,, & \text{if } -\frac{1}{2} \le x \le \frac{1}{2}, \\ +1\,, & \text{if } \frac{1}{2} < x. \end{cases} \tag{4.45}$$

The box-car filter smooths out the sudden jump at $x = 0$ into a ramp.

The convolution in (4.35) shows that the value of $u(x,t)$ is determined by averaging the initial condition of $f(x)$ over an interval of width $\sqrt{\kappa t}$ centered on the point x. The Green's function (4.37) is a Gaussian filter, and the range of the filter grows as $\sqrt{\kappa t}$. Thus as t increases the width of the averaging interval increases, and therefore $u(x,t)$ becomes increasingly smooth. The effect of convolving $\mathrm{sgn}(x)$ with the box-car is the ramp in (4.45) and convolution of $\mathrm{sgn}(x)$ with the Gaussian is the erf in Figure 4.1. The two results are qualitatively similar: the discontinuity is smeared out over a distance comparable to the filter width.

Having the general solution of the infinite-line diffusion problem in (4.35) and (4.37) is very pleasant — particularly because the interpretation as a convolution-average provides intuition. But be aware that (4.35) may not be the easiest way to solve simple problems. For example, consider the initial value problem

$$u_t = \kappa u_{xx}\,, \qquad u(x,0) = \cos kx\,. \tag{4.46}$$

Guessing that the solution is *separable*

$$u = A(t)\cos kx\,, \tag{4.47}$$

we quickly find by substitution that

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\kappa k^2 A\,, \qquad \Rightarrow \qquad u(x,t) = \mathrm{e}^{-\kappa k^2 t}\cos kx\,. \tag{4.48}$$

If we can convince ourselves that the diffusion equation has a unique solution, then we have shown

$$\underbrace{\mathrm{e}^{-\kappa k^2 t}\cos kx}_{\text{separation of variables}} = \underbrace{\int_{-\infty}^{\infty} \cos kx' \,\frac{\mathrm{e}^{-(x-x')^2/4\kappa t}}{\sqrt{4\pi\kappa t}}\,\mathrm{d}x'\,.}_{\text{the Green's function solution}} \tag{4.49}$$

Example: Solve the initial value problem

$$u_t = \kappa u_{xx}\,, \qquad u(x,0) = \mathrm{e}^{-m^2x^2}\,. \tag{4.50}$$

We can guess that if we release a suitably scaled δ-pulse at $t_0 < 0$ then at $t = 0$ this pulse will have evolved into the Gaussian above. So consider

$$u_t = \kappa u_{xx}\,, \qquad u(x,t_0) = A\delta(x)\,, \tag{4.51}$$

with solution

$$u = \frac{A\mathrm{e}^{-x^2/4\kappa(t-t_0)}}{\sqrt{4\pi\kappa(t-t_0)}}\,. \tag{4.52}$$

Evaluating this at $t = 0$ we obtain $t_0 = -1/(4m^2\kappa)$, and we also determine the amplitude A. The final answer is

$$u(x,t) = \frac{\mathrm{e}^{-m^2x^2/(1+4m^2\kappa t)}}{\sqrt{1+4m^2\kappa t}}\,. \tag{4.53}$$

Example: Solve the initial value problem

$$u_t = \kappa u_{xx}\,, \qquad u(x,0) = x\mathrm{e}^{-m^2x^2}\,. \tag{4.54}$$

If u is a solution of the diffusion equation, then so is u_x. So, noting that the initial condition in this example is

$$x\mathrm{e}^{-m^2x^2} = -\frac{1}{2m^2}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-m^2x^2}\,, \tag{4.55}$$

we can employ the solution of the previous example to conclude that

$$u(x,t) = -\frac{1}{2m^2}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{e}^{-m^2x^2/(1+4m^2\kappa t)}}{\sqrt{1+4m^2\kappa t}} = \frac{x\mathrm{e}^{-m^2x^2/(1+4m^2\kappa t)}}{(1+4m^2\kappa t)^{3/2}}\,. \tag{4.56}$$

New solutions from old

In the examples above we've used some quasi-obvious properties of the diffusion equation to manufacture new solutions from old solutions. Just to be clear, let's summarize all these tricks. The most important observation is that the diffusion equation is linear: if u_1 and u_2 are solutions then so is an arbitrary linear combination $a_1u_1 + a_2u_2$.

Moreover, if $u(x,t)$ is a solution of the diffusion equation then:

1. so is $u(-x,t)$, and so is $u(x+x_0, t+t_0)$ where x_0 and t_0 are arbitrary constants;
2. so is $\partial_x^m u(x,t)$, and so is $\partial_t^n u(x,t)$;
3. so is $u(\alpha x, \alpha^2 t)$, where α is an arbitrary constant.

Under the operations above the diffusion equation is transformed into itself.

The transformational properties of differential equations is an important field of mathematics, initiated by Lie over a century ago. This is not the main focus of these lectures. But just to show that Lie group methods lead to non-obvious results, I remark that the transformation

$$t' = t + c\,, \qquad x' = x - 2ct - c^2\,, \qquad u' = u\mathrm{e}^{cx - c^2 t - c^3/3} \tag{4.57}$$

transforms the diffusion equation to itself:

$$u_t = \kappa u_{xx} \qquad \Rightarrow \qquad u'_{t'} = \kappa u'_{x'x'}\,. \tag{4.58}$$

There is nothing quasi-obvious about this transformation.

Uniqueness

So says "The physicist may consider such a proof superfluous; we shall consider it, however, on account of its mathematical elegance and the importance of the method."

To prove that the diffusion equation has a unique solution, suppose to the contrary that the initial value problem

$$v_t = \kappa \triangle v\,, \qquad v(\boldsymbol{x}, 0) = f(\boldsymbol{x})\,. \tag{4.59}$$

has two solutions, $v_1(\boldsymbol{x},t)$ and $v_2(\boldsymbol{x},t)$ say. Then the difference $u = v_1 - v_2$ also satisfies the diffusion equation, but with the initial condition

$$u(\boldsymbol{x}, 0) = 0\,. \tag{4.60}$$

Now let's proceed assuming that u might be non-zero and show that this leads to a contradiction. Multiply the diffusion equation by u and shuffle the result into the following form

$$\tfrac{1}{2}\left(u^2\right)_t = \kappa \triangle \left(u^2\right) - \kappa |\boldsymbol{\nabla} u|^2\,. \tag{4.61}$$

Notice that although u satisfies a conservation equation, u^2-stuff is not conserved: u^2 is destroyed at a rate $2\kappa|\boldsymbol{\nabla} u|^2$. Integrating over the entire space[2] we have

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int u^2\,\mathrm{d}V = -\kappa \int |\boldsymbol{\nabla} u|^2\,\mathrm{d}V\,. \tag{4.62}$$

Now integrate with respect to time

$$\tfrac{1}{2}\int u^2\,\mathrm{d}V = -\kappa \int_0^t \int |\boldsymbol{\nabla} u|^2\,\mathrm{d}V\,\mathrm{d}t\,; \tag{4.63}$$

This is a contradiction: the left is positive and the right is negative.

[2]OK — there is an assumption of rapid decay at infinity so that $\int u \boldsymbol{\nabla} u \cdot \boldsymbol{n}\,\mathrm{d}A$ can be dismissed. In a finite domain, with homogeneous BCs, there is no assumption necessary.

The moment method

Even when guessing doesn't work, one can still extract useful information about the solution without actually solving the diffusion equation.

Suppose, for example, that the initial condition in (4.33) is a hump, such as $f(x) = \operatorname{sech} x$. We can use (4.35) and write down an integral representation of the solution immediately. But if we are content with less than total information we can get a good qualitative feel by considering the *moments* of the solution:

$$m_p(t) = \int_{-\infty}^{\infty} x^p u(x,t)\,\mathrm{d}x\,. \tag{4.64}$$

If we think of $u(x,t)$ as a density then m_0 is the total mass and

$$X(t) \stackrel{\text{def}}{=} \frac{m_1}{m_0} \tag{4.65}$$

is the center of mass. The variance or moment of inertia is

$$\sigma^2 \stackrel{\text{def}}{=} m_0^{-1}\int_{-\infty}^{\infty}(x-X)^2\,u(x,t)\,\mathrm{d}x = (m_2/m_0) - X^2\,. \tag{4.66}$$

If we know m_0, X and σ then we know how much stuff there is, where the stuff is, and how far its spread out.

Multiplying (4.33) by x^p, with $p = 0$, 1 and 2, and integrating from $x = -\infty$ to $x = \infty$ we see

$$\dot m_0 = 0\,, \quad \dot m_1 = 0\,, \quad \dot m_2 = 2\kappa m_0\,. \tag{4.67}$$

The proof is integration by parts; the key intermediate step is

$$\int_{-\infty}^{\infty} x^2 u_{xx}\,\mathrm{d}x = \underbrace{\left[x^2 u_x - 2xu\right]_{-\infty}^{\infty}}_{=0} + \underbrace{2\int_{-\infty}^{\infty} u\,\mathrm{d}x}_{=2m_0}\,. \tag{4.68}$$

Substituting (4.67) into (4.65) and (4.66) we find that both m_0 and X are constant while

$$\sigma^2 = \sigma_0^2 + 2\kappa t\,. \tag{4.69}$$

Thus diffusion leaves the mass and the center of mass fixed but increases the width of the hump linearly with time. This gives us a good qualitative picture for the evolution of an initial pulse of heat.

The maximum principle

We can also show that if $u(x,t)$ is a solution of (4.33) then the maximum (and minimum) of $u(x,t)$ for all x and $t \geq 0$ is at $t = 0$. That is, the largest possible value of u is present in the initial condition $f(x)$.

We can deduce this *maximum principle* from the Green's function solution (4.35). This expresses u at the point (x,t) as an average of f over an interval of width $\sqrt{\kappa t}$ centered on x. If u had a maximum at x with $t > 0$, then this

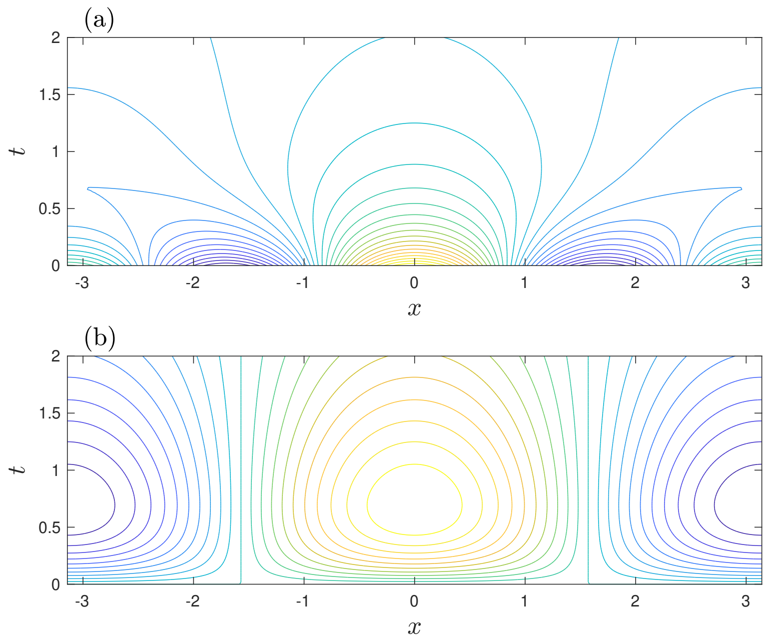


Figure 4.2: One of the panels above shows a solution of $u_t = u_{xx}$: which one is it? `diffSol2019.eps`

u would be larger than every value of f contributing to the average. This is impossible.

Another proof is by contradiction: suppose that there is an isolated maximum of $u(x,t)$ at some time $t > 0$. Then at this point

$$u_x = u_t = 0\,, \qquad \text{and} \qquad u_{xx} < 0\,. \tag{4.70}$$

The conditions above are inconsistent with $u(x,t)$ being a solution of the diffusion equation.

Exercise: One of the functions show in figure 4.2 is a solution of the diffusion equation $u_t = \kappa u_{xx}$. Which one is it?

4.3.2 Diffusion with a source

Consider the forced heat equation

$$u_t - \kappa u_{xx} = s(x,t)\,, \qquad u(x,0) = f(x)\,. \tag{4.71}$$

Without any loss of generality we can take $f(x) = 0$ — we can always use linear superposition to add on the part of the solution due to the initial condition.

If we knew the Green's function

$$\mathcal{G}_t - \kappa \mathcal{G}_{xx} = \delta(x)\delta(t) \tag{4.72}$$

then we could instantly write the solution of (4.72) as

$$u(x,t) = \int_{-\infty}^{\infty} \mathrm{d}x' \int_0^{\infty} \mathrm{d}t'\, \mathcal{G}(x - x', t - t') s(x', t')\,. \tag{4.73}$$

To determine $\mathcal{G}(x,t)$ we integrate (4.72) from $t=0^-$ to $t=0^+$:

$$\mathcal{G}(x,0^+) - \underbrace{\mathcal{G}(x,0^-)}_{=0} - \underbrace{\int_{0^-}^{0^+} \mathcal{G}_{xx}(x,t)\,\mathrm{d}t}_{\text{negligible}} = \delta(x)\,, \tag{4.74}$$

and conclude that

$$\mathcal{G}(x,0^+) = \delta(x)\,. \tag{4.75}$$

In other words, the inhomogeneous Green's function problem is equivalent to the initial value problem solved back in (4.34). Thus we have wasted a perfectly good symbol, $\mathcal{G}$, because:

$$\mathcal{G}(x,t) = g(x,t) = \frac{\mathrm{e}^{-x^2/4\kappa t}}{\sqrt{4\kappa\pi t}} H(t)\,. \tag{4.76}$$

We add the Heaviside factor $H(t)$ to remind us that the Green's function is zero if $t<0$. The Green's function $g(x,t)$ does double-duty: the solution of (4.71) is

$$u(x,t) = \int_{-\infty}^{\infty} f(x')g(x-x',t)\,\mathrm{d}x' + \int_{-\infty}^{\infty}\mathrm{d}x'\int_0^t \mathrm{d}t'\, s(x',t')g(x-x',t-t')\,. \tag{4.77}$$

Example: Solve the forced diffusion equation

$$v_t = \kappa v_{xx} + \exp(-m^2x^2)\,, \qquad v(x,0)=0\,. \tag{4.78}$$

Let's save the formula (4.77) for exercise 4.10. Instead, notice that in an earlier example we considered the initial value problem

$$u_t = \kappa u_{xx}\,, \qquad u(x,0) = \exp(-m^2x^2)\,, \tag{4.79}$$

and obtained the solution in (4.53). Now you can easily show by substitution that

$$v(x,t) = \int_0^t u(x,t')\,\mathrm{d}t' = \int_0^t \frac{\mathrm{e}^{-m^2x^2/(1+4m^2\kappa t')}}{\sqrt{1+4m^2\kappa t'}}\,\mathrm{d}t' \tag{4.80}$$

solves (4.78). We beat this into a standard form involving an exponential integral with the substitution

$$z = \frac{m^2x^2}{(1+4m^2\kappa t')}\,, \qquad \mathrm{d}t' = -\frac{x^2}{4\kappa}\frac{\mathrm{d}z}{z^2}\,; \tag{4.81}$$

thus

$$v(x,t) = \frac{|x|}{4m\kappa}\int_{\frac{m^2x^2}{1+4m^2\kappa t}}^{m^2x^2} \frac{\mathrm{e}^{-z}}{z^{3/2}}\mathrm{d}z\,. \tag{4.82}$$

4.4 Some Stokesian problems

As physical examples of half-line diffusion, we consider two classic problems solved by Stokes in the nineteenth century.

4.4.1 Stokes's first problem

The first problem is

$$T_t = \kappa T_{xx} \,, \qquad \text{with BC} \qquad T(0,t) = A\cos\omega t \,. \tag{4.83}$$

You can think of this as periodic heating the solid Earth (with the coordinate x running downwards from the surface at $x=0$). We're seeking the "forever solution" i.e., the solution at large time after the transients associated with the initial condition have disappeared. (Initial transients can, of course, be included by solving the unforced initial value problem and using linear superposition.)

The diffusion length (skin-depth) is

$$\ell = \sqrt{\kappa/\omega} \,. \tag{4.84}$$

If we're thinking of the annual cycle, and we use κ typical of dry soil: $1\text{yr} \approx \pi \times 10^7\text{s}$

$$\kappa = 2\times 10^{-3}\text{cm}^2 s^{-1} \,, \quad \omega = \frac{2\pi}{\pi\times 10^7\text{s}} \,. \tag{4.85}$$

leading to

$$\ell \approx 1\text{meter} \,. \tag{4.86}$$

Because of *linearity*, the diurnal and annual cycles can be superimposed. The diurnal heating has a penetration depth which is smaller by a factor of $\sqrt{365} \approx 19$ than the annual cycle.

Since

$$\cos\omega t = \tfrac{1}{2}\mathrm{e}^{\mathrm{i}\omega t} + \tfrac{1}{2}\mathrm{e}^{-\mathrm{i}\omega t} \,, \tag{4.87}$$

we can obtain the solution with the subsitution

$$T(x,t) = \mathrm{e}^{-\mathrm{i}\omega t}U(x) + \mathrm{e}^{\mathrm{i}\omega t}U^*(x) \,. \tag{4.88}$$

This gives

$$U'' + \mathrm{i}\ell^{-2}U = 0 \,, \tag{4.89}$$

with boundary conditions

$$U(0) = \tfrac{1}{2}A \,, \qquad \text{and} \qquad \lim_{x\to\infty} U(x) = 0 \,. \tag{4.90}$$

The standard guess $U = \mathrm{e}^{px}$ results in $p^2 + \mathrm{i}\ell^{-2} = 0$, or

$$p = \pm\sqrt{-\mathrm{i}}\ell^{-1} = \pm(1-\mathrm{i})q, \qquad \text{where} \quad q \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}\ell} \,. \tag{4.91}$$

To ensure decay as $x \to +\infty$ we must take $p = -(1-\mathrm{i})q$, so that

$$\begin{aligned} T(x,t) &= A\mathrm{e}^{-qx}\tfrac{1}{2}\left(\mathrm{e}^{\mathrm{i}qx-\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}qx+\mathrm{i}\omega t}\right) \,, \\ &= A\mathrm{e}^{-qx}\cos(\omega t - qx) \,. \end{aligned} \tag{4.92}$$

Notice there is a damped signal propagating downwards from the surface — the phase speed is ω/q.

The phase shift relative to the surface is π at a depth

$$x_* = \sqrt{2}\pi\ell \approx 4.4\,\mathrm{m}\,. \tag{4.93}$$

When it is Summer at the surface, it is Winter at a depth of 4.4 meters. The attenuation factor at this depth is

$$\mathrm{e}^{-qx_*} = \mathrm{e}^{-\pi} \approx \frac{1}{23}\,. \tag{4.94}$$

4.4.2 Stokes's second problem

Again we consider the diffusion equation,

$$T_t = \kappa T_{xx}\,, \tag{4.95}$$

on the half-line $x > 0$. The domain $x > 0$ starts at $T = 0$ but the boundary temperature at $x = 0$ is suddenly maintained at $T = 1$. This thermal signals then diffuses into the domain $x > 0$. Thus the the boundary and initial conditions

$$T(x,0) = 0\,, \qquad \text{and} \qquad T(0,t) = 1\,. \tag{4.96}$$

Dimensional analysis indicates that the solution has the form

$$T(x,t) = T(\eta)\,, \qquad \text{with} \qquad \eta \overset{\text{def}}{=} \frac{x}{2\sqrt{\kappa t}}. \tag{4.97}$$

Substitution into the diffusion equation gives

$$T'' + 2\eta T' = 0\,, \tag{4.98}$$

with

$$T(0) = 1\,, \qquad \text{and} \qquad \lim_{\eta\to\infty} T(\eta) = 0\,. \tag{4.99}$$

Integrating the ODE and applying the boundary conditions we have

$$T = \operatorname{erfc}(\eta)\,. \tag{4.100}$$

$$\operatorname{erfc}(z) \overset{\text{def}}{=} 1 - \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty \mathrm{e}^{-v^2}\,\mathrm{d}v$$

The thermal signal penetrates into solid a distance $\sqrt{\kappa t}$ in time t.

Kelvin's estimate of the age of the Earth

The surface of the Earth is heated by radiation from the sun, but also by diffusion of heat through the crust. Radiation from the sun is about 2×10^{17} Watts, averaging to 4×10^2 Watts m^{-2} on the surface. The geothermal heat flux is 4.4×10^{13} Watts, or on average 8.7×10^{-2} Watts m^{-2} upwards through the surface of the Earth. The Earth "re-radiates" slightly more heat than it receives from the sun because of this geothermal flux. On the other hand, the internal heat production within Jupiter is a substantial fraction of its insolation.

According to **CJ 2.13**, in some deep mines the geothermal temperature gradient — the signature of the diffusive flux of heat through the solid crust — is about 1°C every 24 meters. There is some geographic variation and on land the geothermal gradient Γ is in the range

$$10^{-2}\text{K m}^{-1} < \Gamma < 5 \times 10^{-2}\text{K m}^{-1} .$$

These numbers refer to non-Volcanic regions, and the differences result from local variations in the thermal diffusivity of rocks. No systematic large-scale pattern is evident.

Let's follow Kelvin and use the solution of Stokes's second problem to estimate the age of the Earth. Assume that we start at $t = 0$ with molten rock at a typical melting temperature, determined by laboratory measurements as $T_0 \approx 1500$K. This rock cools by radiation into space because the surface at $x = 0$ is held at $T = 0$. The thermal signal penetrates diffusively into the rock leaving a solid crust behind. We're standing on that crust now.

This crude model is Stokes's second problem, and the solution is

$$T(x,t) = T_0 \operatorname{erf}\left(\frac{x}{2\sqrt{\kappa t}}\right) , \tag{4.101}$$

implying that the surface temperature gradient is

$$T_x(0,t) = \frac{T_0}{\sqrt{\pi \kappa t}} . \tag{4.102}$$

Equating the surface temperature gradient, $T_x(0,t)$ above, to the present surface gradient, Γ, the elapsed time is

$$t = \frac{T_0^2}{\Gamma^2 \pi \kappa} . \tag{4.103}$$

Kelvin took a diffusivity for "average rock" of about $1.18 \times 10^{-6}\text{m}^2\ \text{s}^{-1}$. Using the smallest geothermal gradient Γ from (4.4.2), this gives for the age of the Earth[3]

$$t \sim 193 \times 10^6 \text{years} . \tag{4.104}$$

Note that $\sqrt{\kappa t} \sim 85$ kilometers, so that we can ignore the sphericity of the Earth — after 193 million years, the heat is still being extracted from a thin surface layer.

The age of the Earth is closer to 4.5 billion years and a folk explanation of Kelvin's gross underestimate is that Kelvin was unaware of radioactive heating. However Richter (1985) shows that inclusion of radioactive heating does not substantially increase estimates based on the diffusion equation with a volume source. Richter argues that it is mantle convection, which efficiently mixes the entire heat of the Earth up to the surface, that explains how the geothermal gradient can still be as large as 10 degrees per kilometer after 4.5 billion years.

[3] According to **CJ**, Kelvin's opinion was that the Earth could be no older than 24 million years. Kelvin used $\Gamma = 35$K per kilometer, typical of measurements near Edinburgh. In any event, whether it is 193 million years or 24 million years, this estimate is far too short to be consistent with the geologic record. For an amusing account of the age of the Earth, see *A Short History of Nearly Everything* by Bill Bryson. A textbook account is in *Physics of the Earth* by F.D.Stacey and P.M.Davis.

4.5 Half-line problems and images

4.5.1 The absorption boundary condition

Consider the problem of drunks random walking along the half-line $x > 0$. There is a manhole at $x = 0$ and so the drunks who reach $x = 0$ immediately disappear. This is equivalent to the diffusion problem

$$u_t = \kappa u_{xx} \qquad u(x,0) = f(x)\,, \qquad u(0,t) = 0\,, \tag{4.105}$$

where the dimensions of u are drunks per meter. There is an *absorption* boundary condition at $x = 0$.

The number of surviving drunks at time t is

$$S(t) = \int_0^\infty u(x,t)\,\mathrm{d}x\,, \tag{4.106}$$

and it follows from (4.105) that $f = -\kappa u_x$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\kappa u_x(0,t)\,. \tag{4.107}$$

The interpretation of the relation above is that the population decreases because of flux into the manhole.

One way of describing the demographics of these unfortunate random walkers is to use the histogram

$$P(t)\mathrm{d}t = \text{the number of lifetimes in the interval } (t, t+\mathrm{d}t). \tag{4.108}$$

There is a simple relation between $P(t)$ and $S(t)$ because

$$\begin{aligned} S(t) &= \text{the number of lives longer than } t\,, \\ &= \int_t^\infty P(t')\,\mathrm{d}t'\,. \end{aligned} \tag{4.109}$$

The derivative of this relation is

$$P(t) = -\frac{\mathrm{d}S}{\mathrm{d}t}\,, \tag{4.110}$$

and so from (4.107)

$$P(t) = \kappa u_x(0,t)\,. \tag{4.111}$$

Does this connection between the flux at t and the density of lifetimes make sense to you?

The Green's function of the initial value problem in (4.105), that is *absorption Green's function* $g^{\mathrm{a}}(x,\xi,t)$, is the solution of

$${g^{\mathrm{a}}}_t = \kappa {g^{\mathrm{a}}}_{xx} \qquad g^{\mathrm{a}}(x,0) = \delta(x-\xi)\,, \qquad g^{\mathrm{a}}(0,t) = 0\,. \tag{4.112}$$

Once we have g^{a}, the solution of (4.105) is

$$u(x,t) = \int_0^\infty g^{\mathrm{a}}(x,\xi,t) f(\xi)\,\mathrm{d}\xi\,. \tag{4.113}$$

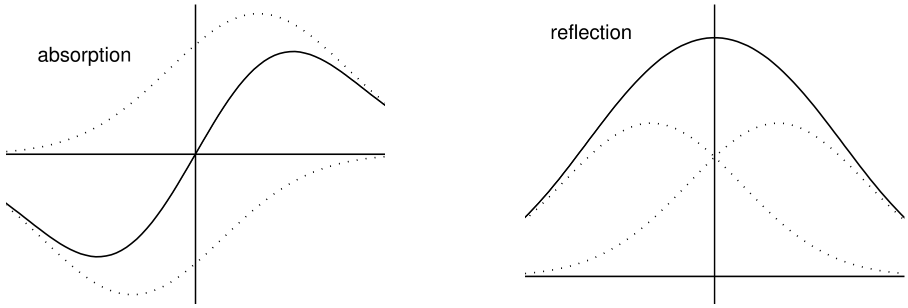


Figure 4.3: The method of images. In the left hand panel the absorption boundary condition is enforced with a negative heat pulse at $x = -\xi$. `imageDiff.eps`

This is a little more complicated than our earlier formulas because g^{a} is a function of x and ξ separately (not just $x - \xi$).

The δ-function at $x = \xi$ in (4.112) might correspond to a bar which closes at $t = 0$. All the patrons are suddenly ejected onto the street and start wandering back and forth. The unlucky ones disappear into the manhole at $x = 0$.

To solve this problem, take the δ-function at $x = \xi$ and reflect it in the boundary ($x = 0$) with a negative sign. Thus on the infinite line the initial condition is odd and the diffusion equation preserves this symmetry so that $g^{\mathrm{a}}(0, \xi, t)$ remains zero at all times. The solution of (4.112) can then be written down using linear superposition:

$$g^{\mathrm{a}}(x, \xi, t) = g(x - \xi, t) - g(x + \xi, t)\,, \tag{4.114}$$

$$= \tfrac{1}{\sqrt{4\pi\kappa t}}\left[\mathrm{e}^{-(x-\xi)^2/4\kappa t} - \mathrm{e}^{-(x+\xi)^2/4\kappa t}\right]\,, \tag{4.115}$$

$$= \tfrac{1}{\sqrt{\pi\kappa t}}\mathrm{e}^{-(x^2+\xi^2)/4\pi\kappa t}\sinh(x\xi/2\kappa t)\,. \tag{4.116}$$

Now we can calculate the number of survivors from the initial δ-pulse:

$$\mathrm{erf}(z) \stackrel{\mathrm{def}}{=} \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-u^2}\mathrm{d}u$$

$$\begin{aligned} S(t) &= \frac{1}{\sqrt{4\pi\kappa t}}\left[\int_0^\infty \mathrm{e}^{-(x-\xi)^2/4\kappa t}\mathrm{d}x - \int_0^\infty \mathrm{e}^{-(x+\xi)^2/4\kappa t}\mathrm{d}x\right]\,, \\ &= \frac{1}{\sqrt{\pi}}\left[\int_{-\xi/\sqrt{4\kappa t}}^\infty \mathrm{e}^{-u^2}\mathrm{d}u - \int_{\xi/\sqrt{4\kappa t}}^\infty \mathrm{e}^{-u^2}\mathrm{d}u\right]\,, \\ &= \mathrm{erf}\left(\frac{\xi}{\sqrt{4\kappa t}}\right)\,. \end{aligned} \tag{4.117}$$

When $\sqrt{4\kappa t} \gg \xi$, the erf simplifies to

$$S(t) \approx \frac{\xi}{\sqrt{\pi\kappa t}}\,. \tag{4.118}$$

Thus death is certain, $S(t) \to 0$, but the average lifetime,

The PDF of lifetimes is
$$P(t) = -\dot{S}$$
and
$$S(0) = N$$

$$\bar{t} = N^{-1}\int_0^\infty tP(t)\,\mathrm{d}t = N^{-1}\underbrace{\int_0^\infty S(t)\,\mathrm{d}t}_{\text{divergent}}\,, \tag{4.119}$$

is infinite. This indicates that a small fraction of the population make large positive excursions before encountering the absorbing boundary.

4.5.2 Inhomogeneous boundary conditions

How to solve the half-line ($x > 0$) initial value problem:

$$u_t = \kappa u_{xx}\,, \qquad u(0,t) = u_0(t)\,, \qquad u(x,0) = 0? \tag{4.120}$$

This mathematical problem corresponds to an isothermal solid medium occupying the half-space $x > 0$. At $t = 0$ the boundary temperature at $x = 0$ is prescribed to be $u_0(t)$. This thermal forcing diffuses into the interior, and we seek the interior temperature distribution $u(x,t)$.

Stokes's second problem is the special case $u_0(t) = H(t)$ and we obtained the erfc similarity solution, written here as

$$u(x,t) = \operatorname{erfc}(\eta) H(t)\,. \tag{4.121}$$

The Heaviside function ensures that $u(x, t < 0) = 0$ i.e., there is no disturbance before the steady surface heating switches on at $t = 0$.

Now consider "top-hat" forcing that switches on at $t = 0$ with an amplitude τ^{-1}, and then switches off at $t = 0$. We can represent this forcing function as

$$u_0(t) = \frac{H(t) - H(t-\tau)}{\tau}\,. \tag{4.122}$$

We've taken the amplitude of the boundary forcing to be τ^{-1}, so that as we take the limit $\tau \to 0$ we obtain a $\delta(t^+)$ i.e., a δ on the positive side of $t = 0$. We can use linear superposition, and the solution in (4.121), to solve the diffusion equation with the boundary temperature in (4.122):

$$u(x,t) = \frac{H(t)\operatorname{erfc}(\eta) - H(t-\tau)\operatorname{erfc}(\eta')}{\tau}\,, \tag{4.123}$$

where $\eta' = x/\sqrt{4\kappa(t-\tau)}$. The solution is shown in Figure 4.4.

Taking $\tau \to 0$ the top-hat in (4.122) becomes a $\delta(t^+)$. The solution (4.123) in this same limit $\tau \to 0$ is:

$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-v^2}\,dv$

$$\mathcal{G}(x,t) = -\eta_t \frac{2}{\sqrt{\pi}} e^{-\eta^2} = \frac{x}{2\sqrt{\pi\kappa t^3}} e^{-x^2/4\kappa t}\,. \tag{4.124}$$

$f(t) - f(t-\tau) \approx \tau f_t(t)$

This is the Green's function for the inhomogeneous half-line problem. The solution of (4.120) is therefore:

$$u(x,t) = \int_0^t u_0(t')\,\frac{x\,e^{-x^2/4\kappa(t-t')}}{2\sqrt{\pi\kappa(t-t')^3}}\,dt'\,. \tag{4.125}$$

The philosophy behind this method is that we approximate the forcing $u_0(t)$ arbitrarily closely by a sequence of narrow top-hats. We know the response to a single top-hat so we use linear superposition to write the response to the

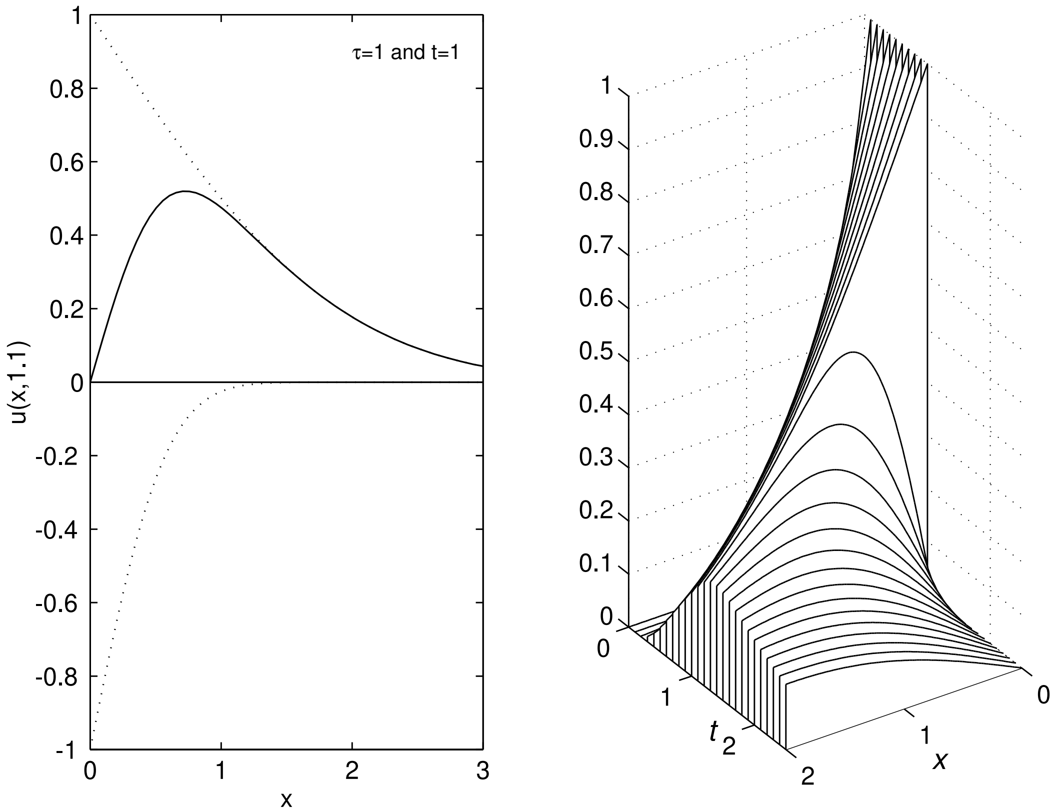


Figure 4.4: The solution in (4.123) with $\tau = \kappa = 1$. The left panel shows the subtraction of the two erfc's. The right panel shows the solution surface above the (x,t)-plane (again with $\tau = 1$). `halfDiff.eps`

sequence of top-hats — that's the guts of (4.125) and every other Green's function formula in these last few lectures. See problem 4.28 for a nice application of (4.125).

If we change variables in (4.125) with $\alpha \stackrel{\text{def}}{=} x/\sqrt{4\kappa(t-t')}$ we obtain the alternative form

$$u(x,t) = \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4\pi\kappa t}}^{\infty} u_0\left(t - \frac{x^2}{4\kappa\alpha^2}\right) \mathrm{e}^{-\alpha^2} \mathrm{d}\alpha \,. \tag{4.126}$$

In this form it is clear that the solution satisfies the initial and boundary conditions.

The Dirichlet to Neumann map

Signalling with diffusion

4.6 Specialized references to diffusion

For an interesting perspective on diffusion problems see

Be *Random Walks in Biology* by H.C. Berg.

A classic handbook on diffusion is

CJ *Conduction of Heat in Solids* by H.S. Carslaw & J.C. Jaeger.

A recent book emphasizing probability theory is

Re *A Guide to First-Passage Processes* by S. Redner.

For similarity solutions, with lots of physical examples, see

Ba *Scaling, Self-similarity, and Intermediate Asymptotics* by G.I. Barenblatt.

For Lie group methods see

Ol *Applications of Lie Groups to Differential Equations* by P.W. Olver. Springer, 1986.

4.7 Problems

Problem 4.1. Reconsider the hopping model in (4.7), assuming that the drunk hops to the right with probability p and to the left with probability q, where $p+q=1$. If $p>q$ then there is a bias towards the right. Find a distinguished limit in which the macroscopic evolution of the slowly varying density $\rho(x,t)$ is determined by the advection-diffusion equation

$$\rho_t + c\rho_x = \kappa\rho_{xx} \,. \tag{4.127}$$

Your answer should include expressions for c and κ in terms of τ, a, p and q.

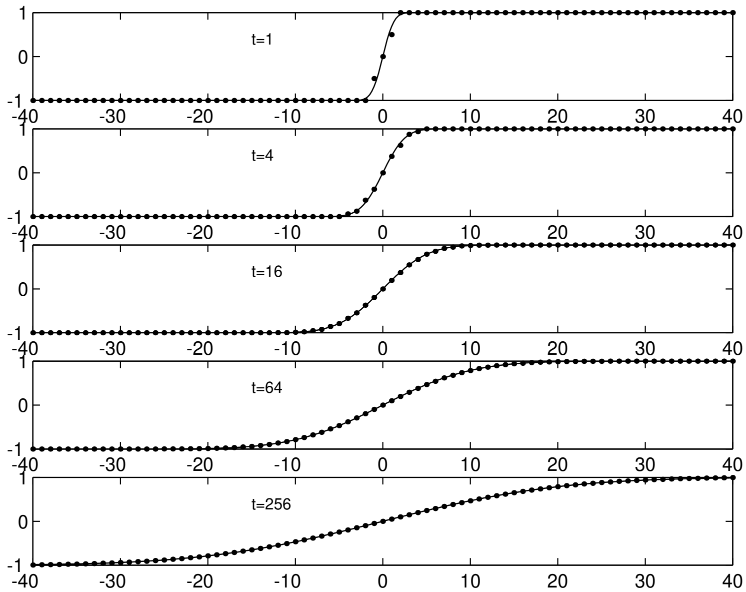


Figure 4.5: The solution of problem 4.2. `diffutz.eps`

Problem 4.2. Write a MATLAB program to iterate (4.7) with $\tau = a = 1$. The initial condition is

$$\rho(x, 0) = \text{sgn}(x)\,,$$

where $x = [-40 : 1 : 40]$. As boundary conditions take $\rho(\pm 40) = \pm 1$. Compare your numerical solution of the discrete system (4.7) with the appropriate erf similarity solution at $t = 1$, 4, 16, 64 and 256.

The most boring part of this problem is getting MATLAB to subplot at the right times and in the right place. Here's some of my code showing how this drudgery is handled. The educational part of the problem is to fill in the ????'s.

```
x =  [-40:1:40];     %% The discrete grid
xx = [-40:0.25:40]; %% For plotting the erf solution
u = sign(x);                        n = length(u);
tplot = [1 4 64 256];              nPlot = 0;
for t=[1:1:256]
?????????
?????????
?????????
if t==4^(nPlot)
nPlot = nPlot + 1;
subplot(5,1,nPlot)
plot(x,u,'.','markersize',10)
hold on
plot(xx,erf(????))
time = num2str(t)
text(-15,0.4,['t='time])
end
end
```

Problem 4.3. Obtain the Green's function for the damped advection-diffusion equation by solving

$$g_t + c g_x = \kappa g_{xx} - \beta g \,, \qquad g(x,0) = \delta(x) \,. \tag{4.128}$$

Hint: transform the damped advection-diffusion equation to the plain old diffusion equation and use results from the lecture.

Problem 4.4. Apply the method of moments to the initial value problem

$$\rho_t + c\rho_x = \kappa \rho_{xx} - \beta \rho \,, \qquad \rho(x,0) = \rho_0(x) \,. \tag{4.129}$$

The initial condition is a compact humplike density e.g., something like $\rho_0(x) = \operatorname{sech}(x)$. Obtain the mass, center of mass and the width of the hump as functions of time.

Problem 4.5. Find a similarity solution of the nonlinear diffusion equation

$$u_t = \kappa \left(u^m\right)_{xx} \,, \qquad u(x,0) = \delta(x) \,. \tag{4.130}$$

Hint: The answer is in section 51 of Landau & Lifshitz *Fluid Mechanics*.

Problem 4.6. Consider the diffusion equation $\theta_t = \kappa\theta_{xx}$ with the initial condition

$$\theta(x,0) = \begin{cases} 1\,, & \text{if} \quad |x| < a\,; \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{4.131}$$

(i) Define nondimensional variables so that $(a,\kappa) \to 1$. *(ii)* Use linearity and the erf-solution in (4.25) to solve this initial value problem. *(iii)* Use MATLAB to illustrate your solution (see figure 4.6).

Problem 4.7. *(i)* Show that if $\theta_1(x,t)$ and $\theta_2(y,t)$ are solutions of the one-dimensional diffusion equations:

$$\theta_{1t} = \kappa\theta_{1xx} \,, \qquad \theta_{2t} = \kappa\theta_{2yy} \,,$$

then $\theta(x,y,t) \stackrel{\text{def}}{=} \theta_1(x,t)\theta_2(y,t)$ is a solution of the two-dimensional diffusion equation, $\theta_t = \kappa(\theta_{xx} + \theta_{yy})$. *(ii)* Solve the 2D diffusion equation with an initial condition that $\theta(x,y,0)$ is one inside the square defined by $|x| < a$ and $|y| < a$ and zero outside.

Problem 4.8. Solve the forced diffusion equation

$$\theta_t - \kappa\theta_{xx} = \cos kx \,, \qquad \text{with IC} \qquad \theta(x,0) = 0 \,. \tag{4.132}$$

Problem 4.9. Solve the forced diffusion equation

$$\theta_t = \theta_{xx} + \mathrm{e}^{-2t}\cos x \,, \qquad \text{with IC} \qquad \theta(x,0) = 0 \,. \tag{4.133}$$

Which panel of figure 4.2 shows this solution?

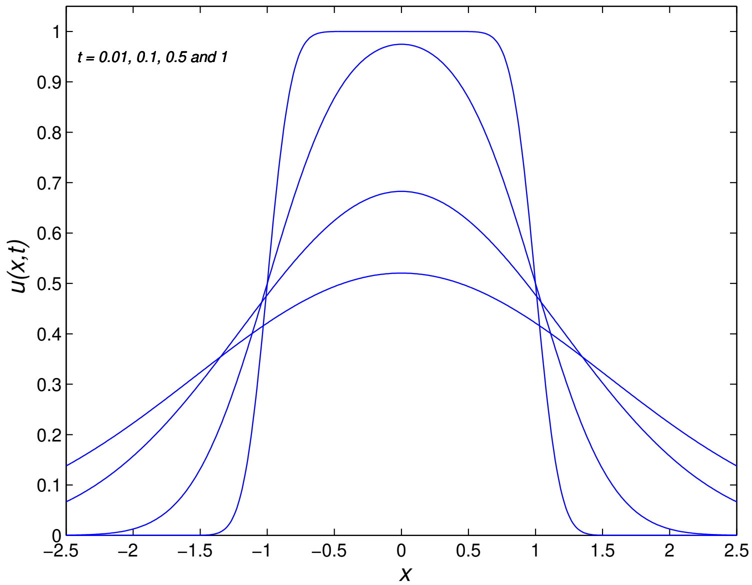


Figure 4.6: Diffusive smoothing of the initial condition in problem 4.6. `diffcompz.eps`

Problem 4.10. *(i)* As an exercise in Gaussian integrals, use the formula (4.77) to show that the solution of

$$u_t = \kappa u_{xx} + h(t)\frac{m\mathrm{e}^{-m^2x^2}}{\sqrt{\pi}}\,, \qquad u(x,0) = 0\,, \tag{4.134}$$

is

$$u(x,t) = \frac{m}{\sqrt{\pi}}\int_0^t \frac{h(t')\,\mathrm{e}^{-m^2x^2/(1+4m^2\kappa(t-t'))}}{\sqrt{1+4\kappa m^2(t-t')}}\,\mathrm{d}t'\,. \tag{4.135}$$

Above, $h(t)$ is some function of time. *(ii)* Consider the limit $m \to \infty$. Show that the heat source is $h(t)\delta(x)$, and that the temperature at the source is

$$u(0,t) = \frac{1}{2\sqrt{\pi\kappa}}\int_0^t \frac{h(t')}{\sqrt{t-t'}}\,\mathrm{d}t'\,. \tag{4.136}$$

(iii) Find a heating function $h(t)$ that makes $u(0,t)$ independent of time.

Problem 4.11. *(i)* Show that the similarity guess

$$u(x,t) = t^a\kappa^b f(\eta)\,, \qquad \eta = x/2\sqrt{\kappa t}\,, \tag{4.137}$$

a possible solution of the problem

$$u_t - \kappa u_{xx} = \delta(x) q t^{q-1}\,, \qquad u(x,0) = 0\,, \qquad (q>0)\,, \tag{4.138}$$

only if $q = a + \frac{1}{2}$ and $b = -1/2$. *(ii)* Substitute the guess into the PDE and show that $f(\eta)$ satisfies the ODE

$$f'' + 2\eta f' - 4af = 0\,. \tag{4.139}$$

(iii) Reduce the ODE above to quadratures in the special case $q = 1$.

Problem 4.12. A function $F(x, y)$ is said to be *homogeneous of degree* n if

$$F(\lambda x, \lambda y) = \lambda^n F(x, y)\,, \tag{4.140}$$

for any constant λ. For example, $F = x/y$ is homogeneous with $n = 0$. *(i)* Give examples of homogeneous functions of degree $n = 0$, 1 and 2. *(ii)* Obtain a PDE satisfied by $F(x, y)$ and solve this equation in terms of an 'arbitrary function'. [*Hint:* differentiate with respect to λ and then set $\lambda = 1$.]

Problem 4.13. After (4.28) I said "As we freely vary both λ and μ the value of u at the same point in space-time is unchanged i.e., (4.28) is always true. This implies that x, t and κ appear only in the combination $x^2/(\kappa t) = x'^2/(\kappa' t')$." Use the method of problem 4.12 to prove this assertion.

Problem 4.14. *(i)* Find the exponents a, b and c which make the similarity guess

$$u(x, t) = t^a \nu^c v(\eta)\,, \qquad \eta \stackrel{\text{def}}{=} x/(\nu t)^b\,, \tag{4.141}$$

(with v non-dimensional) a possible solution of the dispersive wave equation

$$u_t = \nu u_{xxx}\,, \qquad u(x, 0) = \delta(x)\,. \tag{4.142}$$

(ii) Substitute the guess into the PDE and find the ODE satisfied by $v(\eta)$. *(iii)* Solve the ODE in terms of a special function you might find in the appendix of **BO**.

Problem 4.15. Consider the PDE

$$u_t = \kappa u_{xxx}\,, \qquad u(x, 0) = A\,\mathrm{sgn}(x)\,.$$

What are the dimensions of κ? Construct a similarity solution analogous to (4.17), and find the ODE satisfied by $U(\xi)$. Brownie points for solving this ODE.

Problem 4.16. Suppose you are given the solution, $u(x, t)$, of the homogeneous initial value problem:

$$u_t = u_{xxx}\,, \qquad u(x, 0) = \mathrm{e}^{-x^2}\,. \tag{4.143}$$

(i) Express the solution of the forced initial value problem

$$v_t = v_{xxx} + \mathrm{e}^{-x^2}\,, \qquad v(x, 0) = 0\,, \tag{4.144}$$

in terms of $u(x, t)$. *(ii)* Suppose that $\chi(t) = 1$ when $0 < t < 1$, and $\chi(t) = 0$ otherwise. Express the solution of the forced initial value problem

$$w_t = w_{xxx} + \mathrm{e}^{-x^2}\chi(t)\,, \qquad w(x, 0) = 0\,, \tag{4.145}$$

in terms of $v(x, t)$. *(iii)* Express the solution of the forced problem

$$h_t = h_{xxx} - 2x\mathrm{e}^{-x^2}\delta(t)\,, \qquad h(x, t < 0) = 0\,, \tag{4.146}$$

in terms of u.

Problem 4.17. Show that

$$\frac{\mathrm{e}^{-x^2/4\kappa t}}{\sqrt{4\pi\kappa t}} = \int_{-\infty}^{\infty} \mathrm{d}x_1 \, \frac{\mathrm{e}^{-x_1^2/4\kappa t_1}}{\sqrt{4\pi\kappa t_1}} \frac{\mathrm{e}^{-(x-x_1)^2/4\kappa(t-t_1)}}{\sqrt{4\pi\kappa(t-t_1)}} \,, \tag{4.147}$$

where $0 < t_1 < t$. And

$$\frac{\mathrm{e}^{-x^2/4\kappa t}}{\sqrt{4\pi\kappa t}} = \int_{-\infty}^{\infty} \mathrm{d}x_2 \int_{-\infty}^{\infty} \mathrm{d}x_1 \, \frac{\mathrm{e}^{-x_1^2/4\kappa t_1}}{\sqrt{4\pi\kappa t_1}} \frac{\mathrm{e}^{-(x_2-x_1)^2/4\kappa(t_2-t_1)}}{\sqrt{4\pi\kappa(t_2-t_1)}} \frac{\mathrm{e}^{-(x-x_2)^2/4\kappa(t-t_2)}}{\sqrt{4\pi\kappa(t-t_2)}} \,, \tag{4.148}$$

where $0 < t_1 < t_2 < t$. Generalize to $0 < t_1 < t_2 < t_3 \cdots < t_n < t$. (If you understand the concept of a Green's function then no algebra is required.)

Problem 4.18. Solve the half-line ($0 < x < \infty$) diffusion problem

$$u_t = \kappa u_{xx} - \mathrm{e}^{-\alpha x} \cos(\omega t) \,.$$

The boundary condition at $x = 0$ is no flux of heat, $\kappa u_x(0, t) = 0$. Discuss the structure of the solution in the two limits $\alpha^{-1} \gg \sqrt{\kappa/\omega}$ and $\alpha^{-1} \ll \sqrt{\kappa/\omega}$.

Problem 4.19. Solve the PDE

$$u_t + c u_x = \kappa u_{xx} \,, \qquad u(0, t) = \cos \omega t \,,$$

on the half-line $x > 0$. There is a nondimensional parameter, $p \overset{\text{def}}{=} c/\sqrt{\omega\kappa}$. Discuss the structure of the solution in the limits $c \to \infty$ and $c \to -\infty$. Can you obtain these limiting solutions more simply from the original PDE by neglecting the diffusive term and solving the resulting first-order (in x) problem?

To assist grading and discussion of this problem let us agree to use the following notation

$$\nu \overset{\text{def}}{=} \frac{|c|}{2\kappa} \,, \qquad \lambda \overset{\text{def}}{=} \frac{4}{p^2} = \frac{4\omega\kappa}{c^2} \,.$$

The identity

$$\sqrt{1 + \mathrm{i}\lambda} = \pm \left[\sqrt{\frac{r+1}{2}} + \mathrm{i} \sqrt{\frac{r-1}{2}} \right] \,, \quad r \overset{\text{def}}{=} \sqrt{1 + \lambda^2}$$

(which applies only if $\lambda > 0$) might also be useful. If you use the identity, then prove or verify it and explain what happens if $\lambda < 0$.

Problem 4.20. *(i)* Find the *reflection Green's function*

$$g^{\mathrm{r}}{}_t = \kappa g^{\mathrm{r}}{}_{xx} \qquad g^{\mathrm{r}}(x, 0) = \delta(x - \xi) \,, \qquad g^{\mathrm{r}}{}_x(0, t) = 0 \,,$$

using the method of images. *(ii)* The center of mass of the pulse is

$$X(\xi, t) \overset{\text{def}}{=} \frac{\int_0^\infty x g^{\mathrm{r}}(x, \xi, t) \, \mathrm{d}x}{\int_0^\infty g^{\mathrm{r}}(x, \xi, t) \, \mathrm{d}x} \,.$$

Show in the large-time limit (i.e., when $\sqrt{\kappa t} \gg \xi$) that $X(\xi, t) \sim a\sqrt{t}$ and calculate the constant a.

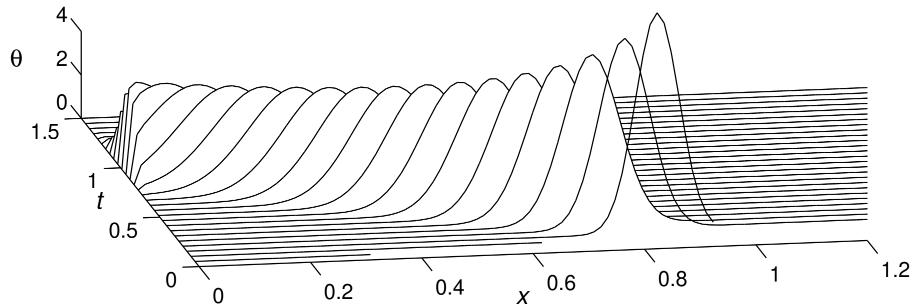


Figure 4.7: Solution of problem 4.21 with $c = -1$, $\xi = 1$ and $\kappa = 1/200$.`advecDiffAbs.eps`

Problem 4.21. *(i)* Show that the Green's function of the advection-diffusion equation

$$g_t + cg_x = \kappa g_{xx}\,, \qquad g(x,0) = \delta(x)\,, \tag{4.149}$$

is

$$g(x,t) = \frac{\mathrm{e}^{-(x-ct)^2/4\kappa t}}{\sqrt{4\kappa t}}\,. \tag{4.150}$$

(ii) Now consider the advection-diffusion equation on a half-line $x > 0$ with an absorbing boundary condition

$$g^{\mathrm{a}}_t + cg^{\mathrm{a}}_x = \kappa g^{\mathrm{a}}_{xx}\,, \qquad g^{\mathrm{a}}(x,0) = \delta(x-\xi)\,, \qquad g^{\mathrm{a}}(0,t) = 0\,. \tag{4.151}$$

If $c > 0$ then the drunks have a bias to walk away from the manhole at $x = 0$. Guess that a solution might be

$$g^{\mathrm{a}}(x,t) = \frac{1}{\sqrt{4\pi\kappa t}}\left[\mathrm{e}^{-(x-\xi-ct)^2/4\kappa t} - \alpha\mathrm{e}^{-(x+\xi-ct)^2/4\kappa t}\right]\,, \tag{4.152}$$

and determine α by substitution. *(iii)* Show that the probability that a drunk escapes the manhole is $1 - \exp(-c\xi/\kappa)$ if $c > 0$ and zero if $c < 0$ (see figure 4.7).

Problem 4.22. Use a Fourier series to solve the half-line ($x > 0$) heating problem:

$$u_t = \kappa u_{xx}\,, \qquad u(0,t) = \mathrm{sqr}(\omega t)\,. \tag{4.153}$$

This is heating the Earth with a uniformly hot Summer followed by a corresponding Winter. Suppose that $\kappa = 2 \times 10^{-3}\mathrm{cm}^2\mathrm{s}^{-1}$ and $T = 2\pi/\omega$=1 year= 3.15×10^7 seconds. Plot the temperature over a yearly cycle at $x = 0$, $x = 1$ meter and $x = 4$ meters.

Problem 4.23. Consider the half-line ($x > 0$) heating problem:

$$\theta_t = \theta_{xx}\,, \qquad \theta(0,t) = \sin t\, H[\sin t]\,. \tag{4.154}$$

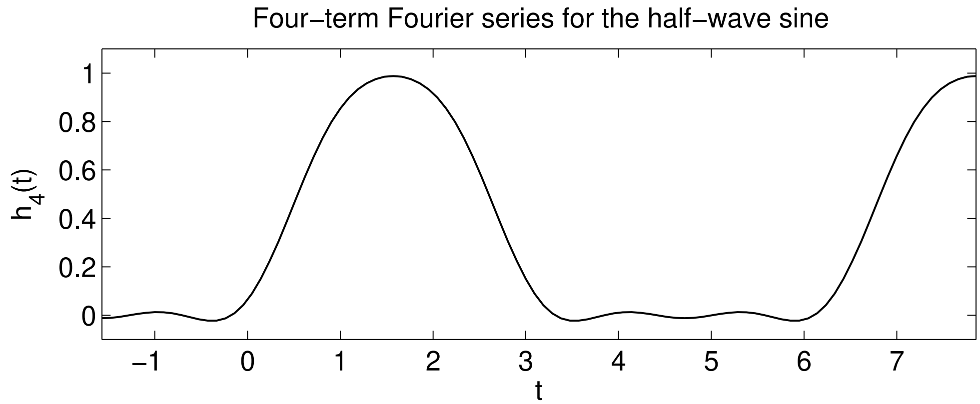


Figure 4.8: The four-term Fourier series from problem 4.23. `halfWave.eps`

($H(s)$ is the Heaviside step function.) *(i)* Show that the four-term Fourier series approximation to the surface temperature, $h(t) = \sin t\, H[\sin t]$, is

$$h_4(t) = 0.318 + 0.5 \sin t - 0.212 \cos 2t - 0.042 \cos 4t\,. \tag{4.155}$$

(You should obtain simple analytic expressions for the numbers above.) *(ii)* Use MATLAB to graph $h_4(t)$ over the interval $(-0.5\pi, 2.5\pi)$ (see figure 4.8). *(iii)* Find the "forever solution" of the PDE and plot the temperature as a function of time at $x = 0.1$, 1 and 4. Discuss the form of the solution at great depth, $x \gg 1$.

Problem 4.24. Solve the hyper-diffusion problem

$$u_t = -u_{xxxx}\,, \qquad u(x, 0) = \mathrm{sqr}(x)\,. \tag{4.156}$$

on the loop $-\pi \le x \le x$. Use MATLAB to display snapshots of the solution at selected times. Does the hyperdiffusion equation have a maximum principle?

Problem 4.25. *(i)* Considering the half-line problem

$$\theta_t = \kappa u_{xx}\,, \qquad \theta(x, 0) = 0\,, \qquad \kappa\theta_x(0, t) = -1\,, \tag{4.157}$$

show that

$$\int_0^\infty \theta(x, t)\mathrm{d}x = t\,. \tag{4.158}$$

(ii) Solve the problem with a similarity ansatz. *(iii)* Adapt the argument in section **4.5.2** to obtain a Green's function solution of the applied flux problem $\eta \overset{\text{def}}{=} \dfrac{x}{2\sqrt{\kappa t}}$

$$\theta_t = \kappa\theta_{xx}\,, \qquad \theta(x, 0) = 0\,, \qquad \kappa\theta_x(0, t) = -f(t)\,. \tag{4.159}$$

(iv) Show that the surface temperature is given in terms of the applied flux by

$$\theta_0(t) = \int_0^t \frac{f(t')}{\sqrt{\pi\kappa(t - t')}}\,\mathrm{d}t'\,. \tag{4.160}$$

(v) Calculate the surface temperature if the flux is $f(t) = t^p$. To check your answer show that θ_0 is constant if $p = -1/2$.
Hint: In part *(ii)* the similarity ansatz results in the ODE $\Theta'' + 2\eta\Theta' - 2\Theta = 0$. Solve this by guessing a special solution, followed by reduction of order.

Problem 4.26. Consider the $x > 0$ forced problem

$$\theta_t = s + \kappa\theta_{xx}\,, \qquad \theta(0,t) = 0\,, \qquad \theta(x,0) = \theta_0\,, \tag{4.161}$$

where the constant s is a uniform source and θ_0 is the uniform initial temperature. *(i)* Solve this diffusion problem. (Hint: you can fool around with the solution of Stokes's second problem to obtain the solution of this new problem. Not much algebra is required.) *(ii)* Show that the surface gradient is

$$\theta_x(0,t) = 2s\sqrt{\frac{t}{\pi\kappa}} + \frac{\theta_0}{\sqrt{\pi\kappa t}}\,. \tag{4.162}$$

Problem 4.27. Give an alternative solution of (4.120) by writing

$$w(x,t) \stackrel{\text{def}}{=} u(x,t) - q(t)\,. \tag{4.163}$$

Thus $w(x,t)$ satisfies a forced diffusion equation with the homogeneous boundary condition $w(0,t) = 0$. Construct a Green's function solution of this equation by applying Duhamel's principle to the absorption Green's function in (4.116). Make sure your answer agrees with (4.125).

Problem 4.28. Considering the problem (4.120), find a compact expression for the "inventory" and the surface flux,

$$h(t) \stackrel{\text{def}}{=} \int_0^\infty u(x,t)\,\mathrm{d}x\,, \qquad \text{and} \qquad f(t) \stackrel{\text{def}}{=} -\kappa u_x(0,t)\,, \tag{4.164}$$

in terms of the applied surface temperature $u_0(t)$. Suppose that $u_0(t)$ decays faster than any power of t at large time e.g., $u_0(t) = \exp(-t)$. Show that at large times

$$h(t) \sim \frac{\alpha_0}{t^{1/2}} + \frac{\alpha_1}{t^{3/2}} + \frac{\alpha_2}{t^{5/2}} + \cdots \tag{4.165}$$

and determine the constants α_0 and α_1 in terms of $u_0(t)$.

Problem 4.29. *(i)* Show that if $\theta_1(x,t)$ and $\theta_2(y,t)$ are solutions of the one-dimensional diffusion equations:

$$\theta_{1t} = \kappa\theta_{1xx}\,, \qquad \theta_{2t} = \kappa\theta_{2yy}\,, \tag{4.166}$$

then $\theta(x,y,t) \stackrel{\text{def}}{=} \theta_1(x,t)\theta_2(y,t)$ is a solution of the two-dimensional diffusion equation. *(ii)* Use this trick to find the two-dimensional diffusion Green's function defined by

$$g_t = \kappa(g_{xx} + g_{yy})\,, \quad g(x,y,0) = \delta(x)\delta(y)\,. \tag{4.167}$$

(iii) Suppose that $\zeta_1(x,t)$ and $\zeta_2(y,t)$ are solutions of the one-dimensional wave equations

$$\zeta_{1tt} = c^2\zeta_{1xx}\,, \qquad \text{and} \qquad \zeta_{2tt} = c^2\zeta_{2yy} \tag{4.168}$$

Is $\zeta(x,y,t) = \zeta_1(x,t)\zeta_2(y,t)$ a solution of the two-dimensional wave equation $\zeta_{tt} = c^2(\zeta_{xx} + \zeta_{yy})$? *(iv)* Find another example of a PDE for which this "multiplication trick" works.

Problem 4.30. Consider an ensemble of $N \gg 1$ random walkers wandering in the half-plane $y > 0$; the walkers are released from the point $(x,y) = (0,\xi)$ at $t = 0$. A cliff runs along the x-axis, so that walkers fall to their death on a first encounter with $y = 0$. *(i)* Formulate and solve the two-dimensional random walk model. *(ii)* Calculate the survival function analogous to (4.106). *(iii)* How does the density of corpses at the foot of the cliff vary with x as $t \to \infty$?

Lecture 5

Fourier series

5.1 Least squares approximation

Consider the periodic function $\text{sqr}(x)$ defined by $\text{sqr}(x) = \text{sgn}[\sin(x)]$. Can we find the "best" approximation of $\text{sqr}(x)$ in terms of $\sin x$,

The "square wave".

$$\text{sqr}(x) \approx b \sin x\,, \tag{5.1}$$

by picking the parameter b? There is no unique answer to this question because any choice will involve compromises somewhere. But here is an answer suggested by Gauss: pick b so that the *mean square error*

$$\varepsilon(b) \stackrel{\text{def}}{=} \int_{-\pi}^{\pi} [\text{sqr}(x) - b\sin x]^2 \,\frac{\mathrm{d}x}{2\pi}\,, \tag{5.2}$$

is as small as possible.

We can figure out b almost *before* evaluating any integrals:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}b} = -2\int_{-\pi}^{\pi} \sin x\,[\text{sqr}(x) - b\sin x]\,\frac{\mathrm{d}x}{2\pi}\,. \tag{5.3}$$

The optimal b is determined by

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}b} = 0\,, \qquad \Rightarrow \qquad b = \frac{\int_{-\pi}^{\pi} \sin x\;\text{sqr}(x)\,\mathrm{d}x}{\int_{-\pi}^{\pi}\sin^2 x\,\mathrm{d}x} = \frac{4}{\pi}\,. \tag{5.4}$$

With (5.1) we can't expect to get a very good approximation. For instance, $4/\pi = 1.27$ and so our approximation has a 27% *pointwise* error at $x = \pi/2$. Another measure of the error is the ratio

$$\int_{-\pi}^{\pi}\sin^2 x\,\mathrm{d}x = \int_{-\pi}^{\pi}\cos^2 x\,\mathrm{d}x = \pi$$

$$\int_0^{\pi}\sin x\,\mathrm{d}x = 2$$

$$\frac{\int_{-\pi}^{\pi} b^2\sin^2(x)\,\mathrm{d}x}{\int_{-\pi}^{\pi}\text{sqr}^2(x)\,\mathrm{d}x} = \frac{8}{\pi^2} = 0.81\,. \tag{5.5}$$

The simple approximation contains 81% of the "energy" in $\text{sqr}(x)$.

Now suppose we use more functions to approximate our target function. The approximation can only improve as we use more basis functions.

Fourier's suggestion is to use sinusoids. Let us consider the general case of approximating a "target function" $f(x)$ on the fundamental interval $-\pi \le x \le \pi$ using a linear combination of sinusoids

$$\begin{aligned} S_n(x) \stackrel{\text{def}}{=} a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx \\ + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx \,. \end{aligned} \tag{5.6}$$

In other words, we hope to pick the coefficients in S_n so that $f(x) \approx S_n(x)$.

We have $2n+1$ adjustable parameters, and we define the mean square error as

$$\varepsilon_n(a_k, b_k) \stackrel{\text{def}}{=} \int_{-\pi}^{\pi} [f(x) - S_n(x)]^2 \, \frac{\mathrm{d}x}{2\pi} \,, \tag{5.7}$$

and we can proceed just as before by taking derivatives with respect to a_k and b_k. We find

$$\begin{aligned} \frac{\mathrm{d}\varepsilon_n}{\mathrm{d}a_k} &= -2 \int_{-\pi}^{\pi} \cos kx \, [f(x) - S_n(x)] \, \frac{\mathrm{d}x}{2\pi} = 0 \,, \\ \frac{\mathrm{d}\varepsilon_n}{\mathrm{d}b_k} &= -2 \int_{-\pi}^{\pi} \cos kx \, [f(x) - S_n(x)] \, \frac{\mathrm{d}x}{2\pi} = 0 \,. \end{aligned} \tag{5.8}$$

These are $2n+1$ equations for the $2n+1$ unknowns a_k and b_k.

The sinusoids have the extraordinary property that if k and m are positive integers

$$\int_{-\pi}^{\pi} \cos kx \, \sin mx \, \mathrm{d}x = 0 \,, \tag{5.9}$$

and

$$\int_{-\pi}^{\pi} \cos kx \, \cos mx \, \mathrm{d}x = \int_{-\pi}^{\pi} \sin kx \, \sin mx \, \mathrm{d}x = \pi \delta_{km} \,, \tag{5.10}$$

where

$$\delta_{km} = \begin{cases} 0 & \text{if} \quad k \neq m \,, \\ 1 & \text{if} \quad k = m \,. \end{cases} \tag{5.11}$$

To prove the results above, you should recall that $\cos kx$ and $\sin kx$ can be written as linear combinations or $\mathrm{e}^{\pm \mathrm{i} kx}$. So a product such as $\cos kx \cos mx$ will contain combinations such as $\mathrm{e}^{\pm \mathrm{i}(k \pm m)x}$, and

$$\int_{-\pi}^{\pi} \mathrm{e}^{\pm \mathrm{i}(k \pm m)x} \, \mathrm{d}x = 0 \,, \quad \text{unless } k = m. \tag{5.12}$$

Using the *orthogonality relations* in the previous paragraph, we determine a_k and b_k without having to solve linear equations. For instance, if $k \ge 1$:

$$\int_{-\pi}^{\pi} S_n(x) \cos kx \, \mathrm{d}x = \sum_{p=1}^{n} a_p \int_{-\pi}^{\pi} \cos kx \cos px \, \mathrm{d}x \,, \tag{5.13}$$

$$= \pi \sum_{p=1}^{n} a_p \delta_{kp} = \pi a_k \,. \tag{5.14}$$

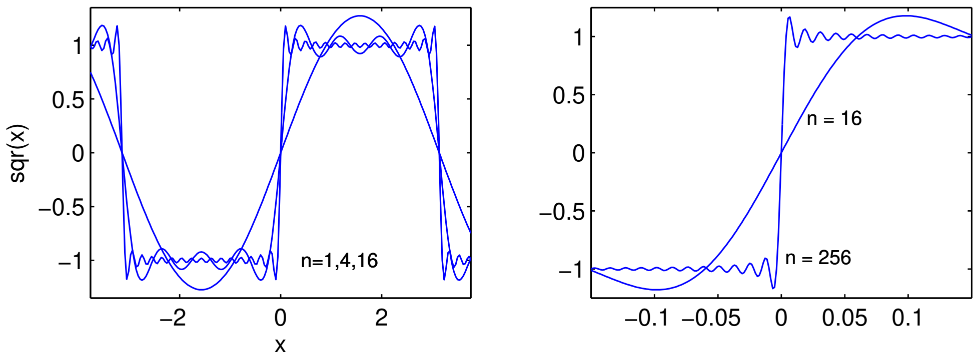


Figure 5.1: Convergence of the Fourier series of sqr(t). The left panel shows the partial sum with 1, 4 and 16 terms. The right panel is an expanded view of the Gibbs oscillations round the discontinuity at $x = 0$. Notice that the overshoot near $x = 0$ does not get smaller if n is increased from 16 to 256. `FourSer4.eps`

We have used the handy rule

$$\sum_{p=1}^{\infty} [\text{anything}]_p \, \delta_{kp} = [\text{anything}]_k \, . \tag{5.15}$$

The optimal choice of the coefficients a_k and b_k in (5.6) is therefore:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, \mathrm{d}x \, , \tag{5.16}$$

and for $k \geq 1$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos kx \, f(x) \, \mathrm{d}x \, , \qquad \text{and} \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin kx \, f(x) \, \mathrm{d}x \, . \tag{5.17}$$

Notice the irritating exception for a_0.

If the target function $f(x)$ is even, then all the b_k's are zero i.e., we approximate an even function using only the cosines, so that our approximation is also even. *Mutatis mutandis* if $f(x)$ is odd.

The complete Fourier series of sqr(x)

Now let's apply the recipe in (5.16) through (5.17) to sqr(x). We begin by observing that because sqr(x) is an odd function, all the a_k's are zero. To evaluate b_k notice that the integrand of (5.17) is even so that we need only integrate from 0 to π

$\cos k\pi = (-1)^k$

$$b_k = \frac{2}{\pi} \int_0^{\pi} \sin kx \, \mathrm{d}x = - \left[\frac{2}{\pi k} \cos kx \right]_0^{\pi} = \left[1 - (-1)^k \right] \frac{2}{\pi k} \, . \tag{5.18}$$

The even b_k's are also zero — this is clear from the anti-symmetry of the integrand above about $x = \pi/2$. A sensitive awareness of symmetry is often a great help in evaluating Fourier coefficients. Thus we have

$$\operatorname{sqr}(x) = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \cdots\right]. \tag{5.19}$$

The wiggly convergence of (5.19) is illustrated in figure 5.1 — more about the wiggles later.

Exercise: Deduce the *Gregory-Leibniz series*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{5.20}$$

from (5.19).

Remarks on Fourier series

As an undergraduate you might have suffered through many enervating problems involving evaluation of the integrals in (5.16) through (5.17) using simple target functions such as $\operatorname{sqr}(x)$. The Fourier series is usually introduced by writing down (5.6) and appealing to the orthogonality relations to deduce (5.16) through (5.17). The important connection to least-squares approximation is not sufficiently emphasized by this approach: the truncated Fourier series is the best least squares approximation to the target using the number of sinusoids in the truncation.

mean square error:

$$\varepsilon_n(a_k, b_k) \stackrel{\text{def}}{=} \int_{-\pi}^{\pi} [f(x) - S_n(x)]^2 \, \frac{\mathrm{d}x}{2\pi}$$

Another important property of the Fourier series is *finality*. Suppose we possess $S_{17}(x)$ and we want to improve the approximation to $S_{18}(x)$ by adding one more term to the series. Then we don't need to recompute the earlier terms: the coefficients are given once and for all in (5.16) through (5.17).

Pointwise convergence of Fourier series

In applications, mean-square convergence of a Fourier is usually all we need. Nonetheless, because pointwise convergence guarantees mean-square convergence, and because nearly all functions you encounter will have pointwise-convergent Fourier series, you should know a little about pointwise convergence.

Mean-square convergence means that the mean-square error, $\varepsilon(a_k, b_k)$, can be made arbitrarily small by increasing the truncation n.

If f is a piecewise continuous function in the fundamental interval $-\pi \le x \le \pi$, and f is also equipped with a piecewise continuous first derivative in this interval, then the Fourier series of f with coefficients in (5.16) through (5.17) converges to f at points where f is continuous. At points of discontinuity, the Fourier series converges to the average

$$\lim_{\epsilon \to 0} \tfrac{1}{2}\big[f(x+\epsilon) + f(x-\epsilon)\big]. \tag{5.21}$$

"Piecewise continuous" in the statement above means that the discontinuities happen at only a finite number of points in the fundamental interval. Via periodic continuation outside of $[-\pi, \pi]$, the Fourier series then extends the definition of f to the whole of the real axis. See section **??** for a proof of this theorem.

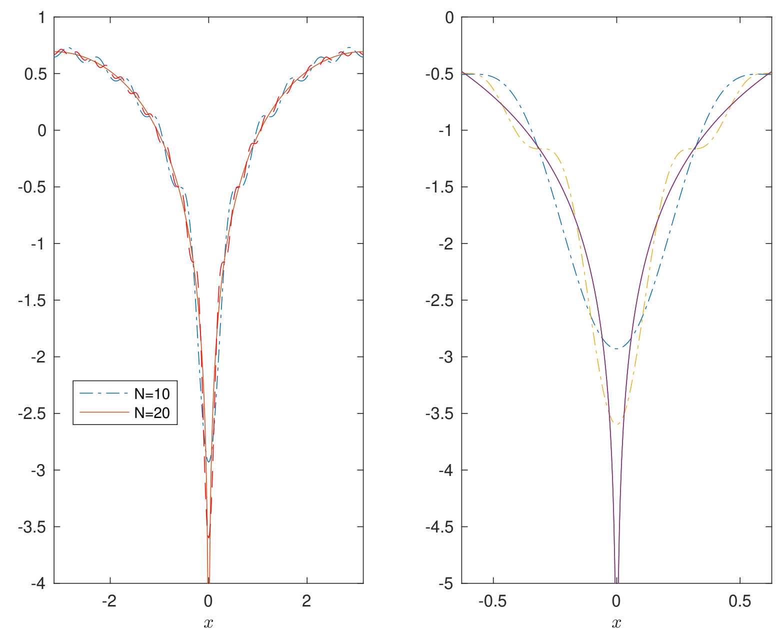


Figure 5.2: Illustration of the Fourier series in (5.22). The figure shows the truncated series with $N = 10$ and 20 terms. The right-hand panel is a zoomed view of the logarithmic singularity.

The example sqr shown in figure 5.1 illustrates the theorem above and we'll see other examples in these notes. Note, however, that the convergence is non-uniform: if you sit at a fixed value of x very close to the discontinuity at $x = 0$ then you'll need to sum a lot of terms before the Fourier series homes in on the target f: see the discussion of Gibbs phenomenon in the next section.

The condition above is sufficient, but by no means necessary, for the convergence of a Fourier series. For example, here is a beautiful Fourier series

See *Discourse on Fourier Series* by Cornelius Lanczos and *Fourier Analysis* by T.W. Körner for readable accounts. Carleson's convergence theorem seems to be the capstone result.

$$\ln\left|2\sin\frac{x}{2}\right| = -\sum_{k=1}^{\infty}\frac{\cos kx}{k}\,. \tag{5.22}$$

The function on the left has a logarithmic singularity at $x = 0$ and this pathology is not considered by the statement surrounding (5.21). Yet, as illustrated in figure 5.2, the Fourier series is converging, albeit slowly close to the singularity. Examples such as these are encompassed by more powerful results of Dirichlet and Riemann.

5.2 Gibbs' phenomenon

Because the coefficients in the Fourier series of $\mathrm{sqr}(x)$ decay rather slowly, as k^{-1}, this series doesn't converge very quickly. It's churlish to complain about slow convergence because we are representing a discontinuous function as a linear superposition of smoothly varying sinusoids: it's amazing that it works

$\mathrm{sqr}(x) = \frac{4}{\pi}[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots]$

at all. Nonetheless, let's examine the convergence in some detail. If we truncate the series with n terms, and define the overshoot

$$\mathcal{O}_n \stackrel{\text{def}}{=} \max_{-\pi\le x\le\pi} \left(\text{sqr}(x) - S_n(x)\right) . \tag{5.23}$$

We find $\mathcal{O}_1 = 0.27$, $\mathcal{O}_3 = 0.2$ and remarkably

$$\lim_{n\to\infty} \mathcal{O}_n = 0.18 . \tag{5.24}$$

In other words, the overshoot $\mathcal{O}_n$ never disappears. Figure 5.1 shows that the overshoot squeezes into the close proximity of points such as $x = 0$ at which $\text{sqr}(x)$ is discontinuous. This behavior means that $S_n(x)$ does not converge *uniformly*[1] to $\text{sqr}(x)$. This problem with non-uniform convergence was discovered by Kelvin and first explained by Gibbs. We discuss *Gibbs' phenomenon* in more detail below.

Gibbs' phenomenon is not inconsistent with $\lim_{n\to\infty} \varepsilon_n = 0$: the small region near $x = 0$ in which there is significant overshoot makes an increasingly feeble contribution to ε_n as n increases. If $\varepsilon_n \to 0$ as $n \to \infty$ we say that the Fourier series *converges in the mean*. In applications we are often concerned with convergence in the mean, rather than pointwise convergence.

The mean square error is:

$$\varepsilon_n \stackrel{\text{def}}{=} \int_{-\pi}^{\pi} [f - S_n]^2 \, \frac{\mathrm{d}x}{2\pi}$$

Truncating the Fourier series of $\text{sqr}(x)$ in (5.19) we obtain the finite sum

$n = 2q + 1$

$$S_{2q+1}(x) = \frac{4}{\pi} \underbrace{\left[\sin x + \frac{1}{3}\sin 3x + \cdots + \frac{1}{2q+1}\sin(2q+1)x\right]}_{(q+1)\text{terms}} . \tag{5.25}$$

This sum be expressed exactly in terms of an integral:

$$S_{2q+1}(x) = \frac{2}{\pi} \int_0^x \frac{\sin[2(q+1)t]}{\sin t} \mathrm{d}t , \tag{5.26}$$

Exercise: Fill in the steps between (5.25) and (5.26). Hint

$$S_{2q+1}(x) = \frac{2}{\pi} \int_0^x \mathrm{d}t \sum_{k=0}^{q} \left[\mathrm{e}^{(2k+1)\mathrm{i}t} + \mathrm{e}^{-(2k+1)\mathrm{i}t}\right] . \tag{5.27}$$

Carefully looking at the right panel in figure 5.1 we are inspired to simplify the integral (5.26) in the neighbourhood of $x = 0$. We can replace $\sin t$ in the denominator of (5.26) by t but we can't touch the $\sin[2(q+1)t]$ in the numerator because q is large. Thus we find that if x is close to zero

$$S_{2q+1}(x) \approx \frac{2}{\pi} \int_0^{2(q+1)x} \frac{\sin u}{u} \, \mathrm{d}u . \tag{5.28}$$

The approximation (5.28) is compared with the sum (5.25) in figure 5.3. Notice that (5.28) is a "similarity solution" in the sense that on the RHS q and x appear only in the combination $(q+1)x$. Thus the truncated Fourier series, S_n, differs

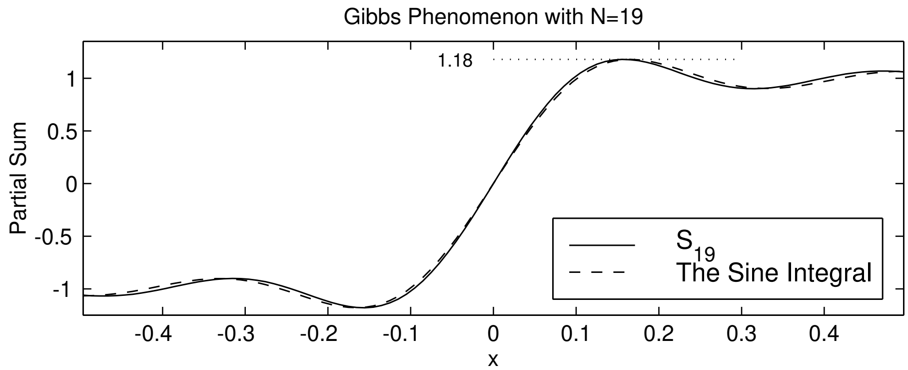


Figure 5.3: Comparison of the approximation (5.28) with the partial sum $S_{19}(x)$. Notice the 18% overshoot. `gibbs.eps`

from the target function $\mathrm{sqr}(x)$ in a neighbourhood of width $\Delta x \sim 1/q$ centered on a point of discontinuity.

It follows from (5.28) that

$$\lim_{q\to\infty}\lim_{x\to 0} S_{2q+1}(x) = 0\,, \tag{5.29}$$

and

$$\lim_{x\to 0}\lim_{q\to\infty} S_{2q+1}(x) = 1\,. \tag{5.30}$$

$$\int_0^\infty \frac{\sin u}{u}\mathrm{d}u = \frac{\pi}{2}$$

The two limits are not exchangeable. We can also see that the maxima of $S_{2q+1}(x)$ are at $2(q+1)x = \pi$, 2π and so on. At the first maximum

$$\begin{aligned} S_{2q+1}\left(\frac{\pi}{2(q+1)}\right) &\approx \frac{2}{\pi}\int_0^\pi \frac{\sin u}{u}\,\mathrm{d}u \\ &= 2\left(1 - \frac{\pi^2}{3\times 3!} + \frac{\pi^4}{5\times 5!} - \frac{\pi^6}{7\times 7!} + \cdots\right)\,, \\ &= 1.18\,. \end{aligned} \tag{5.31}$$

This is the source of the irreducible pointwise error at the discontinuities.

At a simple discontinuity, the truncated Fourier series overshoots the left and right hand limits by 9% of the discontinuous jump. In the case of $\mathrm{sqr}(x)$ the jump is 2 and the overshoot is therefore 0.18.

5.3 Parseval's theorem

The mean square error in (5.6) decreases, or at least can't increase, as we add more terms. And we can pass to the limit $n \to \infty$ and obtain an infinite Fourier series.

mean square error:

$$\varepsilon_n(a_k, b_k) \stackrel{\text{def}}{=} \int_{-\pi}^{\pi} [f(x) - S_n(x)]^2\,\frac{\mathrm{d}x}{2\pi}$$

[1] A sequence of functions $\phi_1(x)$, $\phi_2(x)\cdots$ converges uniformly to $f(x)$ on an interval $[a,b]$ if for each ϵ there is an N_ϵ, **independent of where x is in** $[a,b]$, such that $|f - \phi_{N_\epsilon}| < \epsilon$.

Now that we possess the Fourier coefficients a_n and b_n in (5.16) and (5.17) we can show that provided the target function has a finite square integral,

$$\int_{-\pi}^{\pi} f^2(x)\,\frac{\mathrm{d}x}{2\pi} < \infty\,, \tag{5.32}$$

then the minimum mean square error is

$$\min_{\forall a_k \& b_k} \varepsilon(a_k, b_k) = \int_{-\pi}^{\pi} f^2(x)\,\frac{\mathrm{d}x}{2\pi} - a_0^2 - \tfrac{1}{2}\sum_{k=1}^{n}(a_k^2 + b_k^2)\,. \tag{5.33}$$

Because the mean square error is positive we must have

Bessel's inequality

$$\int_{-\pi}^{\pi} f^2(x)\,\frac{\mathrm{d}x}{2\pi} \geq a_0^2 + \tfrac{1}{2}\sum_{k=1}^{n}(a_k^2 + b_k^2)\,. \tag{5.34}$$

This is *Bessel's inequality*. The sum on the right is non-decreasing and is bounded above by the integral of the squared target function on the left. This means the sum converges as $n \to \infty$, and that the terms in the sum approach zero as $n \to \infty$.

Exercise: Prove (5.33). Hint: show that

$$\int_{-\pi}^{\pi} f(x) S_n(x)\,\frac{\mathrm{d}x}{2\pi} = \int_{-\pi}^{\pi} S_n^2(x)\,\frac{\mathrm{d}x}{2\pi} = a_0^2 + \tfrac{1}{2}\sum_{k=1}^{n}(a_k^2 + b_k^2)\,. \tag{5.35}$$

As n increases the finite Fourier expansion can be imagined as constantly reducing ε_n by piling on more and more orthogonal sinusoids. If in the limit $n \to \infty$ the error $\varepsilon_n(a_k, b_k)$ vanishes then Bessel's inequality becomes Parseval's equality:

Parseval's theorem

$$\int_{-\pi}^{\pi} f^2(x)\,\frac{\mathrm{d}x}{2\pi} = a_0^2 + \tfrac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2)\,. \tag{5.36}$$

We often need to know quadratic integrals of functions (e.g., the energy of a field). Parseval's theorem is an important computational tool because it enables us to calculate these integrals using the Fourier coefficients.

Mathematicians[2] have shown that the sinusoidal basis set

$$\{\cos kx, \sin kx\}$$

is *complete* on the interval $-\pi \leq x \leq \pi$. This means that $\lim_{n\to\infty} \varepsilon_n = 0$ for any $f(x)$ satisfying (5.32) i.e. for any function with finite energy. In other words, with enough terms the Fourier series captures *all* the energy in the target function. Note that the singular function on the right of (5.22) has finite energy and therefore the Fourier series in (5.22) is guaranteed to converge in the mean.

(5.32):

$$\int_{-\pi}^{\pi} f^2(x)\,\frac{\mathrm{d}x}{2\pi} < \infty$$

Example: We illustrate Parseval's theorem by noting that

$$\int_{-\pi}^{\pi} \mathrm{sqr}^2(x)\,\mathrm{d}x = 2\pi\,. \tag{5.37}$$

Recalling the Fourier series for sqr, we see that we see that according to Parseval

$$\frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \cdots \tag{5.38}$$

$$\mathrm{sqr}(x) = \frac{4}{\pi}\Big[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\Big]\,,$$

[2] This is a special case of the Fischer-Riesz Theorem. See also Carleson's theorem.

5.4 The complex form of the Fourier series

Using Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{5.39}$$

the real Fourier series in (5.6) and (5.17) is equivalent to

$$f(x) = \sum_{m=-\infty}^{\infty} f_m e^{imx}\,, \tag{5.40}$$

where

$$f_m = \int_{-\pi}^{\pi} f(x) e^{-imx}\,\frac{dx}{2\pi}\,. \tag{5.41}$$

Notice that to obtain f_m one must multiply $f(x)$ by e^{-imx}. The key to remembering (5.41) (or working it out very quickly) is the orthogonality property of the complex sinusoids:

$$\int_{-\pi}^{\pi} e^{i(p-q)x}\,\frac{dx}{2\pi} = \delta_{pq}\,. \tag{5.42}$$

The complex form of Parseval's theorem is

$$\int_{-\pi}^{\pi} |f(x)|^2\,\frac{dx}{2\pi} = \sum_{m=-\infty}^{\infty} |f_m|^2\,. \tag{5.43}$$

Notice that there is no need to assume that f is real. There are many reasons for preferring the complex form e.g., no irritating exception for the f_0. And in the future lectures we can very quickly obtain the Fourier Integral Theorem starting from (5.41).

It is easy to show that if $f(x)$ is a real function then the Fourier coefficients must satisfy the *reality condition*

$$f_m = f^*_{-m}\,. \tag{5.44}$$

This is is an important check on your algebra.

Exercise: Convince yourself that the complex form and the real form of a Fourier series are equivalent by showing that

$$a_m = f_m + f_{-m} \qquad \text{and} \qquad b_m = i\left(f_m - f_{-m}\right)\,. \tag{5.45}$$

Example: Expand $f(x) = \exp(-\alpha x)$ on the fundamental interval $(-\pi, \pi)$ in a complex Fourier series.

The complex Fourier coefficients are given by

$$\begin{aligned} f_m &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-(\alpha+im)x}\,dx\,,\\ &= -\frac{1}{2\pi}\left[\frac{e^{-(\alpha+im)x}}{\alpha+im}\right]_{-\pi}^{\pi}\,,\\ &= \frac{\sinh\alpha\pi}{\pi}\,\frac{(-1)^m}{\alpha+im}\,. \end{aligned} \tag{5.46}$$

Notice that the reality condition is satisfied. To summarize

$$\mathrm{e}^{-\alpha x} = \frac{\sinh \alpha\pi}{\pi} \sum_{m=-\infty}^{\infty} \frac{(-1)^m \mathrm{e}^{\mathrm{i}mx}}{\alpha + \mathrm{i}m}\,, \qquad -\pi < x < \pi\,. \tag{5.47}$$

At a point of discontinuity of the target function, such as $x = \pi$ in this example, a Fourier series converges to the mean of the target function at the jump. Thus in the example of $\mathrm{sqr}(x)$ the series in (5.19) converges to zero at $x = n\pi$ — this is obvious. But in the present example we find a non-obvious result. The mean of $\mathrm{e}^{\alpha\pi}$ and $\mathrm{e}^{-\alpha\pi}$ is $\cosh \alpha\pi$. Thus putting $x = \pi$ in the right hand side of (5.47) we obtain

$$\cosh \pi\alpha = \frac{\sinh \alpha\pi}{\pi} \sum_{m=-\infty}^{\infty} \frac{1}{\alpha + \mathrm{i}m}\,, \tag{5.48}$$

which can be rearranged to the form

$$\coth \pi\alpha = \frac{1}{\pi\alpha} + \frac{2\alpha}{\pi} \sum_{m=1}^{\infty} \frac{1}{\alpha^2 + m^2}\,. \tag{5.49}$$

This result can also be obtained by application of Parseval's theorem to the Fourier series in (5.47) — see problems.

Notice both $\mathrm{sqr}(x)$ and the periodically extended version of $\mathrm{e}^{-\alpha x}$ have discontinuities and both Fourier series have coefficients $\sim m^{-1}$ as $m \to \infty$. This is a general result: functions with discontinuities have coefficients decaying slowly as m^{-1} and there are always Gibbs oscillations at the discontinuities.

Exercise: Deduce (5.49) by application of Parseval's theorem to the Fourier series in (5.47).

5.5 Examples of Fourier series

We generate new Fourier series from old by the linear operations of differentiation and integration. We start by considering an example of integration. Define the integrated square wave function, isqr, via

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{isqr}(x) = -\,\mathrm{sqr}(x)\,, \qquad \text{and} \qquad \int_{-\pi}^{\pi} \mathrm{isqr}(x)\,\mathrm{d}x = 0\,. \tag{5.50}$$

The minus sign in the first equation above is included so that $\mathrm{isqr}(0) > 0$. On the fundamental interval, the function $\mathrm{isqr}(x)$ is:

$$\mathrm{isqr}(x) = \frac{\pi}{2} - |x|\,. \tag{5.51}$$

Exercise: How was the constant of integration $\pi/2$ in (5.51) determined?

We can also integrate the Fourier series for $\mathrm{sqr}(x)$ term-by-term, to obtain

$$\mathrm{sqr}(x) = \frac{4}{\pi}\Big[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\Big]$$

$$\mathrm{isqr}(x) = \frac{4}{\pi}\Big[\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots\Big]\,, \tag{5.52}$$

$$= \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2} \tag{5.53}$$

Exercise: How was the constant of integration in (5.52) determined?

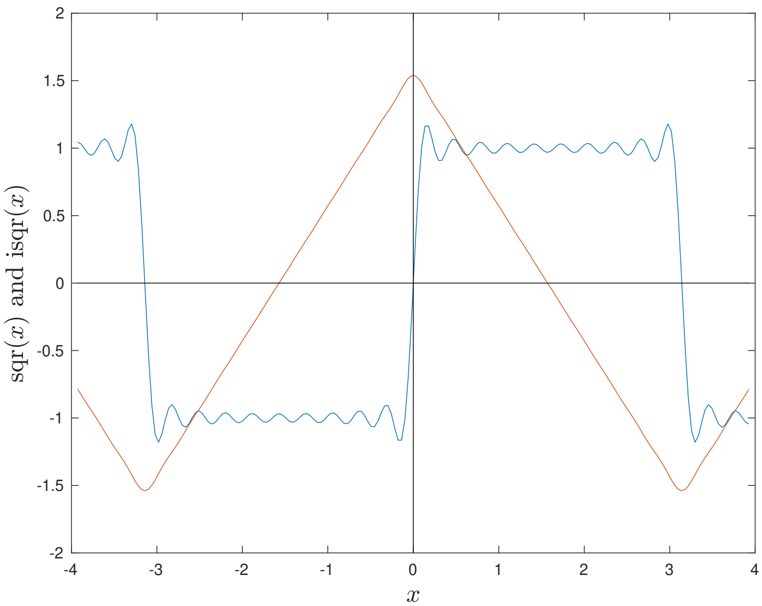


Figure 5.4: Ten-term Fourier series approximations to $\mathrm{sqr}(x)$ and its integral $\mathrm{isqr}(x)$.

Notice that in the right panel of figure 5.4 there is no Gibbs' phenomenon in the truncated series representing isqr: the coefficients in (5.53) are $\sim k^{-2}$ as $k \to \infty$ — this is faster than the $\sim k^{-1}$ decay in the Fourier representation of $\mathrm{sqr}(x)$. Loosely speaking, when we integrate a Fourier series we "pull down" a factor of k^{-1} from $\mathrm{e}^{\mathrm{i}kx}$. Thus the Fourier series of the integral converges faster than the Fourier series of the original function (see problem 5.8).

Example: Consider a square in the (x, y)-plane defined by the four vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$. The square can be represented in polar coordinates as $r = R(\theta)$. Find a Fourier series representation of $R(\theta)$.

Since $R(\theta) = R(-\theta)$ we only need the cosines. But we also have $R(\theta) = R(\theta + \pi/2)$, and this symmetry implies that

$$R(\theta) = a_0 + a_4 \cos 4\theta + a_8 \cos 8\theta + \cdots \tag{5.54}$$

We leave out $\cos\theta$, $\cos 2\theta$, $\cos 3\theta$ etc because these terms reverse sign if $\theta \to \theta + \pi/2$. In the first quadrant of the (x, y)-plane, the square is $x + y = 1$, or $R(\theta) = (\cos\theta + \sin\theta)^{-1}$. The first term in the Fourier series is therefore

$$a_0 = \frac{1}{2\pi} \oint R(\theta)\,\mathrm{d}\theta\,, \tag{5.55}$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\cos\theta + \sin\theta}\,. \tag{5.56}$$

We've used symmetry to reduce the integral to four times the integral over the side in the first quadrant. The MATHEMATICA command

```
Integrate[1/(Sin[x] + Cos[x]), {x, 0, Pi/2}]
```

tells us that

$$a_0 = \frac{2\sqrt{2}}{\pi} \tanh^{-1}\left(\frac{1}{\sqrt{2}}\right) \approx 0.7935\,. \tag{5.57}$$

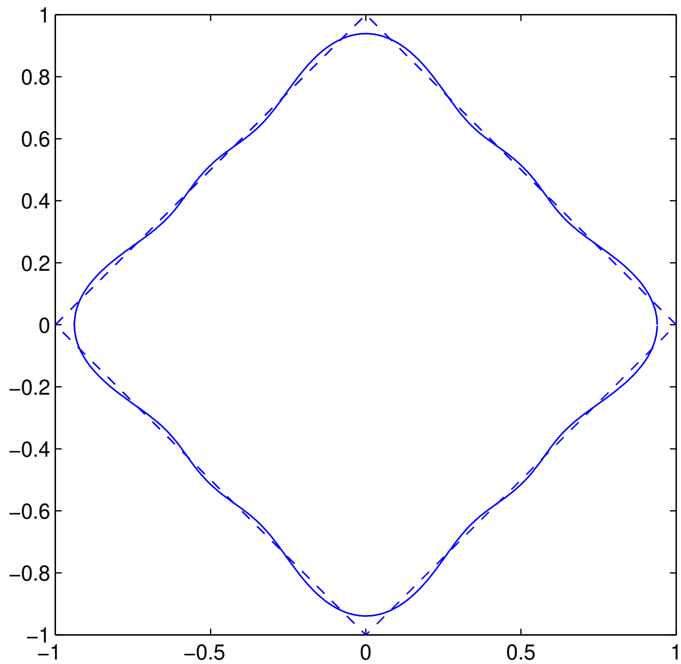


Figure 5.5: Three terms in (5.54) make a rough approximation to the dotted square. `polarsquare.eps`

The higher terms in the series are

$$a_{4k} = \frac{1}{\pi}\oint \cos(4k\theta)R(\theta)\,\mathrm{d}\theta = \frac{4}{\pi}\int_0^{\pi/2} \frac{\cos(4k\theta)\,\mathrm{d}\theta}{\cos\theta + \sin\theta}\,. \tag{5.58}$$

With mathematica, we find

$$a_4 = \frac{4}{\pi}\left[\frac{4}{3} - \sqrt{2}\tanh^{-1}\left(\frac{1}{\sqrt{2}}\right)\right] \approx 0.1106\,, \tag{5.59}$$

$$a_8 = -\frac{4}{\pi}\left[\frac{128}{105} - \sqrt{2}\tanh^{-1}\left(\frac{1}{\sqrt{2}}\right)\right] \approx 0.0349\,. \tag{5.60}$$

MATHEMATICA gives a general formula in terms of hypergeometric series. This is more than we need to know about this example: Figure 5.5 shows that the first three terms in the series can be used to draw a pretty good square. The Fourier series in this example has moderately fast convergence — you're asked to show below that $a_{4k} \sim k^{-2}$ as $k \to \infty$. This k^{-2} decrease is faster than $a_k \sim k^{-1}$ in the sqr Fourier series (5.19).

Exercise: Consider the Fourier coefficients a_{4k} defined by (5.54). Use Parseval's theorem to evaluate

$$a_0^2 + \tfrac{1}{2}\left(a_4^2 + a_8^2 + \cdots\right)\,. \tag{5.61}$$

Hint: with geometric insight there is little or no algebra.

Exercise: Use integration-by-parts (see section 5.7) to show that $a_{4k} \sim 1/2\pi k^2$ as $k \to \infty$. You'll need to integrate-by-parts twice.

Other expansion intervals

Suppose we want to represent a function f defined on an interval $0 \le x \le \ell$. We can expand f as a Fourier series with period ℓ:

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi k x}{\ell}\right) + b_k \sin\left(\frac{2\pi k x}{\ell}\right) . \tag{5.62}$$

We need both the sines and the cosines to represent an arbitrary function f in this fashion. The Fourier coefficients are then

$$a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\,\mathrm{d}x\,, \tag{5.63}$$

and for $k \ge 1$

$$a_k = \frac{1}{\ell}\int_{-\ell}^{\ell} \cos kx\, f(x)\,\mathrm{d}x\,, \qquad \text{and} \qquad b_k = \frac{1}{\ell}\int_{-\ell}^{\ell} \sin kx\, f(x)\,\mathrm{d}x\,. \tag{5.64}$$

Example: On the interval $0 < x < 2$ find the Fourier series of the function $p(x) = 4 - x^2$. This is the case $\ell = 2$ in (5.62). We have

$$a_0 = \frac{1}{2}\int_0^2 4 - x^2\,\mathrm{d}x = \frac{8}{3}\,, \tag{5.65}$$

and

$$a_k = \int_0^2 (4 - x^2)\cos k\pi x\,\mathrm{d}x = -\frac{4}{(k\pi)^2}\,, \tag{5.66}$$

$$b_k = \int_0^2 (4 - x^2)\sin k\pi x\,\mathrm{d}x = \frac{4}{k\pi}\,. \tag{5.67}$$

Thus on the interval $0 < x < 2$,

$$4 - x^2 = \frac{8}{3} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos k\pi x}{k^2} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin k\pi x}{k}\,. \tag{5.68}$$

(I used MATHEMATICA.) By periodic extension, the series on the right defines a function $p(x)$ on the whole real line. At $x = 0$, $p(x)$ is discontinuous and the series converges to the mean of the values $p(0^-) = 0$ and $p(0^+) = 4$. Hence

$$2 = \frac{8}{3} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\,, \tag{5.69}$$

which is equivalent to the famous series

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \tag{5.70}$$

Half-range expansions

Let us continue our discussion of representing an arbitrary function f on the interval $0 \le x \le \ell$. As an alternative to (5.62) we can extend the definition of $f(x)$ to the interval $-\ell \le x \le 0$ by $f(-x) = -f(x)$, and so define an odd function with period 2ℓ. This odd function can then be expanded in sines

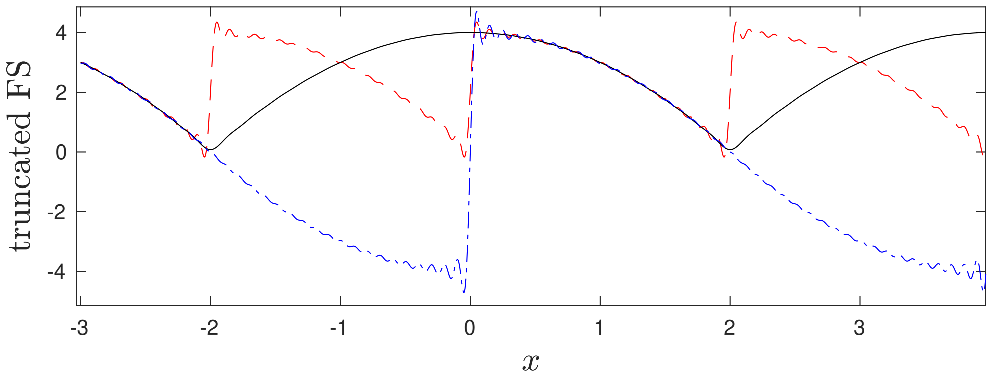


Figure 5.6: Three truncated Fourier series. All three series represent $2-x^2$ on the interval $0<x<2$.

alone. Or we might extend the definition of $f(x)$ to the interval $-\ell \le x \le 0$ by $f(-x)=f(x)$ i.e., we define an even function of period 2ℓ, which can then be expanded in cosines alone.

For example, suppose $\ell=\pi$ i.e., $f(x)$ is given only for $0<x<\pi$ (aka the half-range). Then we can expand *nonuniquely* on this interval $0<x<\pi$ using either a sine series

Example: $f(x)=1$.

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx\,, \qquad b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x\,, \tag{5.71}$$

or a cosine series

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n \cos nx\,, \qquad a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x\,. \tag{5.72}$$

In the first case we used an *odd* extension: $f(x)=-f(-x)$ and in the second case we use the *even* extension $f(x)=f(-x)$.

Exercise: One of the three functions in figure 5.6 is a truncated version of the Fourier series in (5.68). Which one is it?

Of course we can extend the definition of $f(x)$ to $-\pi \le x \le 0$ in many different ways so the two choices above certainly don't exhaust the possibilities of expanding $f(x)$ on an interval $(0,a)$. However the rate of convergence of these various Fourier representations can be very different: often one series is much faster than another.

Example: Represent $4-x^2$ on the interval $0 \le x \le 2$ with a cosine series.

To get a cosine series we use an even extension to the interval $-2 \le x \le 2$, so that

$$4-x^2=a_0+\sum_{k=1}^{\infty} a_k \cos\frac{k\pi x}{2}\,, \qquad \text{for} \quad -2\le x\le 2\,. \tag{5.73}$$

With MATHEMATICA the coefficients are

$$a_0=\underbrace{\frac{1}{4}\int_{-2}^{2} 4-x^2\,\mathrm{d}x}_{8/3}\,, \qquad \text{and} \qquad a_k=\underbrace{\frac{1}{2}\int_{-2}^{2}(4-x^2)\cos\frac{k\pi x}{2}\,\mathrm{d}x}_{(-1)^{k+1}(4/\pi k)^2}\,. \tag{5.74}$$

Example: Represent $4-x^2$ on the interval $0 \le x \le 2$ with a sine series.

To get a sine series we use an odd extension to the interval $-2 \le x \le 2$, so that

$$\operatorname{sgn}(x)\left[4-x^2\right] = \sum_{k=1}^{\infty} b_k \sin\frac{k\pi x}{2}\,, \qquad \text{for} \quad -2 \le x \le 2\,. \tag{5.75}$$

With MATHEMATICA the coefficients are

$$b_k = \underbrace{\frac{1}{2}\int_{-2}^{2} \operatorname{sgn}(x)(4-x^2)\sin\frac{k\pi x}{2}\,\mathrm{d}x}_{8[2-2(-1)^k+(\pi k)^2]/(\pi k)^3}\,. \tag{5.76}$$

Exercise: Identify the truncated Fourier series in figure 5.6 with the half-range cosine and sine series in the examples above.

Exercise: Find a Fourier series representation of the function q defined by $q(x) = (4-x^2)\,\mathrm{H}(x)$ on $-2 \le x \le 2$, and $q(x+4) = q(x)$.

Example: Represent $\cos x$ on $0 \le x \le \pi$ using only sine functions. Sketch the sum of the resulting Fourier series on the interval $[-\pi, 3\pi]$.

Because we're using only sines, we're constructing a function which is odd on the fundamental interval $-\pi \le x \le \pi$ i.e., the target function in this problem is

$$f(x) = \operatorname{sqr}(x)\cos x\,. \tag{5.77}$$

This function coincides with $\cos x$ on $(0,\pi)$ and is an odd function with period 2π. The Fourier series representation is

$$\operatorname{sqr}(x)\cos x = \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{m\sin 2mx}{4m^2-1}\,, \tag{5.78}$$

$$= \frac{8}{\pi}\left[\frac{1}{3}\sin 2x + \frac{2}{15}\sin 4x + \frac{3}{35}\sin 6x + \cdots\right]. \tag{5.79}$$

To obtain the series (5.78), we use the formula (5.71):

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi}\cos x\sin nx\,\mathrm{d}x\,,\\
&= \frac{2}{\pi}\int_0^{\pi}\frac{\left(\mathrm{e}^{\mathrm{i}x}+\mathrm{e}^{-\mathrm{i}x}\right)}{2}\frac{\left(\mathrm{e}^{\mathrm{i}nx}-\mathrm{e}^{-\mathrm{i}nx}\right)}{2\mathrm{i}}\,\mathrm{d}x\,,\\
&= \frac{1}{2\pi\mathrm{i}}\int_0^{\pi}\left(\mathrm{e}^{\mathrm{i}(n+1)x}+\mathrm{e}^{\mathrm{i}(n-1)x}-c.c.\right)\,\mathrm{d}x\,,\\
&= \frac{1}{\pi}\int_0^{\pi}\sin(n+1)x+\sin(n-1)x\,\mathrm{d}x\,,\\
&= \frac{1}{\pi}\left[\frac{1-\cos(n+1)\pi}{n+1}+\frac{1-\cos(n-1)\pi}{n-1}\right].
\end{aligned} \tag{5.80}$$

As one can anticipate using a symmetry argument, all the odd b_n's are zero. If n is even, we write $n = 2m$ so that $b_m = 8m/\pi(4m^2-1)$.

Fourier series without evaluating integrals

Every complex power series you know can be used to obtain a Fourier series at no extra charge. For example, consider

$$\exp(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots \tag{5.81}$$

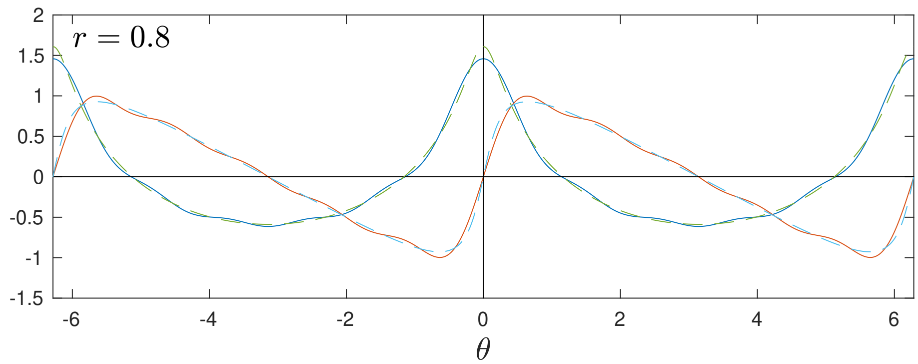


Figure 5.7: Solid curves are five-term truncations of the Fourier series on the left-hand sides (5.91) and (5.92) with $r = 0.8$. The dashed curves are the functions on the right-hand sides.

If we evaluate this with polar coordinates, $z = r\mathrm{e}^{\mathrm{i}\theta}$, then

$$\exp\left(r e^{\mathrm{i}\theta}\right) = 1 + r\mathrm{e}^{\mathrm{i}\theta} + \frac{r^2}{2}\mathrm{e}^{2\mathrm{i}\theta} + \frac{r^3}{6}\mathrm{e}^{3\mathrm{i}\theta} + \cdots \tag{5.82}$$

Taking the real part we have

$$\mathrm{e}^{r\cos\theta}\cos(r\sin\theta) = 1 + r\cos\theta + \frac{r^2}{2}\cos 2\theta + \frac{r^3}{6}\cos 3\theta + \cdots \tag{5.83}$$

The imaginary part of (5.82) produces another monster. It is challenging to obtain these Fourier series using the integration formulas in (5.16) through (5.17).

Example: The function

$$f(z) = \frac{1}{1-z} \tag{5.84}$$

is analytic in the region $|z| < 1$ and therefore inside this disc

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \tag{5.85}$$

With $z = r\mathrm{e}^{\mathrm{i}\theta}$ and separating real and imaginary parts, we have

$$\frac{1 - r\cos\theta}{1 + r^2 - 2r\cos\theta} = \sum_{n=0}^{\infty} r^n \cos n\theta\,, \tag{5.86}$$

$$\frac{r\sin\theta}{1 + r^2 - 2r\cos\theta} = \sum_{n=1}^{\infty} r^n \sin n\theta\,. \tag{5.87}$$

Fiddling about with (5.86) we deduce the Fourier series

$$\frac{1 - r^2}{1 + r^2 - 2r\cos\theta} = 1 + 2r\cos\theta + 2r^2\cos 2x + 2r^3\cos 3\theta + \cdots \tag{5.88}$$

If r is close to one the formula above is a regularized version of the Dirac comb from section 5.6.

Exercise: Sum the Fourier series

$$J(x) = 1 + \sin x + \frac{1}{2!}\sin 2x + \frac{1}{3!}\sin 3x + \frac{1}{4!}\sin 4x + \cdots \tag{5.89}$$

Exercise: Use the series

$$-\ln(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\,, \tag{5.90}$$

to deduce that

$$r\cos\theta + \frac{r^2}{2}\cos 2\theta + \frac{r^3}{3}\cos 3\theta + \cdots = -\frac{1}{2}\ln\left[1 - 2r\cos\theta + r^2\right]\,; \tag{5.91}$$

$$r\sin\theta + \frac{r^2}{2}\sin 2\theta + \frac{r^3}{3}\sin 3\theta + \cdots = \tan^{-1}\left[\frac{r\sin\theta}{1 - r\cos\theta}\right]\,. \tag{5.92}$$

See figure 5.7 for $r = 0.8$.

Exercise: Sum the Fourier series

$$\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta + \frac{1}{4}\sin 4\theta + \cdots \tag{5.93}$$

Exercise: Show that

$$-\ln\left|2\sin\frac{\theta}{2}\right| = \cos\theta + \tfrac{1}{2}\cos 2\theta + \tfrac{1}{3}\cos 3\theta + \tfrac{1}{4}\cos 4\theta + \cdots\,, \tag{5.94}$$

and

$$-\tfrac{1}{2}\ln\left|\tan\frac{\theta}{2}\right| = \cos\theta + \tfrac{1}{3}\cos 3\theta + \tfrac{1}{5}\cos 5\theta + \tfrac{1}{7}\cos 7\theta + \cdots\,. \tag{5.95}$$

5.6 The Dirac comb and the Dirichlet kernel

The Dirac comb

Throwing caution to the winds, we now calculate the Fourier series of the δ-function. Because of the sifting property

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(x)e^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{1}{2\pi}\,. \tag{5.96}$$

Thus the Fourier series is

$$\delta_c(x) = \tfrac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{\mathrm{i}kx}\,, \tag{5.97}$$

$$= \frac{1}{2\pi}\left[1 + 2\cos x + 2\cos 2x + 2\cos 3x + \cdots\right]\,. \tag{5.98}$$

Notice that on the infinite line, as opposed to the fundamental interval $(-\pi,\pi)$, the series on the right hand side of (5.98) is actually the Dirac comb i.e., a sequence of δ-functions at $x = 2m\pi$ where $m = \cdots -1, 0, 1, \cdots$ This is why I have used the subscript c in $\delta_c(x)$. So, to summarize, we have the basic result

$$\delta_c(x) \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty}\delta(x - 2\pi m)\,, \tag{5.99}$$

$$= \tfrac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ikx}\,. \tag{5.100}$$

Exercise: differentiate $\mathrm{sqr}(x)$ and verify that the result is consistent with (5.100).

If we multiply $\delta_c(x-x')$ by a smooth test function $f(x')$ and integrate with respect to x' over the interval $(-\pi,\pi)$ then we reproduce the Fourier series expansion of $f(x)$. Here is the calculation:

$$f(x) = \int_{-\pi}^{\pi} f(x')\delta_c(x-x')\,\mathrm{d}x' \,, \tag{5.101}$$

$$= \int_{-\pi}^{\pi} f(x')\,\frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k(x-x')}\,\mathrm{d}x \,, \tag{5.102}$$

$$= \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} f(x')\mathrm{e}^{-\mathrm{i}kx'}\,\frac{\mathrm{d}x'}{2\pi}\,\mathrm{e}^{\mathrm{i}kx} \,, \tag{5.103}$$

$$= \sum_{k=-\infty}^{\infty} f_k \mathrm{e}^{\mathrm{i}kx} \,. \tag{5.104}$$

So we have a consistent set of results — the manipulations above are an expression of the Fourier series representation of a periodic function.

Exercise: According to (5.47)

$$\mathrm{e}^{-\alpha x} = \frac{\sinh \alpha\pi}{\pi}\sum_{m=-\infty}^{\infty}\frac{(-1)^m \mathrm{e}^{imx}}{\alpha + im} \,, \qquad -\pi < x < \pi \,. \tag{5.105}$$

Show that the derivative of this Fourier series is consistent with $\mathrm{d}\mathrm{e}^{-\alpha x}/\mathrm{d}x = -\alpha \mathrm{e}^{\alpha x}$.

The Dirichlet kernel

Even after these reassuring remarks about Fourier series, you might still be feeling nervous about the formal construction in (5.100). Here is another way of understanding (5.100). If we stop the sum, keeping only $2n+1$ terms, we have a function:

$$D_n(x) \stackrel{\text{def}}{=} \frac{1}{2\pi}\sum_{k=-n}^{n} e^{\mathrm{i}kx} \,, \tag{5.106}$$

$$= \frac{1}{2\pi}\frac{\sin\left[\left(n+\frac{1}{2}\right)x\right]}{\sin(x/2)} \,, \tag{5.107}$$

called the *Dirichlet kernel*. From the expression in (5.107) we see that the series in (5.106) does not converge to a limit at any value of x: if we pick, say $x=1$, and let $n \to \infty$ then $D_n(1)$ bounces back and forth between $\pm\sin(1/2)/2\pi$. But non-convergence of D_n does not imply non-convergence of the integral

$$\int_{-\pi}^{\pi} D_n(x') f(x-x')\,\mathrm{d}x' \,. \tag{5.108}$$

Away from $x'=0$ the wavelength of the oscillations in D_n decreases as n^{-1}, so that there is the destructive interference between the alternately positive and negative small-scale lobes in the integrand of (5.108). Thus as $n \to \infty$ the integral in (5.108) is dominated by contributions from the immediate vicinity of $x'=0$.

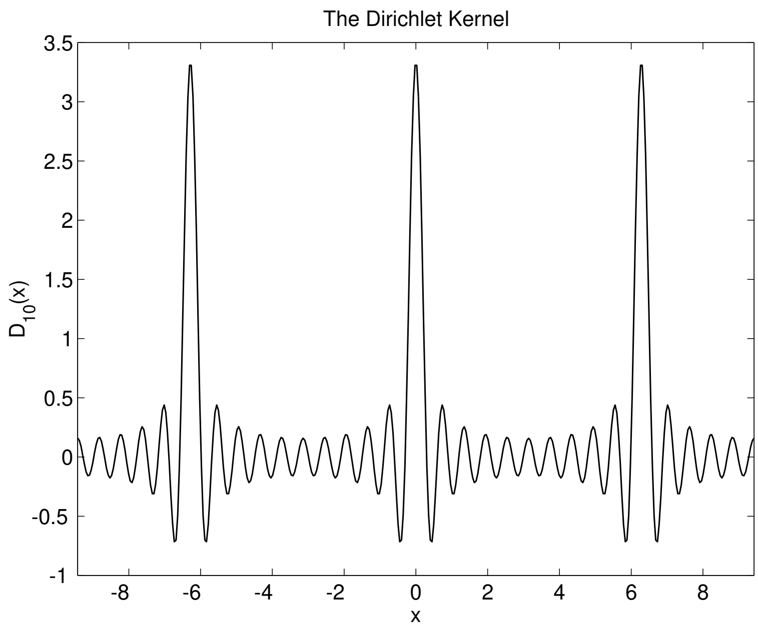


Figure 5.8: The Dirichlet kernel with $n = 10$. `DirichletKernel.eps`

Exercise: Fill in the steps between (5.106) and (5.107). Hint: first show that

$$1 + e^{ix} + e^{2ix} + \cdots + e^{inx} = \frac{1 - e^{i(n+1)x}}{1 - e^{ix}} = \frac{\sin\frac{(n+1)x}{2}x}{\sin\frac{x}{2}} e^{inx/2} \tag{5.109}$$

At $x = 0$ we have $D_n(0) = (2n + 1)/2\pi$ and at $x = \pi/(n + 1/2)$, $D_n = 0$. Both of these results should remind you of a δ-sequence: the peak grows linearly with n and the peak width decreases as $1/n$. Moreover, it is obvious from (5.106) that

$$\int_{-\pi}^{\pi} D_n(x)\, dx = 1\,, \tag{5.110}$$

so the integral of $D_n(x)$ is equal to one, independent of n. The graph of $D_n(x)$ in Figure 5.8 tells the story: as $n \to \infty$ the Dirichlet kernel becomes concentrated around $x = 0$. Thus if $f(x)$ is any reasonably smooth function

$$\int_{-\pi}^{\pi} f(x - x')D_n(x')\, dx' \to f(x)\,, \qquad \text{as } n \to \infty. \tag{5.111}$$

In other words

$$D_n(x) \to \delta(x)\,, \qquad \text{as } n \to \infty, \tag{5.112}$$

where the limiting operations above take place inside an integral.

I feel obliged to prove (5.111) for the smooth functions appearing in many applications [3]. Assume $f(x)$ has at least one derivative in sub-intervals of $(-\pi, \pi)$. At the ends of the sub-intervals, $f(x)$ has a simple jump discontinuities. Thus at every point $x \in (-\pi, \pi)$, $f(x)$ has both a left limit $f(x^-)$ and

[3] The result is true for a much wider class of functions that satisfy *Dirichlet conditions*.

a right limit $f(x^+)$, and moreover the left and right derivatives also exist at every point x. The function $\mathrm{sqr}(x)$ is a simple example. Consider

$$\begin{aligned}
f_n(x) &\stackrel{\text{def}}{=} \int_{-\pi}^{\pi} D_n(y) f(x-y)\,\mathrm{d}y\,, \\
&= \frac{f(x^+)+f(x^-)}{2} \\
&\quad + \int_{-\pi}^{0} \sin\left[(n+\tfrac{1}{2})y\right] \frac{f(x-y)-f(x^-)}{2\pi\sin(y/2)}\,\mathrm{d}y \\
&\qquad + \int_{0}^{\pi} \sin\left[(n+\tfrac{1}{2})y\right] \frac{f(x-y)-f(x^+)}{2\pi\sin(y/2)}\,\mathrm{d}y\,.
\end{aligned} \tag{5.113}$$

Because f has left and right derivatives, the ratios

$$\frac{f(x-y)-f(x^\pm)}{2\pi\sin(y/2)} \tag{5.114}$$

approach finite limits as $y \to 0$.

Now invoke Riemann-Lebesgue lemma. Thus the two integrals on the right-hand side integrals in (5.113) limit to zero as $n \to \infty$, and

$$\lim_{n\to\infty} f_n(x) = \frac{f(x^+)+f(x^-)}{2}\,. \tag{5.115}$$

The Riemann-Lebesgue Lemma

If $\int_a^b |F(t)|\,\mathrm{d}t$ exists then

$$\lim_{\alpha\to\infty} \int_a^b \mathrm{e}^{\mathrm{i}\alpha t} F(t)\,\mathrm{d}t = 0\,. \tag{5.116}$$

See, for example, *Mathematical Analysis* by T.M. Apostol.

Differentiation of Fourier Series

We generate interesting, and even useful, Fourier series by differentiation of earlier results. For example, the derivative of (5.95) produces the unlikely result

$$\frac{1}{2\sin\theta} = \sin\theta + \sin 3\theta + \sin 5\theta + \cdots \tag{5.117}$$

To understand this result we retreat to the truncated sum

$$??? = \sum_{q=0}^{n} \sin[(2q+1)\theta] \tag{5.118}$$

(5.95):

$$-\tfrac{1}{2}\ln\left|\tan\frac{\theta}{2}\right| = \cos\theta + \tfrac{1}{3}\cos 3\theta + \tfrac{1}{5}\cos 5\theta + \cdots$$

As example of a function that is continuous everywhere and differentiable nowhere we mention

$$W(x) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \frac{\sin\left[(k!)^2 x\right]}{k!} \tag{5.119}$$

$$= \sin x + \frac{\sin 4x}{2} + \frac{\sin 36x}{6} + \frac{\sin 576x}{24} + \cdots \tag{5.120}$$

and other lacunary series.

5.7 Asymptotic behaviour of Fourier coefficients

We have encountered Fourier series with very different rates of convergence. The slowest series is that of $\delta_c(x)$ in (5.98). In fact, that series doesn't really converge at all: the oscillations in the Dirichlet kernel become faster, but not smaller, with increasing N. The series for $\mathrm{sqr}(x)$ in (5.19) is a little better, since there $f_\sim k^{-1}$ as $k \to \infty$. This $\mathrm{ord}(k^{-1})$ behavior is typical of sectionally smooth functions i.e., functions that are infinitely different differentiable in finite intervals, with discontinuous jumps at the end-points of the intervals.

$$\mathrm{sqr}(x) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{2k+1}$$

The Fourier coefficients in the isqr-series decrease as $\mathrm{ord}(k^{-2})$, and thus this series converges much faster than that of sqr. This is because isqr is less singular than sqr: a jump in a derivative is not as bad as a jump in the function itself. If integrate isqr we obtain a function with $\mathrm{ord}(k^{-3})$ convergence. This new function has a jump only in its second derivative. The message is that the rate of convergence of a Fourier series is determined by the strength of the singularities in the target function.

$$\mathrm{isqr}(x) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2}$$

Now let's go to the other extreme and consider very rapidly convergent Fourier series, such as

$$\cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos 2x\,. \tag{5.121}$$

For much more information on the rate of convergence of Fourier series see chapter 2 of *Chebyshev and Fourier Spectral Methods* by J.P. Boyd.

Another example of a rapidly convergent Fourier series is

$$\frac{1-r^2}{1+r^2-2r\cos x} = 1 + 2r\cos x + 2r^2\cos 2x + 2r^3\cos 3x + \cdots \tag{5.122}$$

If $|r| < 1$ then the coefficients decrease as $r^k = \mathrm{e}^{k \ln r}$, which is faster than any power of k. The series

$$\mathrm{e}^{\gamma \cos x} = I_0(\gamma) + 2 \sum_{k=1}^{\infty} I_k(\gamma) \cos nx \,, \tag{5.123}$$

with I_k the modified Bessel function of order k, is yet another example of exponentially fast convergence. These series converge quickly because the target function is infinitely differentiable.

The bottom line: Suppose that $f(x)$, $f'(x)$, $f''(x)\,, \cdots f^{(p)}(x)$ all exist and the first $p-1$ derivatives are continuous and differentiable in the *closed* interval $-\pi \le x \le \pi$; the p'th derivative, $f^{(p)}(x)$ might have jump discontinuities at some points. Then for large n the Fourier coefficients are order n^{-p-1}.

Functions such as $\mathrm{sqr}(x)$, with jump discontinuities, correspond to $p = 0$. This is true of all piecewise continuous functions: the Fourier coefficients go to zero as n^{-1}. If $f'(x)$ exists but is piecewise continuous — the function $\mathrm{isqr}(x)$ is an example — then the Fourier coefficients are $\mathrm{ord}(k^{-2})$. Very smooth functions such as (5.121), (5.122) and (5.123) correspond to $p = \infty$.

Integration–by–parts

The results summarized above are obtained by evaluating the Fourier coefficient

$$f_n = \int_{-\pi}^{\pi} f(x) \mathrm{e}^{-\mathrm{i}nx} \, \frac{\mathrm{d}x}{2\pi} \,, \tag{5.124}$$

using integration by parts [4]. Suppose we can break the fundamental interval up into sub-intervals so that $f(x)$ is smooth (i.e., infinitely differentiable) in each subinterval. Non-smooth behavior, such a jump in some derivative, occurs only at the ends of the sub-interval. Then the contribution of the sub-interval (a, b) to f_n involves

$$\begin{aligned} I_n &\stackrel{\text{def}}{=} \int_a^b f(x) \mathrm{e}^{-\mathrm{i}nx} \, \mathrm{d}x \,, \\ &= \frac{1}{-\mathrm{i}n} \int_a^b f(x) \frac{\mathrm{d}\mathrm{e}^{-\mathrm{i}nx}}{\mathrm{d}x} , \mathrm{d}x \,, \\ &= \frac{1}{-\mathrm{i}n} \left[f(x) \mathrm{e}^{-\mathrm{i}nx} \right]_a^b + \frac{1}{\mathrm{i}n} \underbrace{\int_a^b f'(x) \mathrm{e}^{-\mathrm{i}nx} \, \mathrm{d}x}_{\stackrel{\text{def}}{=} J_n} \,. \end{aligned} \tag{5.125}$$

Since $f(x)$ is smooth, we can apply integration by parts to J_n to obtain

$$I_n = \frac{1}{-\mathrm{i}n} \left[f(x) \mathrm{e}^{-\mathrm{i}nx} \right]_a^b - \frac{1}{n^2} \left[f'(x) \mathrm{e}^{-\mathrm{i}nx} \right]_a^b + \frac{1}{n^2} \underbrace{\int_a^b f''(x) \mathrm{e}^{-\mathrm{i}nx} \, \mathrm{d}x}_{\stackrel{\text{def}}{=} K_n} \,. \tag{5.126}$$

[4] Section 6.3 of **BO** is a good reference.

Obviously we can keep going and develop a series in powers of n^{-1}. Thus we can express I_n in terms of the values of f and its derivatives at the end-points.

It is sporting to show that we actually generate an asymptotic series with the approach in (5.126). Looking at (5.126), we should show that the ratio of the remainder, $n^{-2}K_n$, to the previous term limits to zero as n increases. Assuming that f' is not zero at both end points, this requires that

$$\lim_{n\to\infty}\int_a^b f''(x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x = 0\,. \tag{5.127}$$

We can bound the integral easily

$$\Big|\int_a^b f''(x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x\Big| \le \int_a^b |f''(x)||\mathrm{e}^{\mathrm{i}nx}|\,\mathrm{d}x \le \int_a^b |f''(x)|\,\mathrm{d}x\,. \tag{5.128}$$

But this doesn't do the job.

Instead, we can invoke the Riemann-Lebesgue lemma which assures us that the remainder in (5.126) is vanishing faster than the previous term as $n \to 0$ i.e., dropping the remainder we obtain an $n \to \infty$ asymptotic approximation.

An alternative to Riemann-Lebesgue is to change our perspective and think of (5.126) like this:

$$I_n = \frac{1}{\mathrm{i}n}\left[f(x)\mathrm{e}^{\mathrm{i}nx}\right]_a^b \underbrace{- \frac{1}{n^2}\left[f'(x)\mathrm{e}^{\mathrm{i}nx}\right]_a^b + \frac{1}{n^2}\int_a^b f''(x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x}_{\text{the new remainder}}\,. \tag{5.129}$$

The bound in (5.128) then shows that the new remainder is asymptotically less than the first term on the right as $n \to \infty$. We can then continue to integrate by parts and prove asymptoticity by using the last two terms as the remainder.

Example: Consider

$$\mathrm{e}^{-x^2} = \sum_{n=-\infty}^{\infty} f_n \mathrm{e}^{\mathrm{i}nx}\,, \tag{5.130}$$

with Fourier coefficients

$$f_n = \int_{-\pi}^{\pi} \mathrm{e}^{-x^2}\mathrm{e}^{-\mathrm{i}nx}\frac{\mathrm{d}x}{2\pi}\,. \tag{5.131}$$

Example: Consider

$$\mathrm{e}^{\gamma\cos x} = \sum_{k=-\infty}^{\infty} f_n \mathrm{e}^{\mathrm{i}nx} \tag{5.132}$$

with Fourier coefficients

$$f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\gamma\cos x}\mathrm{e}^{-\mathrm{i}nx}\frac{\mathrm{d}x}{2\pi} \tag{5.133}$$

In this example the target function is infinitely differentiable throughout the fundamental interval, and at the endpoints $x = \pm\pi$. Because $f(x)$ and all its derivatives have no jumps, even at $x = \pm\pi$, all the end-point terms vanish. Thus we can integrate–by–parts forever and never generate an end-point term: in this case f_n decreases faster than any power of n e.g., perhaps something like e^{-n}, or $\mathrm{e}^{-\sqrt{n}}$. Integration-by-parts does not provide the asymptotic rate of decay of the Fourier coefficients — we must deploy a more potent method such as steepest descent.

5.8 References for Fourier series

Chapter 14 of **JJ** is a good reference. The slim book

DymMc *Fourier Series and Integrals* by H. Dym & H.P. McKean

is an account of the rigorous theory. I also like

Lan *Discourse on Fourier Series* by C. Lancoz and

Kor *Fourier Analysis* by T.W. Körner.

5.9 Problems

Problem 5.1. Find a least squares approximation

$$\mathrm{sqr}(x) \approx \alpha x \tag{5.134}$$

on the interval $(-\pi, \pi)$ by minimizing

$$\varepsilon(\alpha) = \int_{-\pi}^{\pi} [\mathrm{sqr}(x) - \alpha x]^2 \, \frac{\mathrm{d}x}{2\pi} \, . \tag{5.135}$$

Show that the minimum error is achieved by $\alpha = 3/(2\pi)$. Indicate the difficulties which arise if you attempt to improve this approximation with

$$\mathrm{sqr}(x) \approx \alpha x - \beta x^3 \, . \tag{5.136}$$

Why this approach is inferior to that of Fourier in (5.6)?

Problem 5.2. Develop the least-squares approximation from scratch using the complex sinusoids as a basis. That is, instead of (5.6), start with

$$f(x) \approx \sum_{k=-n}^{n} f_k \mathrm{e}^{\mathrm{i}kx} \, . \tag{5.137}$$

Do not assume that f is real. Show that minimizing

$$\varepsilon_n\left(f_k\right) \stackrel{\text{def}}{=} \int_{-\pi}^{\pi} \left| f(x) - \sum_{k=-n}^{n} f_k \mathrm{e}^{\mathrm{i}kx} \right|^2 \frac{\mathrm{d}x}{2\pi} \tag{5.138}$$

by varying f_k leads to (5.41). Show further that using these optimal coefficients the minimal error is given by

$$\min_{\forall f_k} \varepsilon_n(f_k) = \int_{-\pi}^{\pi} f^2(x) \, \frac{\mathrm{d}x}{2\pi} - \sum_{k=-n}^{n} |f_k|^2 \, . \tag{5.139}$$

Assume that for suitably smooth functions, $f(x)$, the complex sinusoids are complete in the sense that $\varepsilon_n \to 0$ as $n \to \infty$ and deduce the complex form of Parseval's theorem in (5.43)

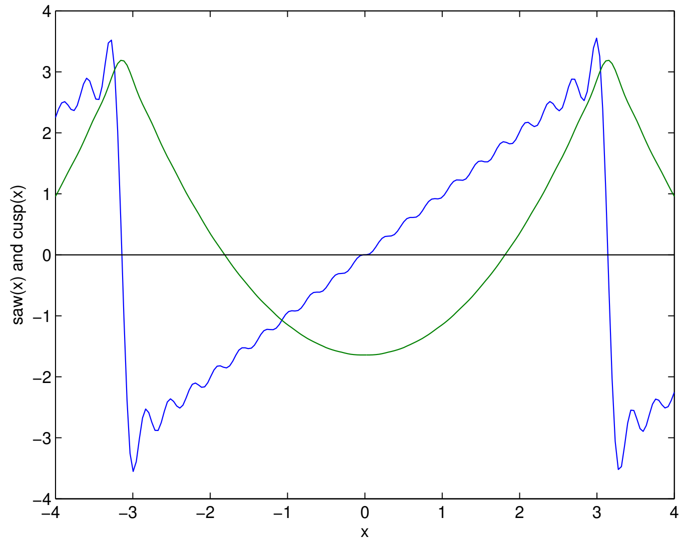


Figure 5.9: The 20-term sum of the saw-series (5.142), and that of its integral, isaw. `sawcusp.eps`

Problem 5.3. *(i)* Consider the 2π-periodic "sawtooth" function $\mathrm{saw}(x)$ defined by

$$\mathrm{saw}(x) = x\,, \qquad \text{if } -\pi < x < \pi, \tag{5.140}$$

and $\mathrm{saw}(-\pi) = \mathrm{saw}(\pi) = 0$. Show that

$$\mathrm{saw}(x) = 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots\right], \tag{5.141}$$

$$= 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx\,. \tag{5.142}$$

(ii) Consider the integrated sawtooth, $\mathrm{isaw}(x)$, defined by

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{isaw}(x) = \mathrm{saw}(x)\,, \qquad \text{and} \qquad \int_{-\pi}^{\pi} \mathrm{isaw}(x)\,\mathrm{d}x = 0\,. \tag{5.143}$$

Obtain an explicit expression for $\mathrm{isaw}(x)$, and obtain the Fourier series for $\mathrm{isaw}(x)$ by integration of (5.142). Illustrate your results as in figure 5.9.

Problem 5.4. Represent the function

$$f(x) = x(\pi - x) \tag{5.144}$$

on the interval $0 \le x \le \pi$ as: *(i)* A sine series; *(ii)* a cosine series. *(iii)* Sum the two series at $x = \pi/2$, keeping only the first three non-zero terms in each. Which series is more rapidly convergent to $f(\pi/2) = \pi^2/4$?

Problem 5.5. Suppose $f(x)$ is defined on the half range $0 \le x \le \pi$ and we extend the definition of $f(x)$ to the entire range $-\pi \le x \le \pi$ by taking

$f(x) = 0$ if $-\pi \le x < 0$. Find the analog of the half-range expansions in (5.71) and (5.72). Write out the three different half-range expansions of the function $f(x) = 1$.

Problem 5.6. Represent the function

$$f(x) = \cos qx \tag{5.145}$$

on the interval $-\ell \le x \le \ell$ as a Fourier series.

Problem 5.7. Find the Fourier coefficients

$$|\sin x| = \sum_{k=-\infty}^{\infty} f_k \mathrm{e}^{\mathrm{i}kx} = \sum_{k=0}^{\infty} a_k \cos kx\,. \tag{5.146}$$

Problem 5.8. Starting with the Fourier series of $\mathrm{isqr}(x)$, deduce the Fourier series representation of the following functions

$$\mathrm{i^2sqr}(x) \stackrel{\text{def}}{=} \frac{\pi^2}{8}x - \frac{\pi}{8}x|x|, \qquad \text{and} \qquad \mathrm{i^3sqr}(x) \stackrel{\text{def}}{=} \frac{\pi}{8}\left[\frac{\pi^3}{12} - \frac{\pi}{2}x^2 - \frac{1}{3}|x|^3\right] \tag{5.147}$$

defined on the interval $-\pi < x < \pi$. Discuss the rate of convergence of these Fourier series and the relationship between rate of convergence and the degree of singularity of the function being represented. As a byproduct of this exercise, show that

$$\tfrac{\pi^3}{32} = 1 - \tfrac{1}{3^3} + \tfrac{1}{5^3} - \tfrac{1}{7^3} + \cdots \qquad \text{and} \qquad \tfrac{\pi^4}{96} = 1 + \tfrac{1}{3^4} + \tfrac{1}{5^4} + \tfrac{1}{7^4} + \cdots \tag{5.148}$$

Problem 5.9. Sum the series

$$1 + \tfrac{1}{2^4} + \tfrac{1}{3^4} + \tfrac{1}{4^4} + \cdots \tag{5.149}$$

Problem 5.10. Suppose that $a(t)$ and $b(t)$ are 2π periodic functions with Fourier series representations

$$a(t) = \sum_{m=-\infty}^{\infty} a_m e^{\mathrm{i}mt}\,, \qquad b(t) = \sum_{m=-\infty}^{\infty} b_m e^{\mathrm{i}mt}\,. \tag{5.150}$$

Find the Fourier series of the convolution

$$a \circ b(t) \stackrel{\text{def}}{=} \int_0^{2\pi} a(t')b(t-t')\,\mathrm{d}t'\,. \tag{5.151}$$

Problem 5.11. *(i)* Show that on the fundamental interval $-\pi \le x \le \pi$ the "box-car" function

$$b(x;\ell) \stackrel{\text{def}}{=} \frac{1}{2\ell}\begin{cases} 1\,, & \text{if } |x| < \ell; \\ 0\,, & \text{if } |x| > \ell; \end{cases} \tag{5.152}$$

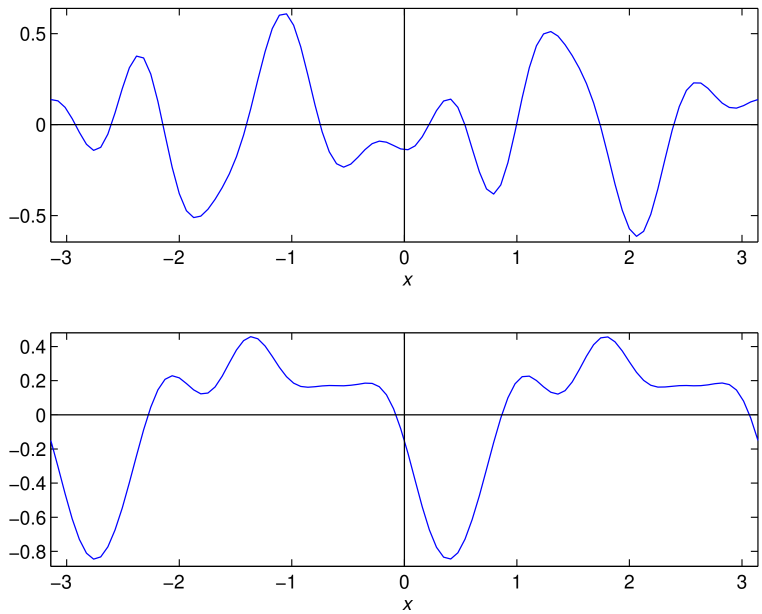


Figure 5.10: Two periodic functions.

has the Fourier series representation

$$b(x;\ell) = \sum_{-\infty}^{\infty} \frac{\sin n\ell}{2n\pi\ell}\, \mathrm{e}^{\mathrm{i}nx} \,. \tag{5.153}$$

(Assume $\ell \le \pi$.) *(ii)* Assuming that $\ell < \pi/2$, compute the convolution

$$\overline{\mathrm{sqr}}(x) \stackrel{\text{def}}{=} \int_{-\pi}^{\pi} \mathrm{sqr}(x') b(x - x';\ell)\, \mathrm{d}x' \,. \tag{5.154}$$

Use matlab to graph of $\overline{\mathrm{sqr}}(x)$ with $\ell = \pi/4$. (The matlab command `switch` might be useful.) *(iii)* Use the results of problem 5.10 to obtain the Fourier series of the smoothed square wave $\overline{\mathrm{sqr}}(x)$. On the same figure as part *(ii)*, plot the partial sum with three non-zero terms. Is there Gibbs phenomenon? *(iv)* Now consider $\ell = \pi/16$. Use matlab and the Fourier method to draw a graph of

$$b \circ b \circ b \circ b \circ \mathrm{sqr}(x) \,,$$

i.e., the quadruply filtered square wave.

Problem 5.12. Figure 5.10 shows two functions of x on the fundamental interval $-\pi < x < \pi$. Here are four possible Fourier series representations of these two functions:

$$f_1(x) = a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + a_4 \cos 4x + \cdots \tag{5.155}$$
$$f_2(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + \cdots \tag{5.156}$$
$$f_3(x) = a_1 \cos x + a_3 \cos 3x + a_5 \cos 5x + \cdots + b_1 \sin x + b_3 \sin 3x + b_5 \sin 5x + \cdots \tag{5.157}$$
$$f_4(x) = a_2 \cos 2x + a_4 \cos 4x + a_6 \cos 6x + \cdots + b_2 \sin 2x + b_4 \sin 4x + b_6 \sin 6x + \cdots \tag{5.158}$$

(i) Which representation might apply to the function shown in the top panel of the figure? *(ii)* Which representation might apply to the function shown in the bottom panel of the figure? Lucky guesses don't count: explain your reasoning.

Problem 5.13. Suppose $f(x)$ is defined on the half-range $0 < x \le \pi$. *(i)* How must we define $f(-x)$ if all of the cosine terms in the Fourier series are to vanish? *(ii)* How must we define $f(-x)$ if all of the sine terms in the Fourier series are to vanish? *(iii)* How must we define $f(-x)$ if all of the even harmonics in the Fourier series are to vanish? *(iv)* How must we define $f(-x)$ if all of the odd harmonics in the Fourier series are to vanish?

Problem 5.14. Here are four complex Fourier series

$$S_1(x) = \frac{\sinh \alpha\pi}{\pi} \sum_{m-=\infty}^{\infty} \frac{(-1)^m \, \mathrm{i}m \, \mathrm{e}^{\mathrm{i}mx}}{\alpha^2 + m^2}\,, \tag{5.159}$$

$$S_2(x) = \frac{\sinh \alpha\pi}{\pi} \sum_{m-=\infty}^{\infty} \frac{(-1)^m \, m \, \mathrm{e}^{\mathrm{i}mx}}{\alpha^2 + m^2}\,, \tag{5.160}$$

$$S_3(x) = \frac{\sinh \pi\alpha}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^m \, \mathrm{i}\alpha \, \mathrm{e}^{\mathrm{i}mx}}{\alpha^2 + m^2}\,, \tag{5.161}$$

$$S_4(x) = \frac{\sinh \pi\alpha}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^m \, \alpha \, \mathrm{e}^{\mathrm{i}mx}}{\alpha^2 + m^2}\,. \tag{5.162}$$

On the interval $-\pi < x < \pi$, which series is equal to $\cosh \alpha x$ and which is equal to $\sinh \alpha x$? Lucky guesses don't count so explain your reasoning (thirty words or less). Considering the end point $x = \pi$, what does the $\cosh \alpha x$ series converge to, and what does the $\sinh \alpha x$ series converge to?

Problem 5.15. Consider a damped oscillator forced by periodic impulses:

$$\ddot{\theta} + \epsilon\dot{\theta} + \sigma^2\theta = F\delta_c(t/T). \tag{5.163}$$

Calculate the mean square displacement $\overline{\theta^2}$ using two methods:

(i) Represent the forcing as a Fourier series, solve the differential equation and use Parseval's theorem.

(ii) Explicit construction of a periodic solution.

Plot the mean square displacement as a function of the forcing frequency $\omega \stackrel{\text{def}}{=} 2\pi/T$. In case *(ii)*, let θ_* and $\dot{\theta}_*$ denote the unknown displacement and velocity *immediately after* the oscillator gets a kick at $t = 0$. Then solve the differential equation as an initial value problem in the interval $0 < t < 2\pi T$. Determine θ_* and $\dot{\theta}_*$ by requiring that the solution is periodic.

Lecture 6

The Fourier Transform

6.1 Definition of the Fourier transform

We begin by defining the Fourier transform of a function $f(x)$

$$\boxed{\mathcal{F}\left[f(x); x \mapsto k\right] \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(x)\,\mathrm{d}x\,.} \tag{6.1}$$

Notice that $\mathcal{F}$ is a *linear operator.*

$$\mathcal{F}[f+g] = \mathcal{F}[f] + \mathcal{F}[g]$$

$$\mathcal{F}[cf] = c\mathcal{F}[f]$$

We start with f, which is a function of x, but the Fourier transform is a function of the transform variable, k. For brevity we will often denote the transform by $\tilde{f}(k)$. The explicit notation in (6.1) is useful if f depends on more than one variable and we need to be precise about which variable we are transforming against.

Lighter notation for the Fourier transform

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(x)\,\mathrm{d}x$$

Example: $\mathcal{F}\left[\mathrm{e}^{-\alpha|x|}; x \mapsto k\right]$ and related transforms

As an example of a Fourier transform we begin by finding $\mathcal{F}\left[e^{-\alpha x}\,\mathrm{H}(x)\right]$. To ensure convergence we assume that the real part of the constant α is positive. Using the definition in (6.1) we quickly find

$$\mathrm{H}(x) \stackrel{\text{def}}{=} \begin{cases} 0\,, & \text{if } x < 0; \\ 1\,, & \text{if } 0 < x. \end{cases}$$

$$\mathcal{F}\left[\mathrm{e}^{-\alpha x}\,\mathrm{H}(x)\right] = \frac{1}{\alpha + \mathrm{i}k}\,. \tag{6.2}$$

Even if α and $f(x)$ are real, the Fourier transform $\tilde{f}(k)$ is complex.

We can also quickly see that

$$\mathcal{F}\left[\mathrm{e}^{\alpha x}\,\mathrm{H}(-x)\right] = \frac{1}{\alpha - \mathrm{i}k}\,. \tag{6.3}$$

Therefore, because

$$\mathrm{e}^{-\alpha|x|} = \mathrm{e}^{\alpha x}\,\mathrm{H}(-x) + \mathrm{e}^{-\alpha x}\,\mathrm{H}(x)\,, \tag{6.4}$$

we can use linearity to find

$$\mathcal{F}\left[\mathrm{e}^{-\alpha|x|}\right] = \frac{1}{\alpha + \mathrm{i}k} + \frac{1}{\alpha - \mathrm{i}k} = \frac{2\alpha}{\alpha^2 + k^2}\,. \tag{6.5}$$

And we can subtract (6.3) from (6.2) to obtain

$$\mathcal{F}\left[\mathrm{sgn}(x)\mathrm{e}^{-\alpha|x|}\right] = \frac{1}{\alpha + \mathrm{i}k} - \frac{1}{\alpha - \mathrm{i}k} = \frac{-2\mathrm{i}k}{\alpha^2 + k^2}\,. \tag{6.6}$$

The reality condition

A useful algebra check is provided by the *reality condition*:

$$\text{if } f(x) \text{ is real, then } \tilde{f}(k) = \tilde{f}^*(-k). \tag{6.7}$$

Notice that the examples in (6.2) through (6.6) pass this test (provided that α is real).

New Fourier transforms from old: punctuation theorems

There are some simple identities obtained by shuffling symbols in the definition of $\mathcal{F}$. Here are the three simplest punctuation identities:

$$\mathcal{F}[f(x-\xi); x \mapsto k] = \mathrm{e}^{-\mathrm{i}k\xi}\tilde{f}(k)\,, \tag{6.8}$$

$$\mathcal{F}[f(\beta x); x \mapsto k] = \frac{1}{|\beta|}\tilde{f}\left(\frac{k}{\beta}\right)\,, \tag{6.9}$$

$$\mathcal{F}[\mathrm{e}^{\mathrm{i}px}f(x); x \mapsto k] = \tilde{f}(k-p)\,. \tag{6.10}$$

Proofs are left as an exercise.

Exercise Prove that if $f(x)$ is a real function then $\tilde{f}(-k)^* = \tilde{f}(k)$.

Exercise: Prove the punctuation identities in (6.8) through (6.10). Make sure you understand where the $|\beta|$ comes from in (6.9): the case $\beta = -1$ is important.

Example: *Parametric differentiation* is often useful: the derivative with respect to α of (6.5) is the Fourier transform

$$\mathcal{F}\left[|x|\mathrm{e}^{-\alpha|x|}\right] = 2\frac{\alpha^2 - k^2}{(\alpha^2 + k^2)^2} \tag{6.11}$$

Example: The limit $\alpha \to 0$ in (6.3) through (6.6) provides useful Fourier transforms. Thus

$$\mathcal{F}[1] = \lim_{\alpha\to 0}\frac{2\alpha}{\alpha^2 + k^2} = 2\pi\delta(k) \tag{6.12}$$

and

$$\mathcal{F}[\mathrm{sgn}(x)] = \lim_{\alpha\to 0}\frac{-2\mathrm{i}k}{\alpha^2 + k^2} = \frac{2\mathrm{pv}}{\mathrm{i}k}\,, \tag{6.13}$$

and

$$\mathcal{F}[\mathrm{H}(\pm x)] = \lim_{\alpha\to 0}\frac{\alpha \mp \mathrm{i}k}{\alpha^2 + k^2} = \pi\delta(k) \pm \frac{\mathrm{pv}}{\mathrm{i}k}\,, \tag{6.14}$$

It may be necessary to retreat to the pre-limit and integration against a test function to interpret divergences in these results. For example, the k^{-1} singularities will result in principal value integrals, denoted pv above.

Example: Consider the indicator function of the interval $-1/2 < x < 1/2$. That is

$$\Pi(x) \stackrel{\text{def}}{=} \begin{cases} 1\,, & \text{if } |x| < \frac{1}{2}\,, \\ 0\,, & \text{if } |x| > \frac{1}{2}\,. \end{cases} \tag{6.15}$$

The Fourier transform of Π is

$$\begin{aligned} \mathcal{F}[\Pi; x \mapsto k] &= \int_{-1/2}^{1/2} \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x\,, \\ &= \frac{2}{k}\sin\left(\frac{k}{2}\right)\,. \end{aligned} \tag{6.16}$$

As a simple check on algebra, notice that

$$\tilde{f}(0) = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x\,. \tag{6.17}$$

The Fourier transform in (6.16) passes this test. We can rescale $\Pi(x)$ so that it becomes the indicator function of the interval $-\ell < x < \ell$:

$$\mathcal{F}\left[\Pi\left(\frac{x}{2\ell}\right); x \mapsto k\right] = 2\frac{\sin(k\ell)}{k}\,. \tag{6.18}$$

Taking the limit $\ell \to \infty$ we find $\mathcal{F}[1] = 2\pi\delta(k)$ (again).

Transforms of derivatives

This

$$\mathcal{F}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] = \mathrm{i}k\tilde{f} \tag{6.19}$$

is a very important result.

Exercise: Assume that $f(x) \to 0$ as $|x| \to \infty$ and use integration-by-parts to prove (6.19).

Exercise: Use (6.19) to find the Fourier transform of $\mathrm{sgn}(x)\mathrm{e}^{-\alpha|x|}$.

$$\mathcal{F}\left[\mathrm{e}^{-\alpha|x|}\right] = \frac{2\alpha}{\alpha^2 + k^2}\,.$$

Notice the signs: in (6.1) we have defined the Fourier transform with $\exp(-\mathrm{i}kx)$ and then the *operational rule* in (6.19) is

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \to +\mathrm{i}k\,.} \tag{6.20}$$

Of course we also have

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \to (\mathrm{i}k)^n\,. \tag{6.21}$$

We defer discussion of transforms of integrals to section 6.4.

6.2 Inverse transforms: the FIT

The Fourier Integral Theorem (FIT) states that if $\tilde{f}(k)$ is the Fourier transform of $f(x)$ then

$$\boxed{f(x) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\tilde{f}(k)\,\frac{\mathrm{d}k}{2\pi}\,.} \tag{6.22}$$

Definition of the Fourier transform in (6.1):

$$\mathcal{F}[f(x); x \mapsto k] \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(x)\,\mathrm{d}x$$

The FIT is

$$\mathcal{F}^{-1}[\tilde{f}(k); k \mapsto x] = \int_{-\infty}^{\infty} \mathrm{e}^{+\mathrm{i}kx}\tilde{f}(k)\,\frac{\mathrm{d}k}{2\pi}$$

Note carefully the difference in the signs of the arguments of the exponentials in (6.1) and (6.22). With (6.1) and (6.22) we can go back and forth between the function f and the transform $\tilde{f}$.

We can combine (6.1) and (6.22) into a single equation

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{\mathrm{i}kx} \left[\int_{-\infty}^{\infty} \mathrm{d}x' \mathrm{e}^{-\mathrm{i}kx'} f(x')\right], \\
&= \int_{-\infty}^{\infty} f(x') \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}k(x-x')}\right] \mathrm{d}x' .
\end{aligned} \tag{6.23}
$$

Comparing the result above with the identity

$$
f(x) = \int_{-\infty}^{\infty} f(x')\delta(x-x')\,\mathrm{d}x' \tag{6.24}
$$

we have the fundamental conclusion that

$$
\boxed{\delta(x-x') = \int_{-\infty}^{\infty} e^{\mathrm{i}k(x-x')} \frac{\mathrm{d}k}{2\pi} .} \tag{6.25}
$$

Equation (6.25) is a concise statement of the FIT.

Example: The Fourier transforms in (6.5) and (6.6) are elementary. But the FIT produces non-elementary integrals

$$
\mathrm{e}^{-\alpha|x|} = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} \frac{2\alpha}{\alpha^2+k^2}\frac{\mathrm{d}k}{2\pi} = \frac{2}{\pi}\int_0^{\infty} \frac{\alpha\cos kx}{\alpha^2+k^2}\,\mathrm{d}k , \tag{6.26}
$$

and

$$
\mathrm{sgn}(x)\,\mathrm{e}^{-\alpha|x|} = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} \frac{-2\mathrm{i}k}{\alpha^2+k^2}\frac{\mathrm{d}k}{2\pi} = \frac{2}{\pi}\int_0^{\infty} \frac{k\sin kx}{\alpha^2+k^2}\,\mathrm{d}k . \tag{6.27}
$$

Example: Recall that in section 6.1 we calculated the indicator function of the interval $-\ell < x < \ell$. The elementary result is

$$
\mathcal{F}\left[\Pi\left(\frac{x}{2\ell}\right); x \mapsto k\right] = 2\frac{\sin k\ell}{k} . \tag{6.28}
$$

Applying the FIT we find an amazing result

$$
\Pi\left(\frac{x}{2\ell}\right) = \int_{-\infty}^{\infty} 2\frac{\sin k\ell}{k}\,\mathrm{e}^{-\mathrm{i}kx}\frac{\mathrm{d}k}{2\pi} , \tag{6.29}
$$

$$
= \frac{2}{\pi}\int_0^{\infty} \frac{\sin k\ell\,\cos kx}{k}\,\mathrm{d}k . \tag{6.30}
$$

In passing from (6.29) to (6.30) we have discarded the imaginary part because

$$
\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin k\ell\,\sin kx}{k}\,\mathrm{d}k = 0 . \tag{6.31}
$$

This might seem obvious because the integrand is odd. But it far from obvious that the integrals in (6.30) and (6.31) converge. We trust the FIT to return reliable results in situations such as this.

6.2.1 Proof of the FIT

Suppose we represent a compact function such as $f(x) = \mathrm{e}^{-x^2}$ on the interval $-L/2 < x < L/2$ using the complex form of a Fourier series. On this interval we represent $f(x)$ via a Fourier series as

To map x onto the fundamental interval, define $X \stackrel{\text{def}}{=} 2\pi x/L$

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}_n \exp\left(+\mathrm{i} n\, \mathrm{d}k\, x\right) , \tag{6.32}$$

$\mathrm{d}k$ is the meat in the $n\,\mathrm{d}k\,x$-sandwich:

$$n\,(\mathrm{d}k\,x) = nX$$

and

$$(n\,\mathrm{d}k)\,x = kx$$

and the "inverse" relation is

$$\hat{f}_n = \frac{1}{L}\int_{-L/2}^{L/2} \exp\left(-\mathrm{i} n\, \mathrm{d}k\, x\right) f(x)\, \mathrm{d}x \,. \tag{6.33}$$

With malice aforethought we are using the notation $\mathrm{d}k \stackrel{\text{def}}{=} 2\pi/L$ in (6.32) and (6.33).

Now let $L \to \infty$, so that $\mathrm{d}k \to 0$. The sequence of wavenumbers

$$\mathrm{d}k \stackrel{\text{def}}{=} \frac{2\pi}{L}$$

$$k_n \stackrel{\text{def}}{=} n\mathrm{d}k \tag{6.34}$$

then becomes very dense on the k-axis i.e., as $L \to \infty$, the difference between adjacent wavenumbers in the Fourier series (6.32) becomes smaller, $\mathrm{d}k \to 0$. We are forming a wavenumber continuum on the k-axis and in (6.32)

$$\mathrm{d}n = 1 = \frac{1}{L} \times \frac{\mathrm{d}k}{2\pi}$$

$$\sum_{n=-\infty}^{\infty} \to \int_{-\infty}^{\infty} \mathrm{d}n = L \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \tag{6.35}$$

But looking at (6.33), we see that as $L \to \infty$, $\hat{f}_n \to 0$. So we are motivated to define $\tilde{f}(k_n)$ via

Consider $f = \exp(-x^2/\sigma^2)$: once $L \gg \sigma$ the integral in (6.33) is independent of L.

$$\tilde{f}(k_n) = L\hat{f}_n \,, \qquad \text{or equivalently} \qquad \hat{f}_n = \tilde{f}(k_n)\frac{\mathrm{d}k}{2\pi} \,. \tag{6.36}$$

As $L \to \infty$, the function $\tilde{f}(k_n)$ remains nonzero and eventually becomes *independent of* L. Thus in this continuum limit the Fourier series in (6.32) and (6.33) is equivalent to the Fourier transform pair

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(k) \exp\left(+\mathrm{i}k\, x\right) \frac{\mathrm{d}k}{2\pi} \,, \tag{6.37}$$

and

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) \exp\left(-\mathrm{i}k\, x\right) \mathrm{d}x \,. \tag{6.38}$$

I like this proof so much that I resist the siren call of symmetry and treat x and k unfairly: I keep 2π under $\mathrm{d}k$ because the ratio $\mathrm{d}k/2\pi$ is the number of Fourier modes per unit length with wavenumber between k and $k + \mathrm{d}k$. Moreover, if we evaluate the formula

See (6.115) and associated discussion for symmetrical definitions of the Fourier transform

$$2\pi\delta(k - k') = \int_{-\infty}^{\infty} \mathrm{e}^{\pm\mathrm{i}(k-k')x}\, \mathrm{d}x \tag{6.39}$$

at $k = k'$ we get

$$2\pi\delta(0) = \int_{-\infty}^{\infty} \mathrm{d}x = L \tag{6.40}$$

where L is the length of the system. In other words, in k-space the apparently divergent quantity $2\pi\delta(0)$ has physical meaning: it is the length of the system. Thus it is good practice to keep the 2π with $\delta(k)$'s.

But note $\delta(0)$ means different things in k-space than in x-space.

6.2.2 Inverting Fourier transforms

Example: inverse transform of $\tilde{f}(k) = 1/(\alpha - \mathrm{i}k)$

Recall (6.3):

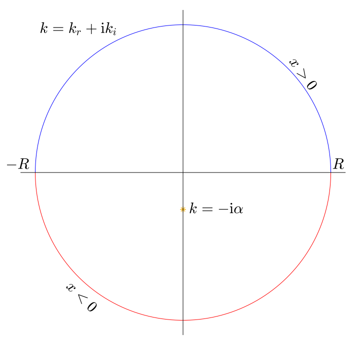
$$\mathcal{F}\left[\mathrm{e}^{\alpha x}\,\mathrm{H}(-x)\right] = \frac{1}{\alpha - \mathrm{i}k}$$

The inverse transform

$$f(x) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kx}}{\alpha - \mathrm{i}k}\frac{\mathrm{d}k}{2\pi} \tag{6.41}$$

can be evaluated by going into the complex plane, $k = k_r + \mathrm{i}k_i$, and using Cauchy's residue theorem.

We first replace $\pm\infty$ in the limits of integration in (6.41) with $\pm R$; we subsequently take $R \to \infty$. The integrand has a simple pole at $k = -\mathrm{i}\alpha$ in the lower-half k-plane. The contour of integration is closed with a big semi-circle of radius R. If $x > 0$ we close in the upper-half and there are no poles within the contour. We use Jordan's lemma to argue that the contribution from the semi-circular arc vanishes as $R \to \infty$. Thus if $x > 0$ then $f(x) = 0$. If $x < 0$ we close in the lower-half plane and enclose the pole at $k = -\mathrm{i}\alpha$. Again we use Jordan's lemma to show there is no contribution from the semi-circle in the limit $R \to \infty$. Thus with $x < 0$ the integral in (6.41) is determined by the residue theorem as:



$$f(x) = -2\pi\mathrm{i} \times \text{residue}\left(\frac{\mathrm{e}^{\mathrm{i}kx}}{\alpha - \mathrm{i}k}; k = -\mathrm{i}\alpha\right)\frac{1}{2\pi} = \mathrm{e}^{\alpha x}\,. \tag{6.42}$$

The minus sign in front is because when we close in the lower half-plane, we're going around the contour in the clockwise (negative) direction. We summarize this result as

$$\underbrace{\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kx}}{\alpha - \mathrm{i}k}\frac{\mathrm{d}k}{2\pi}}_{\mathcal{F}^{-1}\left[\frac{1}{\alpha - \mathrm{i}k}\right]} = \mathrm{H}(-x)\mathrm{e}^{\alpha x} \tag{6.43}$$

Evaluating Fourier transforms and their inverses using duality

(6.2) is:

$$\mathcal{F}\big[\,\mathrm{H}(x)\mathrm{e}^{-\alpha x}\big] = \frac{1}{\alpha + \mathrm{i}k}$$

Applying the FIT to invert the Fourier transform in (6.2) we have

$$\mathrm{e}^{-\alpha x}\,\mathrm{H}(x) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kx}}{\alpha + \mathrm{i}k}\frac{\mathrm{d}k}{2\pi}\,. \tag{6.44}$$

Can we use (6.44) to obtain

$$\mathcal{F}\left[\frac{1}{\alpha + \mathrm{i}x}; x \mapsto k\right]? \tag{6.45}$$

It's easy: in the formula (6.44) replace x by $-k$ and replace k by x and then multiply by 2π, leading to

$$2\pi e^{\alpha k}\, \mathrm{H}(-k) = \underbrace{\int_{-\infty}^{\infty} e^{ix(-k)} \frac{dx}{\alpha + ix}}_{=\mathcal{F}[\frac{1}{\alpha+ix}]} . \tag{6.46}$$

If we play this game with a general inverse Fourier transform,

$$f(x) = \int_{-\infty}^{\infty} e^{ikx} \tilde{f}(k)\, \frac{dk}{2\pi} , \tag{6.47}$$

we obtain

$$2\pi f(-k) = \underbrace{\int_{-\infty}^{\infty} e^{-ikx} \tilde{f}(x)\, dx}_{=\mathcal{F}[\tilde{f}(x)]} . \tag{6.48}$$

I'll call this the *duality trick*. The formula in (6.48) is a general statement of the duality trick. But I find it is very confusing to use (6.48): it's just too easy to bolix the signs. Instead, when confronted with a particular example, I recommend proceeding as we did between (6.44) and (6.46).

(6.13) is:

$$\mathcal{F}[\mathrm{sgn}(x)] = \frac{2}{ik}$$

Exercise: Apply duality to (6.13) to obtain

$$\mathcal{F}\left[\frac{1}{\pi x}\right] = -i\, \mathrm{sgn}(k) . \tag{6.49}$$

Exercise: Use duality to obtain

$$\mathcal{F}\left[\frac{1}{x^2 + a^2}\right] \qquad \text{from} \qquad \mathcal{F}\left[e^{-\alpha|x|}\right] = \frac{2\alpha}{\alpha^2 + k^2} . \tag{6.50}$$

6.3 Parseval's Theorem

First we show that

$$\int_{-\infty}^{\infty} f(x) g(x)\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(-k) \tilde{g}(k)\, dk , \tag{6.51}$$

$$\text{(change variables } k' = -k)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) \tilde{g}(-k)\, dk . \tag{6.52}$$

This identity does not assume that f and g are real functions. To prove (6.51), start with

$$\int_{-\infty}^{\infty} f(x) g(x)\, dx = \int_{-\infty}^{\infty} dx\, f(x) \underbrace{\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{g}(k)}_{g(x)} . \tag{6.53}$$

Changing the order of the integrals we obtain (6.51)

$$\int_{-\infty}^{\infty} f(x)g(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\,\tilde{g}(k) \underbrace{\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}kx} f(x)}_{\tilde{f}(-k)} . \tag{6.54}$$

Parseval's theorem is the special case $g = f^*$ in (6.51):

$$\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2\,\frac{\mathrm{d}k}{2\pi} . \tag{6.55}$$

Via Parseval's theorem every Fourier transform and its inverse provides an interesting integral identity. Some are very non-obvious.

Exercise: Using one of our introductory examples of a Fourier transform, show that

$$\frac{1}{4\alpha^3} = \int_{-\infty}^{\infty} \frac{1}{(\alpha^2+k^2)^2}\,\frac{\mathrm{d}k}{2\pi} . \tag{6.56}$$

6.4 Convolution

We define the *convolution* of two functions by

$$f \circ g \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x-x')g(x')\,\mathrm{d}x' . \tag{6.57}$$

$x'' = x - x'$ in (6.57) is

$$f \circ g = \int_{-\infty}^{\infty} f(x'')g(x-x'')\,\mathrm{d}x'' .$$

This definition should remind you of the Green's function solutions from earlier lectures. Convolution is a linear operation:

$$f \circ (g+h) = f \circ g + f \circ h , \qquad f \circ (cg) = c\, f \circ g , \tag{6.58}$$

where c is a constant and f, g and h are functions of x. Convolution is commutative and associative:

$$f \circ g = g \circ f , \qquad (f \circ g) \circ h = f \circ (g \circ h) . \tag{6.59}$$

Exercise: Verify (6.58) and (6.59).

Exercise: Why is the δ-function the "identity element of the convolution algebra"?

Exercise: Suppose $f(x) = 1$ and g is another function. Compute $f \circ g$. Now suppose $f(x) = x$. Again, what is $f \circ g$?

Convolutions are important because

$$\mathcal{F}\left[f \circ g\right] = \tilde{f}(k)\tilde{g}(k) , \tag{6.60}$$

or equivalently

$$\mathcal{F}^{-1}\left[\tilde{f}\,\tilde{g}\right] = f \circ g . \tag{6.61}$$

In other words, the inverse transform of a product of transforms is the convolution of the two functions in the spatial domain.

To prove the convolution theorem, apply the definition of the Fourier transform to the convolution,

$$\mathcal{F}[f \circ g\,; x \to k] = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} \left(\int_{-\infty}^{\infty} f(x-x')g(x')\,\mathrm{d}x' \right) \mathrm{d}x , \tag{6.62}$$

and follow your nose.

Exercise: Interchange the order of the integrals in (6.62) and prove the convolution theorem.

Example: Convolving the indicator function

$$\Pi(x) = \begin{cases} 1\,, & \text{if } |x| < 1/2\,, \\ 0\,, & \text{if } |x| > 1/2. \end{cases} \tag{6.63}$$

with itself gives a tent function $\Lambda(x)$. The base of the tent extends from -1 to 1 and the height of the tent is 1. The convolution theorem provides the Fourier transform of the tent.

Transforms of integrals

Enquiring minds desire the operational connection between integrating and Fourier transforming. Given that

$$f = \frac{\mathrm{d}g}{\mathrm{d}x}\,, \qquad \Rightarrow \qquad \mathrm{i}k\tilde{g} = \tilde{f}\,, \tag{6.64}$$

it seems likely that integration with respect to x should be related to division by ik. This is, however, is not straightforward. Dividing by ik in (6.64) we find

$$\tilde{g} = \frac{\tilde{f}}{\mathrm{i}k} + c\,\delta(k)\,, \tag{6.65}$$

where c is an undetermined constant. Evidently c in (6.65) is somehow related to the arbitrary lower limit in the integral

$$\int_a^x f(\xi)\,\mathrm{d}\xi\,. \tag{6.66}$$

Let's make a direct assault by taking $a = -\infty$ in (6.66) and applying the definition of the Fourier transform

$$\mathcal{F}\left[\int_{-\infty}^x f(\xi)\,\mathrm{d}\xi\right] = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} \int_{-\infty}^x f(\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,. \tag{6.67}$$

Writing

$$\mathrm{e}^{\mathrm{i}kx} = -\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{e}^{-\mathrm{i}kx}}{\mathrm{i}k} \tag{6.68}$$

and integrating by parts

$$\mathcal{F}\left[\int_{-\infty}^x f(\xi)\,\mathrm{d}\xi\right] = \frac{1}{\mathrm{i}k}\underbrace{\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}kx} f(x)\,\mathrm{d}x}_{\tilde{f}(k)} - \frac{\mathrm{e}^{-\mathrm{i}k(x\to\infty)}}{\mathrm{i}k}\int_{-\infty}^{\infty} f(\xi)\,\mathrm{d}\xi \tag{6.69}$$

There is an embarrassing $\mathrm{e}^{-\mathrm{i}k\infty}$. We easily get rid of this problem only if $\tilde{f}(0) = 0$. But this condition on f is too restrictive and we turn to generalized functions.

Exercise: What happens if you try calculating the inverse transform of $\tilde{f}/\mathrm{i}k$?

J.W. Miles in *Integral Transforms in Applied Mathematics* states that

$$\mathcal{F}^{-1}\left[\frac{\tilde{f}(k)}{\mathrm{i}k}\right] = \int_{\pm\infty}^x f(\xi)\,\mathrm{d}\xi$$

if and only if $\tilde{f}(0) = 0$.
J.S. Schwinger is reputed to have said "just because it's infinite doesn't mean you can neglect it". Miles is following this advice and avoiding generalized functions.

Recall

$$\tilde{f}(0) = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x\,.$$

Here is how we do it with generalized functions. Observe that

$$\int_{-\infty}^{x} f(\xi)\,\mathrm{d}\xi = \underbrace{\int_{-\infty}^{\infty} f(\xi)\,\mathrm{H}(x-\xi)\,\mathrm{d}\xi}_{f\circ\mathrm{H}} . \tag{6.70}$$

Using the convolution theorem

$$\mathcal{F}\left[\int_{-\infty}^{x} f(\xi)\,\mathrm{d}\xi\right] = \mathcal{F}[f\circ\mathrm{H}]\,, \tag{6.71}$$

$$= \tilde{f}(k)\times\left(\pi\delta(k)+\frac{1}{\mathrm{i}k}\right), \tag{6.72}$$

$$= \pi\delta(k)\tilde{f}(0)+\frac{\tilde{f}(k)}{\mathrm{i}k}. \tag{6.73}$$

If $\tilde{f}(0)\neq 0$ then the k^{-1} singularity in the final term should be interepreted as a principal value integral.

Exercise: Suppose that $f_e(x)$ is an even function of x: $f_e(x)=f_e(-x)$. Show that

$$\mathcal{F}\left[\int_{0}^{x} f_e(x)\,\mathrm{d}x'\right] = \frac{\tilde{f}(k)}{\mathrm{i}k}. \tag{6.74}$$

6.5 The Hilbert transform

6.6 Solution of differential equations

The forced transport equation

Let's reconsider the forced linear transport equation

$$u_t + cu_x = f\,, \qquad \text{with initial condition} \qquad u(x,0)=0\,. \tag{6.75}$$

We previously solved this problem with the method of characteristics. Now let's use the Fourier transform. applying $\mathcal{F}$ to (6.75) we obtain

$$\tilde{u}_t + \mathrm{i}kc\tilde{u} = \tilde{f} \qquad \text{with initial condition} \qquad \tilde{u}(k,0)=0. \tag{6.76}$$

This is a linear constant-coefficient first-order ordinary differential equation. Using the integrating factor method, the solution is

$$\tilde{u}(k,t) = \int_0^t \mathrm{e}^{-kc(t-t')}\tilde{f}(k,t')\,\mathrm{d}t'\,. \tag{6.77}$$

Now we write down the inverse transform and exchange the order of the integration so that the t'–integral is last and the k–integral is first

$$u(x,t) = \int_0^t \mathrm{d}t' \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{e}^{-kc(t-t')}\tilde{f}(k,t')\,, \tag{6.78}$$

$$= \int_0^t \mathrm{d}t' \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\mathrm{e}^{\mathrm{i}k(x+ct'-ct)}\tilde{f}(k,t')\,, \tag{6.79}$$

$$= \int_0^t f(x+ct-ct')\,\mathrm{d}t'\,. \tag{6.80}$$

This is the result previously obtained with the method of characteristics.

Airy's equation

Airy's equation,

$$y'' - xy = 0\,, \tag{6.81}$$

is an important second-order differential equation. The two linearly independent solutions, $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$, are shown in figure 6.1. The Airy function, $\mathrm{Ai}(x)$, is defined as the solution that decays as $x \to \infty$, with the normalization

$$\int_{-\infty}^{\infty} \mathrm{Ai}(x)\,\mathrm{d}x = 1\,. \tag{6.82}$$

We obtain an integral representation of $\mathrm{Ai}(x)$ by attacking (6.81) with the Fourier transform.

Using the operational rule

$$\mathcal{F}\left[x f(x)\right] = \mathrm{i}\frac{\mathrm{d}\tilde{f}}{\mathrm{d}k} \tag{6.83}$$

(see problems) we can Fourier transform the term xy in (6.81). We find the transformed differential equation

$$k^2 \tilde{y} + \mathrm{i}\frac{\mathrm{d}\tilde{y}}{\mathrm{d}k} = 0\,. \tag{6.84}$$

Solving this first-order equation, and using the normalization condition (6.82) to determined the constant of integration:

$$\widetilde{\mathrm{Ai}}(k) = \mathrm{e}^{\mathrm{i}k^3/3}\,. \tag{6.85}$$

$\mathcal{F}\left[\mathrm{Ai}(x) : x \mapsto k\right] = \mathrm{e}^{\mathrm{i}k^3}$

Using the Fourier integral theorem

$$\mathrm{Ai}(x) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx + \mathrm{i}k^3/3}\,\frac{\mathrm{d}k}{2\pi}\,, \tag{6.86}$$

$$= \frac{1}{\pi}\int_0^{\infty} \cos\left(kx + \frac{k^3}{3}\right)\,\mathrm{d}k\,. \tag{6.87}$$

Notice that the integral converges at $k = \infty$ because of destructive interference or catastrophic cancellation.

Exercise: Show that $\int_{-\infty}^{\infty} \mathrm{Ai}^2(x)\,\mathrm{d}x = (2\pi)^{-1}$.

Solution of the diffusion equation

Consider once again the initial value problem for the diffusion equation

$$u_t = \kappa u_{xx}\,, \qquad \text{with IC} \qquad u(x,0) = f(x)\,. \tag{6.88}$$

The Fourier transform, $x \mapsto k$, is

(6.20): $\frac{\mathrm{d}}{\mathrm{d}x} \to +\mathrm{i}k$

$$\tilde{u}_t = -\kappa k^2 \tilde{u}\,, \qquad \text{with IC} \qquad \tilde{u}(k,0) = \tilde{f}(k)\,. \tag{6.89}$$

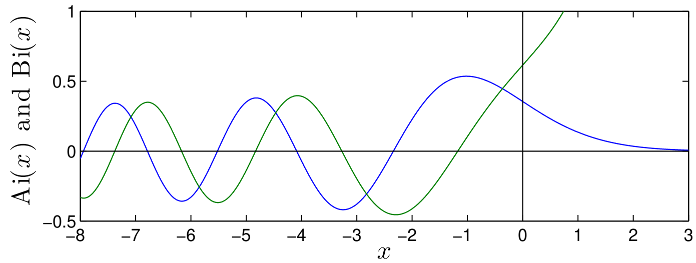


Figure 6.1: The functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$. The Airy function decays rapidly as $x \to \infty$ and rather slowly as $x \to -\infty$.

We have used the operational rule (6.20) twice to get the transformed equation. The transformed problem (6.89) is an ODE with solution

$$\tilde{u}(k,t) = \tilde{f}(k)\mathrm{e}^{-\kappa k^2 t} . \tag{6.90}$$

Now we invoke the FIT to write down the integral representation of the solution

$$u(x,t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx - \kappa k^2 t} \tilde{f}(k) \, \frac{\mathrm{d}k}{2\pi} . \tag{6.91}$$

Exercise: Check by substitution that (6.91) satisfies the diffusion equation.

There is one initial condition for which we can easily invert the transform in (6.91), namely

$$f(x) = \delta(x) , \qquad \text{and therefore} \quad \tilde{f}(k) = 1 . \tag{6.92}$$

We know the answer in this case: it is the Gaussian similarity solution of the diffusion equation, $g(x,t)$. Let's verify this by doing the integral in (6.91) with $\tilde{f}(k) = 1$. The key is to complete the square in the exponential:

$g(x,t) = \dfrac{\mathrm{e}^{-x^2/4\kappa t}}{\sqrt{4\pi\kappa t}}$

$$\kappa k^2 t - \mathrm{i}kx = \kappa t \left(k - \frac{\mathrm{i}x}{2\kappa t}\right)^2 - \kappa t \left(\frac{\mathrm{i}kx}{2\kappa t}\right)^2 . \tag{6.93}$$

Thus, with $\tilde{f}(k) = 1$, and $u \to g$ in (6.91)

$$g(x,t) = \mathrm{e}^{-x^2/4\kappa t} \int_{-\infty}^{\infty} \mathrm{e}^{-\kappa t \left(k - \frac{\mathrm{i}x}{2\kappa t}\right)^2} \, \frac{\mathrm{d}k}{2\pi} . \tag{6.94}$$

To evaluate the integral we translate the contour in the complex k-plane so that it runs along the line $k = (\mathrm{i}x/2\kappa t) + k'$, where k' runs from $-\infty$ to ∞ i.e., the new contour is parallel to the real-k axis. Since the integrand has no singularities the integral is unchanged by this translation, and we find

$$g(x,t) = \mathrm{e}^{-x^2/4\kappa t} \int_{-\infty}^{\infty} \mathrm{e}^{-\kappa t k'^2} \, \frac{\mathrm{d}k'}{2\pi} , \tag{6.95}$$

$$= \frac{\mathrm{e}^{-x^2/4\kappa t}}{\sqrt{4\pi\kappa t}} . \tag{6.96}$$

Now return to (6.91) and write it as

$$u(x,t) = \mathcal{F}^{-1}\left[\tilde{g}(k,t) \times \tilde{f}(k)\right], \tag{6.97}$$

where

$$\tilde{g}(k,t) = \mathrm{e}^{-\kappa k^2 t} = \mathcal{F}\left[g(x,t);\, x \to k\right]. \tag{6.98}$$

In (6.97) we desire the inverse transform of the product of two Fourier transforms. Invoking the convolution theorem we recover the Green's function formula

$$u(x,t) = \underbrace{\int_{-\infty}^{\infty} f(x')\, \frac{\mathrm{e}^{-(x-x')^2}}{4\kappa t}\, \mathrm{d}x'}_{f \circ g}. \tag{6.99}$$

The hyperdiffusion equation

$\Gamma(z) \overset{\text{def}}{=} \int_0^\infty \mathrm{e}^{-t} t^{z-1}\, \mathrm{d}t$

$z\Gamma(z) = \Gamma(z+1)$

Consider the Green's function of the hyperdiffusion equation

$$g_t = -\nu g_{xxxx}, \qquad \text{with IC} \quad g(x,0) = \delta(x). \tag{6.100}$$

With $\partial_x^4 \mapsto k^4$ we see that the Fourier transform is

$$\tilde{g}_t = -\nu k^4 \tilde{g}, \qquad \text{with IC} \quad \tilde{g}(k,0) = 1. \tag{6.101}$$

Solving this ODE and applying the FIT

$$g(x,t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx - \nu k^4 t}\, \frac{\mathrm{d}k}{2\pi}. \tag{6.102}$$

The change of variables $\kappa = (\nu t)^{1/4} k$ puts this in similarity form

$$g(x,t) = (\nu t)^{-1/4} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\kappa\xi - \kappa^4}\, \frac{\mathrm{d}\kappa}{2\pi}, \tag{6.103}$$

$$= \frac{1}{\pi(\nu t)^{1/4}} \underbrace{\int_0^\infty \mathrm{e}^{-\kappa^4} \cos \kappa\xi\, \mathrm{d}\kappa}_{\overset{\text{def}}{=} G(\xi)}, \tag{6.104}$$

where $\xi \overset{\text{def}}{=} x/(\nu t)^{1/4}$. As far as I know, this integral cannot be evaluated analytically. It is, however, easy to evaluate the integral numerically using the MATLAB command `integral`— see figure 6.2.

One can also extract analytic information from the integral representation. For example

$v = \kappa^4$

$\mathrm{d}v = 4v^{3/4}\mathrm{d}\kappa$

$$G(0) = \int_0^\infty \mathrm{e}^{-\kappa^4}\, \mathrm{d}\kappa, \tag{6.105}$$

$$= \Gamma(5/4). \tag{6.106}$$

Had we but world enough, and time, we would use the saddle-point method to obtain the large-ξ asymptotic approximation to $G(\xi)$.

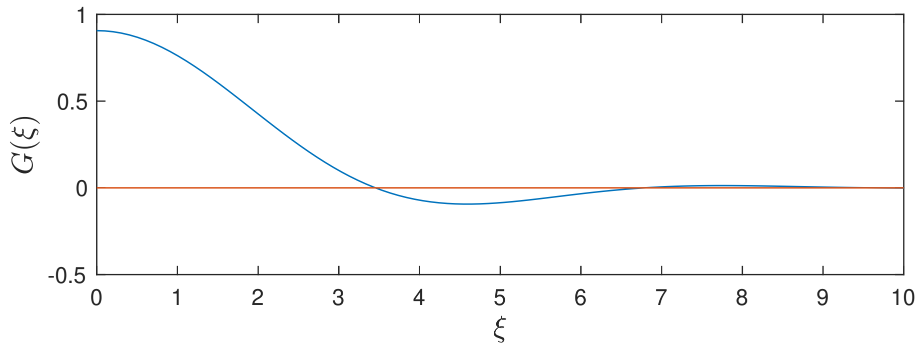


Figure 6.2: The function $G(\xi)$ defined in (6.104). The negative lobe indicates that the hyperdiffusion equations does not possess a maximum principle.

6.7 More examples

To establish Fourier-literacy, in this section we collect some important examples of Fourier transforms.

A Bessel function example

The Bessel function $J_0(x)$ has the integral representation

$$J_0(k) = \frac{1}{\pi}\int_0^\pi \cos(k\cos\theta)\,\mathrm{d}\theta\,. \tag{6.107}$$

Changing variables to $x = \cos\theta$ we obtain

$$\pi J_0(k) = \int_{-1}^{1} \frac{\cos kx}{\sqrt{1-x^2}}\,\mathrm{d}x\,, \tag{6.108}$$

$$= \mathcal{F}\left[\frac{\Pi(x/2)}{\sqrt{1-x^2}}\right]\,. \tag{6.109}$$

The FIT assures us that

$$\frac{\Pi(x/2)}{\sqrt{1-x^2}} = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\pi J_0(k)\,\frac{\mathrm{d}k}{2\pi}\,. \tag{6.110}$$

With $k \mapsto x$ and $x \mapsto -k$ in the formula above

$$\frac{2\Pi(k/2)}{\sqrt{1-k^2}} = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} J_0(x)\,\mathrm{d}x\,, \tag{6.111}$$

$$= \mathcal{F}[J_0(x)]\,. \tag{6.112}$$

Fourier transform of a Gaussian

$$\mathcal{F}[\mathrm{e}^{-\alpha^2x^2}] = \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha^2x^2-\mathrm{i}kx}\,\mathrm{d}x\,, \tag{6.113}$$

$$= \tag{6.114}$$

Other definitions of the Fourier transform, and mathematica

There are several definition of the Fourier transform in circulation. All definitions have the following form

$$\tilde{f}(k;a,b) \stackrel{\text{def}}{=} \sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}bkx}\,\mathrm{d}x\,. \tag{6.115}$$

The inverse transform is

$$f(x) \stackrel{\text{def}}{=} \sqrt{\frac{|b|}{(2\pi)^{1+a}}} \int_{-\infty}^{\infty} \tilde{f}(k)\mathrm{e}^{-\mathrm{i}bkx}\,\mathrm{d}x\,. \tag{6.116}$$

Above $\{a,b\}$ is one of the following

$$\{0,1\}\,, \qquad \{1,-1\}\,, \qquad \{-1,1\}\,, \qquad \{0,1\}\,, \qquad \{0,-2\pi\} \tag{6.117}$$

Our convention is $\{a,b\} = \{1,-1\}$. MATHEMATICA has useful commands for evaluating and inverting Fourier transforms. But MATHEMATICA's default convention is $\{a,b\} = \{0,1\}$. Fortunately there is a handy option, `FourierParameters`, that can be used to choose the convention you prefer. Here is an example that uses our convention to Fourier transform sech$x$:

```
FourierTransform[ Sech[x],x,k,FourierParameters->{1,-1}]
```

MATHEMATICA quickly assures us that

$$\mathcal{F}[\mathrm{sech}x; x \mapsto k] = \pi\,\mathrm{sech}\left(\tfrac{\pi k}{2}\right)\,. \tag{6.118}$$

Now let's try a simple inverse Fourier transform:

```
InverseFourierTransform[1/(a + I k), k ,x , FourierParameters->{1,-1}]
```

MATHEMATICA assumes that a is a complex number and provides a complicated answer with conditional statements. Let's tell MATHEMATICA that a is real and positive:

```
Assuming[a>0,InverseFourierTransform[1/(a + I k), k, x, FourierParameters->{1,-1}]]
```

This produces much simpler output in agreement with (6.2).

Find

$$\mathcal{F}^{-1}\left[\frac{1}{\alpha+\mathrm{i}k}\right] = ???$$

with MATHEMATICA.

Example: Find the Fourier transform of $\mathrm{sech}^n x$ where n is an integer.

Using MATHEMATICA we find the first few transforms:

$$\mathcal{F}[\mathrm{sech}^2 x; x \mapsto k] = k\pi\,\mathrm{cosech}\left(\tfrac{\pi k}{2}\right)\,, \tag{6.119}$$

$$\mathcal{F}[\mathrm{sech}^3 x; x \mapsto k] = \tfrac{1}{2}(1+k^2)\pi\,\mathrm{sech}\left(\tfrac{\pi k}{2}\right)\,, \tag{6.120}$$

$$\mathcal{F}[\mathrm{sech}^4 x; x \mapsto k] = \tfrac{1}{6}k(4+k^2)\pi\,\mathrm{cosech}\left(\tfrac{\pi k}{2}\right)\,, \tag{6.121}$$

$$F[\mathrm{sech}^5 x; x \mapsto k] = \tfrac{1}{24}(1+k^2)(9+k^2)\pi\,\mathrm{sech}\left(\tfrac{\pi k}{2}\right)\,, \tag{6.122}$$

$$\mathcal{F}[\mathrm{sech}^6 x; x \mapsto k] = \tfrac{1}{120}k(4+k^2)(16+k^2)\pi\,\mathrm{cosech}\left(\tfrac{\pi k}{2}\right)\,. \tag{6.123}$$

The pattern is clear and we can use induction. The key step is to show by direct calculation that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,\mathrm{sech}^n x = n^2\,\mathrm{sech}^n x - n(n+1)\,\mathrm{sech}^{n+2} x\,. \tag{6.124}$$

Denoting the Fourier transform of $\mathrm{sech}^n x$ by $\tilde{f}_n(k)$, and Fourier transforming the identity above, we obtain a two-term recurrence relation,

$$n(n+1)\tilde{f}_{n+2} = (n^2+k^2)\tilde{f}_n\,, \tag{6.125}$$

implying the results above from those with $n=1$ and 2. Naturally one wonders what happens if n is not an integer....

Decay of the Fourier transform as $k \to \pm\infty$

The more rapidly $f(x)$ decays as $x \to \pm\infty$, the more high frequency waves it must contain and hence the slower the decay of $\tilde{f}(k)$. Invoking the *Riemann–Lebesgue lemma* one can show that provided

$$\int_{-\infty}^{\infty} |f(x)| \mathrm{d}x < \infty \tag{6.126}$$

then the Fourier transform $\tilde{f}(k)$ exists as a normal function. Moreover, with (6.126), one is sure that $\tilde{f}(k)$ is continuous and

$$|\tilde{f}(k)| \le \int_{-\infty}^{\infty} |f(x)| \mathrm{d}x \,. \tag{6.127}$$

The results above can be illustrated by the Fourier transform of the discontinuous function in (6.3): that Fourier transform is continuous and

$$\left|\frac{1}{\alpha - \mathrm{i}k}\right| = \frac{1}{\sqrt{\alpha^2 + k^2}} < \int_{-\infty}^{0} \left|\mathrm{e}^{\alpha x}\right| \mathrm{d}x = \frac{1}{\alpha} \,. \tag{6.128}$$

The absolute value $||$ in (6.126) is essential. The integral

$$\int_{-\infty}^{\infty} \sin x^2 \, \mathrm{d}x \tag{6.129}$$

is convergent because of the rapid oscillation of the integrand as $x \to \pm\infty$. But the Fourier transform,

$$\mathcal{F}[\sin x^2] = \sqrt{\pi} \cos\left(k^2 + \tfrac{\pi}{4}\right) \,, \tag{6.130}$$

does not decay as $k \to \pm\infty$.

6.8 Problems

Problem 6.1. Solve the ODE:

$$g_{xx} - g = \delta(x) \,, \qquad \lim_{x \to \pm\infty} g(x) = 0 \,,$$

by Fourier transforming with respect to x. Invert the transform and check your answer by substitution. The homogenous problem has exponentially growing solutions $\mathrm{e}^{\pm x}$. But if you use the Fourier method it seems that you never have to worry about these. why is that?

Problem 6.2. Prove the operational rules

$$\mathcal{F}\left[x f; x \mapsto k\right] = \mathrm{i}\frac{\mathrm{d}\tilde{f}}{\mathrm{d}k} \,, \tag{6.131}$$

and

$$\mathcal{F}\left[x\frac{\mathrm{d}f}{\mathrm{d}x}; x \mapsto k\right] = -\frac{\mathrm{d}\big(k\tilde{f}\big)}{\mathrm{d}k} \,. \tag{6.132}$$

Problem 6.3. In the lecture we obtained the Fourier transform

$$\frac{2\alpha}{\alpha^2+k^2} = \int_{-\infty}^{\infty} e^{-\alpha|x|-ikx}\, dx\,. \tag{6.133}$$

Apply "punctuation identities" other tricks to the formula above, and so obtain the Fourier transforms of the following functions

$$f_1(x) = |x|e^{-\alpha|x|}\,, \quad f_2(x) = xe^{-\alpha|x|}\,, \tag{6.134}$$

$$f_3(x) = \text{sgn}(x)e^{-\alpha|x|}\,, \quad f_4(x) = \frac{1-e^{-\alpha|x|}}{|x|}\,, \tag{6.135}$$

$$f_5(x) = \cos(\gamma x)e^{-\alpha|x|}\,, \quad f_6(x) = \sin(\gamma x)e^{-\alpha|x|}\,. \tag{6.136}$$

Your goal is to avoid the honest but tedious work of direct integration.

Problem 6.4. Find the Fourier transform of

$$f_7(x) = \text{sgn}(x)\left(1-e^{-\alpha|x|}\right) \tag{6.137}$$

without evaluating integrals. Hint: read section 6.4

Problem 6.5. *(i)* Use the duality trick on (6.133) to evaluate the Fourier transform

$$\mathcal{F}\left[\frac{1}{a^2+x^2}\right]\,. \tag{6.138}$$

(ii) Evaluate

$$\mathcal{F}^{-1}\left[\frac{1}{(\alpha^2+k^2)^2}\right] \tag{6.139}$$

without integration. *(iii)* Use a Fourier transform to solve

$$\left(\frac{d^2}{dx^2}-1\right)^2 g = \delta(x)\,, \qquad \lim_{x\to\pm\infty} g = 0\,. \tag{6.140}$$

Problem 6.6. Find $\mathcal{F}^{-1}[\tilde{f}(k)/k]$.

Problem 6.7. Find the Fourier transform of the "tent-function"

$$\Lambda(x) = \begin{cases} 1-|x|\,, & |x| \le 1\,; \\ 0\,, & |x| > 1\,. \end{cases} \tag{6.141}$$

Show that

$$\int_0^\infty \left(\frac{\sin y}{y}\right)^4 dy = \frac{\pi}{3}\,. \tag{6.142}$$

Problem 6.8. *(i)* Use the Fourier transform to obtain an integral representation of the solution of the dispersive wave equation

$$\text{Ai}(z) \stackrel{\text{def}}{=} \frac{1}{\pi}\int_0^\infty \cos\left(\frac{v^3}{3}+zv\right) dv$$

$$u_t = \alpha u_{xxx}\,, \qquad u(x,0) = u_0(x)\,, \tag{6.143}$$

where α is a real positive constant. *(ii)* Calculate the inverse transform in the special case $u_0(x) = \delta(x)$ i.e., find the Green's function. Express the answer in terms of the Airy function Ai. *(iii)* In the problems at the end of an earlier lecture you were asked to solve this problem with the similarity method. Make sure your Fourier-transform answer agrees with your similarity solution. *(iv)* Write the general solution of the initial value problem in terms of the Green's function.

Problem 6.9. Show that

$$\int_{-\infty}^{\infty} \mathrm{Ai}(x')\mathrm{Ai}(x-x')\,\mathrm{d}x' = \frac{1}{2^{1/3}}\mathrm{Ai}\left(\frac{x}{2^{1/3}}\right). \tag{6.144}$$

Problem 6.10. Proceeding from the definition of the Fourier transform in (6.1), show that

$$\mathcal{F}\left[\frac{1}{\pi x}\right] = -\mathrm{i}\,\mathrm{sgn}(k)\,. \tag{6.145}$$

Make sure you explain where the sgn(k) comes from.

In problem (6.10) use and prove the result

$$\int_{-\infty}^{\infty} \frac{\sin v}{v}\mathrm{d}v = \pi\,.$$

Hint: $v^{-1} = \int_0^\infty \mathrm{e}^{-vt}\,\mathrm{d}t$ and change the order of the integrals.

Problem 6.11. Evaluate the convolution

$$\int_{-\infty}^{\infty} \frac{1}{a^2+x'^2}\,\frac{1}{b^2+(x-x')^2}\,\mathrm{d}x'\,. \tag{6.146}$$

Problem 6.12. Consider the integro-differential equation

$$g_t + \beta\partial_x \int_{-\infty}^{\infty} \frac{g(x')}{x-x'}\,\frac{\mathrm{d}x'}{\pi} = 0\,, \tag{6.147}$$

with initial condition $g(x,0) = \delta(x)$. *(i)* What are the dimensions of β? What is the possible form of a similarity solution? *(ii)* Solve the equation with a Fourier transform. Verify that your solution has similarity form.

In problem (6.12) the singular integral is a principal value:

$$\lim_{\alpha\to 0}\int_{-\infty}^{\infty} \frac{\xi(x')\,(x-x')}{(x-x')^2+\alpha^2}\,\frac{\mathrm{d}x'}{\pi}$$

Problem 6.13. Solve the ODE:

$$v_{xxxx} + v = \delta(x)\,, \qquad \lim_{x\to\pm\infty} v(x) = 0\,. \tag{6.148}$$

by Fourier transforming with respect to x. Invert the transform to obtain $v(x)$.

Problem 6.14. Consider Laplace's equation

$$u_{xx} + u_{yy} = 0 \tag{6.149}$$

in a strip $-\infty < x < \infty$ and $0 < y < a$, with the condition that $u \to 0$ as $|x| \to \infty$. There are prescribed boundary values $u(x,0) = f(x)$ and $u(x,a) = g(x)$. Express $u(x,y)$ in terms of the Fourier transforms of $f(x)$ and $g(x)$. Suppose that

$$f(x) = 0, \qquad g(x) = \frac{x}{x^2+a^2} - \frac{x}{x^2+9a^2}.$$

Find $u(x,y)$ by inverting the Fourier transform $\tilde{u}(k,y)$.

Problem 6.15. Solve the wave equation

$$\zeta_{tt} - c^2\zeta_{xx} = 0\,, \qquad \zeta(x) = \zeta_0(x)\,, \quad \zeta_t(x) = w_0(x) \tag{6.150}$$

with a FT $x \mapsto k$. Show that you recover D'Alembert's solution.

Problem 6.16. State and prove the convolution theorem for Fourier transforms. Define a linear operator $\sqrt{\mathrm{d}/\mathrm{d}x}$ by

$$\sqrt{\frac{\mathrm{d}}{\mathrm{d}x}} f(x) \stackrel{\text{def}}{=} \int_{-\infty}^{x} \frac{1}{\sqrt{\pi(x-x')}} \frac{\mathrm{d}f}{\mathrm{d}x'}\,\mathrm{d}x'\,. \tag{6.151}$$

Treat the right-hand side as a convolution and find its Fourier transform. Show that

$$\sqrt{\frac{\mathrm{d}}{\mathrm{d}x}}\sqrt{\frac{\mathrm{d}}{\mathrm{d}x}} f = \frac{\mathrm{d}f}{\mathrm{d}x} \tag{6.152}$$

so that $\sqrt{\mathrm{d}/\mathrm{d}x}$ is a "halfth derivative".

Note

$$\int_0^\infty e^{-\mathrm{i}y^2}\,dy = \tfrac{1}{2}\sqrt{\frac{\pi}{\mathrm{i}}}$$

Problem 6.17. Using a Fourier transform, solve the integral equation

$$\mathrm{e}^{-m^2x^2/2} = \int_{-\infty}^{\infty} \mathrm{e}^{-\beta|x-u|} f(u)\,\mathrm{d}u\,. \tag{6.153}$$

Problem 6.18. Solve the elastic wave equation

$$\psi_{tt} + \psi_{xxxx} = 0, \qquad \psi(x,0) = \delta(x), \qquad \psi_t(x,0) = 0\,, \tag{6.154}$$

using the Fourier transform. To invert the transform you'll need to "complete the square". To check your algebra, show that

$$\psi(0,t) = \frac{1}{2\sqrt{2\pi t}}\,. \tag{6.155}$$

The Fresnel integral:

$$\int_0^\infty \mathrm{e}^{\mathrm{i}u}\,\frac{\mathrm{d}u}{\sqrt{u}} = 2\int_0^\infty \mathrm{e}^{\mathrm{i}v^2}\,\mathrm{d}v = \sqrt{\pi}\mathrm{e}^{\mathrm{i}\pi/4}\,.$$

Problem 6.19. Solve the Schrödinger equation

$$\mathrm{i}\psi_t = \psi_{xx}\,, \qquad \psi(x,0) = \mathrm{e}^{-m^2x^2/2}\,. \tag{6.156}$$

Problem 6.20. The boundary value problem

$$Q'' - y^2 Q = y, \qquad Q(0) = 0, \qquad Q(\pm\infty) = 0\,, \tag{6.157}$$

occurs in equatorial oceanography (the Yoshida jet). *(i)* Calculate a few terms in the expansion of $Q(y)$ around $y = 0$ and $y = \pm\infty$. This should convince you that $Q'(0)$ is an unknown constant which must be determined so that $Q(\pm\infty) = 0$. *(ii)* Fourier transform the ODE and solve the transformed equation in terms of the modified Bessel function $K_{1/4}$. Use the resulting integral representation of $Q(y)$ to show that $Q'(0) = -\sqrt{\pi}\Gamma(3/4)/\Gamma(1/4)$.

Useful information:

$$\int_0^\infty x^p K_{1/4}(x)\,dx = 2^{p-1}\Gamma\left(\tfrac{4p+3}{8}\right)\Gamma\left(\tfrac{4p+5}{8}\right)$$

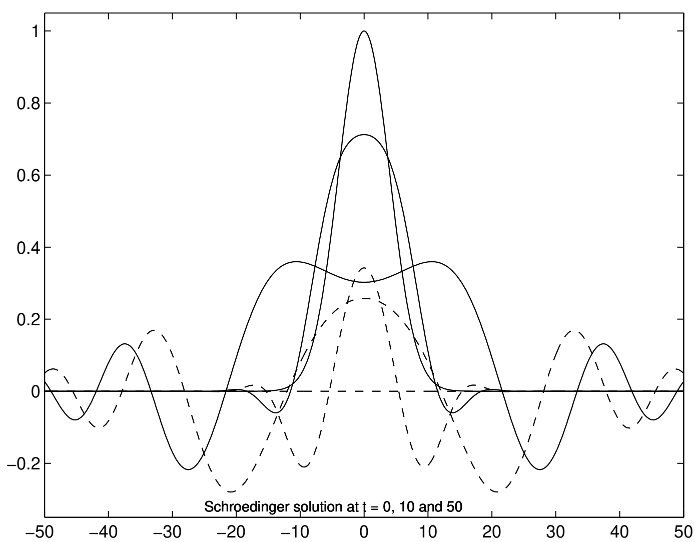


Figure 6.3: Solution of problem 6.19 with $m = 1/4$; the solid curves show $\Re\psi$ and the dashed curves show $\Im\psi$. `schroedingerSoln.eps`

Lecture 7

The 1D wave equation: $\zeta_{tt} = c^2 \zeta_{xx}$

7.1 Waves on a string

I'll derive the equations of motion of a stretched string. A string is a flexible one-dimensional continuum. Flexible means that the string can be bent with no effort — the string has no rigidity.

We assume that in the rest state the string lies along the x-axis. The simplest case is a string stretched between two supports, one at $(x, z) = (0, 0)$ and the other at $(x, z) = (\ell, 0)$. The supports pull on the sting with a force, or *tension*, T_0. There is no gravity so the string does not droop. This is the reference configuration, or rest state, of the string.

Now imagine that the reference configuration is disturbed e.g., by plucking the string, or perhaps switching on gravity. The configuration of the moving string is a curve in the (x, z) plane

$$\boldsymbol{x} = \xi(s, t)\hat{\boldsymbol{x}} + \zeta(s, t)\hat{\boldsymbol{z}} \,. \tag{7.1}$$

In (7.1), s is the x-coordinate of a material element in the rest state; we are using a Lagrangian formulation in which the position $\boldsymbol{x}(s, t)$ of a material element is the main independent variable. We suppose that the string has a variable mass density $\varrho(s)$ (dimensions kilograms per meter). This means that the mass of the string between s_1 and s_2 is

$$\int_{s_1}^{s_2} \varrho(s)\, \mathrm{d}s \,. \tag{7.2}$$

Because the string offers no resistance to bending, the force exerted by one element of string on its neighbour is *tangent* to the string. Thus

$$\text{force} = T\boldsymbol{t} \tag{7.3}$$

where $T(s, t)$ is the *tension* (dimensions are Newtons) and

$$\boldsymbol{t} = \frac{\xi_s\, \hat{\boldsymbol{x}}}{\sqrt{\xi_s^2 + \zeta_s^2}} + \frac{\zeta_s\, \hat{\boldsymbol{z}}}{\sqrt{\xi_s^2 + \zeta_s^2}} \tag{7.4}$$

is the unit tangent to string.

Next, consider the portion of string between s_1 and s_2. The total force acting in the x-direction is

$$\left.\frac{T\xi_s}{\sqrt{\xi_s^2+\zeta_s^2}}\right|_{@s_2} - \left.\frac{T\xi_s}{\sqrt{\xi_s^2+\zeta_s^2}}\right|_{@s_1} \tag{7.5}$$

Thus the x–component of Newton's law for the portion of string between s_1 and s_2 is

$$\partial_t^2 \int_{s_1}^{s_2} \varrho(s)\xi \,\mathrm{d}s = \left.\frac{T\xi_s}{\sqrt{\xi_s^2+\zeta_s^2}}\right|_{@s_2} - \left.\frac{T\xi_s}{\sqrt{\xi_s^2+\zeta_s^2}}\right|_{@s_1} . \tag{7.6}$$

The force balance above is valid for all s_1 and s_2 and letting $s_1 \to s_2$, with s sandwiched in the middle, we have

$$\varrho\xi_{tt} = \partial_s \left(\frac{T\xi_s}{\sqrt{\xi_s^2+\zeta_s^2}} \right) . \tag{7.7}$$

Likewise the z-component of Newton's law results in

$$\varrho\zeta_{tt} = \partial_s \left(\frac{T\zeta_s}{\sqrt{\xi_s^2+\zeta_s^2}} \right) + \varrho f . \tag{7.8}$$

where ϱf is an external vertical force e.g., gravity is $f(s,t) = -g$ where g is a constant. f might also model air resistance — in that case we should also include an external force in the x-component.

We assume that that the tension is related to the local stretching

$$e(s,t) = \sqrt{\xi_s^2+\zeta_s^2} - 1 . \tag{7.9}$$

In other words

$$T = \mathcal{T}(e,s) . \tag{7.10}$$

The function $\mathcal{T}$ describes the elastic properties of the string. With $f = 0$ we have a simple equilibrium solution $(x,z,e) = (s,0,0)$. It follows from (7.7) and (7.8) that in this case the tension must be constant:

$$T_0 = \mathcal{T}(0,s) . \tag{7.11}$$

We have now almost completed the formulation of the problem: (7.7) and (7.8) are coupled PDEs for the displacement. We still need to adopt a model for $\mathcal{T}$ and state initial and boundary conditions. According to Yong, there are at least three "constitutive models" for $\mathcal{T}$:

A perfectly elastic material, such as rubber for which

$$\mathcal{T} = E\sqrt{\xi_s^2+\zeta_s^2} ; \tag{7.12}$$

Linear elasticity:

$$\mathcal{T} = k\left(\sqrt{\xi_s^2 + \zeta_s^2} - 1\right) , \tag{7.13}$$

where k is a "spring constant";

An inextensible string, such as a chain or perhaps piano wire. In this case

$$\sqrt{\xi_s^2 + \zeta_s^2} = 1 , \tag{7.14}$$

and the tension must be determined as part of the problem.

The first case is popular because it immediately makes the wave equations linear and decouples them. But this is probably not realistic.

I'll use the inextensible string model and imagine that the tension is maintained nearly constant and equal to Mg by some contraption involving hanging a large mass M over a pulley.

Linearization

To make further progress we assume that

$$\xi \approx 1 , \qquad \text{and} \qquad \zeta_x \ll 1 . \tag{7.15}$$

In this case, with small transverse displacement, we can use (7.11) and linearize the nonlinear wave equation. The first-order terms are

$$\varrho \zeta_{tt} - T_0 \zeta_{xx} = \varrho f . \tag{7.16}$$

Dividing by ϱ we have

$$\zeta_{tt} - c^2 \zeta_{xx} = f , \tag{7.17}$$

where

$$c = \sqrt{\frac{T_0}{\varrho}} \tag{7.18}$$

is the transverse wave speed. T_0 is the approximately uniform tension.

We should estimate the wave speed c with vaguely musical numbers. Steel has density $\sim 8 \times 10^3$ kg m^{-3}. Suppose the diameter of a steel wire is ~ 0.5mm. The linear density is therefore

$$\varrho = \frac{\pi d^2 \rho}{4} \sim 1.6 \times 10^{-3} \text{ kg m}^{-1}. \tag{7.19}$$

If the tension is ~ 50N then the wave speed is then $c \sim 180$m s^{-1}. These numbers are very rough.

The shallow-water equations: $c = \sqrt{gH}$

As another example of the linear wave equation we consider the shallow water equations.....

7.2 Energy conservation

Now we discuss the energetics of a vibrating string. We also generalize (7.16) by including forcing and dissipation:

$$\varrho\zeta_{tt} + \epsilon\varrho\zeta_t - T_0\zeta_{xx} = \varrho f\,. \tag{7.20}$$

To obtain the energy equation, multiply (7.20) by ζ_t and rearrange it as

$$\mathcal{E}_t + \mathcal{J}_x = -\epsilon\varrho\zeta_t^2 + \varrho f\zeta_t\,, \tag{7.21}$$

where the energy density $\mathcal{E}$ and the energy flux $\mathcal{J}$ are

$$\mathcal{E} \stackrel{\text{def}}{=} \tfrac{1}{2}\varrho\left[\zeta_t^2 + c^2\zeta_x^2\right], \qquad \mathcal{J} \stackrel{\text{def}}{=} -T_0\zeta_t\zeta_x\,. \tag{7.22}$$

To get the total energy,

$$E \stackrel{\text{def}}{=} \tfrac{1}{2}\int_0^L \varrho\left(\zeta_t^2 + c^2\zeta_x^2\right)\,\mathrm{d}x\,, \tag{7.23}$$

we integrate (7.22) over the length of the string, from $x = 0$ to $x = \ell$. If the string is clamped at these end-points then $\mathcal{J}$ is zero and

Clamped BCs: $\zeta(0,t) = \zeta(\ell,t) = 0$

$$E_t = \int_0^\ell \varrho\left(f\zeta_t - \epsilon\zeta_t^2\right)\,\mathrm{d}x\,. \tag{7.24}$$

With no forcing and no dissipation the right hand side of (7.24) is zero and energy is conserved.

Exercise: Find the energy conservation equation for the Klein-Gordon equation

$$\zeta_{tt} - c^2\zeta_{xx} + \sigma^2\zeta = 0\,. \tag{7.25}$$

Also known as the elastically braced spring.

Exercise: Show that with no forcing and dissipation ($f = \epsilon = 0$)

$$\mathcal{J}_t + c^2\mathcal{E}_x = 0\,, \tag{7.26}$$

and therefore

$$\mathcal{E}_{tt} - c^2\mathcal{E}_{xx} = 0\,. \tag{7.27}$$

7.3 The initial value problem on the line

D'Alembert's solution

By inspection there are two very simple solutions of (7.16):

$$\zeta(x,t) = R(x-ct)\,, \qquad \text{and} \qquad \zeta(x,t) = L(x+ct)\,. \tag{7.28}$$

Here R and L are arbitrary functions. $R(x-ct)$ is a right moving wave and $L(x+ct)$ is a left moving wave. Of course (7.16) is linear so that we can use superposition to obtain a solution that is a sum of a left and right moving disturbance

$$\zeta(x,t) = R(x-ct) + L(x+ct). \tag{7.29}$$

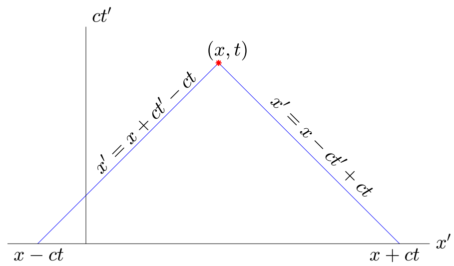


Figure 7.1: Spacetime geometry for the D'Alembert solution. The solution at point (x,t) is determined by initial-value information originating in the interval $x - ct < x' < x + ct$.

Using these solutions we can solve the simplest *initial value problem*. Suppose there is an infinite string and at $t = 0$

$$\zeta(x,0) = \zeta_0(x)\,, \qquad \text{and} \qquad \zeta_t(x,0) = w_0(x)\,. \tag{7.30}$$

Can we find R and L in (7.29) so that at $t = 0$ we satisfy (7.30)? This is straightforward.: we need

$$R + L = \zeta_0\,, \qquad \text{and} \qquad cL' - cR' = w_0\,. \tag{7.31}$$

where the prime denotes a derivative. The second equation is integrated to

$$cL - cR = \int_a^x w_0(x')\,\mathrm{d}x'\,, \tag{7.32}$$

where x_0 is an arbitrary limit of integration. Hence

$$R(x) = \tfrac{1}{2}\zeta_0(x) - \frac{1}{2c}\int_{x_0}^{x} w_0(x')\mathrm{d}x'\,, \tag{7.33}$$

$$L(x) = \tfrac{1}{2}\zeta_0(x) + \frac{1}{2c}\int_{x_0}^{x} w_0(x')\,\mathrm{d}x'\,. \tag{7.34}$$

The constant x_0 cancels when we form the sum $R + L$ to obtain: d'Alembert's solution

$$\zeta(x,t) = \tfrac{1}{2}\zeta_0(x - ct) + \tfrac{1}{2}\zeta_0(x + ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} w_0(x')\,\mathrm{d}x'\,. \tag{7.35}$$

The formula in (7.35) is *d'Alembert's solution* of the wave equation. In applying (7.35) it will help to keep the geometry in figure 7.1 in mind: the solution at (x,t) depends only on the initial condition in the interval $x - ct < x' < x + ct$.

Exercise: Verify that the following functions solve $\zeta_{tt} = \zeta_{xx}$ and write them in the form $R(x - t) + L(x + t)$:

$$\cos x \cos t\,, \qquad \cos\left(\sqrt{7}x\right)\,\sin\left(\sqrt{7}t\right), \qquad x^2 + t^2\,, \qquad t^3 + 3tx^2\,. \tag{7.36}$$

Exercise: Show that if $\zeta = p(x \pm ct)$ then $\mathcal{J} = \pm c\mathcal{E}$.

A string released from rest

As an example of d'Alembert's solution, consider the initial condition

$$\zeta_0 = \mathrm{e}^{-x^2}\,, \qquad \text{and} \qquad w_0(x) = 0\,. \tag{7.37}$$

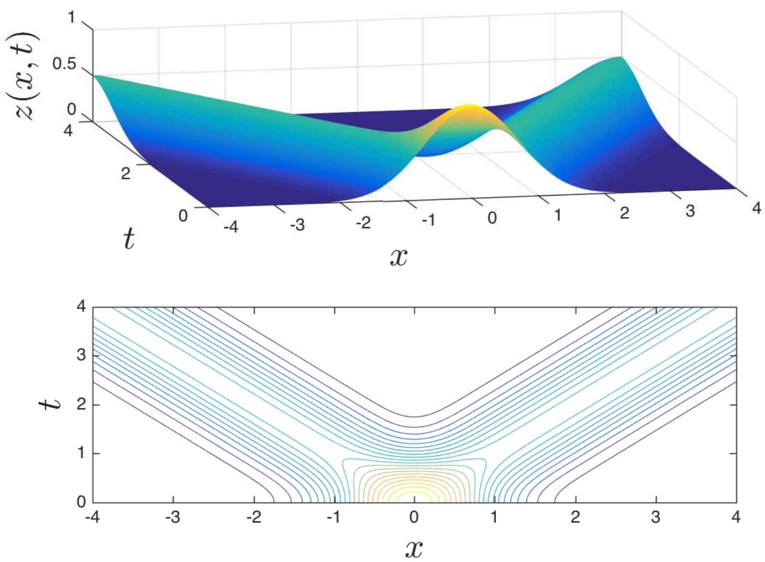


Figure 7.2: The initial Gaussian displacement splits into two Gaussians — a left-going pulse and a right-going pulse. The initial condition is $\zeta_0 = \exp(-x^2)$, $w_0 = 0$ and $c = 1$. Upper panel uses `meshgrid` and the lower panel `contour` to visualize the solution in (7.38).

The initial displacement is a Gaussian and the string is "released from rest" — meaning the initial velocity is zero. The solution of the wave equation with this initial condition is

$$\zeta(x,t) = \tfrac{1}{2}\mathrm{e}^{-(x-ct)^2} + \tfrac{1}{2}\mathrm{e}^{-(x+ct)^2} . \tag{7.38}$$

See figure 7.2.

Another example is the discontinuous initial condition

$$\zeta_0 = \mathrm{sgn}(x) , \qquad \text{and} \qquad w_0 = 0 . \tag{7.39}$$

The D'Alembert solution is equivalent to

$$\zeta(x,t) = \begin{cases} -1 , & \text{if } x < -ct; \\ 0 , & \text{if } -ct < x < ct; \\ +1 , & \text{if } ct < x. \end{cases} \tag{7.40}$$

In contrast to the diffusion equation (recall the erf-solution) the initial discontinuities are preserved.

A string with no initial displacement

With

$$\zeta_0 = 0 , \qquad \text{and} \qquad w_0 = \mathrm{e}^{-x^2} \tag{7.41}$$

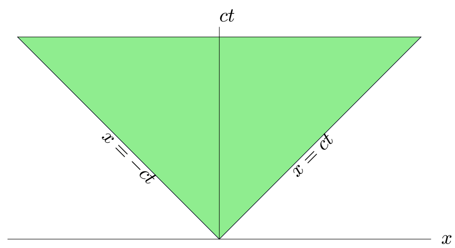


Figure 7.3: Illustration of the Green's function $g(x,t)$ in (7.49). Outside the green region $g(x,t)=0$ and inside $g(x,t) = 1/(2c)$. The green region is the *domain of influence* of a forcing event $\delta(x)\delta(t)$.

the D'Alembert solution is

$$\zeta = \frac{1}{2c}\frac{\sqrt{\pi}}{2}\left[\operatorname{erf}(x+ct) - \operatorname{erf}(x-ct)\right] . \tag{7.42}$$

Wave fronts propagate away from the initial disturbance, leaving a permanent displacement $\zeta = \sqrt{2\pi}/2c$ in their wake.

7.4 The forced string: Duhamel again

7.4.1 The Green's function

How do we solve the wave equation

$$\zeta_{tt} - c^2\zeta_{xx} = f(x,t) , \tag{7.43}$$

driven by an arbitrary distributed force? We assume $\zeta(x, t < 0) = 0$. In other words, when $t < 0$, the force is zero, $f(x, t < 0) = 0$, and the string is motionless. The string is first set into motion at $t = 0$ when the forcing suddenly switches on.

The relevant Green's function is defined by

$$g_{tt} - c^2 g_{xx} = \delta(x)\delta(t) , \tag{7.44}$$

with the initial conditions

$$g(x, 0^-) = 0 , \qquad \text{and} \qquad g_t(x, 0^-) = 0 . \tag{7.45}$$

Above 0^- indicates $t = 0$ minus a little bit. We have $g(x, t < 0^-) = 0$ and then at $t = 0$ there is the impulsive $\delta(x)\delta(t)$-forcing that creates a disturbance.

Once we possess $g(x,t)$ the solution of the forced wave equation (7.43) is

$$\zeta(x,t) = \int_{-\infty}^{\infty}\mathrm{d}x' \int_0^t \mathrm{d}t'\, g(x-x', t-t') f(x', t') . \tag{7.46}$$

Note that the limits in the t'-integration above could be written as $-\infty$ to $+\infty$ because $f(x, t' < 0) = 0$ and $g(x - x', t - t' < 0) = 0$. It is easy to check that (7.46) satisfies (7.43) substitution.

Now to obtain the Green's function we integrate (7.44) from $t = 0^-$ to $t = 0^+$ and see that (7.44) is equivalent to the initial value problem

$$g_{tt} - c^2 g_{xx} = 0 , \tag{7.47}$$

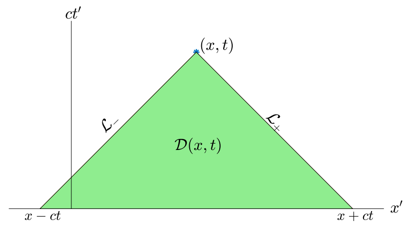


Figure 7.4: The Green's function $g(x-x', t-t')$ in the (x', ct') plane. Outside the green region $g(x-x', t-t') = 0$ and inside $g(x-x', t-t') = 1/(2c)$. The green region, denoted $\mathcal{D}(x,t)$, is the domain of dependence of the field point (x,t).

with the "effective initial conditions":

$$g(x, 0^+) = 0\,, \qquad g_t(x, 0^+) = \delta(x)\,. \tag{7.48}$$

The $\delta(t)$ in the forcing makes g_t discontinuous at $t=0$; but g is continuous. The solution of this initial value problem now follows from D'Alembert's formula:

$$g(x,t) = \frac{1}{2c}\left[\mathrm{H}(x+ct) - H(x-ct)\right]\mathrm{H}(t)\,, \tag{7.49}$$

$$g_t(x,t) = \frac{1}{2}\left[\delta(x-ct) + \delta(x+ct)\right]\,\mathrm{H}(t)\,. \tag{7.50}$$

We've inserted the factor $\mathrm{H}(t)$ above to ensure that the string is at rest when $t<0$. This Green's function is illustrated in figure 7.4. The green region of spacetime, where $g(x,t) = 1/(2c)$, is the *domain of influence* of the impulsive force, $\delta(x)\delta(t)$. Outside the green region, $g(x,t) = 0$ i.e., $\delta(x)\delta(t)$ produces no signal.

Now to apply the formula in (7.46) we have to fold the Green's function $g(x,t)$ in (7.49) into $g(x-x', t-t')$ and present it in the plane integration i.e., the (x', ct') -plane. Thus the domain of of influence in figure 7.4 becomes the *domain of dependence*, denoted $\mathcal{D}(x,t)$ in figure 7.3. The Green's function $g(x-x', t-t')$ can also be written as

$$g(x-x', t-t') = \begin{cases} 1/2c\,, & \text{if } (x',t') \in \mathcal{D}\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{7.51}$$

Thus the integral in (7.46) is equivalent to

$$\begin{aligned}\zeta(x,t) &= \frac{1}{2c}\iint_{\mathcal{D}} f(x',t')\,\mathrm{d}x'\mathrm{d}t'\,,\\ &= \frac{1}{2c}\int_0^t \int_{x-ct+ct'}^{x+ct-ct'} f(x',t')\,\mathrm{d}x'\,\mathrm{d}t'\,. \end{aligned} \tag{7.52}$$

The response at (x,t) is the integral of the forcing over the domain of dependence of (x,t). For similar reasons the forward facing region is the *domain of influence* of (x,t).

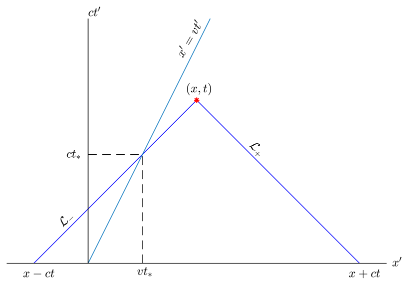


Figure 7.5: Spacetime diagram with $v = c/2$ and $vt < x < ct$. The line $\mathcal{L}_+$ is $x' = x - ct' + ct$ and $\mathcal{L}_-$ is $x' = x + ct' - ct$.

7.4.2 A moving source

As an application of the Green's function, consider the forced wave equation

$$\zeta_{tt} - c^2\zeta_{xx} = q(t)\delta(x - vt)\,, \tag{7.53}$$

with initial conditions $\zeta(x,0) = \zeta_t(x,0) = 0$. We suppose that $0 < v < c$ i.e. the source is moving slower than the wave speed c.

Start by drawing an (x,t)-diagram as in figure 7.5. The domain of dependence of the point (x,t) may or may not contain the trajectory, $x' = vt'$, of the moving source. The easy case is if the trajectory of the forcing does not pass through the domain of dependence $\mathcal{D}(x,t)$:

$$\text{If } x + ct < 0, \text{ or if } x - ct > 0, \text{ then} \qquad \zeta(x,t) = 0\,. \tag{7.54}$$

The more interesting case is when the trajectory of the moving source goes through $\mathcal{D}(x,t)$ and creates a disturbance at (x,t). This divides into two subcases, depending on whether the forcing trajectory strikes the line $\mathcal{L}_-$ or $\mathcal{L}_+$: figure 7.5 shows the case when the forcing trajectory strikes $\mathcal{L}_-$. Then we have to compute the integral

$$\zeta(x,t) = \frac{1}{2c}\int_0^t \mathrm{d}t'\, q(t') \int_{x-ct+ct'}^{x+ct-ct'} \mathrm{d}x'\, \delta(x' - vt')\,, \tag{7.55}$$

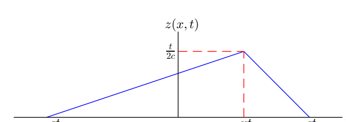
$$= \frac{1}{2c}\int_0^{t_*} q(t')\,\mathrm{d}t'\,. \tag{7.56}$$

In the case shown in figure 7.5 the trajectory $x' = vt'$ intersects $\mathcal{L}_-$ at the point $(x',t') = (vt_*, t_*)$ where

$$t_* = \frac{ct - x}{c - v}\,, \qquad \text{provided that } vt < x < ct. \tag{7.57}$$

The final case occurs if the trajectory $x' = vt'$ intersects $\mathcal{L}_+$ and then in (7.56) the upper limit is

$$t_* = \frac{ct + x}{c + v}\,, \qquad \text{provided that } -ct < x < vt. \tag{7.58}$$



Now let us consider the special case $q(t) = 1$ so that the integral in (7.56) is equal to $t_*(x,t)$. The marginal figure shows a snapshot of the solution

$$\zeta(x,t) = \frac{t_*(x,t)}{2c}\,, \tag{7.59}$$

The triangular pulse propagates with speed v and expands linearly with t i.e., the solution (7.59) can be written in the similarity form

$$\zeta(x,t) = t \times \text{a function of } x/t. \tag{7.60}$$

7.5 The method of images and reflections

What happens when a left moving pulse,

$$\zeta(x,t) = p(x+ct)\,, \tag{7.61}$$

on a semi-infinite string $x > 0$, hits the end at $x = 0$? We suppose that the end of the string is clamped:

$$\zeta(0,t) = 0\,. \tag{7.62}$$

We can construct the solution of this problem using *the method of images*. Here it is

$$\zeta(x,t) = p(x+ct) - p(-x+ct)\,. \tag{7.63}$$

It is easy to verify by substitution that this is a solution of the equation and also by inspection it satisfies the boundary condition at $x = 0$. At large times the image pulse, that starts in the "imaginary" extension of the domain (i.e. $x < 0$), has moved into the real domain and this means that the pulse changes sign on reflection at a fixed end. A picture is worth a thousand words here — see figures 7.6 and 7.7.

Now a thousand words. The problem is posed on $x > 0$ by the PDE

$$\zeta_{tt} = c^2 \zeta_{xx} \tag{7.64}$$

with initial conditions

$$\zeta(x,0) = \zeta_0(x)\,, \qquad \text{and} \qquad \zeta_t(x,0) = w_0(x)\,, \tag{7.65}$$

and the boundary condition

$$\zeta(0,t) = 0\,. \tag{7.66}$$

We extend the initial conditions to whole real line by defining

$$\zeta_0^e(x) \stackrel{\text{def}}{=} \begin{cases} +\zeta_0(x)\,, & \text{if } x > 0; \\ -\zeta_0(-x)\,, & \text{if } x < 0. \end{cases} \tag{7.67}$$

The extended initial velocity, $w_0^e(x)$, is also defined as an odd function of x, equal to $w_0(x)$ if $x > 0$.

The D'Alembert solution of the extended problem is

$$\zeta(x,t) = \tfrac{1}{2}\zeta_0^e(x-ct) + \tfrac{1}{2}\zeta_0^e(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} w_0^e(x')\,\mathrm{d}x'\,. \tag{7.68}$$

This satisfies the wave equation and the initial condition of $x > 0$. To check the $x = 0$ boundary condition, evaluate (7.68) at $x = 0$:

$$\zeta(0,t) = \tfrac{1}{2}\zeta_0^e(-ct) + \tfrac{1}{2}\zeta_0^e(ct) + \frac{1}{2c}\int_{-ct}^{ct} w_0^e(x')\,\mathrm{d}x'\,, \tag{7.69}$$

which is zero because ζ_0^e and w_0^e are odd.

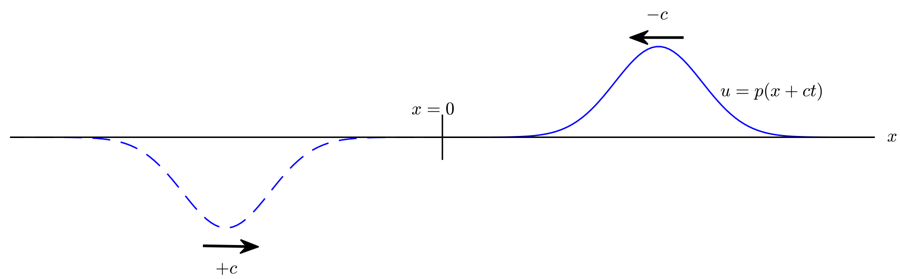


Figure 7.6: Reflection of a Gaussian pulse from a fixed end. The initial condition is a left-moving pulse with profile $p(x+ct)$ and a "reflected" right-moving image pulse. The two pulses linearly superpose so that $z(0,t) = 0$. The pulses pass through each other so the reflected pulse is inverted.

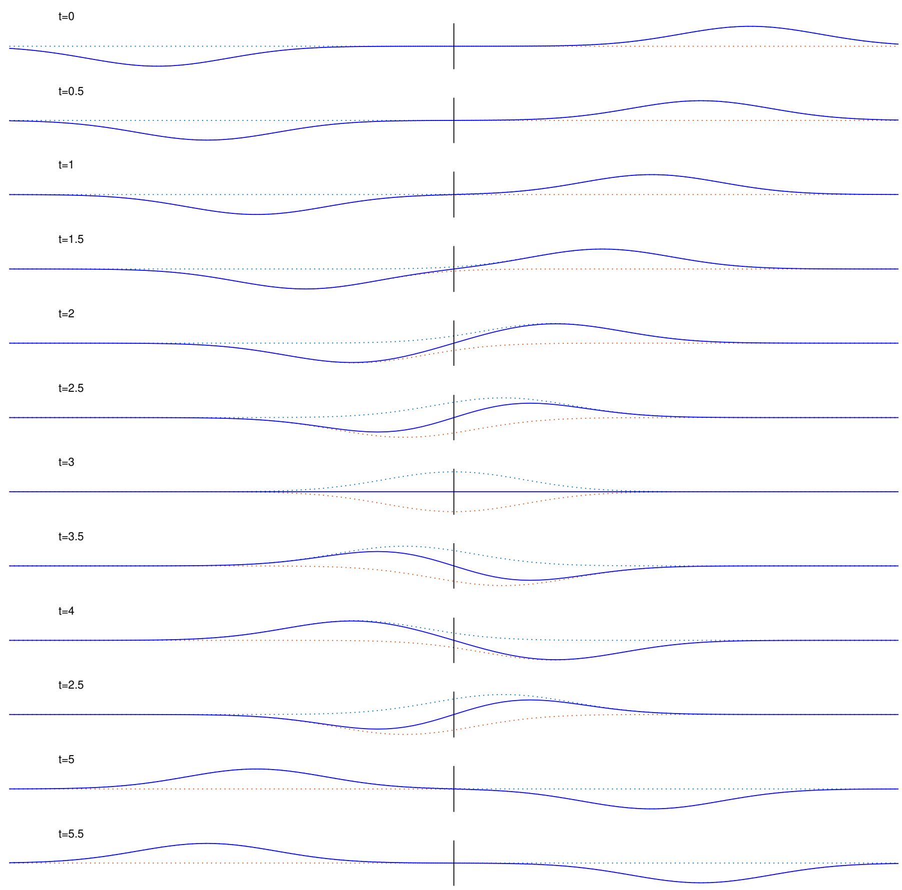


Figure 7.7: Reflection of a Gaussian pulse from a fixed end. The solid curve is the sum of the two Gaussians — the incident pulse and its image. At $t = 3$ the pulses superpose so that $z(x,3) = 0$.

7.6 Some radiation problems

If the initial condition is compact then the solution of

$$\zeta_{tt} - c^2\zeta_{xx} = 0 \tag{7.70}$$

is also compact — the disturbance travels with finite speed c. But when we deal with forced problems,

$$\zeta_{tt} - c^2\zeta_{xx} = f\,, \tag{7.71}$$

where the force $f(x,t)$ has been acting "forever" the waves have had plenty time to reach $x = \pm\infty$. This is true even if $f(x,t)$ is compact in space. In a case like this we don't expect $u(\pm\infty, t) = 0$.

The most important example of this situation is a periodic-in-time, spatially compact force, such as $f = \delta(x)\cos\omega t$. The force was switched on in the distant past and has been radiating waves with frequency ω ever since. If we time-average the energy conservation equation

$$\mathcal{E}_t + \mathcal{J}_x = \varrho\zeta_t f\,, \tag{7.72}$$

we obtain

$$\bar{\mathcal{J}}_x = \varrho\overline{\zeta_t f}\,. \tag{7.73}$$

The overbar denotes the time-average over a period $\tau = 2\pi/\omega$:

$$\bar{\theta}(x) \stackrel{\text{def}}{=} \frac{1}{\tau}\int_{t-\tau}^{t} \theta(x,t)\,\mathrm{d}t\,. \tag{7.74}$$

The average of $\mathcal{E}_t$ is zero because the energy density at a fixed point is a periodic function of time. This is a special case of the general rule

$$\overline{(\text{any stationary function of time})_t} = 0\,. \tag{7.75}$$

As $x \to \pm\infty$, the force $f(x) \to 0$, and the time-averaged energy equation (7.73) reduces to

$$\bar{\mathcal{J}}_x = 0 \qquad \Rightarrow \qquad \bar{\mathcal{J}}(\pm\infty) = \operatorname{sgn}(x)\mathcal{J}_*\,, \tag{7.76}$$

where the constant asymptotic energy flux $\bar{\mathcal{J}}_*$ is positive i.e., energy is strictly radiating away from the source in both directions[1]. This physical condition — the Sommerfeld condition — must be used to resolve mathematical ambiguities in the solution (see next section). The right hand side of (7.73) is the rate of working of the force, and the total rate of working of the source (power) is obtained by integrating (7.73) from $x = -\infty$ to $x = +\infty$:

$$2\mathcal{J}_* \stackrel{\text{def}}{=} \varrho\int_{-\infty}^{\infty} \overline{\zeta_t f}\,\mathrm{d}x\,. \tag{7.77}$$

$\mathcal{J}_*$ is the most important single number characterizing the source $f(x,t)$. Let's calculate $\mathcal{J}_*$.

[1]From the $x \to -x$ symmetry, we anticipate that $\bar{\mathcal{J}}(+\infty) = \bar{\mathcal{J}}(-\infty)$

7.6.1 Radiation from a compact source

A typical radiation problem assumes that

$$f(x,t) = \mathrm{e}^{-\mathrm{i}\omega t} F(x) + \mathrm{e}^{\mathrm{i}\omega t} F^*(x) \,. \tag{7.78}$$

Then we look for solutions of the forced wave equation (7.71) of the form

$$\zeta(x,t) = \mathrm{e}^{-\mathrm{i}\omega t} Z(x) + \mathrm{e}^{\mathrm{i}\omega t} Z^*(x) \,. \tag{7.79}$$

We find that

$$Z'' + \kappa^2 Z = -c^{-2} F \,, \qquad \kappa^2 = \omega^2/c^2 \,. \tag{7.80}$$

Although $Z(x)$ is not growing as $x \to \pm\infty$ it is not decaying either — we expect

$$Z(x) \sim \mathrm{e}^{\pm \mathrm{i}\kappa x} \,, \qquad \text{as} \quad x \to \pm\infty \,. \tag{7.81}$$

This asymptotic behaviour ensures that energy is propagating *away* from the source. For example, suppose $x > 0$. Then in the far-field (a long way from the source) the condition in (7.81) ensures that the solution (7.79) becomes

$$\zeta(x,t) \sim \underbrace{A\mathrm{e}^{\mathrm{i}\kappa x - \mathrm{i}\omega t} + A^* \mathrm{e}^{-\mathrm{i}\kappa x + \mathrm{i}\omega t}}_{\text{a function of } x - ct} \,, \qquad \text{as } x \to +\infty. \tag{7.82}$$

The radiation condition (7.81) ensures that the large-positive-x solution is purely a right going wave.

We can solve (7.80), with the radiation condition (7.81), by finding the Green's function

$$G'' + \kappa^2 G = \delta(x) \,, \qquad \Rightarrow \qquad G(x) = \frac{\mathrm{e}^{\mathrm{i}\kappa |x|}}{2\mathrm{i}\kappa} \,. \tag{7.83}$$

The solution above satisfies the patching conditions at $x = 0$: $G(x)$ is continuous and to balance the $\delta(x)$, the jump in $G_x(x)$ is equal to unity.

Using the Green's function, the solution of the radiation problem in (7.80) is

$$Z(x) = -c^{-2} \int_{-\infty}^{\infty} F(x') \frac{\mathrm{e}^{\mathrm{i}\kappa |x - x'|}}{2\mathrm{i}\kappa} \,\mathrm{d}x' \,, \tag{7.84}$$

$$\to \underbrace{-c^{-2} \int_{-\infty}^{\infty} F(x') \frac{\mathrm{e}^{-\mathrm{i}\kappa x'}}{2\mathrm{i}\kappa} \,\mathrm{d}x'}_{=A} \; \mathrm{e}^{\mathrm{i}\kappa x} \,, \qquad \text{as } x \to +\infty. \tag{7.85}$$

The argument above identifies the asymptotic constant A in (7.82) in terms of the Fourier transform of the source function $F(x)$..

Now that we know the solution in the far-field, we can obtain the large-x energy flux as

$$\mathcal{J}(x = +\infty, t) = -T\zeta_t \zeta_x|_{x=\infty} = \underbrace{2T\omega\kappa A A^*}_{=\mathcal{J}_*} + \text{oscillatory stuff} \,. \tag{7.86}$$

This is only the right-going energy $\mathcal{J}_*$ — there is an equal amount going left. Thus the total rate of working is

Check

$$2\mathcal{J}_* = 4T\frac{\omega^2}{c} A A^* \,. \tag{7.87}$$

$\dim(\mathcal{J}) = \text{Watts}$

$\kappa = \omega/c$

7.6.2 Radiation damping of an oscillator

Consider a semi-infinite string ($x > 0$) coupled to an oscillator at $x = 0$ (see figure 7.8). At $t = 0$ the string is at rest and the oscillator is kicked into motion. what happens? We expect that the oscillator will emit waves and lose energy: this is the simplest model of *radiation damping*. Now let's work out the details.

The displacement of the oscillator away from its equilibrium position is $\zeta(0, t)$. The equation of motion of the oscilator, which is also the $x = 0$ boundary condition for the string, is

$$m\ddot{\zeta}(0,t) + k\zeta(0,t) = T\zeta_x(0,t)\,. \tag{7.88}$$

The RHS is the force of tension tugging on the oscillator.

As a sanity check we now verify that the total mechanical system (string + spring) conserves energy. The energy equation for the oscillator is

$$\frac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\left[m\dot{u}^2 + ku^2\right] = T\zeta_x\zeta_t|_{x=0}\,. \tag{7.89}$$

On the other hand, the wave equation

$$\varrho\zeta_{tt} - T\zeta_{xx} = 0 \tag{7.90}$$

on the half-line $x > 0$ has the energy equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \tfrac{1}{2}(\varrho\zeta_t^2 + T\zeta_x^2)\,\mathrm{d}x = -T\zeta_t\zeta_x|_{x=0}\,. \tag{7.91}$$

Adding (7.88) and (7.91) shows that the total energy is conserved: what the oscillator loses, the string gains.

Now we solve the PDE (7.90) with the initial condition $\zeta(x, 0) = \zeta_t(x, 0) = 0$ and the boundary condition in (7.88). As an initial condition for the oscillator we suppose that

$$u(0,0) = 0\,, \qquad \zeta_t(0,0) = 1\,. \tag{7.92}$$

Thus the string is motionless and the mass at $x = 0$ is kicked into motion.

The problem is not completely posed till we insist that there is no energy impinging on the oscillator from $x = \infty$ i.e., the flow of energy is strictly towards $x = \infty$. This means that the disturbance on the string is travelling only to the right:

$$\zeta(x,t) = U(x - ct) \quad \Rightarrow \quad \zeta_x = -c^{-1}\zeta_t\,. \tag{7.93}$$

Thus the oscillator equation in (7.88) becomes

$$m\ddot{\zeta}(0,t) + Tc^{-1}\zeta_t(0,t) + k\zeta(0,t) = 0\,. \tag{7.94}$$

$c^2 = T/\varrho$

We have managed to get a simple ODE whose solution gives the motion of the end of the string. The key here is the argument in (7.93) that the disturbances on the string go only towards the right.

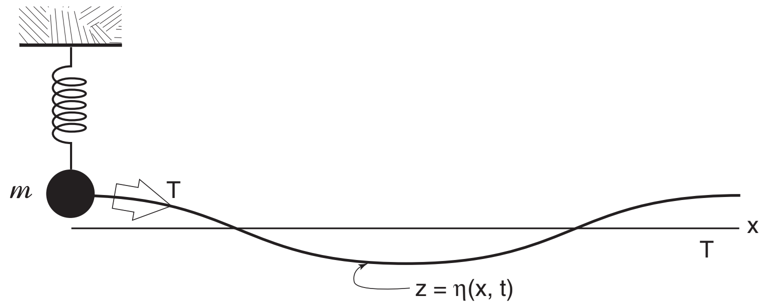


Figure 7.8: `JoFig3.eps`

The solution of the damped oscillator equation (7.94) with the initial condition in (7.92) is

$$\zeta(0,t) = \mathrm{e}^{-\gamma t} \frac{\sin\left[\sqrt{\omega^2-\gamma^2}t\right]}{\sqrt{\omega^2-\gamma^2}} H(t)\,, \tag{7.95}$$

where

$$\omega \stackrel{\text{def}}{=} \sqrt{\frac{k}{m}}\,, \qquad \gamma \stackrel{\text{def}}{=} \frac{T}{2cm}\,. \tag{7.96}$$

7.7 Problems

Problem 7.1. Find the energy conservation equation for the nonlinear wave equation

$$\zeta_{tt} - c^2 \left[\frac{\zeta_x}{\sqrt{1+\zeta_x^2}}\right]_x = 0\,. \tag{7.97}$$

Check that your answer reduces to the linear conservation law if $\zeta_x \ll 1$.

Problem 7.2. Consider the wave equation $\zeta_{tt} = c^2\zeta_{xx}$ in the finite domain $0 < x < \ell$ with boundary conditions $\zeta(0,t) = \zeta(\ell,t) = 0$ and initial conditions with ζ and ζ_t specified. Show that the time-average of the domain-integrated potential energy is equal to the time-average of the domain-integrated kinetic energy. Might this equipartition of energy apply at every point of the domain $0 \le x \le \ell$?

Problem 7.3. *(ii)* Find the general solution of the PDE $x\zeta_{tt} - (x^{-1}\zeta_x)_x = 0$ in terms of "arbitrary functions". *(ii)* Find the energy conservation equation for this wave equation.

Problem 7.4. Solve and visualize the solution of the wave equation $\zeta_{tt} = c^2\zeta_{xx}$ with the initial conditions $\zeta_0(x) = 0$, and $w_0(x) = \mathrm{sgn}(x)$. Draw a graph of ζ as a function of x when $ct = 1$.

Problem 7.5. In the discussion following (7.37) we solved the wave equation using D'Alembert's formula. However following the dimensional reasoning we used in our discussion of the diffusion equation and Laplace's equation one might try the similarity solution

$$\zeta(x,t) = \mathcal{Z}(\eta)\,, \qquad \text{with} \qquad \eta \stackrel{\text{def}}{=} \frac{x}{ct}\,. \tag{7.98}$$

(i) Show that the solution in (7.38) does have the similarity form in (7.98). *(ii)* Find the most general solution of the wave equation having the similarity form in (7.98).

Problem 7.6. *(i)* Solve the *dam-break problem*: $\eta_{tt} = c^2\eta_{xx}$ with initial conditions

$$\eta(x,0) = \mathrm{H}(-x)\,, \qquad \text{and} \qquad \eta_t(x,0) = 0\,. \tag{7.99}$$

(ii) Solve the problem with $\eta(x,0) = 0$ and $\eta_t(x,0) = \mathrm{H}(-x)$.

Problem 7.7. Consider a semi-infinite string initially at rest along the half-line $x > 0$. Solve the initial-boundary value problem $\zeta_{tt} = c^2\zeta_{xx}$ with initial conditions $\zeta(x,0) = 0$ and $\zeta_t(x,0) = 0$, and the boundary condition $\zeta(0,t) = b(t)$. (You'll need to make a physical argument to select a unique acceptable solution from the infinitude of mathematical solutions.)

Problem 7.8. Solve (7.53) in the case $v > c$. Plot a snapshot of the solution if $q(t) = 1$. To check your answer verify that the maximum displacement is $t/(v+c)$ at $x = ct$.

Problem 7.9. *(i)* Solve $\zeta_{tt} = \zeta_{xx} + \sin\omega t\cos x$ with initial conditions $\zeta(x,0) = \zeta_t(x,0) = 0$. Discuss $\omega = 1$. Why is this called *resonant forcing*? *(ii)* Solve $\zeta_{tt} = \zeta_{xx} + \sin\omega t\cos^2 x$ with initial conditions $\zeta(x,0) = \zeta_t(x,0) = 0$. Make sure that you discuss the special value of ω corresponding to resonant forcing.

Problem 7.10. Solve $\zeta_{tt} - \zeta_{xx} = \exp(-x-t)$ with initial conditions $\zeta(x,0) = \zeta_t(x,0) = 0$.

Problem 7.11. Investigate the oscillations observed at a fixed point x if the moving source in (7.53) has $q(t) = \sin\omega t$.

Problem 7.12. *(i)* Show that the PDE

$$\cosh x\, U_{tt} - (\operatorname{sech} x\, U_x)_x = 0\,, \tag{7.100}$$

has an energy conservation law, $\mathcal{E}_t + \mathcal{J}_x = 0$ and find expressions for the energy density $\mathcal{E}$ and flux $\mathcal{J}$ in terms of U_t, U_x, $\cosh x$ etc. *(ii)* Find the general solution of the PDE in terms of two arbitrary functions. *(iii)* Solve the PDE with the initial conditions $U(x,0) = \operatorname{sech} x$ and $U_t(x,0) = 0$. To check your answer, show that $U(0,t) = 1/\sqrt{1+t^2}$.

Problem 7.13. Consider the forced wave equation

$$\zeta_{tt} - \zeta_{xx} = -\,\mathrm{H}(2t - x^2)\,. \tag{7.101}$$

The string is undisturbed at $t = 0$. *(i)* On an (x, t)-diagram sketch the region in which the forcing on the RHS is nonzero. Also indicate the region in which $\zeta(x, t)$ is nonzero. *(ii)* Solve the equation. Check your algebra by showing

$$\zeta(0, t) = \tfrac{1}{3} + t - \tfrac{1}{3}(1 + 2t)^{3/2}\,. \tag{7.102}$$

Problem 7.14. What happens when a left-moving pulse $\zeta(x, t) = f(x+ct)$ on a semi-infinite string $x > 0$, hits the end, $x = 0$, where the boundary condition is $\zeta_x(0, t) = 0$? (This is called a *free* boundary.)

Problem 7.15. Consider the finite $(0 < x < \ell)$ string problem

$$\zeta_{tt} = c^2\zeta_{xx}\,, \qquad \text{with free boundaries } \zeta_x(0, t) = \zeta_x(0, \ell) = 0\,, \tag{7.103}$$

and the initial condition $\zeta(0, x) = \zeta_0(x)$ and $\zeta_t(x, 0) = 0$. Solve the problem by extending the initial condition to whole real line. Discuss the special case $\zeta_0(x, 0) = 1$.

Problem 7.16. Find the Green's function of the simple harmonic oscillator equation

$$\ddot{\theta} + \sigma^2\theta = f(t)\,.$$

Compare this calculation with that in (7.83) and carefully explain why the two Green's function's are not identical even though they satisfy the same ODE.

Problem 7.17. Consider radiation from a compact source on to an "elastically braced" string

$$\zeta_{tt} - c^2\zeta_{xx} + \sigma^2\zeta = F(x)\mathrm{e}^{-\mathrm{i}\omega t} + F^*(x)\mathrm{e}^{\mathrm{i}\omega t}\,.$$

This is the forced Klein-Gordon equation. Find an expression for the radiated energy in terms of $F(x)$.

Problem 7.18. Work through all the calculations in this lecture again, but this time assume that the string is dissipative:

$$\zeta_{tt} + 2\nu\zeta_t - c^2\zeta_{xx} = F(x)\mathrm{e}^{-\mathrm{i}\omega t} + F^*(x)\mathrm{e}^{\mathrm{i}\omega t}\,. \tag{7.104}$$

Now we expect the disturbance to decay at $x = \pm\infty$, and we should not have to make arguments about the direction of energy flux to resolve mathematical ambiguities. Show that this is the case, and that you recover the solution in the lecture by taking $\nu \to 0$.

Problem 7.19. Go back to (7.78), and let $\omega \to \omega + \mathrm{i}\gamma$ where γ is real, positive and very small. In this case we can consider that the forcing has switched on a long time in the past and has been very slowly growing exponentially as $\mathrm{e}^{\gamma t}$. With small but nonzero γ we can require that the solution will be small at $x = \pm\infty$. Solve the problem with non-zero γ and show that you recover the solution in the lecture in the $\gamma \to 0$ limit.

Problem 7.20. Solve the initial value problem

$$\zeta_{tt} - c^2\zeta_{xx} = 0\,, \qquad \zeta(x,0) = \zeta_t(x,0) = 0\,, \quad \zeta(0,t) = \cos(\omega t)\,.$$

The string is initially at rest and then the end at $x = 0$ is set into motion by shaking the end. Hint: draw the x-t diagram and carefully distinguish between characteristics originating at $x = 0$ and those at $t = 0$.

Problem 7.21. At a fixed time t sketch the displacement, u as a function of x, produced by the boundary condition in (7.95).

Problem 7.22. Consider a mass-spring system attached to a string at $x = 0$; the mass is m, the spring constant is K and the oscillator frequency is $\sigma = \sqrt{m/K}$. A steady wave with frequency ω is incident from the left ($x = -\infty$). The energy is partly reflected back to $x = -\infty$ and partly transmitted to $x = +\infty$. Find the reflection and transmission coefficients. Check your answer by showing that you recover the result in lectures if $K \to 0$. You should also find that there is no reflected wave if $\omega = \sigma$.

Problem 7.23. Consider an "elastically braced" string attached to an oscillator. The problem is the same as section 7.6.2 except that the PDE is

$$\zeta_{tt} - c^2\zeta_{xx} + \sigma_0^2\zeta = 0\,.$$

Find the damping rate of the oscillator.

Lecture 8

Nonlinear steepening and shocks

8.1 The nonlinear advection equation

We turn now to a detailed study of an important nonlinear PDE. This is the nonlinear advection equation

$$u_t + u u_x = 0\,. \tag{8.1}$$

This PDE can equivalently be written in the form of a conservation law

$$u_t + \left(\tfrac{1}{2}u^2\right) = 0\,. \tag{8.2}$$

In this form the density is u, and the flux $u^2/2$.

Comparing the constant c linear advection equation

$$u_t + c u_x = 0\,, \tag{8.3}$$

with nonlinear advection equation (8.1) we see that in the nonlinear case the speed of the disturbance is $c = u$. Thus, loosely speaking, we anticipate that in the nonlinear case bigger disturbances will travel more rapidly.

We use the method of characteristics to solve (8.1) with the initial condition that

$$u(x, 0) = F(x)\,, \tag{8.4}$$

where $-\infty < x < \infty$. My discussion follows Whitham, though I am solely responsible for the myrmecophilia.

Imagine an ant moving along some curve $x = x(t)$ in the (x, t)-plane. The solution of (8.1), $u(x, t)$, can be visualized as a surface lying above that plane. The value of $u(x, t)$ observed by the moving ant is obtained from the "total derivative" of $u(x, t)$, namely

Also know as the "advective derivative"

$$\frac{\mathrm{d}}{\mathrm{d}t} u\left(x(t), t\right) = u_t\left(x(t), t\right) + \frac{\mathrm{d}x}{\mathrm{d}t} u_x\left(x(t), t\right)\,. \tag{8.5}$$

We now consider a mathematically inclined ant, starting at $x(0) = \xi$, who adjusts her trajectory so that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u(x,t)\,, \quad x(0) = \xi\,. \tag{8.6}$$

Because $u(x,t)$ satisfies the advection equation (8.1) this ant observes that

On characteristics in (8.6), PDE (8.1) reduces to ODE (8.7).

$$\frac{\mathrm{d}}{\mathrm{d}t} u\left(x(t), t\right) = 0\,. \tag{8.7}$$

In other words, on a curve determined by (8.6), $u(x,t)$ is constant. In fact, since u satisfies the initial condition (8.4), the constant value of u is just

$$u(x,t) = F(\xi)\,. \tag{8.8}$$

In other words, the initial value of u is $F(\xi)$ and if ant picks her trajectory according to (8.6) then u doesn't change.

Now that we realize u is constant on the trajectory, it is trivial to determine that trajectory by integrating (8.6):

$$x = \xi + ut\,. \tag{8.9}$$

Finally, eliminate $\xi = x - ut$ between (8.8) and (8.9) to get u as a function of x and t. This gives

$$u = F(x - ut)\,. \tag{8.10}$$

Given F we can, in principle, solve the equation above for $u(x,t)$ (examples follow). After sorting out some notational distractions, you can also obtain this solution using the quasilinear recipe earlier from an earlier lecture.

It is instructive to check by substitution that (8.10) solves the PDE (8.1). Taking an x-derivative of (8.10) we get:

$$u_x = (1 - u_x t)F'(x - ut)\,, \quad \Rightarrow \quad u_x = \frac{F'(\xi)}{1 + tF'(\xi)}\,. \tag{8.11}$$

And the t-derivative of (8.10) is

$$u_t = -(tu)_t F'(x - ut)\,, \quad \Rightarrow \quad u_t = -\frac{uF'(\xi)}{1 + tF'(\xi)} = -uu_x\,. \tag{8.12}$$

Example: Solve the initial value problem $u_t + uu_x = t$ with the initial condition $u(x,0) = x$. In this case the characteristic equations are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u\,, \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = t\,. \tag{8.13}$$

We solve the second equation first

$$u = \tfrac{1}{2}t^2 + A(\xi)\,, \tag{8.14}$$

where the constant of integration A is a function of the characteristic coordinate ξ. We define ξ to be the value of x at which the characteristic curve intersects the x

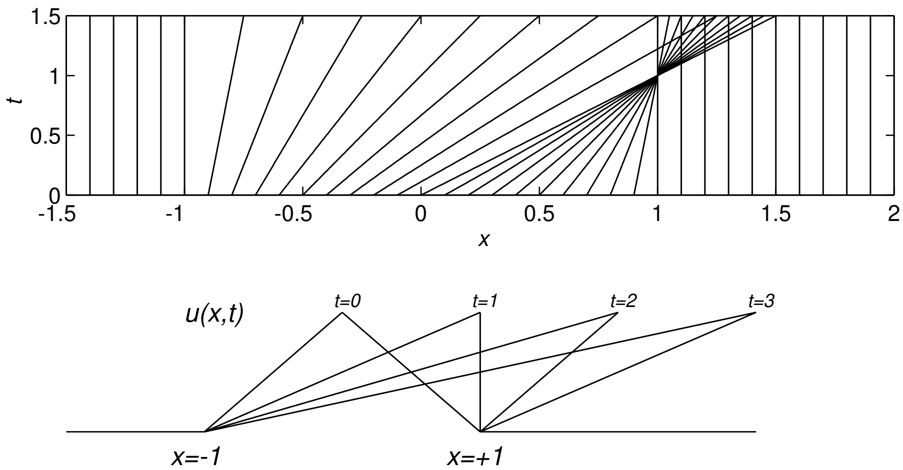


Figure 8.1: The top panel shows the characteristics with the piecewise linear initial condition, F in (8.20). The bottom panel shows the steepening wave. The characteristics first cross, and the solution becomes multivalued, at $(x,t) = (1,1)$. `kink.eps`

axis. Although we don't yet know the characteristic curves, we can determine A by applying the initial condition to (8.14):

$$u = \tfrac{1}{2}t^2 + \xi\,. \tag{8.15}$$

Turning to the first equation in (8.14) we now have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \tfrac{1}{2}t^2 + \xi\,, \tag{8.16}$$

which integrates to

$$x = \xi(1+t) + \tfrac{1}{6}t^3\,. \tag{8.17}$$

Notice that in (8.17) we have applied the initial condition that $x = \xi$ at $t = 0$. In this problem we can easily invert (8.17) to obtain a nice expression for ξ:

$$\xi = \frac{x - \frac{1}{6}t^3}{1+t}\,. \tag{8.18}$$

Substituting (8.18) into (8.15) we obtain the explicit solution

$$u = \frac{x - \frac{1}{6}t^3}{1+t} + \tfrac{1}{2}t^2\,. \tag{8.19}$$

A kinky initial condition

Now consider the implicit solution in (8.10) with the initial condition

$$u = \begin{cases} 1 - |x|\,, & \text{if } |x| < 1, \\ 0\,, & \text{if } |x| > 1,. \end{cases} \tag{8.20}$$

The solution is shown in figure 8.1. Big values of u overtake the small values of u; this is *nonlinear steepening.*

To obtain the solution we can deal with the algebraic result in (8.10) or we can use the geometric construction:

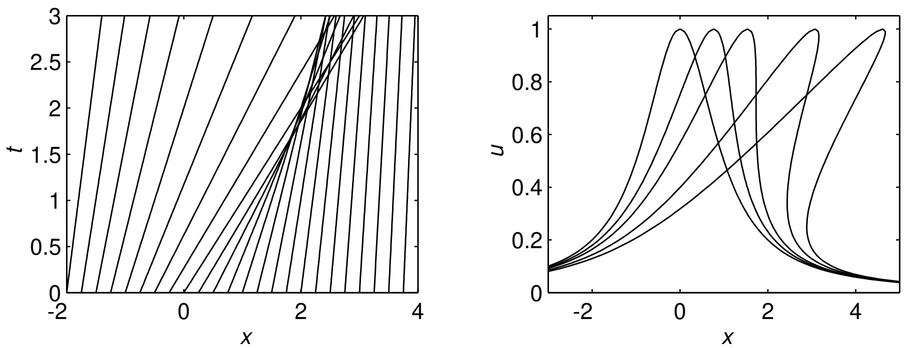


Figure 8.2: The left panel shows the characteristics obtained from (8.22). The right panel shows the steepening wave, calculated using (8.24). The plots are at $t = [0\ 1/2\ 1\ 2\ 3]t_{\mathrm{s}}$, where $t_{\mathrm{s}} = 8\sqrt{3}/9$ is the shock time. `advecEqn.eps`

- Draw the graph of the initial condition, $u = F(x)$, in the (x, u)-plane;
- Take each point on the initial curve and slide it sideways a distance $F(x)t$;
- the resulting curve is the graph of the solution $u(x, t)$.

As you see in figure 8.1, this construction gives rise to a multivalued solution when $t > 1$. There is a region of the (x, t)-plane in which there are three values of u.

(8.10): $u = F(x - ut)$

Exercise: Solve (8.10) with F in (8.20) and so find simple expressions for the moving line segments in the lower panel of figure 8.1.

A witchy initial condition

As an another example, suppose that we want to solve (8.1) with the initial condition

$$F(x) = \frac{1}{1 + x^2}. \tag{8.21}$$

This initial condition is the famous Witch of Agnesi (also known as the Lorentzian). In this case, from (8.9), the characteristics comprise the family of lines

$$x = \xi + \frac{t}{1 + \xi^2}. \tag{8.22}$$

Using MATLAB we visualize this family by specifying ξ in (8.22) and then plotting x versus t (see figure 8.2). We see that characteristics intersect, so that there are some points in the (x, t)-plane at which there are three values of u.

Maria Gaetana Agnesi (1718-1799) was an Italian mathematician and philosopher. She was the first woman to be appointed as a mathematics professor at a university — specifically the University of Bologna. "Witch" is either a mistranslation or a pun on Agnesi's name for this curve *averisera*, meaning "versed sine curve".

To determine $u(x,t)$ we must now eliminate ξ from (8.8) and (8.9):

$$u = \frac{1}{1+\xi^2}\,, \quad x = \xi + ut \quad \Rightarrow \quad u = \frac{1}{1+(x-ut)^2}\,. \tag{8.23}$$

Equation (8.23) is a cubic which defines u as an implicit function of x and t. We can visualize the solution without solving this cubic by specifying u and t then solving (8.23) for x in terms of u and t:

$$x_{\pm} = ut \pm \sqrt{\frac{1-u}{u}}\,. \tag{8.24}$$

In figure 8.2 we show u as a function of x at fixed times calculated from (8.24) and plotted with MATLAB. The shock first rears its ugly head at $t_s = 8\sqrt{3}/9 \approx 1.54$ when there is a point on the forward face of the pulse at which the slope u_x is infinite (and negative).

```
% the witch figure
close all
clc

%%%%%% The characteristic diagram %%%%%%
xo=[-2:0.25:4];
t=linspace(0,3);
for xx=xo
    subplot(2,2,1)
    plot(xx+t./(1+xx.^2),t);
    hold on
end
axis([-2 4 0 3])
xlabel('{\it x}')
ylabel('{\it t}')
%%%%%%%% now the right panel %%%%%%%%%
ts=8*sqrt(3)/9
u=linspace(0+eps,1);
hold on
for t=[0 ts/2 ts 2*ts 3*ts]
    xp=u*t+sqrt((1-u)./u);
    xm=u*t-sqrt((1-u)./u);
    subplot(2,2,2)
    plot(xm,u,xp,u)
    hold on
    axis([-3 5 0 1.05])
end
xlabel('{\it x}')
ylabel('{\it u}')
```

8.2 Shocks, caustics and multivalued solutions

How did we determine the shock time $t_s = 8\sqrt{3}/9$ in the example of figure 8.2? When $t < t_s$, $u(x,t)$ is a single-valued function of x and the derivative u_x is finite everywhere. But because of nonlinear steepening, u_x becomes infinitely negative at a finite time, known as the shock-time and denoted t_s. When $t > t_s$ the solution is multivalued and there are two locations at which $u_x = \infty$ — see the right hand panel of figure 8.2.

To determine t_s we could go back to our earlier expression for u_x in (8.11) and figure out when and where the infinity first appears. However there is an alternative route which lets us admire some different scenery. Let $v(x,t) \stackrel{\text{def}}{=} u_x(x,t)$. Differentiating the advection equation (8.1) we have

$$v_t + v^2 + u v_x = 0\,. \tag{8.25}$$

Now we notice that (8.25) evaluated on a characteristic curve is

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -v^2\,, \quad \Rightarrow \quad v(\xi,t) = \frac{F'(\xi)}{1+F'(\xi)t}\,. \tag{8.26}$$

We see that if $F'(\xi) < 0$ then $v = u_x$ will become infinite in a finite time. The singularity first forms on the characteristic ξ_s which originates at the point of most-negative slope and so

$$t_{\rm s} = \min_{\forall\xi} \left[-1/F'(\xi)\right] = -1/F'(\xi_{\rm s})\,. \tag{8.27}$$

I like this alternative route because it is obvious from the ODE (8.26) that the slope $v = u_x$ is monotonically decreasing as we move along each characteristic. Also one can calculate $t_{\rm s}$ *before* solving the PDE.

After the shock forms $u(x,t)$ is a multivalued function: when $t > t_{\rm s}$ there are three values of u at a single location x. The function $u(x,t)$, viewed as a surface above the (x,t) plane, is folded and the shock location $(x_{\rm s}, t_{\rm s})$ is the "point" of the fold. There are two creases (or caustics) which originate at $(x_{\rm s}, t_{\rm s})$. The caustic curves, $x = x_{\rm c}(t)$, are located by the condition that $u_x = \infty$, or from (8.11)

$$1 + tF'(\xi_{\rm c}) = 0\,. \tag{8.28}$$

We have to solve the equation above, together with $x_{\rm c} = \xi_{\rm c} + F(\xi_{\rm c})t$, to determine the caustic location (example below).

In many cases the appearance of a multivalued solution at $(x_{\rm s}, t_{\rm s})$ indicates that the PDE model is failing. For example, if $u(x,t)$ is traffic density it doesn't make sense that there are three different values of u at the same location. In later lectures we discuss how the "solution" of a PDE can be extended using physical arguments to describe evolution once $t > t_{\rm s}$.

The witch again

In the witchy example:

$$F = \frac{1}{1+\xi^2}\,, \qquad F'(\xi) = -2\xi F^2\,, \qquad F''(\xi) = 2F^2\left(4\xi^2 F - 1\right)\,. \tag{8.29}$$

The most negative value of F' is at $\xi_{\rm s}$, where $F''(\xi_{\rm s}) = 0$; a short calculation using F'' in (8.29) then gives $\xi_{\rm s} = 1/\sqrt{3}$. Thus

$$F'(\xi_{\rm s}) = -3\sqrt{3}/8\,, \qquad \text{and from (8.27):} \qquad t_{\rm s} = 8\sqrt{3}/9\,. \tag{8.30}$$

Exercise: Find the location, $x_{\rm s}$, at which the shock forms in figure 8.3.

Let's complete this witch example by finding the location of the caustics. We have to find the curves in the (x,t) plane along which $u_x = \infty$. In the case of the witch, this leads to the system

$$x_{\rm c} = \xi_{\rm c} + \frac{t}{1+\xi_{\rm c}^2}\,, \qquad \frac{2\xi_{\rm c}}{(1+\xi_{\rm c}^2)^2} = \frac{1}{t}\,. \tag{8.31}$$

The second equation is the condition that $u_x = \infty$, or equivalently $F'(\xi_{\rm c}) = -1/t$. We must eliminate $\xi_{\rm c}$ and find $x_{\rm c}$ as a function of t. This can't be done algebraically. But, once again, it is easy to specify $\xi_{\rm c}$, then find t and finally obtain $x_{\rm c}$.

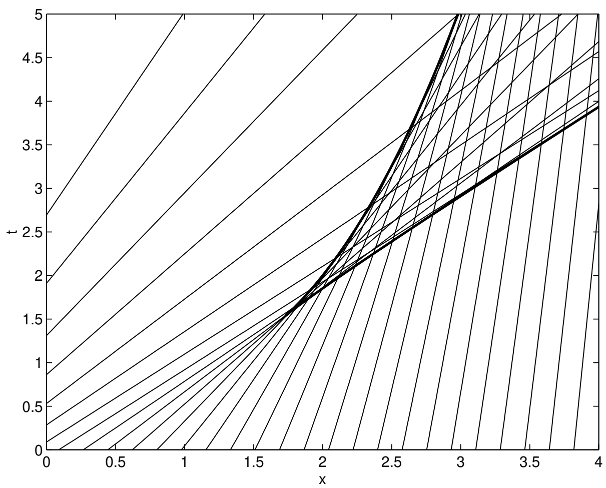


Figure 8.3: The caustic curves of the witch problem. Inside the caustics the characteristics cross and there are three values of u at each (x, t). `caust.eps`

8.3 Shock propagation

Superiority of the integral form of the conservation law

What do multivalued solutions mean? In physical problems the multi-valuedness usually signals the failure of the continuum approximation. In an earlier lecture we started with the integral conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \rho(x,t)\,\mathrm{d}x + f(b,t) - f(a,t) = 0\,, \tag{8.32}$$

and then we took the limit $a \to b$. We *assumed* that ρ is continuous and f differentiable. These assumptions are essential in obtaining the differential statement of the conservation law:

$$\rho_t + f_x = 0\,. \tag{8.33}$$

The transition from (8.32) to (8.33) is *not* possible once the solution becomes nonsmooth e.g., by forming a shock. On physical grounds, after the shock forms, we expect that further evolution involves the dynamics of discontinuities called *shock waves.*

To determine the motion of a shock wave we insist that the integral form of the conservation law is valid at all time. We also assume that if $\rho(x,t)$ is discontinuous at $x = s(t)$ then there are still well defined left and right limits, $\rho(s_+, t)$ and $\rho(s_-, t)$ — see figure 8.4. To determine $s(t)$ we break-up (8.32) like this:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_a^s \rho(x,t)\,\mathrm{d}x + \int_s^b \rho(x,t)\,\mathrm{d}x\right] + f(b,t) - f(a,t) = 0\,. \tag{8.34}$$

Then, recalling Leibnitz's rule for differentiating integrals, we have

$\frac{\mathrm{d}}{\mathrm{d}t}\int_{p(t)}^{q(t)} g(x,t)\,\mathrm{d}x = ?$

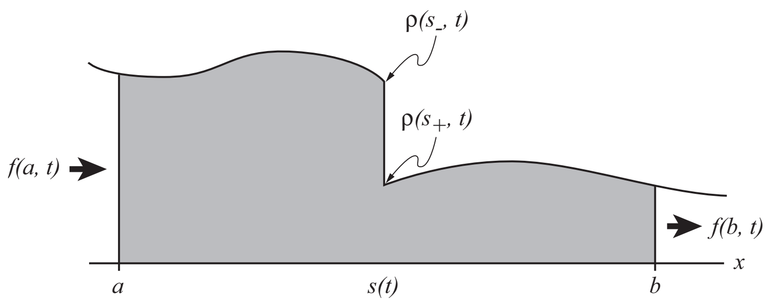


Figure 8.4: Derivation of the shock condition in (8.36). `shockCond.eps`

$$\int_a^s \rho_t(x,t)\,\mathrm{d}x+\int_s^b \rho_t(x,t)\,\mathrm{d}x+\dot{s}\,[\rho(s_-,t)-\rho(s_+,t)]+f(b,t)-f(a,t)=0\,. \tag{8.35}$$

Now we let a and b approach $s(t)$. The first two terms in (8.35) vanish because ρ is bounded. Therefore we get

$[\theta] \stackrel{\text{def}}{=} \theta(s_+)-\theta(s_-)$

$$-\dot{s}[\rho]+[f]=0\,,\qquad \text{or}\qquad \boxed{\dot{s}=\frac{[f]}{[\rho]}}\,. \tag{8.36}$$

Equation (8.36) is called the *shock condition*; it relates the speed of a shock to the conditions immediately behind and in front of the shock. The jump in ρ, $[\rho]$, is the *strength* of the shock and the spacetime curve $x=s(t)$ is the *shock path.*

Two examples of shock propagation

Consider the nonlinear advection equation

$$u_t+uu_x=0\,,\qquad \text{with initial condition}\quad u(x,0)=H(-x)\,. \tag{8.37}$$

The shock is already present in the initial condition at $x=0$. Since $f=u^2/2$ the shock speed is

$u_\pm \stackrel{\text{def}}{=} u(s_\pm,t)$

$$\dot{s}=\frac{\frac{1}{2}u_+^2-\frac{1}{2}u_-^2}{u_+-u_-}=\tfrac{1}{2}(u_++u_-)\,. \tag{8.38}$$

The initial condition has $u_+=0$ and $u_-=1$ so the shock speed is $\dot{s}=1/2$ and the shock path is $s=t/2$. The characteristic diagram is shown in figure 8.5 Notice particularly that *characteristics flow into the shock from both sides.* This is an essential feature of shock waves: shocks swallow characterisitics and prevent the solution becoming multivalued.

Now consider a slightly different nonlinear advection equation

$$v_t+v^2v_x=0\,,\qquad \text{with initial condition}\qquad v(x,0)=H(-x)\,. \tag{8.39}$$

Again, the shock forms instantly because the initial condition is discontinuous. In this case, with $f=v^3/3$, the shock speed is

$$\dot{s}=\frac{\frac{1}{3}v_+^3-\frac{1}{3}v_-^3}{v_+-v_-}=\tfrac{1}{3}(v_+^2+v_+v_-+v_-^2)\,. \tag{8.40}$$

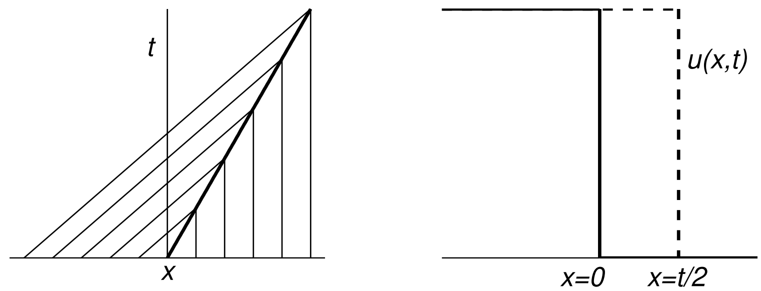


Figure 8.5: Left panel: the characteristic diagram of the problem in (8.37); the shock path, $x = t/2$, is indicated by the heavy line. Right panel: the discontinuity in u advances to the right. `heaviShock.eps`

Since the initial condition has $v_+ = 0$ and $v_- = 1$ the shock speed is now $\dot{s} = 1/3$ and the shock path is $s = t/3$.

A problem

Suppose we multiply the PDE in (8.39) by $2v$. The result can be written as

$$(v^2)_t + v^2(v^2)_x = 0\,. \tag{8.41}$$

or, with $w \overset{\text{def}}{=} v^2$,

$$w_t + w w_x = 0\,. \tag{8.42}$$

The initial condition is $w(x, 0) = v^2(x, 0) = H(-x)$. It seems that the PDE for $w(x, t)$ is the same as the PDE for $u(x, t)$ in (8.37): we are tempted to conclude that

$$w(x, t) \overset{?}{=} u(x, t)\,. \tag{8.43}$$

But this can't be true: the v-shock travels with speed $\dot{s} = 1/3$, while the u-shock travels with speed $\dot{s} = 1/2$!

The point is that the shock condition (8.36) provides additional physical information which removes the mathematical ambiguity which occurs if the density is discontinuous. In other words, we can't guess the correct shock condition by staring at a naked PDE: the integral form of the conservation law provides fundamental physical information which is simply lost in the transition from the integral formulation in (8.32) to the PDE conservation law in (8.33).

8.4 Shocks from continuous initial conditions

A kinky initial condition

In our earlier example (8.37) the shock was already present in the initial condition. Now let's consider the kinky initial condition:

$$u = \begin{cases} 1 - |x|\,, & \text{if } |x| < 1, \\ 0\,, & \text{if } |x| > 1. \end{cases} \tag{8.44}$$

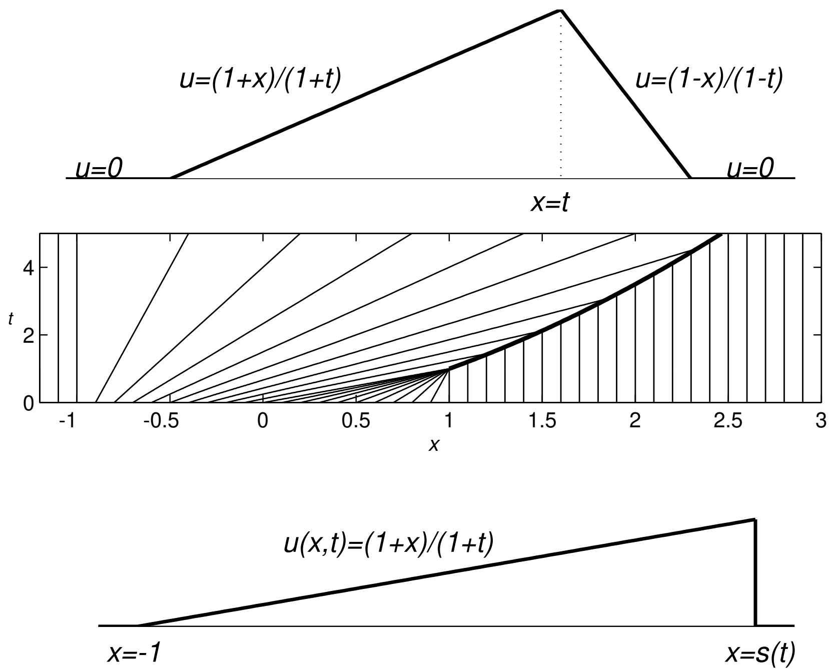


Figure 8.6: Upper panel: the steepening profile at $t = 1/2$ before shock formation. Middle panel: the characterisitic diagram showing the shock path. Bottom panel: the triangular wave. `kinkShock.eps`

This initial condition is continuous but nondifferentiable at three points. We know that a shock forms at $(x_{\rm s}, t_{\rm s}) = (1, 1)$ (see figure 8.1). Let's now find the shock path and strength.

Before the shock forms, the solution is shown in the upper panel of figure 8.6. At $t = 1$ the right face of the triangular wave becomes vertical and the shock appears. This vertical face (a discontinuity in u) then propagates to the right, into the region where $u = 0$. Thus, ahead of the shock $u(s^+, t) = 0$. Behind the shock we can continue to use the solution in the upper panel of figure 8.6:

$$u(x, t) = \frac{1 + x}{1 + t}\,, \tag{8.45}$$

so that $u(s^-, t) = (s + 1)/(1 + t)$. Ahead of the shock $u(s^+, t) = 0$. Therefore the shock condition is

$$\dot{s} = \tfrac{1}{2}\frac{s + 1}{t + 1}\,. \tag{8.46}$$

We have to solve this ODE for $s(t)$ subject to the initial condition that $s(1) = 1$. Multiplying by the integrating factor $1/\sqrt{1 + t}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{s}{\sqrt{1 + t}}\right) = \frac{1}{2(1 + t)^{3/2}}\,. \tag{8.47}$$

Integrating from $t' = 1$ to $t' = t$, and applying $s(1) = 1$, we find

$$s(t) = \sqrt{2(1 + t)} - 1\,. \tag{8.48}$$

The shock path is the heavy curve in the characteristic diagram shown in the middle panel of figure 8.6; characteristics flow into both sides of the shock. The solution is the *triangular wave* shown in the bottom panel of figure 8.6.

The strength of the shock is

$$[\![u]\!] = \frac{1+s}{1+t} = \sqrt{\frac{2}{1+t}}\,. \tag{8.49}$$

Thus we can easily see that the area under the triangular wave in the bottom panel of figure 8.6 is constant

$$\int_{-\infty}^{\infty} u(x,t)\,\mathrm{d}x = \tfrac{1}{2} \times [s(t)+1] \times [\![u]\!] = 1\,. \tag{8.50}$$

Of course this must be the case since $u(x,t)$ is a conservative density. On the other hand, we can also compute the 'energy' of the solution

$$\int_{-\infty}^{\infty} u^2\,\mathrm{d}x = \int_{-1}^{s(t)} \frac{(1+x)^2}{(1+t)^2}\,\mathrm{d}x = \frac{4}{3}\frac{1}{\sqrt{2(1+t)}} \qquad (\text{if } t>1). \tag{8.51}$$

Thus once, the shock forms, when $t > 1$, the energy decreases monotonically. How does the energy evolve before $t = 1$?

Smooth initial condition and Whitham's geometric construction

Now suppose the initial condition is $u(x,0) = F(x)$ where F is a completely differentiable function, such as our earlier example, $F(x) = 1/(1+x^2)$. In that example the shock forms at $(x_\mathrm{s}, t_\mathrm{s}, \xi_\mathrm{s}) = (\sqrt{3}, 8\sqrt{3}/9, 1/\sqrt{3})$. We also know that this is a generic property of the evolution of smooth initial humps: nonlinear steepening produces a shock at $(x_\mathrm{s}, t_\mathrm{s}, \xi_\mathrm{s})$ and these three unknowns are determined from

$$F''(\xi_\mathrm{s}) = 0\,, \quad 1 + t_\mathrm{s}F'(\xi_\mathrm{s}) = 0\,, \quad x_\mathrm{s} = \xi_\mathrm{s} + t_\mathrm{s}F(\xi_\mathrm{s})\,. \tag{8.52}$$

Where does the shock go after that? At time t the shock is swallowing two characteristics, $\xi_-(t)$ behind the shock and $\xi_+(t)$ ahead of the shock. The velocities before and after the shock are therefore

$$u(s^-,t) = F(\xi_-)\,, \qquad \text{and} \qquad u(s^+,t) = F(\xi_+)\,. \tag{8.53}$$

Thus to determine the shock path we must solve the following system

$$s(t) = \xi_+ + tF(\xi_+)\,, \qquad s(t) = \xi_- + tF(\xi_-)\,, \tag{8.54}$$

and

$$\dot{s} = \tfrac{1}{2}\left[F(\xi_-) + F(\xi_+)\right] \tag{8.55}$$

for the three unknowns $[s(t), \xi_-(t), \xi_+(t)]$. This is a much more difficult problem than our earlier examples because we cannot easily solve for ξ_- and ξ_+ in terms of $s(t)$. (Try it with $F(\xi) = 1/(1+\xi^2)$ and see how far you can get!)

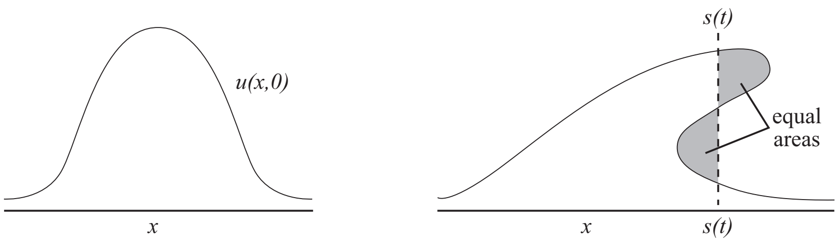


Figure 8.7: Whitham's equal area rule. `eqArea.eps`

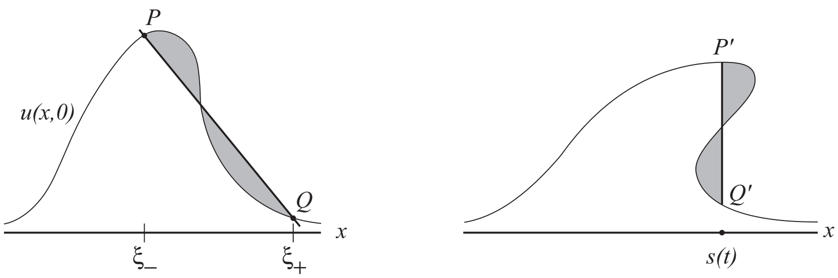


Figure 8.8: Whitham's chordal construction. `chord.eps.eps`

Fortunately, Whitham (1974) has formulated a set of geometric arguments which reveal the structure of the solution without getting involved in algebraic details. There are two principles. First there is the *equal-area* rule: on the multi-valued solution at time t the shock is located by drawing a vertical line which cuts off equal area lobes (see figure 8.7).

The second principle is the *chordal construction*: To find ξ_- and ξ_+ at time t draw a chord of slope $-1/t$ on the graph of $F(\xi)$. Translate this line vertically till it cuts-off equal-area lobes (see figure 8.8).

One interesting result is that as $t \to \infty$ the hump evolves into a triangular wave, essentially identical to the bottom panel in figure 8.6. Thus with Whitham's rules we have a complete understanding of how a hump evolves even without suffering through the solution of (8.54) and (8.55).

8.5 The Lighthill-Whitham traffic model

A Google search using "Lighthill" and "Whitham" indicates that one of the most popular applications of quasilinear PDEs is the theory of traffic flow developed by Lighthill & Whitham.

Formulating the model

$\rho(x,t)$ is the density of cars on a highway; with this definition $\rho(x,t)$ cannot become multivalued (or negative). We start with the differential form of the conservation law

$$\rho_t + f_x = 0\,, \tag{8.56}$$

and argue that the flux is

$$f = \rho u(\rho)\,, \tag{8.57}$$

where $u(\rho)$ is the average speed of cars if the traffic density is ρ. We make the assumption that drivers slow down as the density increases, and that they stop completely once the density exceeds ρ_{J}, where the subscript J stand for "jam".

A simple model with the behavior above is

$$u = \begin{cases} u_{\mathrm{m}}\left(1 - \frac{\rho}{\rho_{\mathrm{J}}}\right)\,, & \text{if } \rho < \rho_{\mathrm{J}}\,, \\ 0\,, & \text{if } \rho > \rho_{\mathrm{J}}\,. \end{cases} \tag{8.58}$$

The maximum speed, characteristic only of low densities, is u_{m}. With this assumption about human behaviour, the relation between flux to density is

$$f = u_{\mathrm{m}}\rho\left(1 - \rho/\rho_{\mathrm{J}}\right)\,, \quad \text{provided that } \rho < \rho_{\mathrm{J}}\,. \tag{8.59}$$

The flux is a maximum,

$$f_{\mathrm{m}} = \tfrac{1}{4} u_{\mathrm{m}} \rho_{\mathrm{J}}\,, \tag{8.60}$$

at $\rho = \rho_{\mathrm{J}}/2$. According Haberman, in the Lincoln Tunnel the maximum flux is observed to be about 1600 cars per hour at a speed of around 20 miles per hour and a density of 80 cars per mile. If ρ is greater than this optimal density we say that the traffic is *heavy* and if ρ is less than the optimal then the traffic is *light*. The maximum flux is the *capacity* of the road (cars per second).

(The Lincoln Tunnel connects New York to New Jersey under the Hudson River. The tunnel is about 1.5 miles long. Construction was funded by the New Deal's Public Works Administration. Construction started in 1934 and, after a delay due to WWII, concluded in 1957.)

To summarize: we use the simple model in (8.59) and rewrite the conservation law (8.56) as

$$\rho_t + c(\rho)\rho_x = 0 \qquad c(\rho) \stackrel{\text{def}}{=} f_\rho = u_{\mathrm{m}}\left(1 - 2\frac{\rho}{\rho_{\mathrm{J}}}\right)\,. \tag{8.61}$$

$c(\rho)$ is the speed with which small disturbances propagate through a stream of traffic with uniform density ρ. Notice that $c(\rho)$ is different from the speed of a car — that's $u(\rho)$. Figure 8.9 displays both $c(\rho)$, $u(\rho)$ and $f(\rho)$ graphically. This is the "fundamental diagram" of road traffic.

Small disturbances on a uniform stream

We study small disturbances propagating through a uniform stream (density ρ_0) by linearizing (8.61)

$$\rho = \rho_0 + \eta\,, \quad \Rightarrow \quad \eta_t + c(\rho_0)\eta_x + O(\eta^2) = 0\,. \tag{8.62}$$

Neglecting the quadratic terms, the disturbance η satisfies the linear wave equation and consequently:

$$\eta = \eta(x - c_0 t)\,. \tag{8.63}$$

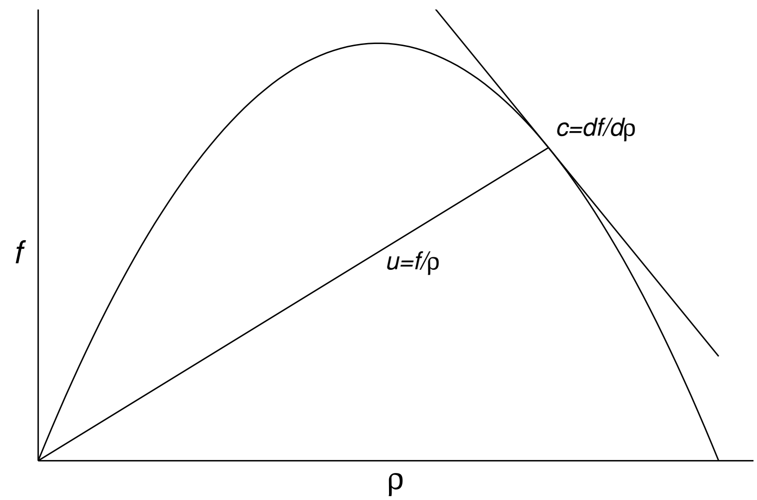


Figure 8.9: The fundamental diagram of road traffic: if we plot the flux f as a function of ρ then both $u = f/\rho$ and $c = \mathrm{d}f/\mathrm{d}\rho$ are the slopes of the lines shown above. Heavy traffic is when the density is greater than the optimal density. `fundDiag.eps`

The speed $c_0 \overset{\text{def}}{=} c(\rho_0)$ can be either positive or negative depending on whether the traffic is either *light* ($\rho_0 < \rho_J/2$) or *heavy* ($\rho_0 > \rho_J/2$).

Notice that $u(\rho) > c(\rho)$. This agrees with the experience of highway drivers: one overtakes disturbances in the traffic stream. (You can see the wave of red brake lights moving towards you.) And in heavy traffic the density wave actually moves backwards relative to the road (i.e., $c(\rho_0) < 0$) even though no car is moving backwards.

These remarks emphasize that there are *two* important velocities $c(\rho)$ and $u(\rho)$.

The green light problem

Suppose that there is a block of cars stopped at a red light located at $x = 0$. At $t = 0$ the light turns green. What happens? The initial density in this situation is

$$F(x) = \begin{cases} \rho_J \,, & \text{if } -\ell < x < 0 \,, \\ 0 \,, & \text{otherwise.} \end{cases} \tag{8.64}$$

The initial condition is discontinuous at two points: $x = 0$ and $x = -\ell$.

Lagrange's equations,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c(\rho) \qquad \text{and} \qquad \frac{\mathrm{d}\rho}{\mathrm{d}t} = 0 \,, \tag{8.65}$$

show that the characteristics are straight lines in the (x, t)-plane and the implicit solution of (8.61) is

$$\rho(x, t) = F[x - c(\rho)t] \,. \tag{8.66}$$

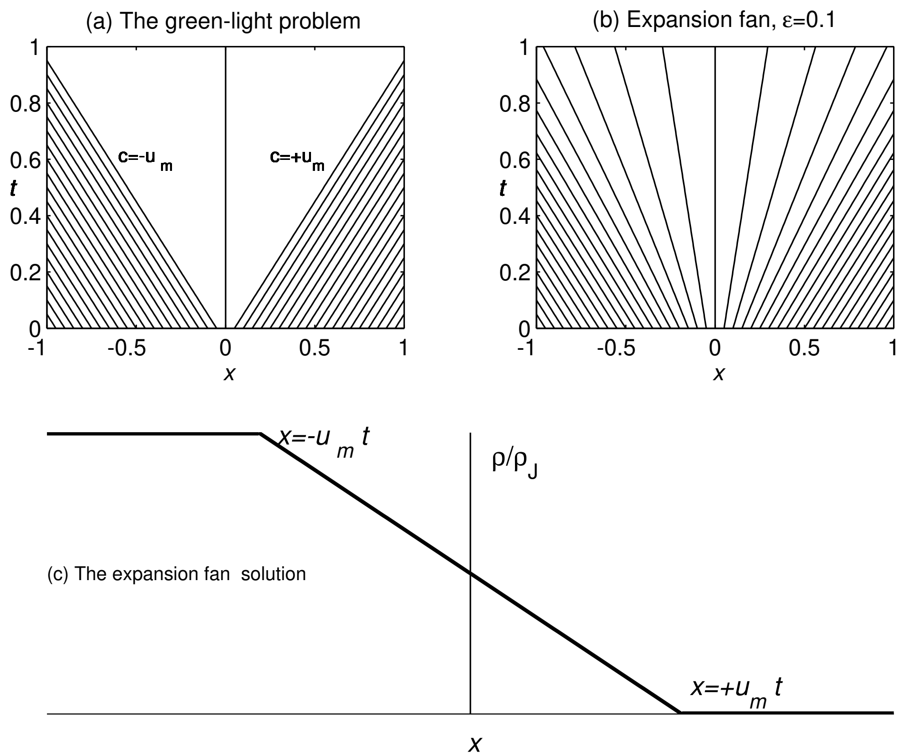


Figure 8.10: The expansion fan. `expFan.eps`

We draw the characteristics of the PDE in (8.61): these are straight lines

$$\xi = x - c(\rho)t\,. \tag{8.67}$$

At $t = 0$ $x = \xi$ and $c = \pm u_{\mathrm{m}}$ — see figure 8.10. There is a wedge-shaped region, originating at $(x,t) = (0,0)$, and within the wedge there are no characteristics. We have not previously encountered this situation.

The leading expansion fan

To understand how to fill in the wedge we soften the discontinuous initial condition by smoothing out the jump. For example, instead of a discontinuous $F(x)$ in (8.64), let us use

$$F(x,\epsilon) = \frac{1}{1 + \mathrm{e}^{x/\epsilon}}\,, \tag{8.68}$$

and then take the limit $\epsilon \to 0$. Now when we can plot the characteristics (see figure 8.10) we see that the wedge is filled by an *expansion fan* or *rarefaction wave*. Inside the fan

$$c(\rho) = \frac{x}{t} + \text{some small correction proportional to } \epsilon. \tag{8.69}$$

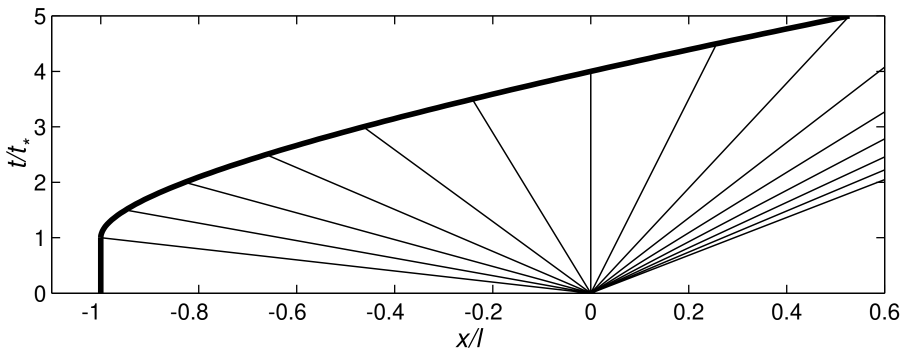


Figure 8.11: The forward expansion fan and the trailing shock. The shock is stationary at $x = -\ell$ till the fan arrives at $t = t_*$. `green.eps`

Taking $\epsilon \to 0$, we obtain the solutionρ within the expansion fan from (8.69). The complete solution in the three regions shown in figure 8.10(a) is assembled in

$$\rho(x,t) = \begin{cases} \rho_{\rm J}\,, & x < -u_{\rm m} t\,, \\ \tfrac{1}{2}\rho_{\rm J}\left(1 - \tfrac{x}{t\,u_{\rm m}}\right)\,, & -u_{\rm m} t < x < u_{\rm m} t\,, \\ 0\,, & x > u_{\rm m} t\,. \end{cases} \tag{8.70}$$

The trailing shock

With the initial density in (8.64), the car at $x = -\ell$ remains stationary until the time

$$t_* \stackrel{\rm def}{=} \ell/u_{\rm m} \tag{8.71}$$

at which the expansion fan reaches the end of the line. The arrival of the fan at $x = -\ell$ sets the hitherto stationary rear shock into motion. We denote the position of this trailing shock by $x = s(t)$. Evidently the initial condition for the shock position $s(t)$

$$s(0 < t < t_*) = -\ell\,. \tag{8.72}$$

To figure out the subsequent position of the shock we use

$$\dot s = \tfrac{f(s^+,t) - f(s^-,t)}{\rho(s^+,t) - \rho(s^-,t)}\,, \tag{8.73}$$

$$= u_{\rm m}\left(1 - \tfrac{\rho_+ + \rho_-}{\rho_{\rm J}}\right)\,. \tag{8.74}$$

Behind the shock $\rho^- = f^- = 0$ and so (8.74) reduces to

$$\dot s = u_{\rm m}\left[1 - \frac{\rho(s^+,t)}{\rho_{\rm J}}\right]\,. \tag{8.75}$$

As one intuitively anticipates, the shock moves with the same speed as the final car.

Ahead of the shock we can use the expansion fan solution in (8.70) to obtain

$$\rho(s^+,t) = \tfrac{1}{2}\rho_{\rm J}\left(1 - \tfrac{s}{t u_{\rm m}}\right)\,. \tag{8.76}$$

Substituting (8.77) into (8.75) we obtain an ODE for the shock position

Integrating factor

$$\dot{s} = \frac{u_{\rm m}}{2}\left(1 + \frac{s}{t u_{\rm m}}\right)\,, \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{s}{\sqrt{t}} = \tfrac{1}{2}\frac{u_{\rm m}}{\sqrt{t}}\,. \tag{8.77}$$

$\frac{1}{\sqrt{t}}$

Integrating (8.77) from t_* to t we get the shock position as

$$s(t) = \ell\left(\frac{t}{t_*} - 2\sqrt{\frac{t}{t_*}}\right)\,. \tag{8.78}$$

The shock, and the final car, passes the traffic light at $t = 4t_*$.

It's a good exercise to sketch "snapshots" of the solution: ρ as a function of x at times both before and after t_*. Notice that once $t > t_*$, ρ is a "triangular wave".

Remark: the flux of cars through a green light

Notice that in the expansion wave solution (8.70), $\rho(0,t) = \rho_{\rm J}/2$ so that the flux at the light is $f = f(\rho_{\rm J}/2) = f_{\rm m}$, where $f_{\rm m}$ is the maximum flux. The result that $f(\rho(0,t)) = f_{\rm m}$ is independent of the detailed form of $f(\rho)$: when the light turns green the piled-up traffic flushes through the intersection at the maximal rate, $f_{\rm m}$. To prove this, denote the position at which $f = f_{\rm m}$ by $x(t)$ and the corresponding density $\rho_{\rm m}$. Then

$$f\left[\rho(x_{\rm m},t)\right] = f_m\,, \text{and} \quad \rho[x_{\rm m}(t),t] = \rho_{\rm m}\,. \tag{8.79}$$

Also $c(\rho_{\rm m}) = 0$ because the flux is a maximum at $\rho_{\rm m}$. Thus evaluating the conservation equation

$c = \frac{\mathrm{d}f}{\mathrm{d}\rho}$

$$\rho_t + c\rho_x = 0\,, \tag{8.80}$$

on the trajectory $(x_{\rm m}(t),t)$ we get

$$\rho_t(x_{\rm m},t) = 0\,. \tag{8.81}$$

But differentiating the second relation in (8.79) we have

$$\underbrace{\rho_t(x_{\rm m},t)}_{=0} + \frac{\mathrm{d}x_{\rm m}}{\mathrm{d}t}\rho_x(x_{\rm m},t) = 0\,. \tag{8.82}$$

From (8.81) and (8.82) we conclude that either

$$\frac{\mathrm{d}x_{\rm m}}{\mathrm{d}t} = 0\,, \quad \text{or} \quad \rho_x(x_{\rm m},t) = 0\,. \tag{8.83}$$

Thus as long as there is a density gradient at the light ($\rho_x \neq 0$) the flux at the light is f_m.

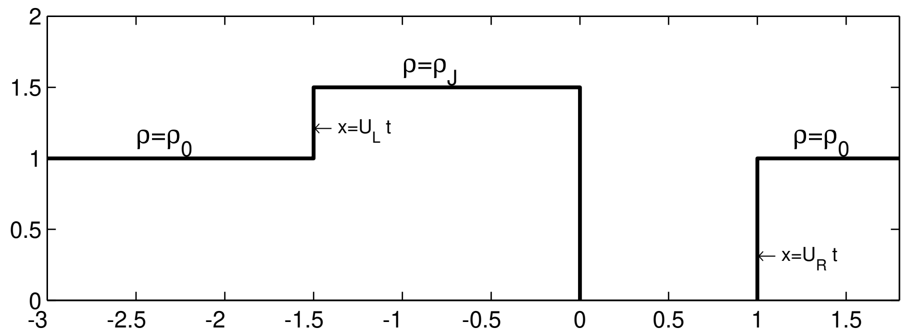


Figure 8.12: The red light stops traffic with density ρ_0 and creates two shocks. The left going shock leaves jammed traffic behind it. `red.eps`

The red light problem

Suppose we have a uniform stream of traffic with density ρ_0 and suddenly we turn on a red light at $x = 0$. We assume that the first car stops at the red light instantly and the following cars stop, building a jam with density ρ_J.

The problem is therefore

$$\rho_t + c(\rho)\rho_x = 0\,, \quad \rho(x,0) = \rho_0 < \rho_J\,, \quad f(0,t) = 0\,. \tag{8.84}$$

There are two shocks propagating away from $x = 0$. Using the standard model, both shock trajectories can be obtained from

$$\dot{s} = u_m \left(1 - \frac{\rho_+ + \rho_-}{\rho_J}\right)\,. \tag{8.85}$$

At the left-moving shock $\rho^+ = \rho_J$ and $\rho^- = \rho_0$. Therefore the left moving shock is at position

$$L(t) = -\frac{\rho_0}{\rho_J} u_m t\,. \tag{8.86}$$

This shock goes backwards into the incident traffic stream, leaving a jam in its wake.

At the right-moving shock $\rho^- = 0$ and $\rho^+ = \rho_0$, and the shock is at

$$R(t) = u_m \left(1 - \frac{\rho_0}{\rho_J}\right) t\,. \tag{8.87}$$

and

$$\rho_- = 0\,, \quad \rho_+ = \rho_0$$

$$\dot{s} = \frac{f(\rho_0)}{\rho_0}$$

The last guy is congratulating himself on making it through, and the speed of his car, $u(\rho_0)$, is also the speed on the shock. Again, one should sketch snapshots of ρ and show that cars are conserved: see figure 8.12.

Red light and green light

Now suppose we have the situation in figure 8.12 and the light turns back to green at $t = \tau$. What happens?

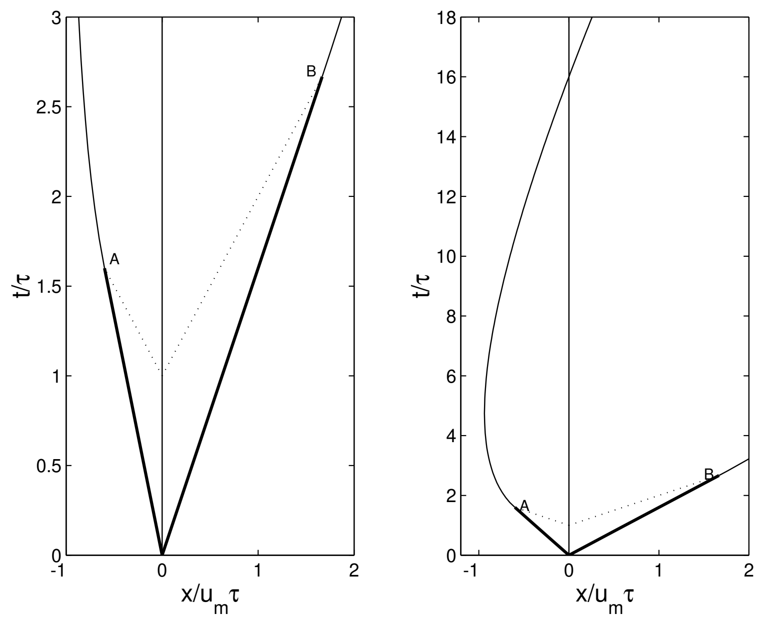


Figure 8.13: Trajectories of the shocks in the red light and green light problem. Before the expansion fan arrives the shocks move with constant speed (these are the heavy line segments). After the arrival of the expansion fan (at points A and B) the shock paths curve. The expansion fan produced by the Green light is centered on the point $(x,t)=(0,\tau)$. In the solution above $\rho_0 = 3\rho_{\rm J}/8$ and the shock finally makes it through the light at $t=16\tau$. `redgreen.eps`

The key is to realize that there is an expansion fan solution centered on the point $(x,t)=(0,\tau)$. This fan solution is

$$\rho_{\rm F}(x,t) = \frac{\rho_{\rm J}}{2}\left(1-\frac{x}{(t-\tau)u_{\rm m}}\right), \qquad \text{if} \quad |x| < u_{\rm m}(t-\tau)\,. \tag{8.88}$$

The fan first interacts with the left shocks at point A in figure 8.13:

$$(s_A, \tau_A) = \left(-\frac{\rho_0 u_{\rm m}\tau}{\rho_{\rm J}-\rho_0}, \frac{\rho_{\rm J}\tau}{\rho_{\rm J}-\rho_0}\right). \tag{8.89}$$

And the fan reaches the right shock at point B in figure 8.13:

$$(s_B, \tau_B) = \left(\frac{u_0\rho_{\rm J}\tau}{\rho_0}, \frac{\rho_{\rm J}\tau}{\rho_0}\right). \tag{8.90}$$

You can obtain these results by drawing the characteristic diagram and figuring out where the edges of the expansion fan at $x = \pm u_{\rm m} t$ first hit the shocks in (8.86) and (8.87).

Once the fan starts to deflect the shocks the shock speed is given by

$$\dot{s} = \frac{u_{\rm F}\rho_{\rm F} - u_0\rho_0}{\rho_{\rm F}(s,t)-\rho_0} = u_{\rm m}\left(1-\frac{\rho_0+\rho_{\rm F}}{\rho_{\rm J}}\right). \tag{8.91}$$

Thus to determine the position of both the left and the right shock we must integrate the ODE:

$$\dot{s} - \tfrac{1}{2}\frac{s}{t-\tau} = u_{\mathrm{m}}\left(\frac{1}{2} - \frac{\rho_0}{\rho_{\mathrm{J}}}\right) . \tag{8.92}$$

The solution of (8.92) is

$$s(t) = u_{\mathrm{m}}\left(1 - 2\frac{\rho_0}{\rho_{\mathrm{J}}}\right)(t-\tau) + B\sqrt{t-\tau}\,, \tag{8.93}$$

where B is the constant of integration. We can determine B using the initial conditions in (8.89) and (8.90).

For the left hand shock we find after some brutal algebra that

$$L(t) = u_{\mathrm{m}}\left(1 - 2\frac{\rho_0}{\rho_{\mathrm{J}}}\right)(t-\tau) - 2u_{\mathrm{m}}\sqrt{\left(1-\frac{\rho_0}{\rho_{\mathrm{J}}}\right)\frac{\rho_0}{\rho_{\mathrm{J}}}\tau(t-\tau)}\,, \quad \text{if } t > \tau_L\,. \tag{8.94}$$

For the right hand shock we find

$$R(t) = u_{\mathrm{m}}\left(1 - 2\frac{\rho_0}{\rho_{\mathrm{J}}}\right)(t-\tau) + 2u_{\mathrm{m}}\sqrt{\left(1-\frac{\rho_0}{\rho_{\mathrm{J}}}\right)\frac{\rho_0}{\rho_{\mathrm{J}}}\tau(t-\tau)}\,, \quad \text{if } t > \tau_R\,. \tag{8.95}$$

If the traffic is light ($\rho_0 < \rho_{\mathrm{J}}/2$) then the left-shock reverses direction and eventually goes through the intersection (as shown in figure 8.13). This happens at

$$\tau_{\mathrm{c}} = \frac{\tau}{(1 - 2\rho_0/\rho_{\mathrm{J}})^2}\,. \tag{8.96}$$

After the *clearance time*, τ_{c}, the congestion behind the light has dissipated. An observer sitting at the light notices that at $t = \tau_{\mathrm{c}}$ the density drops from $\rho_{\mathrm{J}}/2$ back to ρ_0. The clearance time can be distressingly long e.g., in figure 8.13 $\tau_{\mathrm{c}} = 16\tau$ so a stopage lasting five minutes produces a jam requiring an hour and a quarter to clear.

References

My discussion of traffic closely follows

Ha *Mathematical Models: Mechanical Vibrations, Population Dynamics and Traffic flow* by Richard Haberman.

8.6 Burgers' equation and traveling waves

Following the discussion of shock formation in lecture 3, perhaps you are alarmed by the ambiguity inherent in PDE models? Fortunately in most physical applications it is easy to understand which variables are conservative. Our favourite equations usually contain *dissipative* terms. For instance, as an analog of fluid mechanics, Burgers introduced the model equation

$$u_t + uu_x = \nu u_{xx}\,. \tag{8.97}$$

If the viscosity ν is very small we might be tempted to drop the diffusive term on the right hand side. Then we get the advection equation as an approximation of Burgers' equation. But the advection equation by itself is ambiguous because, for instance,

$$u_t + uu_x = 0 \qquad \Leftrightarrow \qquad (\tfrac{1}{2}u^2)_t + (\tfrac{1}{3}u^3)_x = 0\,. \tag{8.98}$$

The different forms have different shock conditions. But if the original problem was (8.97) then the correct shock condition is obtained using the flux $f = u^2/2$ in the u-equation (not the u^2-equation).

If we do form the u^2-equation from (8.97), here is what happens:

$$(\tfrac{1}{2}u^2)_t + (\tfrac{1}{3}u^3)_x = \nu(uu_x)_x - \nu u_x^2\,. \tag{8.99}$$

Considering a humplike initial condition, so that the fluxes all vanish at $x = \pm\infty$, we see that Burgers' equation dissipates 'energy':

$$E(t) \stackrel{\text{def}}{=} \tfrac{1}{2}\int_{-\infty}^{\infty} u^2 \mathrm{d}x\,. \tag{8.100}$$

From (8.99)

$$\dot{E} = -\nu \int_{-\infty}^{\infty} u_x^2\,\mathrm{d}x\,, \tag{8.101}$$

and it turns out that $\dot{E}$ is nonzero even in the limit $\nu \to 0$ (see the problems).

The traveling wave solution

We can understand the role of the right hand side in (8.44) by looking for a traveling wave solution of Burgers' equation. We have in mind a smooth solution, as shown in figure 8.14, in which u varies gradually between $u(-\infty, t) = L$ to $u(+\infty, t) = R$. It is important that $L > R$. With $L > R$ we expect from the nonlinear advection equation a solution analogous to a shock traveling with speed $(R+L)/2$. On the other hand, with $R > L$, we expect a rarefaction wave — and thus we do not anticipate finding a traveling wave solution of Burgers equation in this case.

We introduce the traveling wave guess

$$u = U(z)\,, \qquad z \stackrel{\text{def}}{=} x - ct\,, \tag{8.102}$$

into (8.97) and so obtain

$$-cU' + UU' = \nu U''\,. \tag{8.103}$$

Integrating (8.103) we have

$$A - cU + \tfrac{1}{2}U^2 = \nu U'\,. \tag{8.104}$$

Now the unknowns A and c are now determined by requiring that

$$\lim_{x\to\infty} U = R\,, \qquad \text{and} \qquad \lim_{x\to-\infty} U = L\,. \tag{8.105}$$

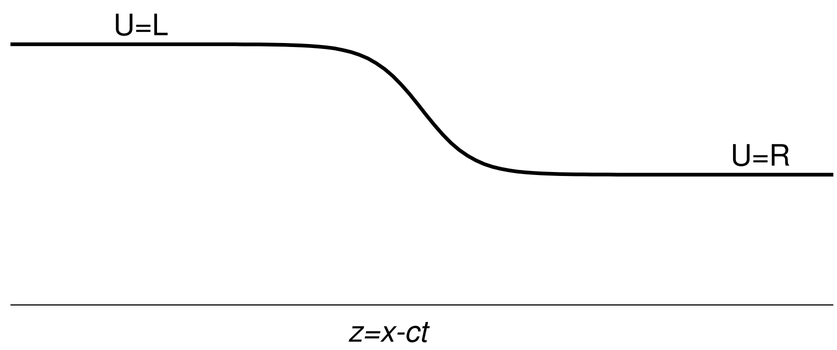


Figure 8.14: The traveling wave solution of Burgers' equation. Note that $L > R$ implies that $U_z < 0$. `burShock.eps`

Thus

$$A - cR + \tfrac{1}{2}R^2 = 0\,, \qquad \text{and} \qquad A - cL + \tfrac{1}{2}R^2 = 0\,. \tag{8.106}$$

Solving these relation for A and c, we obtain

$$A = \frac{RL}{2}\,, \qquad \text{and} \qquad c = \frac{R+L}{2}\,. \tag{8.107}$$

At this point the expression for the speed, c, should remind you of the shock condition for the nonlinear advection equation.

With A and c determined, we must now solve

$$LR - (L+R)U + U^2 = 2\nu U'\,. \tag{8.108}$$

Before getting involved in the algebra of separating variables, one should use a phase-line analysis to show that ODE (8.108) does, in fact, have a solution corresponding to figure 8.14. In that figure $L > R$ and $U' < 0$. This is consistent with (8.108): when $L > U > R$ we see that $U' < 0$. On the other hand, considering the possibility that $R > L$, we see that in this case U' is positive, which is in contradiction to (8.108).

We press on by separating variables in (8.108) and writing the result as

$$\left(\frac{1}{L-U} + \frac{1}{U-R}\right) \mathrm{d}U = -(L-R)\frac{\mathrm{d}z}{2\nu}\,. \tag{8.109}$$

Note all factors above, $L-U$, $U-R$ and $L-R$, are positive. Thus integration gives

$$\ln\frac{U-R}{L-U} = -\mu(z - z_*)\,, \tag{8.110}$$

where z_* is a constant of integration and

$$\mu \stackrel{\text{def}}{=} \frac{L-R}{2\nu}\,. \tag{8.111}$$

Solving (8.110) for U, we finally have

$$u(x,t) = \frac{L + R\mathrm{e}^{\mu(x-ct+z_*)}}{1+\mathrm{e}^{\mu(x-ct+z_*)}}\,. \tag{8.112}$$

The width of the transition region between L and R is

$$\mu^{-1} = \frac{2\nu}{L-R}\,. \tag{8.113}$$

Thus as $\nu \to 0$ the traveling-wave solution becomes a discontinuous shock. In other words, as $\nu \to 0$, the transition between $u = L$ and $u = R$ is an internal boundary layer. If we have a gentleperson's agreement that the details of this boundary layer are of no interest then we can set the right hand side of (8.97) to zero and solve the nonlinear advection equation using

$$\dot{s} = \tfrac{1}{2}u_+ + \tfrac{1}{2}u_- \tag{8.114}$$

to track the location of the shocks. We are arguing heuristically that if ν is very small then $u(x,t)$ will vary slowly relative to the shock thickness in (8.113).

The Cole-Hopf transformation

First substitute $u = \phi_x$ into Burgers' equation (8.97). We can then integrate with respect to x to obtain:

$$\phi_t + \tfrac{1}{2}\phi_x^2 = \nu\phi_{xx}\,. \tag{8.115}$$

Now we substitute $\phi = \alpha \ln \theta$; notice

$$\phi_x = \alpha\frac{\theta_x}{\theta}\,, \quad \phi_{xx} = \alpha\frac{\theta_{xx}}{\theta} - \alpha\left(\frac{\theta_x}{\theta}\right)^2 = \alpha\frac{\theta_{xx}}{\theta} - \frac{1}{\alpha}\phi_x^2\,. \tag{8.116}$$

We find then

$$\alpha\frac{\theta_t}{\theta} + \left(\tfrac{1}{2} + \frac{\nu}{\alpha}\right)\phi_x^2 = \nu\alpha\frac{\theta_{xx}}{\theta}\,. \tag{8.117}$$

Picking $\alpha = -2\nu$ we destroy the nonlinear term and find that θ satisfies the heat equation

$$\theta_t = \nu\theta_{xx}\,. \tag{8.118}$$

To summarize, the substitution

$$u = \phi_x = -2\nu\frac{\theta_x}{\theta} \tag{8.119}$$

enables us to move back and forth between Burgers' equation and the heat equation.

As an example, it is easy to guess that

$$\theta = x^2 + 2\nu t \tag{8.120}$$

is a solution of the heat equation. Thus it seems that

$$u = -2\nu\frac{2x}{x^2+2\nu t} \tag{8.121}$$

must be a not-so-obvious solution of Burgers' equation.

8.7 Problems

Problem 8.1. Solve $u_t + uu_x = 1$ with the initial condition $u(x,0) = x$.

Problem 8.2. *(i)* Does the PDE

$$u_t + uu_x = 0\,, \quad u(x,0) = \frac{\mathrm{e}^x}{1+\mathrm{e}^x}\,. \tag{8.122}$$

form a shock? If so, calculate t_s without solving the PDE. *(ii)* Find an implicit solution for $u(x,t)$ and use MATLAB to graph this solution.

Problem 8.3. *(i)* Does the PDE

$$u_t + uu_x = 0\,, \quad u(x,0) = \frac{\mathrm{e}^{-x}}{1+\mathrm{e}^{-x}}\,, \tag{8.123}$$

form a shock? If so, calculate t_s without solving the PDE. *(ii)* Find an implicit solution for $u(x,t)$ and use MATLAB to graph this solution.

Problem 8.4. Find an implicit solution of

$$\phi_t + p(\phi)\phi_x + q(\phi)\phi_y = 0 \tag{8.124}$$

with the initial condition $\phi = f(x,y)$ at $t = 0$.

Problem 8.5. *(i)* Find an implicit solution, analogous to (8.10), of

$$u_t + uu_x = -\alpha u\,, \qquad \text{with IC} \qquad u(x,0) = \frac{1}{1+x^2}\,. \tag{8.125}$$

Make sure your answer reduces to (8.10) if $\alpha \to 0$. Draw a figure, like 8.2, showing both characteristics and snapshots of $u(x,t)$. (Use several values of α so that you understand how this parameter changes the solution). *(ii)* Calculate t_s as a function of α. *(iii)* Find the smallest value of the damping α which is sufficient to prevent shock formation.

Problem 8.6. Find an expression for the shock time t_s, analogous to (8.27), for the PDE

$$u_t + u^2 u_x = 0\,, \quad u(x,0) = F(x)\,. \tag{8.126}$$

Also give a simple formula for x_s.

Problem 8.7. *(i)* Find a solution analogous to (8.10) of the PDE

$$u_t + c(u)u_x = 0\,, \quad u(x,0) = F(x)\,. \tag{8.127}$$

(ii) Check your answer by substitution. *(iii)* Considering the special case $c = u^2$ and $F = x$, show that once $t > 0$ there are either two real values of $u(x,t)$ or no real values of $u(x,t)$ at each point in spacetime. Locate the curve in the (x,t) plane which separates these two behaviours.

Problem 8.8. Consider

$$u_t + u^3 u_x = 0\,, \quad u(x,0) = \sin x\,. \tag{8.128}$$

At what time, t_s, and location, x_s, does the solution $u(x,t)$ first become singular?

Problem 8.9. Solve the quasilinear system

$$u_t + u u_x = a\,, \qquad a_t + u a_x = -a\,, \tag{8.129}$$

for $u(x,t)$ and $a(x,t)$, with the initial conditions

$$u(x,0) = 0\,, \qquad \text{and} \qquad a(x,0) = x\,. \tag{8.130}$$

Problem 8.10. Assuming that u is a conserved density, find the location and the strength of the shock which forms from

$$u_t + u u_x = 0\,, \qquad \text{with intial condition} \qquad u(x,0) = -\frac{x}{1+x^2}\,. \tag{8.131}$$

Problem 8.11. Assuming that u is a conserved density, find the locations and the strengths of the shocks which forms from

$$u_t + u u_x = 0\,, \qquad u(x,0) = -\sin x\,. \tag{8.132}$$

Problem 8.12. Consider

$$u_t + (u^2)_x = 0\,, \qquad \text{with the initial condition} \qquad u(x,0) = H(-x)\sqrt{-x}\,. \tag{8.133}$$

(Notice the wave speed is $2u$ in this problem.) Assume $u(x,t)$ is a conserved density. *(i)* Show that

$$\int_0^\infty u(x,t)\,\mathrm{d}x = \int_0^t u^2(0,t')\,\mathrm{d}t'\,. \tag{8.134}$$

(ii) Draw the characteristic diagram and find where the shock first forms. *(iii)* Solve the PDE behind and ahead of the shock. *(iv)* Find an ODE which determines the shock position, $s(t)$, and solve this equation. (I found it was easy to guess the solution of the ODE.) *(v)* Check your answer by verifying the result from part *(i)*.

Problem 8.13. Suppose that the initial condition $F(x)$ is a smooth function. Investigate the solution of (8.54) and (8.55) close to the initial condition $(x,t,\xi) = (x_\mathrm{s}, t_\mathrm{s}, \xi_\mathrm{s})$. Show that if $t = t_\mathrm{s} + \tau$, with $\tau \ll 1$, then

$$s(t) \approx x_\mathrm{s} + F_\mathrm{s}\tau + \sqrt{\frac{8F_\mathrm{s}F_\mathrm{s}'^3}{3F_\mathrm{s}'''}}\,\tau^{3/2} + \cdots \tag{8.135}$$

where $F_\mathrm{s} \stackrel{\text{def}}{=} F(\xi_\mathrm{s})$, $F_\mathrm{s}' \stackrel{\text{def}}{=} F'(\xi_\mathrm{s})$ etc.

Problem 8.14. Consider the nonlinear signaling problem with $x > 0$ and $t > 0$:

$$u_t + uu_x = 0\,, \quad u(x,0) = 1\,, \quad u(0,t) = 1 + t\,. \tag{8.136}$$

(i) Sketch the characteristic diagram and show that a shock forms at $(\xi_s, t_s) = (1,1)$. *(ii)* Assuming that $u(x,t)$ is a conserved density, show that the shock path is $s(t) = (t+3)(3t+1)/16$, with $t > 1$.

Problem 8.15. Consider the signaling problem with $x > 0$ and $t > 0$:

$$u_t + uu_x = 0\,, \qquad u(x,0) = 0\,, \qquad u(0,t) = a(t)\,. \tag{8.137}$$

Let us use three different models for the boundary condition at $x = 0$:

$$a_1(t) = 1\,, \qquad a_2(t) = \frac{1}{1+t}\,, \qquad a_3(t) = t\,. \tag{8.138}$$

(i) Assuming that $u(x,t)$ is a conserved density, show that:

$$\int_0^\infty u(x,t)\,\mathrm{d}x = \tfrac{1}{2}\int_0^t a^2(t')\,\mathrm{d}t'\,. \tag{8.139}$$

(ii) Construct a single-valued solution using the shock condition to ensure that the global conservation law above is satisfied. *(iii)* Find another $a(t)$ for which you can calculate $s(t)$.

Solution. The location of the shock, $x = s(t)$, in the three cases is

$$s_1(t) = t/2\,, \qquad s_2(t) = \sqrt{1+t} - 1\,, \qquad s_3(t) = 3t^2/16\,. \tag{8.140}$$

Problem 8.16. The expansion fan (8.70) is a similarity solution in which x and t appear only in the combination $\eta = x/u_{\mathrm{m}}t$. Substitute $\rho(x,t) = R(\eta)$ into

$$\rho_t + c(\rho)\rho_x = 0\,,$$

and show that you obtain (8.69).

Problem 8.17. Calculate the function ρ_1 in the expansion (**??**). Obtain an explicit expression for ρ_1, assuming that F is given by (8.68). Is $\epsilon\rho_1 \ll \rho_0$ for all x as $\epsilon \to 0$ i.e., is the expansion uniformly valid?

Problem 8.18. Solve the expansion-fan problem

$$u_t - u^2 u_x = 0\,, \quad u(x,0) = H(-x)\,.$$

Sketch both the characteristic diagram and $u(x,1)$.

Problem 8.19. Solve the expansion-fan problem

$$u_t + uu_x = 0\,, \quad u(x,0) = H(x)\,.$$

Sketch both the characteristic diagram and $u(x,1)$.

Problem 8.20. Paint flowing down a wall has a thickness $\eta(x,t)$ governed by

$$\eta_t + \eta^2 \eta_x = 0 \,. \tag{8.141}$$

A stripe of paint is applied at $t = 0$ with

$$\eta(x,0) = \begin{cases} 1\,, & 0 < x < 1\,, \\ 0\,, & \text{otherwise.} \end{cases} \tag{8.142}$$

Find $\eta(x,t)$.

Problem 8.21. For the green light solution shown in figure 8.11 draw $\rho(x,t)$ as a function of x at $t =$ 0, 1/2, 1, 2, 4 and 5. Compute the strength of the trailing shock as a function of time.

Problem 8.22. Solve the traffic problem

$$\rho_t + c(\rho)\rho_x = 0\,, \quad \rho(x,0) = \rho_{\mathrm{J}} H(x)\,.$$

Draw the characteristic diagram.

Problem 8.23. Consider a more general traffic flow model with the flux function

$$f(\rho) = u_{\mathrm{m}}\rho \left[1 - (\rho/\rho_{\mathrm{J}})^n\right] \,, \tag{8.143}$$

where $n > 0$ is a parameter. The special case $n = 1$ is the standard model. Solve the green light problem, starting at (8.64), using this more general model. (You can check your answer by taking $n = 1$ and recovering the results from the lecture.) Suppose the last car (at $x = -\ell$) starts moving at t_* and passes through the light ($x = 0$) at $t = t_{**}$. Find the ratio t_{**}/t_* as a function of n and discuss the limits $n \to 0$ and $n \to \infty$.

Problem 8.24. Suppose that

$$f(\rho) = 120\sqrt{\rho}(1 - \sqrt{\rho})\,,$$

where ρ is cars per mile. At $t = 0$ (Noon) the caravan is 2 miles long. The airport is 15 miles away and the velocity of the lead car is

$$U = 120t\,, t \text{ measured in hours past noon.}$$

The position of the lead car is therefore $X(t) = 60t^2$ and so it reaches the airport at half past noon. The plane takes off at 12:35PM. Does the last car in the caravan arrive in time for the occupants to board the plane?

Problem 8.25. Draw $\rho(x,t)$ as a function of x for the solution in figure 8.13. (Note that in this figure $\rho_0 = 3\rho_{\mathrm{J}}/8$.) Choose values of t/τ to illustrate how the structure of the solution changes with time.

Problem 8.26. Repeat problem 1 assuming that $\rho_0 = 5\rho_{\mathrm{J}}/8$ (heavy traffic). The analog of figure 8.13 is figure 6.3.

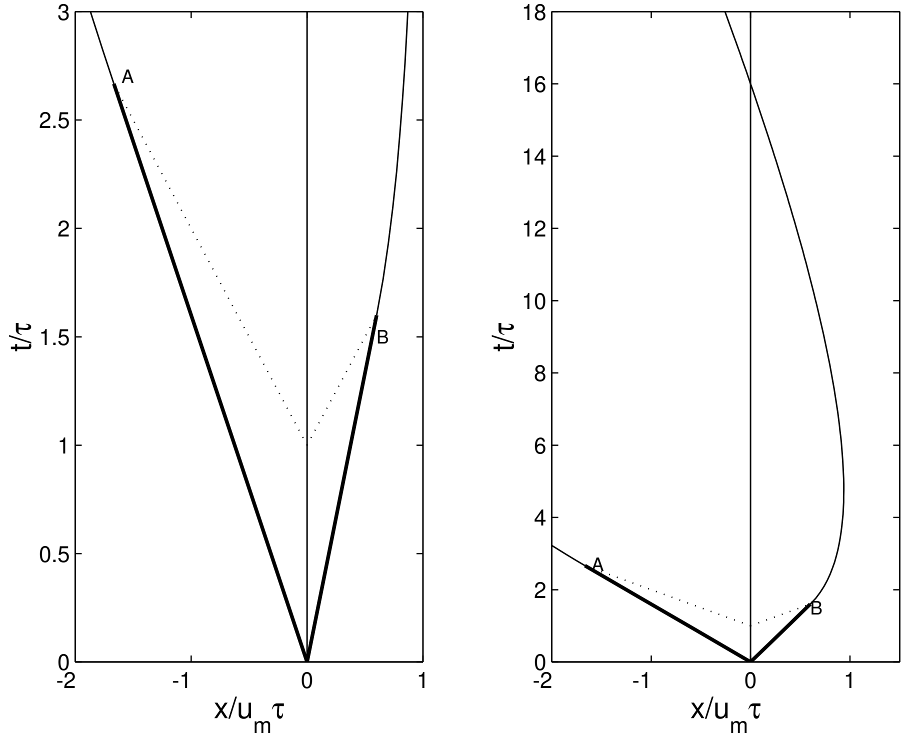


Figure 8.15: Shock trajectories if $\rho_0 = 5\rho_\mathrm{J}/8$ (heavy traffic). `rgprob.eps`

Problem 8.27. Let us develop an alternative theory for the speed of an ρ_J-plug behind a red light. Suppose the cars are all initially moving at $u(\rho_0)$ and consider car $n+1$ behind the light. The initial distance between this car and the light is then n/ρ_0. *(i)* How far will this car be from the light when it is forced to stop? Call this stopping distance X_n. *(ii)* Assuming instant decceleration from u_0 to rest, how long it take car $n+1$ to stop? Call this stopping time T_n. *(iii)* Eliminate n between X_n and T_n and show that the line of stopped vehicles moves backwards with the speed U_L in (8.86).

Problem 8.28. Give a simpler derivation of the clearence time in (8.96). First show that

$$\text{Number of cars through the light between } \tau \text{ and } \tau_\mathrm{c} = f_\mathrm{m}(\tau_\mathrm{c} - \tau)\,.$$

Also show that

$$\text{Number of cars slowed by the light} = f(\rho_0)\tau_\mathrm{c}\,.$$

Equating these two numbers show that

$$\tau_\mathrm{c} = \frac{f_\mathrm{m}}{f_\mathrm{m} - f_0}\,. \tag{8.144}$$

Show that for the standard model (8.144) reduces to (8.96).

Problem 8.29. Consider Burgers' equation

$$u_t + uu_x = \nu u_{xx}\,, \tag{8.145}$$

with the kinky initial condition

$$u = \begin{cases} 1 - |x|\,, & \text{if } |x| < 1, \\ 0\,, & \text{if } |x| > 1. \end{cases} \tag{8.146}$$

If we take the limit $\nu \to 0$ we obtain the problem discussed in section 8.4. *(i)* Find the thickness of the internal boundary layer which smooths out the shock at $x = s(t)$. *(ii)* We saw in (8.51) that $E(t) \propto t^{-1/2}$ as $t \to \infty$. On the other hand, from (8.100)

$$\dot{E} = -\nu \int_{-\infty}^{\infty} u_x^2 \,\mathrm{d}x\,. \tag{8.147}$$

Explain carefully how $E(t) \propto t^{-1/2}$ is consistent with (8.147) as $\nu \to 0$.

Problem 8.30. The hamburger equation

$$u_t + u^2 u_x = \epsilon u_{xx}\,, \tag{8.148}$$

with boundary conditions

$$\lim_{x\to+\infty} u(x,t) = 0\,, \qquad \text{and} \qquad \lim_{x\to-\infty} u(x,t) = 1\,, \tag{8.149}$$

has a traveling wave solution in which $u(x,t) = U(\xi)$ with $\xi \stackrel{\text{def}}{=} x - ct$. *(i)* Find the value of c and show that

$$-U + U^3 = 3\epsilon U' \tag{8.150}$$

where the prime denotes an ξ-derivative. *(ii)* Solve the ODE above and give an explicit expression for $U(\xi)$.

Problem 8.31. *(i)* Show by substitution that if $u(x,t)$ is a solution of Burgers' equation then so is $v(x,t) = u(x - ct, t) + c$. *(ii)* Show that the problem

$$u_t + uu_x = \nu u_{xx}\,, \tag{8.151}$$

with boundary conditions

$$\lim_{x\to-\infty} u(x,t) = L\,, \qquad \lim_{x\to-\infty} u(x,t) = R\,, \tag{8.152}$$

can be reduced to the parameterless form:

$$U_t + UU_X = U_{XX}\,, \qquad \text{with} \qquad \lim_{X\to\pm\infty} U(X,T) = \pm 1\,. \tag{8.153}$$

(iii) Use this transformation to deduce that the shock thickness is given by (8.113) without solving any PDEs.

Problem 8.32. Consider the cheeseburger equation

$$u_t + uu_x = -\nu u_{xxxx}\,, \tag{8.154}$$

where $\nu > 0$. *(i)* Explain the sign of the RHS. *(ii)* Slavishly follow the Burgers' travelling wave calculation — starting in (8.102) — and see how far you can get in finding a travelling wave solution of the cheeseburger equation. Make sure that show that the wave speed is given by $c = (R+L)/2$. *(iii)* Use the method in problem 8.31 to reduce the cheeseburger equation to a parameterless form. Deduce the thickness of the cheeseburger shock in terms of R, L and ν.

Problem 8.33. Construct some polynomial solutions of the heat equation, beginning with

$$\theta = x^3 + \dots\,, \qquad \text{and} \qquad \theta = x^4 + \cdots \tag{8.155}$$

Use the Cole-Hopf transformation to generate some less obvious solutions of Burgers equation.

Problem 8.34. Show that

$$\theta(x,t) = A\mathrm{e}^{\kappa q^2 t - qx} + B\mathrm{e}^{\kappa t p^2 - px} \tag{8.156}$$

is a solution of the diffusion equation. Use the Cole-Hopf transformation to find a solution of Burgers equation.

Problem 8.35. Consider the problem of finding a travelling-wave solution, $u(x,t) = u(z)$, with $z \stackrel{\text{def}}{=} x - ct$, of the elastic-medium equation

$$u_{tt} = u_{xx} + u_x u_{xx} + u_{xxxx}\,, \qquad \lim_{x\to-\infty} u = L\,, \qquad \lim_{x\to\infty} u = R\,. \tag{8.157}$$

(i) Show that $v \stackrel{\text{def}}{=} u_z$ satisfies

$$v_z^2 = (c^2-1)v^2 - \tfrac{1}{3}v^3\,. \tag{8.158}$$

(ii) Solve the ODE above by substituting

$$v = a\,\mathrm{sech}^2(pz) \tag{8.159}$$

and determining a and p in terms of the wave speed, c. *(iii)* Finally, determine c by ensuring that u has the correct limiting behaviour as $x \to \pm\infty$.

Lecture 9

Diffusion on a finite interval

9.1 Diffusion of a heat around a wire loop

Consider a loop of very thin wire. We suppose that the length L of the loop is much greater than the thickness of the wire, so that heat conduction in loop is governed by the one-dimensional diffusion equation[1]

$$U_T = \kappa U_{XX}\,, \qquad u(X,0) = U_0(X)\,. \tag{9.1}$$

The coordinate $0 < X < L$ runs around the loop. The shape of the loop is unimportant: it might be a circle or a square. What is important is that the point $X = 0$ is the same as the point $X = L$, and the solution is periodic with period L

$$U(X,T) = U(X+L,T)\,. \tag{9.2}$$

We can crack problems like this using Fourier series. To use the tools we've developed so far, we first change the coordinate so that the problem is posed on the fundamental interval $(-\pi, \pi)$ This is accomplished with the change of variable

$$x = \pi\frac{2X-L}{L}\,, \qquad t = \kappa\left(\frac{2\pi}{L}\right)^2 T\,. \tag{9.3}$$

As X goes from 0 to L, x runs from $-\pi$ to π. We write $u(x,t) = U(X,T)$. Then on the fundamental interval, the diffusion problem is

$$u_t = u_{xx}\,, \qquad u(x,0) = u_0(x)\,, \tag{9.4}$$

with the periodicity condition

$$u(x,t) = u(x+2\pi,t)\,. \tag{9.5}$$

It is easy to see that

$$\mathrm{e}^{inx-n^2t}\,, \tag{9.6}$$

[1] A more detailed description would require solving the three-dimensional diffusion equation. But since the thickness of the wire is small, the temperature rapidly becomes uniform in the transverse (y and z) directions. We're interested in the slower process of heat diffusion around the loop.

with n an integer, satisfies the diffusion equation and the periodicity condition. Therefore so does the Fourier series

$$u(x,t) = \sum_{n=-\infty}^{\infty} \tilde{u}_n \mathrm{e}^{\mathrm{i}nx - n^2 t} . \tag{9.7}$$

To fulfill the initial condition we use the Fourier method to determine the constants $\tilde{u}_n$ above:

$$\tilde{u}_n = \int_{\pi}^{\pi} \mathrm{e}^{-\mathrm{i}nx} u_0(x) \, \frac{\mathrm{d}x}{2\pi} . \tag{9.8}$$

For example, if the initial condition is $u_0(x) = \delta(x)$ then $\tilde{u}_n = 1/(2\pi)$ and we thus have the solution

$$u(x,t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}nx - n^2 t} . \tag{9.9}$$

Figure 9.1 shows this solution with the sum running from -20 to 20. The initial condition is therefore the Dirichlet kernel

$$D_{20}(x) = \frac{\sin\left[\left(20 + \frac{1}{2}\right) x\right]}{2\pi \sin(x/2)} . \tag{9.10}$$

The diffusion quickly removes the high wavenumbers from the sum so that by $t = 0.01$ the rapidly oscillatory initial condition has evolved into a beautiful smooth Gaussian.

There is another way of solving this partial differential equation. We can "unwrap" the periodic initial condition onto the whole of the real line so that the initial condition is

$$u_0(x) = \sum_{p=-\infty}^{\infty} \delta(x - 2\pi p) . \tag{9.11}$$

This initial condition is the Dirichlet comb: a periodic array of δ-functions separated with an interval of $\Delta x = 2\pi$. We know how to solve the diffusion equation with a δ-function initial condition, and we can then superimpose these solutions to obtain the complete solution

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \sum_{p=-\infty}^{\infty} \exp\left[-\frac{(x - 2\pi p)^2}{4t}\right] . \tag{9.12}$$

Notice that (9.12) is welcoming in the small-time limit: with $t \ll 1$ we need only one term in the series to obtain a good approximation to the solution. On the other hand, the Fourier series in (9.9) is effective at large time: with $t \gg 1$

$$u \approx \tfrac{1}{2\pi} \left[1 + 2\mathrm{e}^{-t} \cos x\right] . \tag{9.13}$$

We've neglected a term proportional to e^{-4t}. At $t = 1$ this term is smaller by a factor $\mathrm{e}^{-3} \approx 1/20$ than the term $\mathrm{e}^{-t} \cos x$.

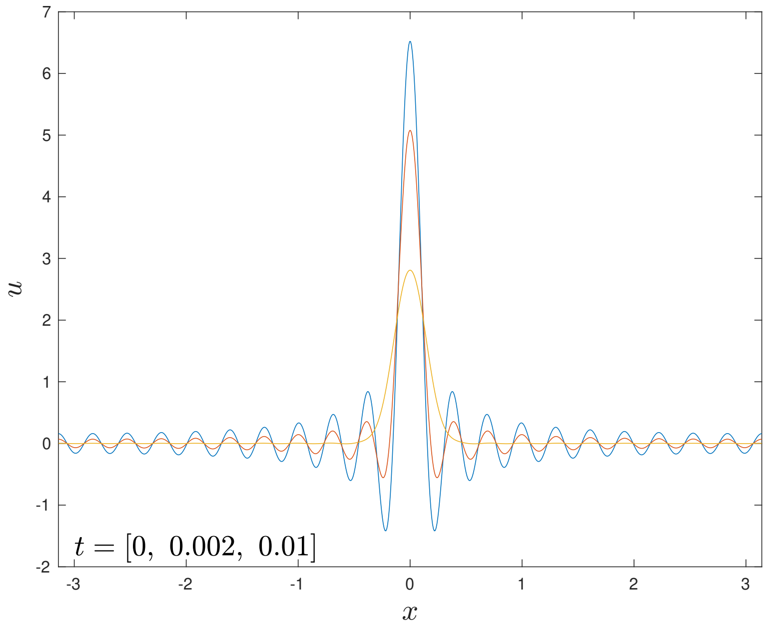


Figure 9.1: Three snapshots of the solution in (9.9) using 41 terms in the sum i.e., the sum runs from -20 to 20. The initial condition is the Dirichlet kernel. Diffusion acts quickly to smooth out the rapid oscillations. `dirKernelDiff.eps`

9.2 Other loopy partial differential equations

Let's consider other loopy PDEs defined on $-\pi < x < \pi$: we identify $x = -\pi$ with $x = +\pi$ and look for periodic solutions

$$u(x,t) = u(x+\pi,t)\,. \tag{9.14}$$

Some examples are

$$u_t + cu_x = 0\,, \qquad u_t = \beta u_{xxx}\,, \qquad u_t = -\nu u_{xxxx}\,. \tag{9.15}$$

We suppose that these PDEs come equipped with an initial conditions, $u(x,0) = f(x)$ with

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \mathrm{e}^{\mathrm{i}nx}\,. \tag{9.16}$$

Because the coefficients in (9.15) are constant, it is easy to see that the solution for each complex sinusoid in the sum has the form

$$u(x,t) = \mathrm{e}^{\mathrm{i}nx - \mathrm{i}\omega t}\,. \tag{9.17}$$

In the three examples in (9.15), the "dispersion relation" is

$$\omega = cn\,, \qquad \omega = \beta n^3\,, \qquad \omega = -\mathrm{i}\nu n^4\,. \tag{9.18}$$

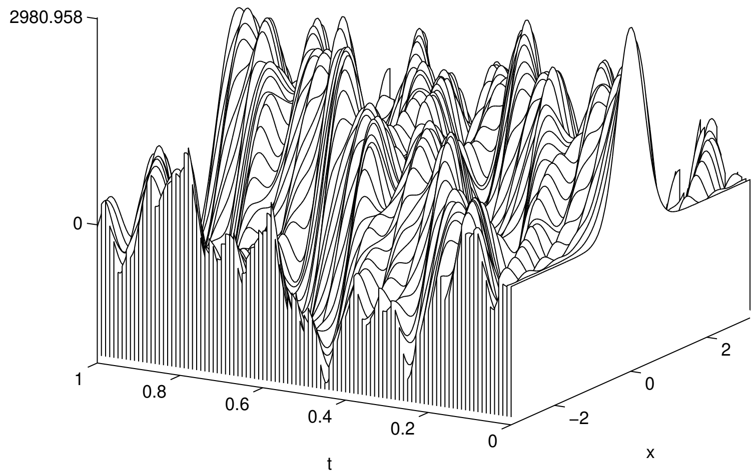


Figure 9.2: A solution of the dispersive wave equation with $\beta = 1$ and $\gamma = 8$. `dispersivewaterfall.eps`

The expressions above show that to solve the PDE there has to be a connection between the frequency ω and the wavenumber n i.e., the frequency is a function of the wavenumber. Once we have the dispersion relation, the solution of the PDE with the initial condition in (9.16) is simply

$$u(x,t) = \sum_{n=-\infty}^{\infty} f_n e^{inx - i\omega t} . \tag{9.19}$$

Example Solve $u_t = \beta u_{xxx}$ on the loop $-\pi \le x \le \pi$ with the initial condition

$$u(x,0) = e^{\gamma \cos x} . \tag{9.20}$$

Let's use a complex Fourier series to represent the initial condition. The coefficients in this series are

$$f_n = \int_{-\pi}^{\pi} e^{-inx+\gamma\cos x} \frac{dx}{2\pi} = I_n(\gamma) , \tag{9.21}$$

where I_n is the modified Bessel function. Thus the solution of the PDE is

$$\begin{aligned} u(x,t) &= \sum_{n=-\infty}^{\infty} I_n(\gamma) e^{i(nx-\beta n^3 t)} , \\ &= I_0(\gamma) + 2\sum_{n=1}^{\infty} I_n(\gamma)\cos\left(nx - \beta n^3 t\right) . \end{aligned} \tag{9.22}$$

We've used $I_n(\gamma) = I_{-n}(\gamma)$ to write the series in terms of cosines. The solution is shown in Figure 9.2.

```
% Solution of the dispersive wave equation on a loop
beta=1; gamma = 8; X = linspace(-pi,pi,100); T = [0:0.01:1];
N=40; % truncation of the Fourier series
udata = zeros(length(T),length(X));
```

```
nloop = 0;
for tt = T
    nloop = nloop + 1;
    u = besseli(0,gamma)*ones(1,length(X));
    for n = 1:N
        u = u + 2*besseli(n,gamma)*cos(n*X - beta*n^3*tt);
    end
    udata(nloop,:)=u;
end
waterfall(X,T,udata), xlabel('x'), ylabel('t')
colormap([0 0 0]), axis([-pi pi 0 max(T) ]), grid off
set( gca,'ztick',[-0.1*exp(-gamma) exp(gamma)] )
view(-60,15),  pbaspect([ 1 1 0.75])
```

To compute the solution in Figure 9.2, the Fourier series in (9.22) has been truncated at $n = 40$ terms. Figure 9.3 indicates that $n = 40$ is overkill: $I_n(\gamma)$ decreases very rapidly with n. This fast convergence is because the initial condition in (9.20) is smooth i.e., infinitely differentiable.

9.3 Diffusion on the interval $0 \le x \le \ell$: the initial value problem

Suppose we need to solve the diffusion equation

$$u_t = \kappa u_{xx} \,, \tag{9.23}$$

on the interval $0 < x < \ell$ with boundary conditions

$$u(0,t) = u(\ell,t) = 0 \,, \tag{9.24}$$

and some initial condition

$$u(x,0) = f(x) \,. \tag{9.25}$$

$x' = \pi x/\ell$,
$t' = \pi^2 \kappa t/\ell^2$

We suppose that $f(x)$ is square integrable so that there is no problem expanding in a Fourier series. With suitable nondimensionalization we can take $\ell \to \pi$ and $\kappa \to 1$.

Now we ignore the initial condition for a just a bit and find solutions of the PDE which also satisfy the boundary conditions. We can do this using *separation of variables*:

$$u(x,t) = \phi(x)\psi(t) \,, \qquad \Rightarrow \qquad \frac{\psi'}{\psi} = \frac{\phi''}{\phi} = -\lambda^2 \,. \tag{9.26}$$

We get a Sturm-Liouville eigenproblem for $\phi(x)$:

$$\phi'' + \lambda^2 \phi = 0 \,, \qquad \phi(0) = \phi(\pi) = 0 \,. \tag{9.27}$$

There is a complete set of eigenfunctions

$$\phi_n(x) = \sin(nx) \,, \qquad \lambda = n \,. \tag{9.28}$$

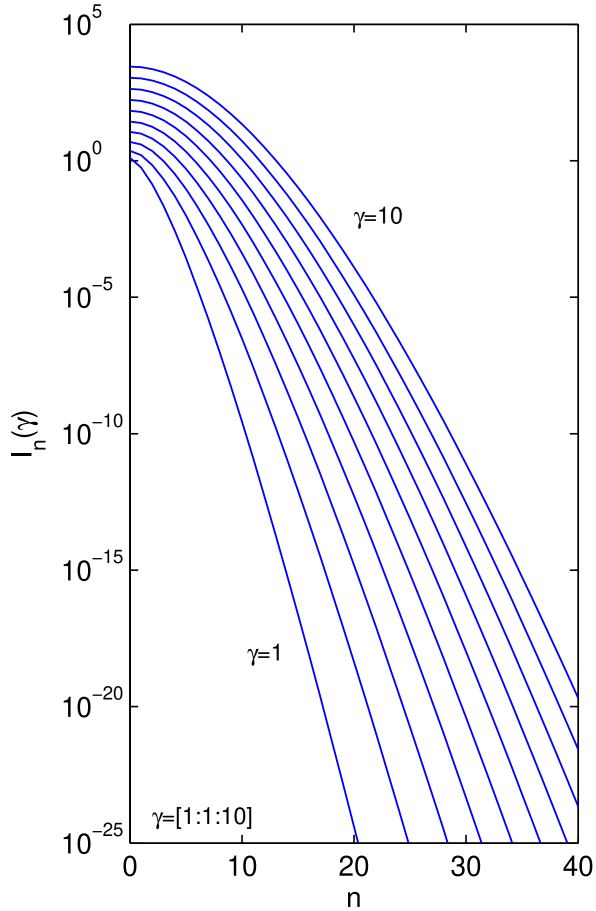


Figure 9.3: The modified Bessel function $I_n(\gamma)$ as a function of n at fixed values of γ between 1 and 10. This shows that coefficients of the Fourier series in (9.22) decrease very quickly with n i.e., the series converges very quickly. To better estimate the rate of convergence you can apply the saddle point method to analyze the integral in (9.21) for $n \gg 1$ (or look-up the relevant asymptotic expansion). `besselconverge.eps`

Thus we have an infinite number of solutions (one for each n) of our diffusion problem:

$$u(x,t) = \mathrm{e}^{-n^2 t} \sin nx \,. \tag{9.29}$$

You can check in seconds by substitution that this is a solution of the diffusion equation for every n.

We are solving a linear problem and so we can use superposition to write down a more complicated solution as a half-range Fourier expansion

$$u(x,t) = \sum_{n=1}^{\infty} f_n \mathrm{e}^{-n^2 t} \sin nx \,. \tag{9.30}$$

Our hope is by picking the coefficients f_n we can ensure that the solution above satisfies the initial condition $\zeta(x,0) = f(x)$. This leads to

$$f_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, \mathrm{d}x \,. \tag{9.31}$$

Notice we are working on the half range with a sine series so we have implicitly used an *odd* extension of $f(x)$ to $-\pi < x < 0$.

As soon as t is a little bit greater than zero we can happily plug (9.30) into the PDE by blithely differentiating under the $\sum$ -sign — because of the $\exp(-n^2 t)$ this series converges very quickly. Hence $u(x,t)$ in (9.30) satisfies the PDE, the boundary conditions and the initial condition.

Example: $f(x) = 1$

Using the Fourier series for $\mathrm{sqr}(x)$, the solution of (9.23) through (9.25) with $f(x) = 1$ is

$$u(x,t) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin[(2k+1)x]}{2k+1} \mathrm{e}^{-(2k+1)^2 t} \,. \tag{9.32}$$

This solution, using 20 terms in the series (9.32), is shown in figure 9.4. The original problem back in (9.23) through (9.25) was posed on the interval $0 \le x \le \pi$; in figure 9.4 I have plotted (9.32) on a larger interval so you can see how we've used symmetry to extend the half-range to the full range.

Because we can use only a finite number of terms in the series the initial condition exhibits Gibbs' phenomenon. But these rapid oscillations disappear very quickly because of diffusion. Again, the lesson is that a little bit of diffusion is an effective low-pass filter.

AKA a "Gaussian filter".

Inconsistent boundary conditions and initial conditions

Suppose, as in the example above, that either $f(0)$ or $f(\pi)$ is nonzero. Then there is a 'corner layer' at $(x,t) = (0,0)$ or at $(x,t) = (\pi,t)$ as the initial condition adjusts to the boundary condition. (Locally we have something which

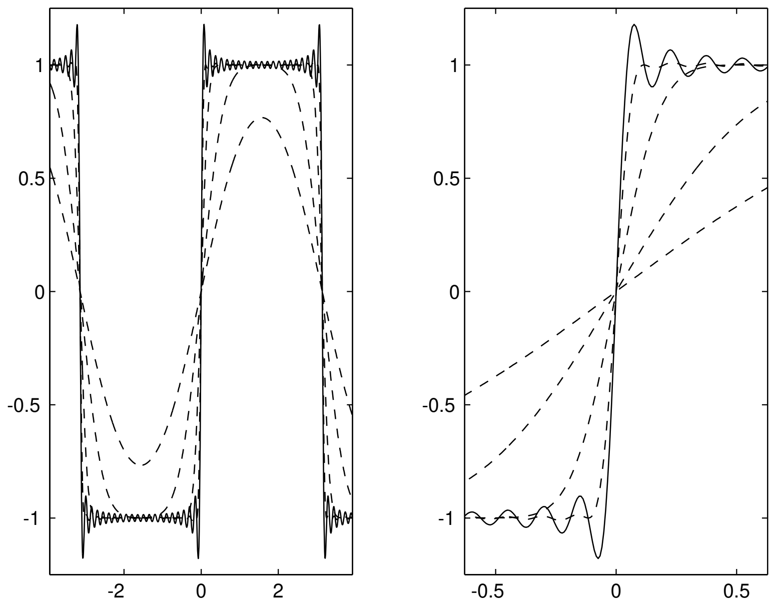


Figure 9.4: Solution of $u_t = u_{xx}$ with the initial condition $u(x,0) = \mathrm{sqr}(x)$. The initial square wave is represented approximately taking 20 terms in (9.32). The solution is shown at $t = 0.001$, 0.01, 0.1 and 0.5. Viewed on the half-range $0 < x < \pi$ this is the solution of a problem with inconsistent initial and boundary data. `sqrDiff.eps`

looks just like the erf solution from an earlier lecture.) How quickly do the Fourier coefficents in (9.31) decrease? We use integration by parts to estimate

$$
\begin{aligned}
f_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,\mathrm{d}x\,, &(9.33)\\
&= -\frac{2}{\pi}\int_0^\pi f(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos nx}{n}\right)\mathrm{d}x\,,\\
&= \frac{2}{n\pi}\left[f(0) - (-1)^n f(\pi)\right] + \frac{2}{n\pi}\int_0^\pi f'(x)\cos nx\,\mathrm{d}x\,. &(9.34)
\end{aligned}
$$

Invoking the Riemann-Lebesgue lemma the second term on the RHS is much less than the first as n beomes large. Thus $b_n \sim n^{-1}$ as $n \to \infty$. The Fourier series of $\mathrm{sqr}(x)$ illustrates this slow convergence.

Suppose $f(0) = f(\pi) = 0$ but $f''(0)$ or $f''(\pi)$ is nonzero. Notice that if we evaluate the PDE at $x = 0$ or π, where $u = u_t = 0$, we conclude that $u_{xx} = 0$. Thus we still have an inconsistency between initial and boundary data at the corners of the domain. In this case repeated integration by parts shows that $b_n \sim n^{-3}$. Because of this improved convergence there is no Gibbs phenomenon in our representation of $f(x)$. However if we calculate $f''(x)$ by twice differentiating the Fourier series we again encounter the problem. It is clear that this process can be continued to show that Gibbs phenomenon and nonuniform convergence are associated with the corner layers which are created by inconsistent boundary and initial data.

Example:

$$f(x) = x(\pi - x)$$

.

Exercise: With repeated integration by parts find the next non-zero term in (9.34). You'll find that this term is $\sim n^{-3}$ (not $\sim n^{-2}$).

9.4 Inhomogeneous boundary conditions

Another way to solve the initial value problem in (9.23) through (9.25) is to write down a Fourier series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin nx \,, \tag{9.35}$$

(9.23) – (9.25):

$$u_t = u_{xx}$$

$$u(0,t) = 0 \,, \quad u(\pi,t) = 0$$

$$u(x,0) = f(x)$$

and plug it into the PDE. We blithely "differentiate under the Σ sign" and quickly see that *modal amplitudes* $u_n(t)$ satisfy simple ODEs

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = -n^2 u_n \,. \tag{9.36}$$

We obtain $u_n(0)$ by applying the initial condition so we recover the earlier solution in (9.30).

The procedure above seems natural provided $u(x,t)$ satisfies homogenous boundary conditions at $x = 0$ and π. But suppose instead we have inhomogenous boundary conditions

$$u(0,t) = a(t) \,, \qquad u(\pi,t) = b(t) \,. \tag{9.37}$$

It might seem that the guess in (9.35) is wrong because this function *always* vanishes on the boundary. But apart from a few isolated embarrassments at points of discontinuity we can expand *any* function in a Fouier series. So our guess in (9.35) *must* work! The problem is that we can't take two x derivatives "under the Σ sign" because the resulting series doesn't converge.

To get around this problem, we use the method of Galerkin projection. To galerk we multiply the diffusion equation by $2 \sin nx/\pi$ and integrate from $x = 0$ to $x = \pi$:

$$\underbrace{\frac{2}{\pi}\int_0^{\pi} u_t \sin nx \,\mathrm{d}x}_{\mathrm{d}u_n/\mathrm{d}t} = \frac{2}{\pi}\int_0^{\pi} u_{xx} \sin nx \,\mathrm{d}x \tag{9.38}$$

Fourier coefficient:

$$u_n(t) = \frac{2}{\pi}\int_0^{\pi} u \sin nx \,\mathrm{d}x$$

The right-hand side is handled with integration-by-parts (twice):

$$\frac{2}{\pi}\int_0^{\pi} u_{xx} \sin nx \,\mathrm{d}x = \frac{2}{\pi}\left[\sin nx \, u_x - n \cos nx \, u\right]_0^{\pi} - n^2 u_n \,, \tag{9.39}$$

$$= na(t) - n(-1)^n b(t) - n^2 u_n \,. \tag{9.40}$$

$\cos(n\pi) = (-1)^n$

$\sin(n\pi) = 0$

Hence, with inhomogeneous boundary conditions, the modal-amplitude equation (9.38) becomes an inhomogenous ODE:

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} + n^2 u_n = \frac{2n}{\pi}\left[a - (-1)^n b\right] \,. \tag{9.41}$$

This is OK, but not great because convergence is slow — for large n there is a dominant balance

$$n^2 u_n \approx \frac{2n}{\pi}\left[a - (-1)^n b\right] , \tag{9.42}$$

implying that $u_n \sim n^{-1}$. In fact, this slow convergence is precisely why our solution manages to satisfy the inhomogeneous BC as we approach $x = 0$ and $x = \pi$ from the *interior* of the interval.

Improvement of convergence

We can improve convergence by reformulating the PDE so that it has homogenous boundary conditions:

$$u(x,t) = a(t) + (b(t) - a(t))\,\frac{x}{\pi} + w(x,t)\,. \tag{9.43}$$

The function $w(x,t)$ defined above vanishes at both $x = 0$ and $x = \pi$. Throwing this into the PDE shows that

$$w_t - w_{xx} = -\dot{a}(t)\left(1 - \frac{x}{\pi}\right) - \dot{b}(t)\frac{x}{\pi}\,, \tag{9.44}$$

with boundary conditions

$$w(0,t) = w(\pi,t) = 0\,. \tag{9.45}$$

Now we have an inhomogeneous equation rather than inhomogeneous boundary conditions. We use a Fourier series to represent w:

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t)\sin nx\,, \tag{9.46}$$

with

$$w_n(t) = \frac{2}{\pi}\int_0^{\pi} w(x,t)\sin nx\,\mathrm{d}x\,. \tag{9.47}$$

This is progress because the w series converges faster than the u-series: $w_n \sim n^{-3}$, while $u_n \sim n^{-1}$.

Proof is integration by parts (twice). All of the terms which fall outside the integral are zero. Invoke Riemann-Lebesgue. Then we expect $w_n \sim n^{-3}$ — a factor n^{-1} faster than the Fourier expansion of $u(x,t)$. This improved convergence is because w satisfies homogenous boundary conditions.

Example: Solve $u_t = u_{xx}$ with boundary conditions $u(0,t) = 0$ and $u(\pi,t) = 1 - \exp(-\alpha t)$.

9.5 Inhomogeneous equations

We still have to determine $w_n(t)$ in (9.44) and (9.46). We might as well consider the general case of an inhomogeneous diffusion equation

$$w_t - w_{xx} = h(x,t)\,, \tag{9.48}$$

with the homogeneous boundary conditions

$$w(0,t) = w(\pi,t) = 0\,. \tag{9.49}$$

Again we Galerk by projecting the PDE (9.48) onto $\sin nx$: multiply the PDE by $\sin nx$ and integrating we have

$$\int_0^\pi \sin nx\,[w_t - w_{xx}]\,\mathrm{d}x = \int_0^\pi \sin nx\, h(x,t)\,\mathrm{d}x\,. \tag{9.50}$$

We get

$$\dot{w}_n + n^2 w_n = h_n\,, \tag{9.51}$$

where $w_n(t)$ and $h_n(t)$ are the coefficients of the Fourier sine-series of $w(x,t)$ and $h(x,t)$.

A crucial point is that in the Galerkin procedure we *never* take ∂_x^2 under the Σ-sign.

OK, so now we have an ODE, (9.51), for $w_n(t)$ — how fast does w_n decrease as $n \to \infty$? That is how rapidly does (9.46) converge? Since $w(x,t)$ has homogeneous boundary conditions we expect $w_n \sim n^{-3}$. One way to see this to argue that as $n \to \infty$ the dominant balance in (9.51) is

$$w_n \sim h_n/n^2\,. \tag{9.52}$$

Thus if $h_n \sim n^{-p}$ then $w_n \sim n^{-p-2}$. Typically if the source is nonzero at $x = 0$ and π then $p = 1$ and $w_n \sim n^{-3}$. This means we can differentiate the Fourier series twice and still maintain convergence. As in (9.44)

Example: inhomogenous boundary conditions again

For the problem back in (9.44) and (9.46) we have the source term

$$h(x,t) = -\dot{a}(t)\left(1 - \frac{x}{\pi}\right) - \dot{b}(t)\frac{x}{\pi}\,, \tag{9.53}$$

and (9.51) is

$$\dot{w}_n + n^2 w_n = \frac{2}{n\pi}\left[(-1)^n \dot{b} - a\right]\,. \tag{9.54}$$

With $n \gg 1$ there is a two-term dominant balance in (9.54) between the second term on the right-hamd side and the forcing on the left and thus

$$w_n \sim \frac{2}{n^3\pi}\left[(-1)^n \dot{b} - a\right]\,, \qquad \text{as } n \to \infty. \tag{9.55}$$

Thus the w-series is more rapidly convergent than the u-series with $u_n \sim n^{-1}$.

(9.44): $$w_t - w_{xx} = -\dot{a}(t)\left(1 - \frac{x}{\pi}\right) - \dot{b}(t)\frac{x}{\pi}$$

We gild the lily and produce an even more rapidly convergent series by returning to (9.44) and just for the moment dropping the term w_t. We can then integrate the remaining terms,

$$w_{xx} \stackrel{?}{=} \dot{a}(t)\left(1 - \frac{x}{\pi}\right) + \dot{b}(t)\frac{x}{\pi}\,, \tag{9.56}$$

to obtain

$$w \stackrel{?}{=} -\frac{\dot{a}}{6\pi}\left(2\pi^2 x - 3\pi x^2 + x^3\right) - \frac{\dot{b}}{6\pi}\left(\pi^2 x - x^3\right) . \tag{9.57}$$

Above we have applied the boundary conditions that $w = 0$ at $x = 0$ and π

Thus we are motivated to represent the solution of the full version of (9.44) as the approximate solution in (9.57) plus a residual:

$$w = -\frac{\dot{a}}{6\pi}\left(2\pi^2 x - 3\pi x^2 + x^3\right) - \frac{\dot{b}}{6\pi}\left(\pi^2 x - x^3\right) + y(x,t) . \tag{9.58}$$

Substituting (9.58) into (9.44) we have

$$y_t - y_{xx} = \frac{\ddot{a}}{6\pi}\left(2\pi^2 x - 3\pi x^2 + x^3\right) + \frac{\ddot{b}}{6\pi}\left(\pi^2 x - x^3\right) . \tag{9.59}$$

When we galerk (9.59) we anticipate that the Fourier coefficients of the right-hand side will be $\sim n^{-3}$ as $n \to \infty$ and thus the Fourier coefficients of $y(x,t)$ will be $\sim n^{-5}$ — this is faster by a factor of n^{-2} than $w_n \sim n^{-3}$. We can, moreover, continue gilding and pick up an extra factor of n^{-2} each time.

Exercise: Explain why, in advance of any calculation, expects the Fourier coefficients of the right-hand side of (9.59) to be $\sim n^{-3}$.

I suspect, however, that we may have reached the point of diminishing returns. To summarize the solution of $u_t = u_{xx}$ with inhomogenous boundary conditions in (9.37) can be written as

$$u(x,t) = \frac{a}{\pi}(\pi - x) - \frac{\dot{a}}{6\pi}\left(2\pi^2 x - 3\pi x^2 + x^3\right) + \frac{b}{\pi}x - \frac{\dot{b}}{6\pi}\left(\pi^2 x - x^3\right) + y(x,t) , \tag{9.60}$$

where

$$y = \sum_{n=1}^{\infty} y_n \sin nx \tag{9.61}$$

and

$$\dot{y}_n + n^2 y_n = \tag{9.62}$$

9.6 A perverse solution of the initial value problem

Let's return to the initial value problem from section 9.3. Using dimensionless variables the diffusion equation is

$$u_t = u_{xx} . \tag{9.63}$$

The problem is posed on the interval $0 < x < \pi$ with boundary conditions

$$u(0,t) = u(\pi,t) = 0 , \tag{9.64}$$

and an initial condition

$$u(x,0) = f(x) . \tag{9.65}$$

Let's solve this problem, representing the solution as a cosine series

$$u(x,t) = a_0(t) + \sum_{k=1}^{\infty} a_k(t)\cos(kx)\,, \tag{9.66}$$

where

$$a_0 = \frac{1}{\pi}\int_0^\pi u(x,t)\,\mathrm{d}x\,, \qquad \text{and} \qquad a_{k\geq 1} = \frac{2}{\pi}\int_0^\pi u(x,t)\,\cos kx\,\mathrm{d}x \tag{9.67}$$

Each term term in the series on the right of (9.68) violates the boundary condition (9.64). But the cosine functions are complete on the interval $0 \leq x \leq \pi$ so it is always possible the solution of this problem in the form (9.68). The series satisfies the boundary condition provided that

$$0 = a_0(t) + \sum_{k=1}^{\infty} a_k(t)\,, \qquad \text{and} \qquad 0 = a_0(t) + \sum_{k=1}^{\infty}(-1)^k a_k(t)\,. \tag{9.68}$$

We obtain the evolution equations for a_k by Galerking. For $k = 0$ we find

$$\dot{a}_0 = \frac{1}{\pi}\left[\theta_x(\pi) - \theta_x(0)\right]\,, \tag{9.69}$$

and for $k \geq 1$

$$\dot{a}_k + k^2 a_k = \frac{2}{\pi}\left[(-1)^k\theta_x(\pi) - \theta_x(0)\right]\,. \tag{9.70}$$

With (9.68), (9.69) and (9.70) we have enough equations to determine all the a_k's.

Now let's restrict attention to initial conditions that are even about $\pi/2$ e.g. $f(x) = 1$. Then, from symmetry, all the odd terms in on the right of (9.68) are zero. We consider the remaining even terms and write $k = 2m$ so that

The general initial condition can be written as the sum of an even and odd functions. With linear superposition there is no loss of generality in considering the odd and even parts separately.

$$\dot{a}_0 = \frac{2}{\pi}\theta_x(\pi)\,, \tag{9.71}$$

and

$$\dot{a}_{2m} + (2m)^2 a_{2m} = \frac{4}{\pi}\theta_x(\pi)\,, \tag{9.72}$$

$$= 2\dot{a}_0\,, \tag{9.73}$$

$$= -2\sum_{m=1}^{\infty} a_{2m}\,, \tag{9.74}$$

where we have used the boundary conditions in (9.68) in passing from (9.73) to (9.74).

Incomplete

9.7 Problems

Problem 9.1. Use a Fourier series to solve the diffusion equation $u_t = u_{xx}$ on the periodic domain $\pi < x \le \pi$ with the initial condition $u(x, 0) = \exp(x)$. Truncate the Fourier series and visualize this solution with MATLAB; make sure your figures show the disappearance of Gibbs oscillations at very small time.

Problem 9.2. Solve (9.23) through (9.25) with $f(x) = x(\ell - x)$.

Problem 9.3. Consider a diffusion problem defined on the interval $0 \le x \le \ell$:

$$u_t = \kappa u_{xx}\,, \qquad u(0, t) = 0\,, \qquad u_x(\ell, t) = 0\,, \tag{9.75}$$

with initial condition $u(x, t) = 1$. *(i)* If you use separation of variables then it is easy to anticipate that you'll find a Sturm-Liouville eigenproblem with sinusoidal solutions. Sketch the first two eigenfunctions *before* doing this algebra. Explain why you are motivated to nondimensionalize so that $0 \le x \le \pi/2$ (a *quarter-range* expansion). *(ii)* With $\ell \mapsto \pi/2$ and $\kappa \mapsto 1$, work out the Sturm-Liouville algebra and find the eigenfunctions and eigenvalues just as we did in (9.26) through (9.29). *(iii)* With $f(x) = 1$ find the solution as a Fourier series and use MATLAB to visualize the answer.

Problem 9.4. Consider the inhomogeneous diffusion equation

$$\theta_t - \theta_{xx} = \frac{1}{\sqrt{x(\pi - x)}}\,, \tag{9.76}$$

with boundary conditions $\theta(0, t) = \theta(\pi, t) = 0$ and initial condition $\theta(x, 0) = 0$. *(i)* Using a Fourier sine series,

$$\theta(x, t) = \sum_{n=1}^{\infty} \theta_n(t) \sin nx\,, \tag{9.77}$$

and the Galerkin procedure, find the ODE's satisfied by the modal amplitudes $\theta_n(t)$. *(ii)* Your answer will involve the integral

$$h_n \stackrel{\text{def}}{=} \frac{4}{\pi} \int_0^{\pi/2} \frac{\sin nx}{\sqrt{x(\pi - x)}} \mathrm{d}x\,, \qquad \text{with } n \text{ odd.} \tag{9.78}$$

Using techniques from part B, find the $n \to \infty$ asymptotic expansion of h_n above. *(iii)* Solve the initial value problem. At large times, $\theta_n \sim n^{-q}$ as $n \to \infty$. Find q.

Problem 9.5. Continuing with the previous problem, show that

$$h_n = 2 \sin\left(\frac{n\pi}{2}\right) J_0\left(\frac{n\pi}{2}\right)\,, \tag{9.79}$$

where J_0 is the zeroth order Bessel function. Use MATLAB to sum the truncated Fourier series for the source $[x(\pi - x)]^{-1/2}$ and the solution $\theta(x, t)$ with 100

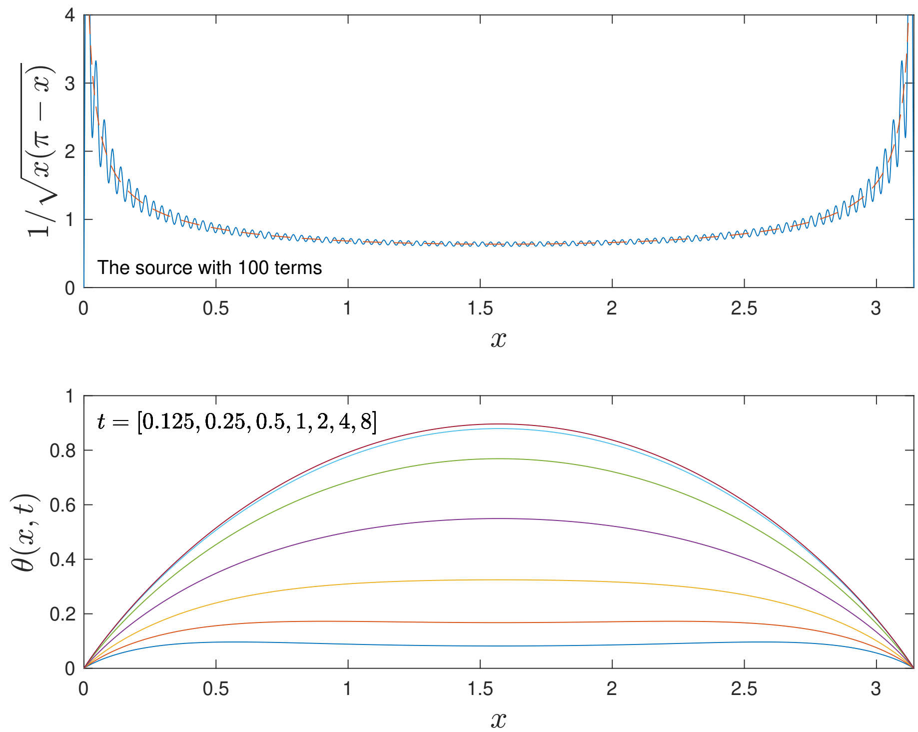


Figure 9.5: Fourier series solution of the forced diffusion equation (9.76). The top panel shows the source and the bottom panel the solution at selected times.

terms — see figure 9.5. Check your result by finding the final ($t \to \infty$) steady solution of (9.76) and comparing it with the Fourier series solution.

Hint: change variables in (9.78) and recall the integral representation

$$J_0(z) = \frac{2}{\pi}\int_0^{\pi/2} \cos(z\cos v)\,\mathrm{d}v\,. \tag{9.80}$$

Problem 9.6. Use a Fourier series expansion to solve the loopy problem

$$u_t = u(x,t) - u(x-\ell,t)\,, \qquad u(x,0) = \exp[\gamma\cos x]\,. \tag{9.81}$$

The domain is $-\pi < x < \pi$ and $u(x+2\pi,t) = u(x,t)$; ℓ is a "delay-length". Use MATLAB to visualize the solution for selected values off ℓ. Discuss the special cases $\ell = \pi$ and $\ell = 2\pi$.

Problem 9.7. A rod occupies $1 \le x \le 2$ and the thermal conductivity depends on x so that diffusion equation is

$$u_t = (x^2 u_x)_x\,. \tag{9.82}$$

The boundary and initial conditions are

$$u(1,t) = u(2,t) = 0\,, \qquad u(x,0) = 1\,. \tag{9.83}$$

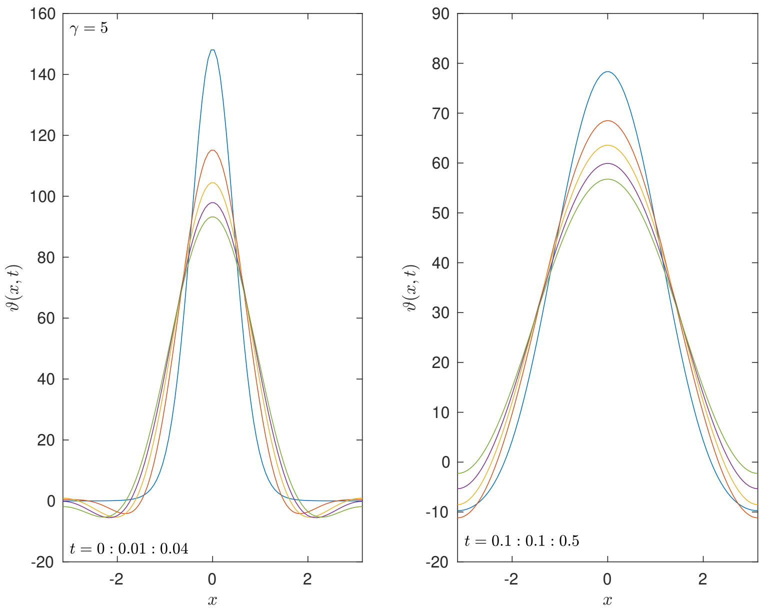


Figure 9.6: Fourier series solution of the hyperdiffusion equation in problem 9.8. Left panel shows short time evolution and right is longer time.

(i) The total amount of heat in the rod is

$$H(t) = \int_1^2 u(x,t)\,\mathrm{d}x\,. \tag{9.84}$$

Show that $H(0) = 1$ and

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 4u_x(2,t) - u_x(1,t)\,. \tag{9.85}$$

Physically interpret the two terms on the right hand side above. What is the sign of the $u_x(2,t)$ and the sign $u_x(1,t)$? *(ii)* Before solving the PDE, show that roughly 61% of the heat escapes through $x = 2$. (There is a simple analytic expression for the fraction $0.61371\cdots$ which you should find.) *(iii)* Use separation of variables to show that the eigenfunctions are

$$\phi_n(x) = \frac{1}{\sqrt{x}} \sin\left(\frac{n\pi \ln x}{\ln 2}\right)\,. \tag{9.86}$$

Find the eigenvalue which is associated with the n'th eigenfunction. *(iv)* Use modal orthogonality to find the series expansion of the initial value problem.

Problem 9.8. Solve the hyper-diffusion problem $\vartheta_t = -\vartheta_{xxxx}$ with initial condition $\vartheta(x,0) = \exp[\gamma\cos x]$ on the loop $-\pi < x \le x$. With $\gamma = 5$, use MATLAB to show snapshots of the solution at selected times. Does the hyperdiffusion equation have a maximum principle?

Lecture 10

The equations of Laplace and Poisson

10.1 Examples of the Poisson's equation

The diffusion equation with a source function $s(\boldsymbol{x}, t)$ is

$$\theta_t = \kappa \triangle \theta + s \,, \tag{10.1}$$

where, in three-dimensional space, the Laplacian is

$$\triangle = \partial_x^2 + \partial_y^2 + \partial_z^2 \,. \tag{10.2}$$

Let's consider the case of unbounded space and a steady source i.e., s depends only on $\boldsymbol{x} = (x, y, z)$. Then we can look for a steady solution of (10.1) i.e., $u_t = 0$ and

$$\kappa \triangle \theta + s = 0 \,. \tag{10.3}$$

This is Poisson's equation.

As another example, the gravitational potential $\varphi(\boldsymbol{x})$ associated with a mass density $\rho(\boldsymbol{x})$ is obtained by solving

$$\triangle \varphi = 4\pi G \rho \,, \tag{10.4}$$

where G is the gravitational constant. Once we possess φ, the force of gravity on a mass m is $-m\boldsymbol{\nabla}\varphi$. In electrostatics Poisson's equation also determines the electrical potential associated with a charge density.

Yet another example is incompressible irrotational fluid mechanics in which the fluid velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ has two important properties

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \,, \qquad \text{and} \qquad \boldsymbol{\nabla} \times \boldsymbol{u} = 0 \,. \tag{10.5}$$

Because the vorticity is zero we can represent the velocity as a the gradient of a velocity potential, $\boldsymbol{u} = \boldsymbol{\nabla}\phi$. Then, because of incompressibility, the potential satisfies Laplace's equation:

$$\triangle \phi = 0 \,. \tag{10.6}$$

What is the Laplacian, and why does it occur so frequently?

10.2 The δ-function in $d = 3$

Suppose we have a small, intense source of heat sitting at the origin of a conducting medium e.g., a hot radioactive pellet at $\boldsymbol{x} = 0$ surrounded by concrete. We model this heat source as $s(\boldsymbol{x}) = q\delta(\boldsymbol{x})$, so that steady-state Poisson's equation is

$$\kappa \triangle \theta = -q\delta(\boldsymbol{x}) \,. \tag{10.7}$$

The constant q is the total strength of the heat source and $\delta(\boldsymbol{x})$ is a three-dimensional δ-function i.e, a quantity localized at the point $\boldsymbol{x} = (x, y, z) = 0$, but with non-zero integral. Another example, used in theory of gravity, is a point mass i.e., an infinitesimally small particle with non-zero mass. To model this situation we use the mass density $\rho(\boldsymbol{x}) = m\delta(\boldsymbol{x})$ in (10.4). Apart from notation the resulting PDE is the same as (10.7).

Before solving (10.7) let's discuss how this idealized function $\delta(\boldsymbol{x})$ is constructed. To make a three-dimensional δ-function we consider a sequence of functions with a parameter ϵ:

$$\delta_\epsilon(\boldsymbol{x}) = \frac{1}{\epsilon^3} F\left(\frac{r}{\epsilon}\right) \,. \tag{10.8}$$

Notice that ϵ has the dimensions of length. The function $F(s)$ is normalized by requiring that

$$4\pi \int_0^\infty F(s) s^2 \, \mathrm{d}s = 1 \,. \tag{10.9}$$

The conditions above ensure that if we integrate over all space then

$$\int \delta_\epsilon(\boldsymbol{x}) \, \mathrm{d}V = 1 \,, \qquad \text{independent of } \epsilon. \tag{10.10}$$

$\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$

$\mathrm{d}V = \mathrm{d}\Omega \, r^2 \mathrm{d}r$

$\int \mathrm{d}\Omega = 4\pi$

Some popular choices for $F(s)$ are the Gaussian

$$F(s) = \pi^{-3/2} \mathrm{e}^{-s^2} \,, \tag{10.11}$$

or the top-hat

$$F(s) = \frac{3}{4\pi} \begin{cases} 1 \,, & \text{if } s < 1; \\ 0 \,, & \text{if } s > 1. \end{cases} \tag{10.12}$$

Any equation involving $\delta(\boldsymbol{x})$ can be interpreted — or *must* be interpreted — by backing up to the non-singular function $\delta_\epsilon(\boldsymbol{x})$ and taking the limit as $\epsilon \to 0$. This is how **BO** explain the one-dimensional δ-function in their section **1.5** — see also lecture 4. It is the same here, except that in $d = 3$ we have a factor ϵ^{-3} in (10.8). In $d = 3$ the exponent -3 is required so that the normalization in (10.10) is maintained as we dial ϵ all the way down to zero. It doesn't matter whether we use the Gaussian or the top-hat for F: the internal details of $F(s)$ should be irrelevant in the limit $\epsilon \to 0$.

Following the prescription above, the equation

$$\boxed{\delta(\boldsymbol{x}) = 0 \qquad \text{if } \boldsymbol{x} \neq 0,} \tag{10.13}$$

really means

$$\lim_{\epsilon \to 0} \delta_\epsilon(\boldsymbol{x}) = 0, \qquad \text{provided that } \boldsymbol{x} \neq 0. \tag{10.14}$$

It is easy to see this is true if we use either the Gaussian or top-hat for F.

A most important property of the three-dimensional δ-function is "sifting". If $f(\boldsymbol{x})$ is a continuous function then:

$\mathrm{d}V' = \mathrm{d}x'\mathrm{d}y'\mathrm{d}z'$

$$\boxed{f(\boldsymbol{x}) = \int f(\boldsymbol{x}')\delta(\boldsymbol{x} - \boldsymbol{x}')\,\mathrm{d}V' = \int f(\boldsymbol{x} - \boldsymbol{x}')\delta(\boldsymbol{x}')\,\mathrm{d}V'\,.} \tag{10.15}$$

The value of f at the location of the δ-singularity is sifted out of the integral. This crucial result should be obvious if you think about replacing δ by δ_ϵ in the integrand, and then taking the limit $\epsilon \to 0$. Another main property of the three-dimensional δ-function is

$$\int_{\mathcal{D}} \delta(\boldsymbol{x})\,\mathrm{d}V = \begin{cases} 1\,, & \text{if } \boldsymbol{x} = 0 \text{ is in the domain } \mathcal{D}; \\ 0\,, & \text{if } \boldsymbol{x} = 0 \text{ is not in } \mathcal{D}. \end{cases} \tag{10.16}$$

Above we have used a spherically symmetric function $F(r)$ to construct the 3D Green's function. This is not necessary: a formula such as

$$\delta(\boldsymbol{x}) = \delta(x)\delta(y)\delta(z) \tag{10.17}$$

is interpreted by backing up to

$$\delta_\epsilon(\boldsymbol{x}) = \delta_\epsilon(x)\delta_\epsilon(y)\delta_\epsilon(z)\,, \tag{10.18}$$

where the three one-dimensional δ-functions on the right are constructed following the recipe in an earlier lecture. It is easy to verify that this product of three one-dimensional δ-functions satisfies our three-dimensional requirements in the boxed equations above.

For example, the indicator function of the interval $-1 < x < 1$ is

$$\Pi \stackrel{\text{def}}{=} \begin{cases} 1\,, & \text{if } -1 < x < 1; \\ 0\,, & \text{otherwise;} \end{cases} \tag{10.19}$$

To construct a 1D δ-function sequence we can consider

$$\delta_\epsilon = \frac{1}{2\epsilon}\Pi\Big(\frac{x}{\epsilon}\Big) \tag{10.20}$$

and take ϵ to zero. If we multiply three of the 1D δ_ϵ's together, as in (10.18), we obtain a 3D function that is non-zero inside a cube of side 2ϵ. As $\epsilon \to 0$ this limits to $\delta(bx)$.

Exercise: Prove that

$$\delta(\boldsymbol{x}) = \delta(x)\delta(y)\delta(z) = \frac{\delta(r^+)}{4\pi r^2}\,. \tag{10.21}$$

What's the analog of this result in $d = 2$?

10.3 The Green's function of Poisson's equation in unbounded space

The Green's function of Poisson's equation is the solution of

$$\triangle g = \delta(\boldsymbol{x}) \,. \tag{10.22}$$

We're considering unbounded three-dimensional space and we seek a solution with

$$\lim_{|\boldsymbol{x}|\to\infty} g = 0 \,. \tag{10.23}$$

Once we possess $g(\boldsymbol{x})$ the solution of the Poisson equation

$$\triangle \phi = \varrho \,, \tag{10.24}$$

is

$$\phi(\boldsymbol{x}) = \int \varrho(\boldsymbol{x} - \boldsymbol{x}')g(\boldsymbol{x}')\,\mathrm{d}V' \,. \tag{10.25}$$

$\mathrm{d}V' = \mathrm{d}x'\mathrm{d}y'\mathrm{d}z'$

If we integrate (10.22) over any domain $\mathcal{D}$ containing the origin then, with the divergence theorem,

$$\int_{\partial D} \boldsymbol{\nabla} g \boldsymbol{\cdot} \hat{\boldsymbol{n}}\,\mathrm{d}S = 1 \,. \tag{10.26}$$

In the case of heat diffusion away from a hot pellet this says that in steady state all the heat released at $\boldsymbol{x} = 0$ has to diffuse through the control surface $\partial\mathcal{D}$.

Suppose we pick a special domain which is a sphere of radius of r denoted this $\mathcal{S}_r$. From symmetry, g is a function only of r, and thus

$r = |\boldsymbol{x}| = \sqrt{x^2 + y^2 + z^2}$

$$\boldsymbol{\nabla} g \boldsymbol{\cdot} \hat{\boldsymbol{n}} = \frac{\mathrm{d}g}{\mathrm{d}r} \,. \tag{10.27}$$

Hence (10.26) is

$$\frac{\mathrm{d}g}{\mathrm{d}r} = \frac{1}{4\pi r^2} \,, \tag{10.28}$$

and

$$g = -\frac{1}{4\pi r} \,. \tag{10.29}$$

The result above can be obtained more formally. Using spherical coordinates, (10.7) is

$$\kappa \underbrace{\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}r^2\frac{\mathrm{d}}{\mathrm{d}r}}_{\triangle} g = \underbrace{\frac{\delta(r^+)}{4\pi r^2}}_{=\delta(\boldsymbol{x})} \tag{10.30}$$

The r^{-2} on left and right cancel, and $\delta(r^+)$ is concentrated just on the positive side of $r = 0$ so that $\int_0^r \delta(r_1^+)\mathrm{d}r_1 = 1$. Then integrating with respect to r we quickly get back to (10.28). If this is the first time you've seen

$$\delta(\boldsymbol{x}) = \frac{\delta(r^+)}{4\pi r^2} \,, \tag{10.31}$$

then this might seem mysterious. (See the exercises.) In any event, using either of the methods above, we arrive at (10.29) and then the solution of Poisson's equation in (10.25) can be written more expansively as

$$\phi(\boldsymbol{x}) = -\frac{1}{4\pi}\iiint \frac{\varrho(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\,\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\,, \tag{10.32}$$

where the integrals are over all space.

10.4 Other dimensions

Now consider the problem of finding the spherically symmetric Green's function in $d = 1$, 2 and 3. This means solving the Poisson equation

$$\triangle g = \delta(\boldsymbol{x})\,. \tag{10.33}$$

with

$$\triangle = \sum_{n=1}^{d} \partial_{x_n}^2\,. \tag{10.34}$$

The answer is

$$g(\boldsymbol{x}) = \begin{cases} \frac{1}{2}|x|\,, & \text{in } d=1; \\ -\frac{1}{2\pi}\ln\frac{1}{r}\,, & \text{in } d=2; \\ -\frac{1}{4\pi r}\,, & \text{in } d=3. \end{cases} \tag{10.35}$$

These solutions are not unique: one can add any constant to the solutions above.

The case $d = 3$ is from the previous section. The $d = 1$ and $d = 2$ cases follow in the examples.

Example: Find a "radially symmetric" solution of the one-dimensional Poisson equation

$$\frac{\mathrm{d}^2 g}{\mathrm{d}x^2} = \delta(x)\,. \tag{10.36}$$

Away from the singularity at $x = 0$, we have the one-dimensional Laplace equation $g_{xx} = 0$, with general solution

$$g(x) = \begin{cases} A_- + B_- x\,, & \text{if } x < 0, \\ A_+ + B_+ x\,, & \text{if } x > 0. \end{cases} \tag{10.37}$$

We also have jump conditions, obtained by integrating (10.36) across the δ-function at $x = 0$. This gives

$$\frac{\mathrm{d}g}{\mathrm{d}x}(0^+) - \frac{\mathrm{d}g}{\mathrm{d}x}(0^-) = 1\,. \tag{10.38}$$

There is no jump in $g(x)$ at $x = 0$ i.e., $g(x)$ is continuous, or $g(0^+) = g(0^-)$. These conditions give

$$A_- = A_+\,, \qquad B_+ - B_- = 1\,. \tag{10.39}$$

But we're looking for a symmetric solution, so $B_- = -B_+$, so $B_+ = 1/2$ and $B_- = -1/2$. Thus the Green's function is

$$g(x) = A + \tfrac{1}{2}|x|\,. \tag{10.40}$$

There is no way to determine A, except perhaps by considering the initial value problem.

Regarding the assumption of symmetry above: think of $g(x)$ as the temperature at a distance $|x|$ from a point source of heat. Diffusion will transmit heat away from $x = 0$ symmetrically.

Example: Consider the $d = 1$ diffusion problem

$$\kappa\theta_{xx} = -s\,, \tag{10.41}$$

where $s(x)$ is a steady source. Verify by substitution that the Green's function $g(x)$ from the previous example provides the solution

$$\kappa\theta(x) = -\int_{-\infty}^{\infty} s(x')g(x-x')\,\mathrm{d}x'\,. \tag{10.42}$$

Using g in (10.40), we have the integral representation

$$\kappa\theta(x) = -A\int_{-\infty}^{\infty} s(x')\,\mathrm{d}x' - \tfrac{1}{2}\int_{-\infty}^{x}(x-x')s(x')\,\mathrm{d}x' + \tfrac{1}{2}\int_{x}^{\infty}(x'-x)s(x')\,\mathrm{d}x'\,. \tag{10.43}$$

Now take the first x-derivative. The contribution from differentiating the x in the limit of integration is zero (both of them). We also destroy the constant term involving the undetermined constant A. Thus

$$\kappa\theta_x = -\tfrac{1}{2}\int_{-\infty}^{x} s(x')\,\mathrm{d}x' + \tfrac{1}{2}\int_{x}^{\infty} s(x')\,\mathrm{d}x' \tag{10.44}$$

Taking the second x-derivative we recover (10.41).

Example: Find a "radially symmetric" solution of the two-dimensional Poisson equation

$$g_{xx} + g_{yy} = \delta(x)\,. \tag{10.45}$$

Integrate over the area enclosed by a circle of radius r, and perimeter $2\pi r$, to obtain

$$\oint g_r \mathrm{d}\ell = 1\,. \tag{10.46}$$

Since the solution is axisymmetric

$$g_r = \frac{1}{2\pi r}\,, \qquad \text{or } g = \frac{\ln r}{2\pi} + A. \tag{10.47}$$

Again there is an undetermined constant of integration A.

We might as well record the general case: in dimension d the Laplacian Green's function is

$$g(\boldsymbol{x}) = \frac{1}{(2-d)\Omega_d r^{d-2}}\,, \quad (d \geq 3)\,, \tag{10.48}$$

where Ω_d is the area of a d-dimensional unit sphere (see the exercises).

10.5 The d-dimensional diffusion Green's function

In dimension d, with $\boldsymbol{x} = [x_1, x_2, \cdots x_d]$, the diffusion Green's function $G(\boldsymbol{x}, t)$ is determined by

$$G_t = \kappa\underbrace{\left(G_{x_1x_1} + G_{x_2x_2} + \cdots + G_{x_dx_d}\right)}_{\triangle G}, \tag{10.49}$$

with the initial condition

$$G(\boldsymbol{x},0) = \delta(\boldsymbol{x})\,. \tag{10.50}$$

Since the d-dimensional δ-function can be written as product of one-dimensional δ-functions,

$$\delta(\boldsymbol{x}) = \delta(x_1)\delta(x_2)\cdots\delta(x_d)\,, \tag{10.51}$$

it is easy to verify that the solution of (10.50) and (10.50) is

$$G(\boldsymbol{x},t) = g(x_1,t)g(x_2,t)\cdots g(x_d,t)\,, \tag{10.52}$$

$$= (4\pi\kappa t)^{-d/2}\exp(-r^2/4\kappa t)\,, \tag{10.53}$$

where $g(x_1,t)$ is the one-dimensional Green's function in (4.37) and $r^2 = x_1^2 + x_2^2 + \cdots + x_d^2$.

Once we have $G(\boldsymbol{x},t)$, the solution of the d-dimensional initial value problem

$$\theta_t = \kappa\triangle\theta\,, \qquad \theta(\boldsymbol{x},0) = f(\boldsymbol{x})\,, \tag{10.54}$$

is given by the convolution

$$\theta(\boldsymbol{x},t) = \int G(\boldsymbol{x}-\boldsymbol{x}',t)f(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}'\,. \tag{10.55}$$

Once again we counsel against reflexive use of this formula....

10.6 The efficiency of diffusion depends on d

You may have noticed in (10.35) that only in $d = 3$ does the Laplacian Green's function decay to zero as $r \to \infty$. With $d = 1$ or 2 the Laplacian Green's function diverges as $r \to \infty$. This is an indication that diffusion is more efficient in higher-dimensional spaces — heat gets away from the $\delta(\boldsymbol{x})$-source faster in $d = 3$. A supporting observation is that according to (10.53), $G(0,t) \propto t^{-d/2}$ i.e., the central value of a diffusive pulse decays faster with time in large dimensions.

To illustrate the difference between diffusion in $d = 2$ and $d = 3$, consider the problem of a hot radioactive rod embedded in a large mass of concrete, with uniform temperature, $\theta = \theta_\infty$, at large distances from the rod.

If the rod is very long and stretches along the z-axis, then we might expect that the problem is effectively two dimensional, and we could try to find the steady-state temperature distribution by writing

$$\theta(r,t) = \theta_\infty + v(s)\,, \tag{10.56}$$

where $s = \sqrt{x^2+y^2}$ is cylindrical radius. The steady temperature anomaly must satisfy

$$\kappa\left(v_{xx} + v_{yy}\right) + q\delta(x)\delta(y) = 0\,, \qquad \text{with } \lim_{s\to\infty} v(s) = 0. \tag{10.57}$$

But the general $d = 2$ axisymmetric solution of Laplace's equation is $v = A + B\ln s$, and this cannot satisfy the condition of decay as $s \to \infty$. The best we can do is to ignore the condition at ∞ and determine B by balancing the heat input with diffusion, so that

$$v \stackrel{?}{=} A + \frac{\ln(s^{-1})}{2\pi} . \tag{10.58}$$

We just can't satisfy the condition at $s = \infty$. There are at least two ways of resolving this problem:

1. Recognize that the rod has finite length, $2a$, and solve a steady three-dimensional diffusion problem with the source

$$q\delta(x)\delta(y)\chi(z) ,$$

where $\chi(z) = 1$ if $|z| < a$ and zero otherwise.

2. Remain in $d = 2$ and solve an unsteady diffusion equation in which the rod is inserted into an isothermal medium at $t = 0$.

Both problems are instructive, and provide good applications of Green's method. Let's explore them in turn.

The steady $d = 3$ problem

In this case we use the $d = 3$ Green's function to write down the solution

$$v(x, y, z) = \frac{q}{4\pi\kappa} \int_{-a}^{a} \frac{\mathrm{d}z'}{\sqrt{x^2 + y^2 + (z - z')^2}} . \tag{10.59}$$

Feeding this command

```
Assuming[r > 0 && z > 0 && a > 0,
 Integrate[1/Sqrt[r^2 + (z - u)^2], {u, -a, a}]]
```

into mathematica[1], produces

```
Log[-((a + Sqrt[r^2 + (a - z)^2] - z)/(a + z - Sqrt[r^2 + (a + z)^2]))]
```

(It is clearer on the computer screen.) In other words, the three-dimensional temperature is

$$v(x, y, z) = \frac{q}{4\pi\kappa} \ln\left[\frac{z - a - R_+}{z + a - R_-}\right] , \tag{10.60}$$

where

$$R_\pm \stackrel{\text{def}}{=} \sqrt{s^2 + (a \mp z)^2} . \tag{10.61}$$

$R_\pm$ is the distance between the field point $\boldsymbol{x}$ and the end of the rod at $z = \pm a$. Figure 10.1 shows the potential.

$s \stackrel{\text{def}}{=} \sqrt{x^2 + y^2}$

$r \stackrel{\text{def}}{=} \sqrt{x^2 + y^2 + z^2}$

[1] See also classic texts on potential theory such as *Foundations of Potential theory* by O.D. Kellogg, or *Newtonian Attraction* by A. S. Ramsey, where this integral, and many others, are done analytically.

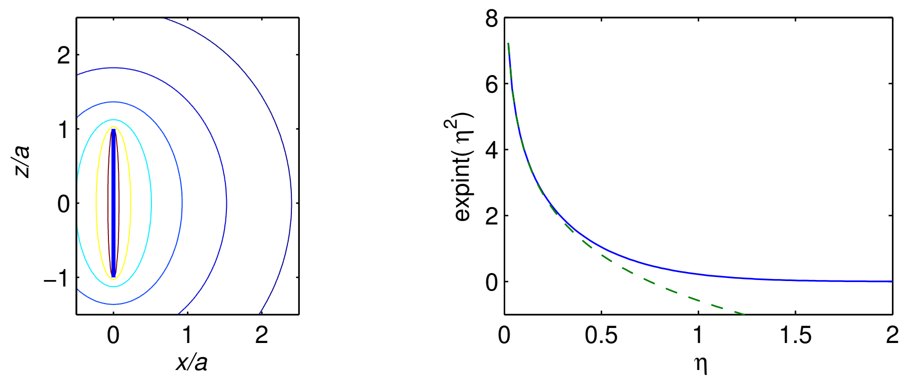


Figure 10.1: Left panel shows isotherms round the hot rod in the (x, z)-plane computed from (10.60). The contour interval is logarithmic. Right panel shows the $\mathrm{E}_1(\eta^2)$ (the solid curve) and the small-r approximation $-2\ln\eta - 0.57721$ (the dashed curve). `hotRod.eps`

This solution has the expected behaviour. For example, on the plane $z = 0$ the temperature simplifies to

$$v(x, y, 0) = \frac{q}{4\pi\kappa} \ln\left[\frac{\sqrt{s^2 + a^2} + a}{\sqrt{s^2 + a^2} - a}\right] . \tag{10.62}$$

From this expression we see that close to the rod $G \propto \ln s$, and far from the rod one recovers the r^{-1} decay of the $d = 3$ Green's function (see exercises).

Exercise: Simplify $v(x, y, 0)$ in (10.62) in the limits $r \ll a$ and $r \gg a$. Calculate the first two or three terms so that you understand the corrections to the expected asymptotic forms $\ln r$ and r^{-1}.

The unsteady $d = 2$ problem

In this case we solve the two-dimensional initial value problem

$$v_t = \kappa\,(v_{xx} + v_{yy}) + q\delta(x)\delta(y)\,, \qquad \text{with } v(\boldsymbol{x}, 0) = 0. \tag{10.63}$$

The method of section 4.3.2 gives the integral representation

$$v(r, t) = q\int_0^t \frac{\mathrm{e}^{-s^2/4\kappa\tau}}{4\pi\kappa\tau}\,\mathrm{d}\tau\,, \tag{10.64}$$

where $s \stackrel{\text{def}}{=} \sqrt{x^2 + y^2}$ is the cylindrical radius. We beat this integral into a standard from with the change of variables $w = s^2/4\kappa\tau$. Thus

$$\frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}w}{w}$$

$$v(r, t) = \frac{q}{4\pi\kappa}\int_{s^2/4\kappa t}^{\infty} \frac{\mathrm{e}^{-w}}{w}\,\mathrm{d}w\,, \tag{10.65}$$

$$= \frac{q}{4\pi\kappa}\,\mathrm{E}_1\left(\eta^2\right)\,, \tag{10.66}$$

where E_1 is the exponential integral (e.g., **BO** section 6.2, and `expint` in MATLAB.). Also in (10.66), $\eta \stackrel{\text{def}}{=} s/2\sqrt{\kappa t}$ is our favourite similarity variable. Close to the rod, where $\eta \ll 1$, we use the expansion of E_1 given by **BO** in their (6.2.8):

Euler's constant is $\gamma_{\mathrm{E}} = 0.57721\cdots$

$$v(r,t) = \frac{q}{4\pi\kappa}\left[2\ln\left(\eta^{-1}\right) - \gamma_{\mathrm{E}} + \eta^2 + O\left(\eta^4\right)\right] ,$$
$$\sim \frac{q}{2\pi\kappa}\ln\frac{1}{s} + \text{a term with } \ln t. \tag{10.67}$$

The solution is shown in the right panel of Figure 10.1. In the near field of the rod the steady $\ln s^{-1}$ solution is a dominant part of the answer. But there is also the slowly growing $\ln t$. This slow increase in temperature is an indication that two-dimensional diffusion cannot keep up with the steady influx of heat from the rod.

10.7 Some 3D sphere problems

10.8 Problems

Problem 10.1. One face of a slab, say $x = 0$, is held at uniform temperature $\theta(0) = 0$ and the other face at $x = \ell$ is held at $\theta(\ell) = \theta_1$. Assume that the conductivity k is a function only of x. Find the temperature profile and the steady heat flux through the slab.

Solution. In steady state the temperature depends only on x and the diffusion equation (4.1) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}x}k\frac{\mathrm{d}\theta}{\mathrm{d}x} = 0\,, \qquad \Rightarrow \qquad \frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{f}{k(x)}\,. \tag{10.68}$$

The constant of integration f is the heat flux (Watts per square meter). To satisfy the boundary conditions we require that the integral from $x = 0$ to ℓ of $\mathrm{d}\theta/\mathrm{d}x$ is equal to θ_1, or

$$\theta_1 = f\int_0^\ell \frac{\mathrm{d}x}{k(x)}\,. \tag{10.69}$$

With f determined by this formula, the temperature profile is

$$\theta(x) = f\int_0^x \frac{\mathrm{d}x'}{k(x')}\,. \tag{10.70}$$

If k is constant then the temperature varies linearly with x — this is the second simplest solution of Laplace's equation in $d = 1$. The simplest solution is just constant u. Note a small region in which $k(x)$ is very small (an insulating layer) makes a large reduction in the heat flux f determined from (10.69).

Problem 10.2. The temperature in (10.29) is singularly large as $r \to 0$. This is an unphysical artifact of assuming that the radioactive pellet is a point source. So consider a new and improved model with a spherical pellet

$$\kappa \triangle \theta = -\frac{q}{\epsilon^3} F\left(\frac{r}{\epsilon}\right) , \tag{10.71}$$

where ϵ is a length (the radius of the pellet) and F is the top-hat in (10.12) Solve (10.71) and verify that the central temperature $\theta(0)$ is finite if $\epsilon > 0$.

Problem 10.3. Show that in dimension d, with $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$,

$$\triangle f(r) = \frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{d-2}{r}\frac{\mathrm{d}f}{\mathrm{d}r} . \tag{10.72}$$

Solve the previous hot pellet problem with $d \geq 3$. What happens in $d = 1$ and $d = 2$?

Problem 10.4. Solve the 1D diffusion equation

$$\theta_t = \kappa \theta_{xx} + \delta(x) , \qquad \text{with IC } \theta(x, 0) = 0. \tag{10.73}$$

Put the answer in the form

$$\kappa \theta = \tfrac{1}{2} \sqrt{\frac{\kappa t}{\pi}}\, E_{3/2}\left(\eta^2\right) \tag{10.74}$$

where $\eta = x/\sqrt{4\kappa t}$ is the usual similarity variable and

$$E_p(z) \stackrel{\text{def}}{=} z^{p-1} \int_z^\infty \frac{\mathrm{e}^{-\tau}}{\tau^p}\, \mathrm{d}\tau \tag{10.75}$$

is the exponential integral defined in the **DLMF** and implemented in `matlab`. The solution is shown in the figure. Using properties of $E_{3/2}$ given in **DLMF**, show that as $t \to \infty$, with x fixed,

$$\kappa \theta(x, t) \to \sqrt{\frac{\kappa t}{\pi}} - \tfrac{1}{2}|x| + \operatorname{ord}\left(x^2/\sqrt{\kappa t}\right) . \tag{10.76}$$

Discuss the relation between this solution and the 1D Laplacian Green's function in (10.40).

Problem 10.5. Make sure you've done the previous problem. Now consider the initial value problem

$$\theta_t = \kappa \triangle \theta + q\delta(\boldsymbol{x}) , \qquad v(\boldsymbol{x}, 0) = 0 , \tag{10.77}$$

in arbitrary d. Write the answer in terms of exponential integrals. As $t \to \infty$, with $|\boldsymbol{x}|$ fixed, does the solution evolve to the Laplacian Green's function?

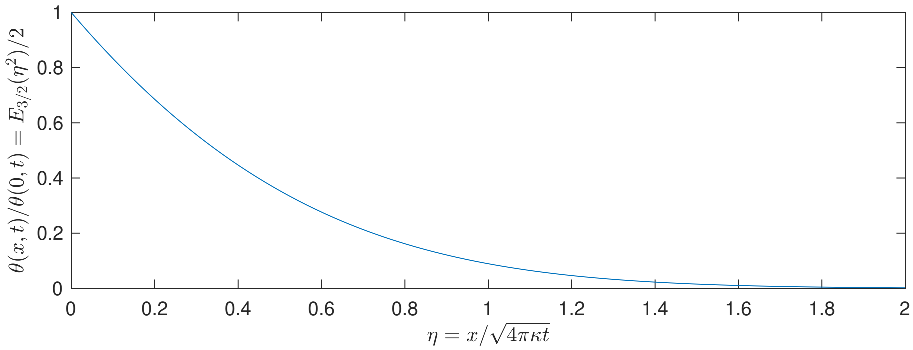


Figure 10.2: The solution in (10.74). `oneDGfunc.eps`

Problem 10.6. Consider

$$v(r) \stackrel{\text{def}}{=} -\frac{1}{4\pi(r+\epsilon)}, \tag{10.78}$$

where $r = \sqrt{x^2+y^2+z^2}$ is the $d=3$ radial coordinate. *(i)* What are the dimensions of ϵ and v? *(ii)* Calculate $\triangle v$ and show that

$$\lim_{\epsilon\to 0} \triangle v = \delta(\boldsymbol{x}). \tag{10.79}$$

(iii) Find a $d=2$ version of this regularized Green's function.

Problem 10.7. What are the dimensions of $\delta(\boldsymbol{x})$ in $d = 1$, 2 and 3?

Problem 10.8. Find the spherically symmetric Green's function defined by

$$\triangle g - k^2 g = \delta(\boldsymbol{x}) \tag{10.80}$$

in $d = 1$, 2 and 3.

Problem 10.9. Obtain the d-dimensional Laplacian Green's function in (10.48).

Problem 10.10. *(i)* Check that G in (10.53) is normalized to unity by performing the d-dimensional integral in Cartesian coordinates. *(ii)* Integrate $G(r,t)$ in (10.53) over all space using spherical coordinates. In these coordinates the volume of the shell between r and $r + \mathrm{d}r$ is

$$\mathrm{d}\boldsymbol{x} = \Omega_d r^{d-1} \mathrm{d}r, \tag{10.81}$$

where Ω_d is the area of a d-dimensional unit sphere. Deduce

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \qquad \text{where } \Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t. \tag{10.82}$$

Problem 10.11. *(i)* Assume that the solution of (10.49) and (10.50) is spherically symmetric i.e., that $G = G(r,t)$. *(ii)* Show that the axisymmetric diffu-

$r \stackrel{\text{def}}{=} \sqrt{x_1^2 + \cdots + x_d^2}$

sion equation can be written as

$$r^{d-1}G_t = \kappa \left(r^{d-1}G_r\right)_r . \tag{10.83}$$

Interpret this last equation in terms of flux through a sphere with area $\Omega_d r^{d-1}$ etc. *(iii)* Solve this PDE with the similarity method, and show that you recover (10.53).

Problem 10.12. A laser zaps a big copper sheet and instantly deposits Q Joules of heat at a point the origin. Treat the sheet as two-dimensional, with density ϱ kilograms per square meter and heat capacity C joules per kilogram and thickness Δz that is much smaller than all other length scales of interest. Suppose that the sheet initially has uniform temperature. Then the excess temperature u satisfies (4.1). Solve this PDE with an initial condition that models the zap, and show that Q Joules eventually diffuse through any circle of radius r centered on the origin.

Lecture 11

Laplace's equation in $d = 2$

11.1 Discrete harmonic functions

Tom and Jerry toss a fair coin. If the toss is a 'head' then Tom gives Jerry \$1 and if it is a 'tail' then vice versa. Tom starts with x dollars and Jerry with $n-x$ dollars. Find the probability that Tom wins all the money.

Consider

$$P(x) \stackrel{\text{def}}{=} \text{Probability Tom wins starting with } x \text{ dollars} \,. \tag{11.1}$$

Clearly $P(0) = 0$ and $P(n) = 1$ After the first toss Tom has either $x-1$ dollars or $x+1$ dollars, both with probability $1/2$. These events are mutually exclusive and exhaustive so

$$P(x) = \tfrac{1}{2}P(x-1) + \tfrac{1}{2}P(x+1) \,. \tag{11.2}$$

We have used the following idea from probability theory: suppose E is an event (Tom wins) and F and G be mutually exclusive subevents. Then

$$\text{prob}(E) = \text{prob}(F) \times \text{prob}(E \text{ given } F) + \text{prob}(G) \times \text{prob}(E \text{ given } G) \tag{11.3}$$

In (11.2) F is the event that the first toss is a head and G is the event that the first toss is a tail.

The function $P(x)$ is said to be *harmonic* if it has the *averaging* property in (11.2): the solution at x is the average of the solution at the neighbouring two points.

By considering small values of n it is easy to convince oneself that the solution of the difference equation (11.2) is

$$P(x) = \frac{x}{n} \,. \tag{11.4}$$

The averaging property is apparent e.g. $17 = (16 + 18)/2$. If you look at the month of May in a calendar you'll see that each interior entry (e.g., May 23) is an average of the four entries to the North, South, East and West (e.g., May 16, 30, 24 and 22). This leads us to the two-dimensional case...

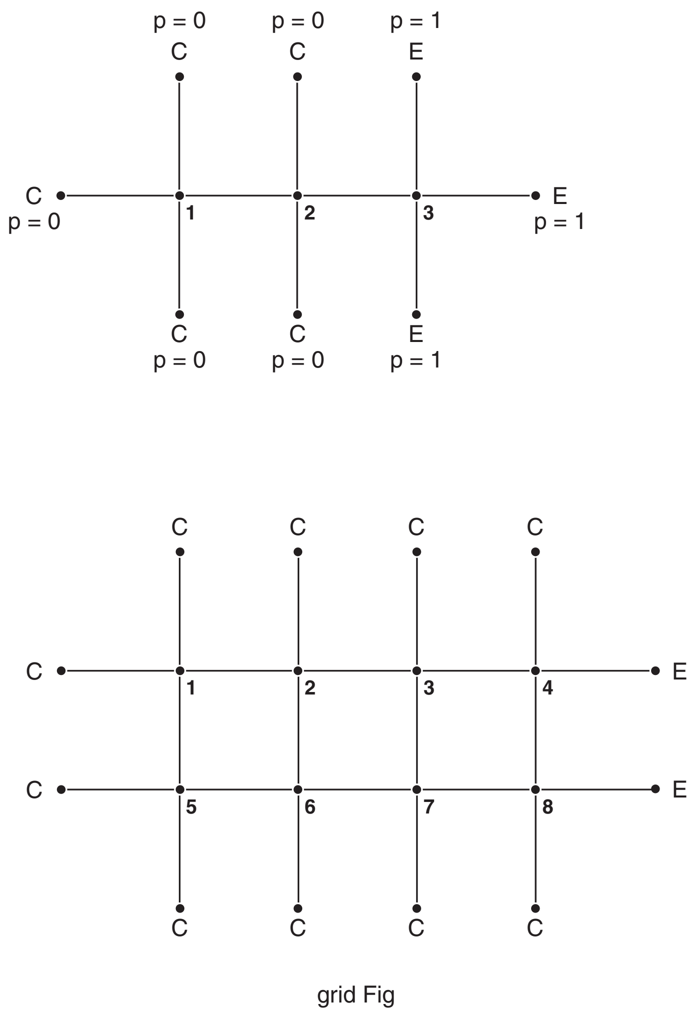


Figure 11.1: Two discrete grids. The boundary points are labelled C if the walker is captured and E if she escapes. `JoFig1.eps`

Random walks $d = 2$

Suppose we play a similar game in two dimensions (see figure) in which a criminal random walks on an integer grid. Find

$$P(x, y) \stackrel{\text{def}}{=} \text{Probability of escaping starting at } (x, y) \tag{11.5}$$

In this case the analog of (11.2) is

$$P(x, y) = \tfrac{1}{4}\left[P(x-1, y) + P(x+1, y) + P(x, y-1) + P(x, y+1)\right] , \tag{11.6}$$

where (x, y) is an interior point. Once again, notice that (11.6) says that the value of $P(x, y)$ at an *interior point* is the average of p at the four neighbouring points. At the *boundary points* p is either zero (capture) or one (escape). The solution of this problem is a two-dimensional discrete harmonic function.

Example: Determine P at the three interior grid points in the top panel of figure 11.1.

In this case the averaging-equations are

$$P_1 = \tfrac{1}{4}P_2\,, \quad P_2 = \tfrac{1}{4}(P_1 + P_3)\,, \quad P_3 = \tfrac{1}{4}(P_2 + 3)\,. \tag{11.7}$$

After some scribbling we find the solution of this 3×3 linear system is

$$P_1 = \frac{3}{56}\,, \quad P_2 = \frac{12}{56} \quad P_3 = \frac{45}{56}\,. \tag{11.8}$$

The maximum principle

The averaging property in (11.6) implies that the maximum and minimum values of $P(x,y)$ must be at the boundaries of the domain: an interior point cannot have a greater value of p than all four of its neighbours.

The maximum principle implies *uniqueness*. Suppose we have two solutions $P_1(x,y)$ and $P_2(x,y)$ satisfying (11.13) with the same boundary conditions. Then the difference $V = P_2 - P_1$ also satisfies (11.13) with $V = 0$ on the boundary. Thus

$$\max(V) = \min(V) = 0\,, \quad \Rightarrow \quad V = 0\,. \tag{11.9}$$

This is a justification for guessing solutions — as if we need justification! It may be some comfort to know that if you can find a solution than you can be sure you have the *only* solution.

Minimization of the Dirichlet functional

We now show that the solution of the discrete Laplace equation minimizes the Dirichlet functional

$$\mathcal{D}\,[\boldsymbol{p}] = \sum_{\text{nnp}} (P_m - P_n)^2\,, \tag{11.10}$$

where 'nnp' stands for 'nearest neighbour pairs' and $\boldsymbol{p} = [P_1, P_2 \cdots P_{n+m}]$ is a vector containing the value of p at the n interior points and the m boundary points.

The obvious way to minimize $\mathcal{D}$ is to make $\boldsymbol{p}$ constant: then $\mathcal{D}[\boldsymbol{p}] = 0$. But this trivial solution is not consistent with the boundary conditions. The functional $\mathcal{D}$ is a measure of how much the different elements of $\boldsymbol{p}$ vary, and minimizing $\mathcal{D}[\boldsymbol{p}]$ makes $\boldsymbol{p}$ as close to constant as possible, consistent with the variations imposed by the boundary points. To find the true minimum we differentiate $\mathcal{D}$ with respect to the n interior points. For instance, suppose P_7 is an interior point. Now in $d = 2$ there are four terms in $\mathcal{D}$ containing P_7:

$$\mathcal{D}[\boldsymbol{p}] = (P_7 - P_N)^2 + (P_7 - P_S)^2 + (P_7 - P_E)^2 + (P_7 - P_W)^2 + \cdots \tag{11.11}$$

where N, S, E and W stands for the four nearest neighbour points to point 7. To find the minimum

$$\frac{\partial \mathcal{D}}{\partial P_7} = 0\,, \qquad \Rightarrow \qquad P_7 = \tfrac{1}{4}\left(P_N + P_S + P_E + P_W\right)\,. \tag{11.12}$$

Thus for every interior point we recover the discrete Laplace equation in (11.6).

11.2 From the discrete to the continuous

If we leap from the discrete to the continuous — which is justified if we have an enormous grid with lots of points — then we arrive at

$$
\begin{aligned}
(11.2) \quad &\Rightarrow \quad \frac{\mathrm{d}^2 P}{\mathrm{d}x^2} = 0\,, \\
(11.6) \quad &\Rightarrow \quad \boxed{P_{xx} + P_{yy} = 0}\,.
\end{aligned}
\tag{11.13}
$$

Thus Laplace's equation can also be regarded as a continuum approximation to this probabilistic escape problem. In the remainder of this lecture we use the escape problem to illustrate solutions of Laplace's equation.

Connection with our earlier results

Let's derive (11.13) starting with the continuous formulation of an ensemble of random walkers in a 2D domain $\mathcal{R}$ with an absorbing boundary condition at the boundary $\partial\mathcal{R}$. We suppose that $\partial\mathcal{R}$ is divided into two parts

$$
\partial\mathcal{R} = \mathcal{E} \cup \mathcal{C} \tag{11.14}
$$

where $\mathcal{E}$ denotes escape and $\mathcal{C}$ is capture. The diffusion equation for the concentration is

$$
c_t = \kappa \triangle c\,, \tag{11.15}
$$

with initial condition

$$
c(x, y, 0) = c_0(x, y)\,, \tag{11.16}
$$

and a Dirichlet boundary condition

$$
c(\partial\mathcal{R}, t) = 0\,. \tag{11.17}
$$

We use the normalization

$$
1 = \int_{\mathcal{R}} c_0(x, y)\, \mathrm{d}A\,. \tag{11.18}
$$

The integral of (11.15) over $\mathcal{R}$ is

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{R}} c\, \mathrm{d}A = \kappa \oint_{\partial\mathcal{R}} \boldsymbol{\nabla} c \boldsymbol{\cdot} \boldsymbol{n}\, \mathrm{d}\ell\,, \tag{11.19}
$$

$$
= \underbrace{\kappa \oint_{\mathcal{E}} \boldsymbol{\nabla} c \boldsymbol{\cdot} \boldsymbol{n}\, \mathrm{d}\ell}_{\text{escaping walkers}} + \underbrace{\kappa \oint_{\mathcal{C}} \boldsymbol{\nabla} c \boldsymbol{\cdot} \boldsymbol{n}\, \mathrm{d}\ell}_{\text{captured walkers}}\,. \tag{11.20}
$$

To obtain the probability of eventual escape we integrate the escape term in (11.20) from $t = 0$ to $t = \infty$ and obtain

$$
\text{probability of eventual escape} = -\kappa \oint_{\mathcal{E}} \boldsymbol{\nabla} C \boldsymbol{\cdot} \boldsymbol{n}\, \mathrm{d}\ell\,, \tag{11.21}
$$

where

$$C(x,y) \stackrel{\text{def}}{=} \int_0^\infty c(x,y,t)\,\mathrm{d}t\,. \tag{11.22}$$

But integrating (11.15) from $t = 0$ to $t = \infty$ we see that C satisfies the Poisson equation

$$\kappa \triangle C = -c_0\,. \tag{11.23}$$

So, if we can solve the Poisson equation (11.23), with boundary condition $C(\partial\mathcal{R}) = 0$, then we obtain the probability of escape by computing the line integral on the right of (11.21).

The procedure above is more involved than solving the Laplace equation, $P_{xx} + P_{yy} = 0$, with the Dirichlet boundary condition

$$P(\mathcal{E}) = 1\,, \qquad \text{and} \qquad P(\mathcal{C}) = 0\,. \tag{11.24}$$

To see that the results are equivalent, multiply (11.23) by P and integrate over $\mathcal{R}$. Using Green's second theorem we have

$$\kappa \oint_{\partial\mathcal{R}=\mathcal{E}+\mathcal{C}} P\,\boldsymbol{\nabla} C\boldsymbol{\cdot}\boldsymbol{n} - \underbrace{C}_{=0}\boldsymbol{\nabla} P\boldsymbol{\cdot}\boldsymbol{n}\,\mathrm{d}\ell + \kappa\int_{\mathcal{R}} C\underbrace{\triangle P}_{=0}\,\mathrm{d}A = -\int_{\mathcal{R}} c_0 P\,\mathrm{d}A\,. \tag{11.25}$$

The only surviving non-zero term on the LHS of (11.25) is minus probability of escape in (11.21) and thus we conclude that

$$\text{probability of eventual escape} = \int_{\mathcal{R}} c_0 P\,\mathrm{d}A \tag{11.26}$$

Taking the initial condition $c_0 = \delta(x-\xi)\delta(y-\eta)$ we see that $P(\xi,\eta)$ is the probability of eventual escape given that an ensemble of drunken criminals flees from the initial point (ξ,η) in $\mathcal{R}$.

11.3 The Dirichlet and Neuman problems in $d = 2$

We can visualize two-dimensional solutions of Laplace's equation

$$u_{xx} + u_{yy} = 0 \tag{11.27}$$

in a simply connected region[1] of the (x,y) plane, $\mathcal{R}$, as steady-state temperature distribution in sheet of metal shaped like $\mathcal{R}$. To prevent diffusion from making the temperature uniform, suppose that the temperature on the boundary $\partial\mathcal{R}$ is held in some non-uniform condition. The problem of solving Laplace's equation $\triangle u = 0$, subject to a boundary condition with u is specified on $\partial\mathcal{R}$ is the *Laplace-Dirichlet problem.*

Exercise: Is the product of two harmonic functions harmonic?

[1] Simply connected means that every closed curve in $\mathcal{R}$ can be contracted to a point without passing outside of $\mathcal{R}$. A disc is simply connected and an annulus is not.

The probabilistic introduction is motivation for considering the Laplace-Dirichlet problem with the simple boundary condition that P is either zero or one on $\partial\mathcal{R}$. Another motivation is the *minimal surface* problem, leading to consideration of the minimal surface equation

$$(1+h_x^2)h_{xx} - 2h_x h_y h_{xy} + (1+h_y^2)h_{yy} = 0\,. \tag{11.28}$$

with a boundary condition $h(\partial\mathcal{R}) = h_0(\ell)$. If the boundary displacement h_0 is small then the linearization of (11.28) is Laplace's equation.

There is a second problem, the *Laplace-Neuman problem*, which is to solve Laplace's equation within $\mathcal{R}$ with the normal derivative $\hat{\boldsymbol{n}}\boldsymbol{\cdot}\boldsymbol{\nabla}u$ specified on $\partial\mathcal{R}$. In the case of heat conduction, the Laplace-Neumann problem amounts to specifying the heat flux through the boundary $\partial\mathcal{R}$.

$\hat{\boldsymbol{n}}$ is the outward unit normal to ∂R.

It is usually impossible to solve Laplace's equation within $\mathcal{R}$ if both u and $\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\nabla}u$ are specified. Using steady-state heat conduction as an example, it is not possible to simultaneously specify both the temperature and the heat flux at the boundary $\partial\mathcal{R}$. If the boundary temperature is specified then the boundary heat flux is part of the answer, and vice versa.

Example: Consider Laplace's equation $\triangle u = 0$ in the upper half plane (UHP) $y > 0$ with two boundary conditions

$$u(x,0) = \cos kx\,, \qquad u_y(x,0) = \cos kx\,. \tag{11.29}$$

This is the Laplace-Cauchy problem. The only solution is

$$u = \mathrm{e}^{ky}\cos kx\,. \tag{11.30}$$

The blow-up as $y\to\infty$ will usually mean that this solution is unacceptable. On the other hand, the Laplace-Dirichlet problem, $\triangle v = 0$ with the single boundary condition

$$u(x,0) = \cos kx \tag{11.31}$$

has acceptable solutions $v = \mathrm{e}^{-ky}\cos kx$ in the UHP and $\mathrm{e}^{ky}\cos kx$ in the LHP.

Solutions of (11.27) have a remarkable property. Suppose $\mathcal{C}$ is any closed curve lying completely in $\mathcal{R}$. Integrating Laplace's equation (11.27) over the area contained within $\mathcal{C}$, and using Gauss's theorem, we have

$$\iint_{\mathcal{C}} \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}\,\mathrm{d}^2\boldsymbol{x} = \oint_{\mathcal{C}} \boldsymbol{v}\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,\mathrm{d}\ell$$

$$\oint_{\mathcal{C}} \boldsymbol{\nabla}u\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,\mathrm{d}\ell = 0\,, \tag{11.32}$$

where the line integral is around $\mathcal{C}$, and $\hat{\boldsymbol{n}}$ is the outward unit normal to $\mathcal{C}$. A physical interpretation of (11.32) is that in steady state the flux of heat into and out of a closed region must be zero.

Considering the Laplace-Dirichlet problem with boundary condition

$$\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\nabla}u = f(\ell) \tag{11.33}$$

on $\partial\mathcal{R}$, we see from (11.32) that in order for the problem to have a solution the specified boundary flux f must satisfy the *solvability condition*

$$\oint_{\partial\mathcal{R}} f(\ell)\,\mathrm{d}\ell = 0\,. \tag{11.34}$$

The maximum principle

From the intuitive picture of steady-state heat distributions we can anticipate that u cannot have maximum within $\mathcal{R}$: otherwise heat would continually flow away from the hot-spot. In other words the maximum (and minimum) temperatures are on $\partial\mathcal{R}$. Let's prove this maximum principle by contradiction: suppose that there is a interior point $\boldsymbol{x} \in \mathcal{R}$ at which u achieves a local maximum. This implies that $\boldsymbol{x}$ is surrounded by an isothermal contour $\mathcal{C}$, whose temperature is less than $u(\boldsymbol{x})$. The outward normal to this hypothetical isotherm $\mathcal{C}$ is

$$\hat{\boldsymbol{n}} = -\boldsymbol{\nabla} u / |\boldsymbol{\nabla} u| \,. \tag{11.35}$$

With this expression for $\hat{\boldsymbol{n}}$, (11.32) becomes:

$$\oint_{\mathcal{C}} |\boldsymbol{\nabla} u| \, \mathrm{d}\ell = 0 \,. \tag{11.36}$$

This a contradiction: the integrand on the left is positive definite so the contour integral cannot be zero. We conclude that we cannot find an interior point of $\mathcal{R}$ at which the temperature is a local maximum.

Exercise: Why is there a minus sign on the right of (11.35)?

The mean value theorem

Now we show that the temperature at a point inside $\mathcal{R}$ is the average of the temperature at surrounding points. Define an "average temperature" at a point $\boldsymbol{x} \in \mathcal{R}$ as

$$U(\boldsymbol{x}, r) \stackrel{\text{def}}{=} \oint u(\boldsymbol{x} + \boldsymbol{r}) \, \frac{\mathrm{d}\theta}{2\pi} \,, \tag{11.37}$$

where $\boldsymbol{r} \stackrel{\text{def}}{=} r(\cos\theta, \sin\theta)$. We can pick any value of r so long as $\boldsymbol{x} + \boldsymbol{r}$ lies within $\mathcal{R}$ for every value of θ. That is, $\boldsymbol{x}$ is surrounded by a circular contour $\mathcal{C}$ of radius r lying properly within $\mathcal{R}$.

Taking the derivative with respect to r we have

$$\frac{\partial U}{\partial r} = \oint \boldsymbol{\nabla} u \boldsymbol{\cdot} \hat{\boldsymbol{n}} \, \frac{\mathrm{d}\theta}{2\pi} \,. \tag{11.38}$$

But from (11.32), the right hand side is zero i.e., U is independent of r. Thus we can evaluate U by taking $r = 0$ i.e.,

$$U(\boldsymbol{x}, r) = u(\boldsymbol{x}) \,. \tag{11.39}$$

This is the mean value theorem for harmonic functions.

Uniqueness for the Dirichlet problem

It is easy to prove that the Dirichlet problem in a bounded domain $\mathcal{R}$ has a unique solution. Suppose we have two solutions of Laplace's equation (11.27) that agree on the boundary. Then the difference of the two solutions — call it

$v(x,y)$ — is also a solution of (11.27) that vanishes on the boundary. Multiply $\triangle v = 0$ by v and integrating over the domain $\mathcal{R}$ we have

$$\underbrace{\oint_{\partial\mathcal{R}} v\boldsymbol{\nabla} v\boldsymbol{\cdot}\boldsymbol{n}\,\mathrm{d}\ell}_{=0} = \iint_{\mathcal{R}} |\boldsymbol{\nabla} v|^2\,\mathrm{d}A \tag{11.40}$$

The integral on the left is zero because $v = 0$ on the boundary $\partial\mathcal{R}$. Hence $\boldsymbol{\nabla} v$ must bezero. Perhaps v is a constant? But that's not consistent with $v = 0$ on the boundary. We have reached a contradiction and so we conclude that the Laplace-Dirichlet problem has a unique solution.

The same argument shows that the Laplace-Neumann problem has a unique solution *up to an additive constant*: if $u(x,y)$ is a solution of the Laplace-Neumann problem, then so is u plus any constant.

Mininization of the Dirichlet functional

Consider a set of functions $\{v\}$ defined on the domain $\mathcal{R}$, all of which are equal to a specified f on the boundary of $\partial\mathcal{R}$. As a measure of roughness we use the Dirichlet functional

$$\mathcal{D}[v] \stackrel{\text{def}}{=} \iint_{\mathcal{R}} |\boldsymbol{\nabla} v|^2\,\mathrm{d}A\,. \tag{11.41}$$

Then the function that minimizes $\mathcal{D}[v]$ is the harmonic function

$$\triangle u = 0\,, \qquad \text{with } u = f \text{ on } \partial\mathcal{R}. \tag{11.42}$$

The harmonic function is the smoothest function that is consistent with the variations imposed by the boundary condition.

11.4 Separation of variables in Cartesian coordinates

With separation of variables in Cartesian coordinates we substitute

$$u = X(x)Y(y) \tag{11.43}$$

into Laplace's equation. We quickly find that

$$\mathrm{e}^{\pm kx}\cos ky\,, \qquad \sinh kx\cos ky\,, \quad \sin kx\sinh k(y-a)\,, \tag{11.44}$$

$$\mathrm{e}^{\pm ky}\cos kx\,, \qquad \sinh ky\cos kx\,, \quad \sin ky\sinh k(x-a)\,, \quad \text{et cetera}\,, \tag{11.45}$$

are solutions of Laplace's equation.

Example: Suppose $\mathcal{R}$ is the rectangle $0 < x < a$ and $0 < y < b$, solve

$$\triangle P^S = 0 \tag{11.46}$$

with the boundary condition

$$P^S(x,0) = S(x)\,, \quad P^S(x,b) = P^S(0,y) = P^S(a,y) = 0\,. \tag{11.47}$$

P^S is non-zero on the southern boundary.

We expand the southern boundary condition in a Fourier series

$$S(x) = \sum_{n=1}^{\infty} S_n \sin\left(\frac{n\pi x}{a}\right) , \tag{11.48}$$

and then the solution of this Laplace-Dirichlet problem is

$$P^S(x,y) = \sum_{n=1}^{\infty} S_n \sin\left(\frac{n\pi x}{a}\right) \frac{\sinh \frac{n\pi(b-y)}{a}}{\sinh \frac{n\pi b}{a}} . \tag{11.49}$$

For example, if $S(x) = 1$ we have

$$P^S(x,y) = \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\sin \frac{(2m+1)\pi x}{a}}{2m+1} \frac{\sinh \frac{(2m+1)\pi(b-y)}{a}}{\sinh \frac{n\pi b}{a}} . \tag{11.50}$$

On $0 < X < \pi$:

$$1 = \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\sin(2m+1)X}{2m+1}$$

Example: Above we specified a non-zero boundary value on the southern boundary. More generally we might have

$$P(x,0) = S(x) , \quad P(x,b) = N(x) , \quad P(0,y) = W(y) , \quad P(a,y) = E(x) . \tag{11.51}$$

We can use linearity to write the solution as the sum of four subproblems

$$P = P^S + P^N + P^W + P^E . \tag{11.52}$$

For each of the four subproblems we use a different set of basis functions. For the southern subproblem we use P^S in (11.49) and for the eastern subproblem

$$P^E = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi y}{b}\right) \frac{\sinh \frac{n\pi x}{b}}{\sinh \frac{n\pi a}{b}} . \tag{11.53}$$

Exercise: Construct the solutions of the northern and western subproblems.

Exercise: By inspection, solve $\triangle P = 0$, with $P = 1$ on all four sides. If you can prove that the solution is unique, then you obtain an impressive identity using the method in (11.52).

Example: Consider the Laplace-Dirichlet problem on the strip $0 < x < \infty$ and $0 < y < b$ with non-zero P only on the western boundary

$$P(y,0) = W(y) , \qquad P(x,0) = P(y,b) = 0 . \tag{11.54}$$

The solution is

$$P(x,y) = \sum_{n=1}^{\infty} W_n \mathrm{e}^{-n\pi x/b} \sin\left(\frac{n\pi y}{b}\right) . \tag{11.55}$$

Naturally we consider our favourite example, $W(y) = 1$, leading to

$$P(x,y) = \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\mathrm{e}^{-(2m+1)\pi x/b}}{2m+1} \sin\left(\frac{(2m+1)\pi y}{b}\right) . \tag{11.56}$$

This solution is shown in (11.2). The series in (11.56) can be summed, resulting in

$$P(x,y) = \frac{2}{\pi} \arctan\left(\frac{\sin \pi y/b}{\sinh \pi x/a}\right) . \tag{11.57}$$

Use

$$\arctan \zeta = \zeta + \frac{\zeta^3}{3} + \frac{\zeta^5}{5} + \cdots$$

with

$$\zeta = \mathrm{e}^{\frac{\pi}{b}(x+\mathrm{i}y)}$$

This is very pleasant, but notice that if you want to calculate the escape probability at rather moderate distances from the boundary, say $x = b$, then the first term in the series (11.56) is immediately informative:

$$P(b,y) \approx \underbrace{\frac{4}{\pi}\mathrm{e}^{-\pi}}_{\approx 1/18} \sin \frac{\pi y}{b} \tag{11.58}$$

The next term in the series will be smaller by a factor $\mathrm{e}^{-2\pi} = 1/535.5$

Example: Consider the Laplace-Neumann problem on the strip $0 < x < \infty$ and $0 < y < b$ with

$$P_x(y,0) = 1 , \qquad P_y(x,0) = P_y(y,b) = 0 . \tag{11.59}$$

The solution is $P(x,y) = x$, plus an arbitrary constant. The boundary condition violates the solvability condition, but heat diffuses out to $x = \infty$.

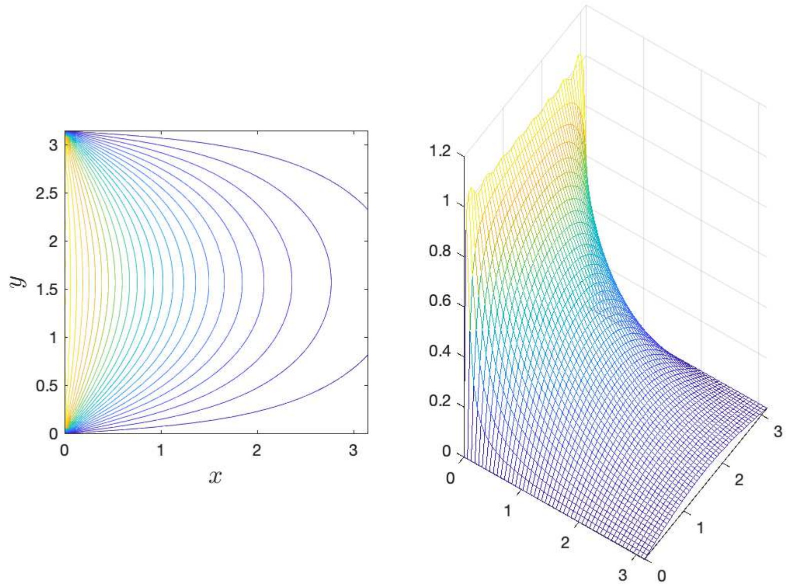


Figure 11.2: Two views of the solution in (11.56) with $b = \pi$. `westBndryLap2020.eps`

11.5 Laplace's equation via analytic functions

It is very easy to manufacture solutions of Laplace's equations in $d = 2$. To do this, we can change variables from x and y to

$$z \overset{\text{def}}{=} x + \mathrm{i}y\,, \qquad \text{and} \qquad z^* \overset{\text{def}}{=} x - \mathrm{i}y \tag{11.60}$$

Any function of x and y can be written equivalently as function of z and z^*.

With a one-line substitution you can show that

$$u(x, y) = \tfrac{1}{2}a(z) + \tfrac{1}{2}b(z^*)\,, \tag{11.61}$$

with a and b arbitrary functions, is a solution of Laplace's equation. If u is to be a real function then a and b aren't independent: we must have

$\Re$ is the real part.

$$u(x, y) = \tfrac{1}{2}a(z) + \tfrac{1}{2}a^*(z^*) = \Re a(z)\,. \tag{11.62}$$

In other words, the real (and also the imaginary) parts of an analytic function are solutions of the 2D Laplace equation. But because of boundary conditions, it is still not easy to solve the Dirichlet problem.

Example: Write $f_1 = x(x^2 + y^2)$ and $f_2 = x^3 - 3y^2x$ as functions of z and z^*.

Eliminate x and y using

$$x = \tfrac{1}{2}(z + z^*)\,, \qquad y = \tfrac{1}{2\mathrm{i}}(z - z^*)\,. \tag{11.63}$$

This gives

$$f_1 = \tfrac{1}{2}(z + z^*)\,zz^*\,, \qquad f_2 = \tfrac{1}{2}\left(z^3 + z^{*3}\right)\,. \tag{11.64}$$

Note f_2 is the real part of z^3 and is therefore a solution of Laplace's equation.

Example: Find a bounded solution of Laplace's equation in the UHP $y > 0$ satisfying the boundary condition

$$u(x,0) = \frac{1}{a^2 + x^2} \,. \tag{11.65}$$

Let's guess that $u(x,y)$ might be equal to

$$v(x,y) \stackrel{\text{def}}{=} \Re \frac{1}{a^2 + z^2} = \frac{a^2 + x^2 - y^2}{(a^2 + x^2 - y^2)^2 + 4x^2y^2} \,. \tag{11.66}$$

This function satisfies the boundary condition, and also Laplace's equation. But unfortunately $v(x,y)$ doesn't solve the problem because there is a singularity in the upper half plane at $z = \mathrm{i}a$: $v(0,a) = \infty$. So our guess has failed. The solution to this problem is in (11.81).

11.6 Laplace's equation in the upper half plane (UHP)

11.6.1 The Dirichlet problem

The harmonic function θ is very handy if we want to solve the general Dirichlet problem in the UHP $y > 0$. Just to be clear, the problem is

$$\triangle = \partial_x^2 + \partial_y^2 = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$$

$$u_{xx} + u_{yy} = 0 \,, \qquad u(x,0) = f(x) \,. \tag{11.67}$$

Above, $f(x)$ is some specified boundary value. We desire a non-singular solution which is bounded as $y \to \infty$. The Green's function solution of this problem is given below in (11.90). On the way to this beautiful formula we admire some harmonic scenery.

$$\theta \stackrel{\text{def}}{=} \arctan(y/x)$$
$$\triangle\theta = 0$$

We begin with discontinuous boundary data:

Compare with $\mathrm{sgn}(x)$ solutions of the wave equation and the diffusion equation.

$$f(x) = \mathrm{sgn}(x) \,. \tag{11.68}$$

Since the boundary data does not provide a length scale, similarity reasoning suggests that the solution must have the form:

$$u = f\left(\frac{y}{x}\right) = g(\theta) \,. \tag{11.69}$$

But the most general solution of Laplace's equation which is independent of r is just $u = a + b\theta$. Fiddling around with the boundary condition we quickly see that the solution of the problem is

$$\theta = 0: \quad +1 = a$$
$$\theta = \pi: \quad -1 = a + b\pi$$

$$u = 1 - \frac{2\theta}{\pi} \,. \tag{11.70}$$

The discontinuity in the $\mathrm{sgn}(x)$ boundary condition is removed as soon as y is a little bit positive. (This smoothing is like the erf-solution of the diffusion equation.)

Exercise: A drunken criminal flees from the initial point (x,y) with $y > 0$. She keeps random walking till she arrives at $y = 0$. The police arrest her if she arrives at $y = 0$ with $x > 0$, while she escapes capture if she arrives at $y = 0$ with $x < 0$. Find the probability of escape as a function of (x,y)

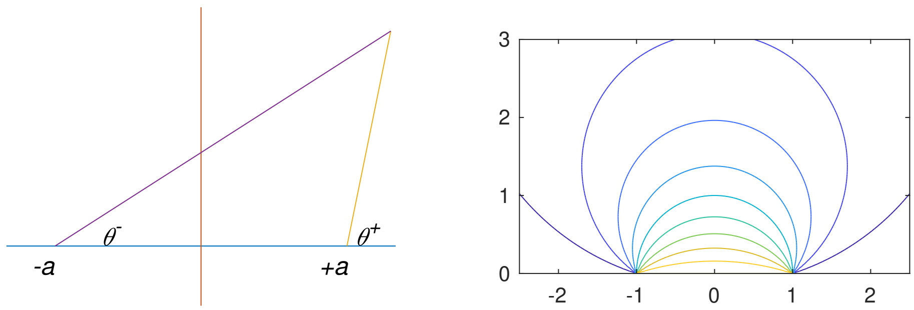


Figure 11.3: The solution in (11.72) with $a = 1$. `dirichlet.eps`

Exercise: Find a UHP solutions of Laplace's equation with boundary condition $u_1(x, 0) = \arctan(x/\ell)$ and $u_2 = \arctan(\ell/x)$. Hint: use a polar coordinate system centered on $(0, -\ell)$.

Now consider the upper half-plane Dirichlet problem with

$\Pi(x)$ is the indicator function of the interval $-1 < x < 1$.

$$\Pi(x/a) = \begin{cases} 1\,, & \text{if } -a < x < a; \\ 0\,, & \text{if } |x| > a. \end{cases} \tag{11.71}$$

I'm sure at this point you can supply a probabilistic interpretation. And using linear superposition you can also quickly see that the solution is

$$u = \frac{\theta^+ - \theta^-}{\pi}\,, \tag{11.72}$$

where the angles θ^+ and θ^- are defined by

$$\theta^\pm = \tan^{-1}\left(\frac{y}{x \mp a}\right)\,. \tag{11.73}$$

This solution is illustrated in Figure 11.3.

Now we take a δ-limit in which $a \to 0$ in (11.71). We also have to divide by $2a$ so that the boundary condition is

(11.71):

$$\Pi(x/a) = \begin{cases} 1\,, & \text{if } -a < x < a; \\ 0\,, & \text{if } |x| > a. \end{cases}$$

$$\lim_{a \to 0} \frac{\Pi(x/a)}{2a} = \delta(x)\,. \tag{11.74}$$

The solution of this new problem — the Green's function of the upper half plane Dirichlet problem — is

A short-cut to g in (11.77): ∂_x (11.70)= 2×(11.77). The factor of two results from $\partial_x \operatorname{sgn}(x) = \delta(x)$.

$$g(x, y) = \lim_{a \to 0} \frac{1}{2a\pi}\left(\arctan\left(\frac{y}{x-a}\right) - \arctan\left(\frac{y}{x+a}\right)\right)\,, \tag{11.75}$$

$$= -\frac{1}{\pi}\theta_x\,, \tag{11.76}$$

$$= \frac{1}{\pi}\frac{y}{x^2 + y^2}\,. \tag{11.77}$$

With the Green's function in hand, the solution of the half-plane Dirichlet problem posed in (11.67) is

$$u(x, y) = \int_{-\infty}^{\infty} \frac{y f(\xi)}{(x - \xi)^2 + y^2}\frac{\mathrm{d}\xi}{\pi}\,. \tag{11.78}$$

Exercise: Find the Green's function for the lower half plane (LHP) Dirichlet problem.

Example: Solve the UHP Dirichlet problem with boundary condition $u(x,0) = 1/(x^2 + a^2)$. Using the Green's function in (11.90)

$$\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} \frac{1}{a^2 + \xi^2} \frac{\mathrm{d}\xi}{\pi} . \tag{11.79}$$

I believe we can evaluate this integral by splitting up the integrand with partial fractions. The lazy alternative is MATHEMATICA, which gives

$$u = \frac{1}{a} \frac{a+y}{x^2 + (a+y)^2} . \tag{11.80}$$

Head scratching indicates that this solution can also be written as

$$u = -\frac{1}{a} \Im \left[\frac{1}{z + \mathrm{i}a} \right] . \tag{11.81}$$

There is a pole in the lower half plane (LHP) at $z = -\mathrm{i}a$. The takeaway here is that one can construct UHP harmonic function by putting singularities in the LHP.

Exercise: Solve the UHP Dirichlet problem with boundary condition

$$u(x,0) = \frac{\alpha_1}{(x-x_1)^2 + a_1^2} + \frac{\alpha_2}{(x-x_2)^2 + a_2^2} \tag{11.82}$$

Exercise: Solve the UHP Dirichlet problem with $u(x,0) = (x^2 + a^2)^{-2}$. Hint: parametric differentiation.

Example: Consider a big sheet of metal, which we take to be the half-plane $y > 0$. Suppose that the temperature at $y = 0$ is prescribed to be

$$f(x) = \cos kx . \tag{11.83}$$

Find the steady-state temperature in the sheet.

The fastest way to solve this Dirichlet problem is separation of variables: substituting $u(x,y) = U(y)\cos kx$ we have

$$U'' - k^2 U = 0 , \qquad \Rightarrow \qquad U = \underbrace{A}_{=1} \mathrm{e}^{-ky} + \underbrace{B}_{=0} \mathrm{e}^{ky} . \tag{11.84}$$

We take $k > 0$ and discard the exponentially growing solution. Thus the upper half plane solution is

$$u(x,y) = \cos kx \, \mathrm{e}^{-ky} = \Re \mathrm{e}^{\mathrm{i}kz} . \tag{11.85}$$

$z = x + \mathrm{i}y$

We might have guessed

$$v(x,y) = \Re \cos kz = \cos kx \cosh ky \tag{11.86}$$

as a solution of this problem. Indeed, v satisfies the boundary condition on $y = 0$, and v is a harmonic function. But v is not bounded as $y \to \infty$, and so our guess is wrong.

Heat flows into the plate where the temperature is high and flows out where it is cold. For instance, heat fluxes into the plate through the boundary segments where $\cos kx > 0$ and out of the plate where $\cos kx < 0$. The *net* flow of heat through $y = 0$ is zero — what goes in at one place comes out somewhere else. The thermal disturbance decays at large distances from the boundary, and the decay length, or penetration distance, is $k^{-1} = \lambda/2\pi$ where λ is the wavelength on the boundary.

In solving this problem we haven't used the Green's function formula in (11.90). So it's sporting to independently check that

$$\cos kx \, \mathrm{e}^{-ky} = \int_{-\infty}^{\infty} \frac{y \cos k\xi}{(x-\xi)^2 + y^2} \frac{\mathrm{d}\xi}{\pi} . \tag{11.87}$$

After a noticeable pause on my computer, MATHEMATICA verifies the formula above with the command

```
Assuming[{k > 0, x > 0, y > 0},
 Integrate[ y*Cos[k*a]/((x - a)^2 + y^2), {a, -Infinity, Infinity}]]
```

One way to do the integral yourself, is to notice that

$$\frac{1}{(x-\xi)^2+y^2} = \int_0^\infty \mathrm{e}^{-\beta[(x-\xi)^2+y^2]}\,\mathrm{d}\beta\,. \tag{11.88}$$

Using this trick one can move the denominator in (11.87) into an exponential. Then switching the order of integration produces standard Gaussian integrals. Dutifully performing this exercise will convince you that separation of variables is the best way to solve this problem.

The Dirichlet-to-Neumann map

Here is a very handy trick: one can use the Green's function solution in (11.90) to obtain a compact expression for the normal derivative at the boundary, $u_y(x,0)$. In many physical problems one does need all details of the interior solution $u(x, y>0)$: the normal derivative at the boundary can be used to calculate important quantities.

By direct calculation from (11.90)

$$u_y(x,y) = \int_{-\infty}^{\infty} \frac{(x-\xi)^2 - y^2}{[(x-\xi)^2+y^2]^2} f(\xi)\,\frac{\mathrm{d}\xi}{\pi}\,, \tag{11.89}$$

$$= -\partial_x \int_{-\infty}^{\infty} \frac{(x-\xi)}{(x-\xi)^2+y^2} f(\xi)\,\frac{\mathrm{d}\xi}{\pi}\,. \tag{11.90}$$

To get the normal derivative on the boundary we take the limit $y \downarrow 0$:

$$u_y(x,0) = -\partial_x\, \mathrm{pv}\!\int_{-\infty}^{\infty} \frac{f(\xi)}{x-\xi}\,\frac{\mathrm{d}\xi}{\pi}\,, \tag{11.91}$$

$$= -\partial_x \mathcal{H}[f]\,, \tag{11.92}$$

where pv denotes the principal value integral and $\mathcal{H}$ is the Hilbert transform. We refer to (11.92) as the Dirichlet–to–Neumann map because it converts the Dirichlet boundary condition $u(x,0) = f(x)$ to the Neumann boundary condition $u_y(x,0)$.

Solution with Fourier transform

Now let's solve the Laplace-Dirichlet problem in (11.67) with a Fourier transform in x:

$$\tilde{u}(k,y) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} u(x,y)\,\mathrm{d}x\,. \tag{11.93}$$

With $\partial_x \mapsto \mathrm{i}k$, the transformed problem is

$$\tilde{u}_{yy} - k^2\tilde{u} = 0\,, \tag{11.94}$$

with boundary conditions $\tilde{u}(k,0) = \tilde{f}(k)$ and $\tilde{u}(k,y) \to 0$ as $y \to \infty$. The solution is

$$\tilde{u}(k,y) = \tilde{f}(k)\mathrm{e}^{-|k|y}\,. \tag{11.95}$$

Note particularly the $|k|$ in the exponential above: we want $\tilde{u}(k,y) \to 0$ as $y \to \infty$ for *all* k's, including $k < 0$.

The FIT now provides an integral representation of the solution:

$$u(x,y) = \mathcal{F}^{-1}\left[\tilde{f}(k)\mathrm{e}^{-|k|y}; k \mapsto x\right], \tag{11.96}$$

$$= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\tilde{f}(k)\mathrm{e}^{-|k|y}\,\frac{\mathrm{d}k}{2\pi}. \tag{11.97}$$

To show that (11.97) is equivalent to our earlier integral representation in (11.90), we use the convolution theorem:

$$\mathcal{F}^{-1}\left[\tilde{f}(k)\tilde{g}(k); k \mapsto x\right] = \underbrace{\int_{-\infty}^{\infty} f(x')g(x-x')\,\mathrm{d}x'}_{=f\circ g}. \tag{11.98}$$

In the problem at hand $\tilde{g}(k) = \mathrm{e}^{-|k|y}$ is the Fourier transform of the Green's function. In x-space, the Green's function is

$$g(x) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx-|k|y}\,\frac{\mathrm{d}k}{2\pi} = \frac{1}{\pi}\frac{y}{x^2+y^2}. \tag{11.99}$$

Thus we recover (11.90) from the inverse Fourier transform in (11.97).

11.6.2 The Neuman problem

The Neuman problem is

$$u_{xx} + u_{yy} = 0, \qquad u_y(x,0) = g(x). \tag{11.100}$$

Above, $g(x)$ is some specified boundary value. We desire a non-singular solution which is bounded as $y \to \infty$. However our desires are likely frustrated if

$$\int_{-\infty}^{\infty} g(x)\,\mathrm{d}x \neq 0. \tag{11.101}$$

Using the heat-condition analogy, the condition above means there is a net flux of heat into the sheet. In steady state, this has to be conducted away to $r = \infty$ and this probably requires $u \sim \ln r$. So we must reduce our Neuman expectations to saying that $u(x,y)$ grows no faster than $\ln(r)$.

Example: Use the Dirichlet solution in (**??**) to construct a Neuman solution.

Suppose we calculate

$$g(x) \stackrel{\text{def}}{=} \partial_y\left(\frac{\theta^- - \theta^+}{\pi}\right), \text{ evaluated at } y = 0. \tag{11.102}$$

Once we have this $g(x)$ we can amaze our friends by exhibiting a soluble Neuman problem....

11.7 The disk: Poisson's formula

Recall

$$\mathrm{sqr}(\theta) = \mathrm{sgn}(\sin\theta)\,,$$
$$= \frac{4}{\pi}\left[\sin\theta + \tfrac{1}{3}\sin 3\theta + \cdots\right]$$

That is, the top half of the disk ($0 < \theta < \pi$) is hot and the lower half ($-\pi < \theta < 0$) is cold.

Consider a circular disk which heated non-uniformly only at the edge $r = a$. The problem is

$$\triangle u = 0\,, \qquad \text{with BC} \qquad u(a,\theta) = f(\theta)\,. \tag{11.103}$$

It seems like a good idea to represent the boundary value $f(\theta)$ using a Fourier series

$$f(\theta) = \sum_{m=-\infty}^{\infty} f_m \mathrm{e}^{\mathrm{i}m\theta}\,, \tag{11.104}$$

$f_m = f_{-m}^*$

where

$$f_m = \oint f(\theta)\mathrm{e}^{-\mathrm{i}m\theta}\,\frac{\mathrm{d}\theta}{2\pi}\,. \tag{11.105}$$

Then we hope to represent the solution as a Fourier series

$$u(r,\theta) = \sum_{m=-\infty}^{\infty} u_m(r)\,\mathrm{e}^{\mathrm{i}m\theta}\,. \tag{11.106}$$

With linear superposition in view, we first consider the simpler sub-problem with just one term

$$\triangle\left(u_m(r)\,\mathrm{e}^{\mathrm{i}m\theta}\right) = 0\,, \qquad u_m(a) = f_m\,. \tag{11.107}$$

Using the Laplacian in cylindrical coordinates, we find that $u_m(r)$ satisfies the ODE

$$u_m'' + r^{-1}u_m' - m^2 r^{-2} u_m = 0\,. \tag{11.108}$$

I'm sure you immediately recognize that the above is an Euler equation and that the substitution

$$u_m = r^q\,, \tag{11.109}$$

produces the general solution:

$$q = \pm m\,, \qquad \text{or} \qquad u_m = Ar^m + Br^{-m}\,. \tag{11.110}$$

The temperature is unlikely to be singular at $r = 0$ and so if $m > 0$ we must take $B = 0$. We secure the $r = a$ boundary condition by taking $A = a^{-m}$ and so:

$$u_m(r) = f_m\left(\frac{r}{a}\right)^m \mathrm{e}^{\mathrm{i}m\theta}\,, \qquad \text{if } m > 0. \tag{11.111}$$

If $m < 0$, then the non-singular solution is

$$u_m(r) = f_m\left(\frac{r}{a}\right)^{-m} \mathrm{e}^{\mathrm{i}m\theta}\,, \qquad \text{if } m < 0. \tag{11.112}$$

We should also mention $m = 0$. In this case the two solutions are u is a constant and $u \propto \ln r$. We reject the logarithm because of its singularity at $r = 0$.

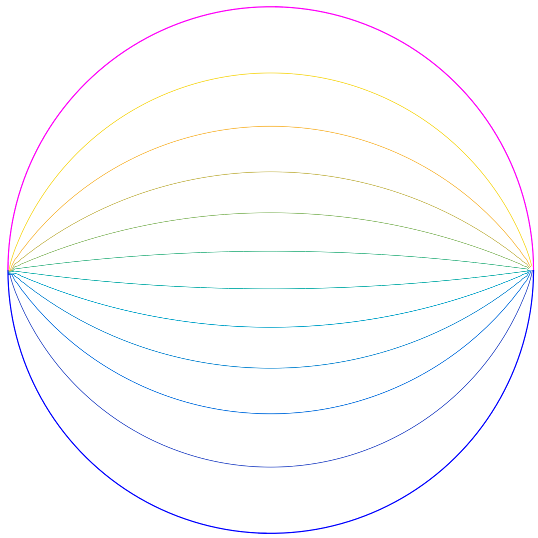


Figure 11.4: Solution of (11.103) obtained by evaluation of the Poisson integral in (11.116) with $f = \mathrm{sqr}(\theta)$.

With the boundary condition in (11.104) we can write down the solution

$$u(r,\theta) = \sum_{m=-\infty}^{\infty} f_m \left(\frac{r}{a}\right)^{|m|} \mathrm{e}^{\mathrm{i}m\theta} . \tag{11.113}$$

The $|m|$ above takes care of both $m > 0$ and $m < 0$. Something glorious happens if we recall that f_m is given by the integral in (11.105) and we press on by exchanging integration and summation:

(11.105) $$f_m = \oint f(\theta)\mathrm{e}^{-\mathrm{i}m\theta}\,\frac{\mathrm{d}\theta}{2\pi}$$

Poisson's integral

$$u(r,\theta) = \sum_{m=-\infty}^{\infty} \left[\oint \mathrm{e}^{-\mathrm{i}m\theta'} f(\theta')\,\frac{\mathrm{d}\theta'}{2\pi}\right] \left(\frac{r}{a}\right)^{|m|} \mathrm{e}^{\mathrm{i}m\theta} , \tag{11.114}$$

$$= \oint f(\theta') \left[\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\theta-\theta')} \left(\frac{r}{a}\right)^{|m|}\right] \frac{\mathrm{d}\theta'}{2\pi} , \tag{11.115}$$

(see exercise below)

$$= \oint f(\theta') \left[\frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta-\theta')}\right] \frac{\mathrm{d}\theta'}{2\pi} . \tag{11.116}$$

This is Poisson's integral — we can get the solution everywhere inside the disk from the boundary function $f(\theta)$. The limit $r \to a$ is singular — the Poisson kernel goes to $2\pi\delta(\theta-\theta')$ in this limit.

Exercise: Show that if $|\rho| < 1$

$$\sum_{m=-\infty}^{\infty} \rho^{|m|}\mathrm{e}^{\mathrm{i}m\phi} = 1 + \frac{\rho\mathrm{e}^{-\mathrm{i}\phi}}{1-\rho\mathrm{e}^{-\mathrm{i}\phi}} + \frac{\rho\mathrm{e}^{\mathrm{i}\phi}}{1-\rho\mathrm{e}^{\mathrm{i}\phi}} , \tag{11.117}$$

$$= \frac{1-\rho^2}{1+\rho^2-2\rho\cos\phi} . \tag{11.118}$$

Evaluating (11.116) at $r = 0$, we see that the central temperature $u(0,\theta)$ is equal to the average of $f(\theta)$. Once again, this is the mean value theorem for harmonic functions.

With the particular boundary condition $f = \mathrm{sqr}(\theta)$ we can evaluate the integral in (11.116) — see problems — and the solution is displayed in figure 11.4.

11.8 Minimization of the Dirichlet functional

The solution of Laplace's equation in the domain $\mathcal{R}$ minimizes the Dirichlet functional

$$\mathcal{D}[v] = \int_{\mathcal{R}} |\boldsymbol{\nabla} v|^2 \, \mathrm{d}A \,. \tag{11.119}$$

Here $v(x,y)$ is *any* function whose boundary value is the same as that of the harmonic function $u(x,y)$.

Example: As an example, consider

$$v(x,y) = \mathrm{e}^{-\alpha y} \cos kx \tag{11.120}$$

in the half-plane $y > 0$. The 'trial function' $v(x,y)$ is always equal to $\cos kx$ on the boundary $y = 0$. Now we calculate the Dirichlet functional

$$\mathcal{D}[v] = \tfrac{1}{4}\lambda \left(\frac{k^2}{\alpha} + \alpha \right) . \tag{11.121}$$

($\lambda = 2\pi/k$ is one wavelength of the periodic-in-x solution.) Sure enough, the minimum is achieved if $\alpha = k$, and this makes v the solution of Laplace's equation obtained previously by separation of variables.

Minimizing the Dirichlet functional is a good example of the calculus of variations. Suppose the minimum value of $\mathcal{D}[v]$ is achieved at $u(x,y)$, and consider how the functional changes if $v = u + \delta v$. Here $\delta v(x,y)$ is a small variation away from the optimal $u(x,y)$. We insist that u and v satisfy the same boundary condition, and therefore δv must vanish on the boundary $\partial\mathcal{R}$. Now

$$\mathcal{D}[u+\delta v] = \mathcal{D}[u] + \underbrace{2\int_{\mathcal{R}} \boldsymbol{\nabla} u \boldsymbol{\cdot} \boldsymbol{\nabla} \delta v \, \mathrm{d}A}_{\text{the first variation}} + \underbrace{\int_{\mathcal{R}} |\boldsymbol{\nabla} \delta v|^2 \, \mathrm{d}A}_{\text{the second variation}} \,. \tag{11.122}$$

At the optimal u, the first variation must vanish for all possible δv's. But integrating by parts

$$2\int_{\mathcal{R}} \boldsymbol{\nabla} u \boldsymbol{\cdot} \boldsymbol{\nabla} \delta v \, \mathrm{d}A = \underbrace{\oint_{\partial\mathcal{R}} \delta v \boldsymbol{\nabla} u \boldsymbol{\cdot} \hat{\boldsymbol{n}} \, \mathrm{d}\ell}_{=0} - \int_{\mathcal{R}} \delta v \triangle u \, \mathrm{d}A \,. \tag{11.123}$$

To make the first variation zero, the final integral must be zero for all δv, including δv's which are non-zero only in the close neighbourhood of an arbitrary point in $\mathcal{R}$. The only way to do this is to make the integrand zero at every point, which is achieved by $\triangle u = 0$.

11.9 References

For much more on the discrete Laplace equation, see the little book:

DS P.G. Doyle & J.L. Snell *Random Walks and Electric Networks*, Carus Mathematical monographs.

11.10 Problems

Problem 11.1. Let $m(x)$ be the expected number of coin-tosses before someone wins in Tom and Jerry's game. *(i)* Show that

$$m(x) = \tfrac{1}{2}m(x-1) + \tfrac{1}{2}m(x+1) + 1\,. \tag{11.124}$$

(ii) Solve this difference equation using the appropriate boundary conditions at $x = 0$ and N.

Problem 11.2. In Tom and Jerry's game, suppose that the probability of a head is a and the probability of a tail is b with $a + b = 1$. *(i)* Show that the generalization of (11.2) is

$$p(x) = ap(x-1) + bp(x+1)\,. \tag{11.125}$$

(ii) Solve this difference equation (with the appropriate boundary conditions at $x = 0$ and n) by looking for solutions of the form

$$p = r^x\,. \tag{11.126}$$

(See page 41 of BO for a discussion of constant coefficient difference equations.) *(iii)* Develop an approximate second-order ODE, analogous to (11.13), assuming that n is a large $p(x)$ changes only slightly when x changes by ± 1. *(iv)* Solve the ODE and numerically compare the approximate solution with the exact solution of the difference equation for selected values of a, b and n.

Problem 11.3. Find the escape probabilities for the criminal random walking on the lattice in the bottom panel of figure 11.1.

Problem 11.4. Calculate the integral

$$\oint_{\mathcal{C}} 3 + 11x + x^3 - 3xy^2 \,\mathrm{d}\ell\,, \tag{11.127}$$

where $\mathcal{C}$ is the circle $(x-1)^2 + (y - 3^{-1/2})^2 = 7$.

Notation: $\hat{\boldsymbol{n}}$ is the outward unit normal to the contour $\mathcal{C}$.

Problem 11.5. Find a vector field $\boldsymbol{v}$ for which:

$$\text{Area enclosed by any closed curve } \mathcal{C} = \oint_{\mathcal{C}} \boldsymbol{v}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}\ell\,. \tag{11.128}$$

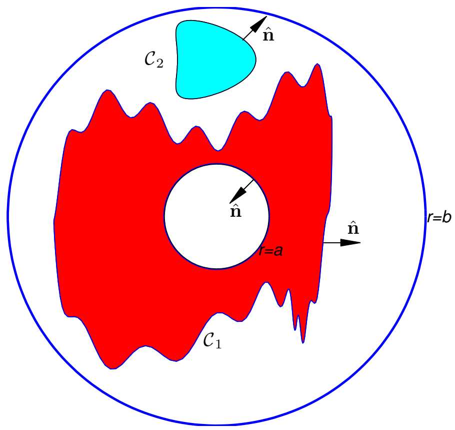


Figure 11.5: Figure for problem 11.8. `blob.eps`

Problem 11.6. If $\mathcal{C}$ is a closed curve in the (x, y)-plane, calculate the contour integrals

$$J_1 = \oint_{\mathcal{C}} \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}\ell \,, \qquad J_2 = \oint_{\mathcal{C}} (x\hat{\boldsymbol{x}} - y\hat{\boldsymbol{y}}) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}\ell \,. \tag{11.129}$$

Problem 11.7. Real solutions of Laplace's equation can be written in the form (11.62). Verify that the functions below are solutions of Laplace's equation, and find $a(z)$:

$$u_1 = \mathrm{e}^{-ky} \cos kx \,, \qquad u_2 = x^4 - 6x^2y^2 + y^4 \,, \qquad u_3 = \frac{x+1}{(x+1)^2 + y^2} \,. \tag{11.130}$$

Calculate the area integrals

$$I_n = \iint_{\mathcal{C}} u_n \, \mathrm{d}^2\boldsymbol{x} \,, \tag{11.131}$$

where $\mathcal{C}$ is the unit circle.

Problem 11.8. Consider an annular sheet of metal with the temperature of the inner boundary $(r = a)$ fixed at temperature θ_0 and the outer boundary $(r = b > a)$ at temperature θ_1 (see Figure 11.5). *(i)* Solve Laplace's equation and thus find $\theta(\boldsymbol{x})$. *(ii)* Apply Gauss's theorem to the two shaded areas in Figure 11.5 and thus calculate

$$J_n \stackrel{\text{def}}{=} \oint_{\mathcal{C}_n} \boldsymbol{\nabla}\theta \cdot \hat{\boldsymbol{n}} \, \mathrm{d}\ell \,. \tag{11.132}$$

Problem 11.9. An ensemble of random walkers is released at a radius r in the annular domain shown in Figure 11.5. Those that first reach the inner boundary at $r = a$ are captured, while those that first reach the outer boundary at $r = b$ escape. Find the probability of escape.

Problem 11.10. *(i)* Show that both

$$u_1 \stackrel{\text{def}}{=} \cos kx \cosh ky\,, \qquad \text{and} \qquad u_2 \stackrel{\text{def}}{=} \cos kx\, \mathrm{e}^{-ky} \tag{11.133}$$

are UHP solutions of the Dirichlet problem $\triangle u = 0$ with $u(x, 0) = \cos kx$. *(ii)* Why doesn't this example contradict the uniqueness proof?

Problem 11.11. Consider heat conduction in a uniformly heated 2D domain $\mathcal{D}$. Poisson's equation for the steady-state temperature distribution, $T(x, y)$, is

$$0 = \kappa \triangle T + h\,, \tag{11.134}$$

where the constant $h > 0$ is the uniform heating. Steady state is maintained by heat flux out through the boundary of the domain $\mathcal{D}$, where the boundary condition $T = 0$ is applied. *(i)* First consider the special case in which $\mathcal{D}$ is the disc $0 < \sqrt{x^2 + y^2} < a$. Solve (11.134).

For the remainder of this problem take $\mathcal{D}$ to be an arbitrary domain i.e., no longer restrict attention to the special case in which the domain is a disc.

Define an "average temperature" at a point $\boldsymbol{x}$ within the arbitrary domain $\mathcal{D}$ as

$$\bar{T}(\boldsymbol{x}, r) \stackrel{\text{def}}{=} \oint T(\boldsymbol{x} + \boldsymbol{r})\, \frac{\mathrm{d}\theta}{2\pi}\,, \tag{11.135}$$

where $\boldsymbol{r} \stackrel{\text{def}}{=} r(\cos\theta, \sin\theta)$. The radius r is restricted so that $\boldsymbol{x} + \boldsymbol{r}$ is always within $\mathcal{D}$. *(ii)* On physical grounds, do you expect $\bar{T}(\boldsymbol{x}, r)$ to be greater or less than $T(\boldsymbol{x})$? *(iii)* Prove that

$$\bar{T}(\boldsymbol{x}, r) = T(\boldsymbol{x}) - \frac{hr^2}{4\kappa}\,. \tag{11.136}$$

Problem 11.12. Let $\phi(x, y)$ be the angle subtended by a line segment in the plane at the point $\boldsymbol{x} = (x, y)$. Show that ϕ is harmonic (except at the end-points of the segment).

Problem 11.13. *(i)* Solve the Dirichlet-Poisson problem

$$\triangle u = \mathrm{e}^{-y} \cos kx\,, \qquad u(x, 0) = 0\,, \tag{11.137}$$

in the upper half plane. *(ii)* Solve the Neuman-Poisson problem

$$\triangle u = \mathrm{e}^{-y} \cos kx\,, \qquad u_y(x, 0) = 0\,. \tag{11.138}$$

Make sure you discuss the case $k = 0$.

Problem 11.14. *(i)* Consider heat conduction in a circular disk of metal, $0 < r < a$. Solve Laplace's equation for the steady-state temperature distribution:

$$\triangle T = 0\,, \tag{11.139}$$

assuming that the boundary condition at $r = a$ is

$$T_1(a, \theta) = 1\,, \qquad T_2(a, \theta) = \cos\theta\,, \qquad T_{3r}(a, \theta) = 1\,, \qquad T_{4r}(a, \theta) = \cos\theta\,. \tag{11.140}$$

In one of the four cases the problem "obviously" has no solution — so don't be surprised if you can only solve three of the four. *(ii)* After you have identified the case with no steady solution, continue to find an unsteady solution of the diffusion equation

$$T_t = \kappa \triangle T \tag{11.141}$$

with the offending boundary condition.

Problem 11.15. *(i)* Find two real solutions of Laplace's equation by fiddling around with the complex function

$$A(z) = \frac{1}{z}, \qquad z = x + \mathrm{i}y\,. \tag{11.142}$$

(ii) Using the solutions from part *(i)* as a building block, find a bounded solution of Laplace's equation in the half-plane $y > 0$ with the boundary condition

$$u(x, 0) = \frac{1}{a^2 + x^2}\,. \tag{11.143}$$

(iii) Find a bounded solution of Laplace's equation in the half-plane $y > 0$ with the boundary condition

$$u(x, 0) = \frac{1}{(a^2 + x^2)^2}\,. \tag{11.144}$$

Problem 11.16. *(i)* Find an approximate solution $v(x, y)$ of the half-plane $(y > 0)$ Dirichlet problem

$$\triangle u = 0\,, \qquad u(x, 0) = \frac{1}{x^2 + a^2}\,, \tag{11.145}$$

by minimizing the Dirichlet function with an optimal choice of q in the test function

$$v(x, y) = \frac{\mathrm{e}^{-qy}}{x^2 + a^2}\,. \tag{11.146}$$

(ii) If you've also solved the previous problem, compare the exact and approximate values of the Dirichlet functional i.e., calculate $\mathcal{D}[u]$ and $\mathcal{D}[v]$. To check your answer show that $\mathcal{D}[v] = \sqrt{2}\mathcal{D}[u]$. *(iii)* Bonus if you can find a simple test function $w(x, y)$ with $\mathcal{D}[w] < \mathcal{D}[v]$.

Problem 11.17. *(i)* Show that if $u(x, y)$ is a solution of either the Dirichlet-Laplace or Neumann-Laplace problems in a region $\mathcal{R}$, then

$$\mathcal{D}[u] = \oint_{\partial\mathcal{R}} u \boldsymbol{\nabla} u \cdot \hat{\boldsymbol{n}}\, \mathrm{d}\ell\,. \tag{11.147}$$

(ii) Suppose $v(x, y)$ satisfies the same Neumann or Dirichlet boundary condition as the harmonic function $u(x, y)$ on $\partial\mathcal{R}$. Starting from

$$\int_{\mathcal{R}} |\boldsymbol{\nabla}(u - v)|^2\, \mathrm{d}A \geq 0\,, \tag{11.148}$$

show that $\mathcal{D}[v] \geq \mathcal{D}[u]$.

Problem 11.18. *(i)* Solve Laplace's equation, $\triangle u = 0$, in the disc $0 < r < a$ with the boundary condition $u(a, \theta) = x^4$. *(ii)* With the boundary condition $u(a, \theta) = \sin^3 \theta$. *(iii)* Solve Laplace's equation, $\triangle \phi = 0$, outside of the circle $r = a$ with the boundary condition $u(a, \theta) = x^3$. (Find the solution with $\lim_{r\to\infty} u(r, a) = 0$.)

Problem 11.19. Suppose $u(r, \theta)$ is the solution of Laplace's equation on a disc with $0 < r < a$, with boundary value $u(a, \theta) = f(\theta)$. Find

$$\bar{u}(r) \overset{\text{def}}{=} \oint u(r, \theta) \frac{\mathrm{d}\theta}{2\pi} \tag{11.149}$$

in terms of $f(\theta)$.

Problem 11.20. Suppose $u(r, \theta)$ is the solution of Laplace's equation on a disc with $r < a$, with boundary value $u(a, \theta) = f(\theta)$, and $v(r, \theta)$ is defined by the Poisson equation

$$\triangle v = u\,, \qquad v(a, \theta) = 0\,. \tag{11.150}$$

Find

$$\bar{v}(r) \overset{\text{def}}{=} \oint v(r, \theta) \frac{\mathrm{d}\theta}{2\pi} \tag{11.151}$$

in terms of $f(\theta)$.

Problem 11.21. *(i)* Solve Laplace's equation $\triangle u = 0$ for $r < a$ with the boundary condition $u(a, \theta) = x^4$. *(ii)* With the boundary condition $u(a, \theta) = \sin^3 \theta$.

Problem 11.22. Consider $f(\theta) = 1$ in Poisson's integral. From the solution of Laplace's equation deduce that

$$\oint \frac{\mathrm{d}\alpha}{A + B\cos\alpha} = \frac{2\pi}{(A^2 - B^2)^{1/2}}\,. \tag{11.152}$$

Find other results of this nature by choosing $f(\theta)$ so that you can easily solve Laplace's equation.

Problem 11.23. Show that the solution of (11.103) is

$$u(r, \theta) = \frac{2}{\pi} \tan^{-1}\left(\frac{2ar\sin\theta}{a^2 - r^2}\right)\,. \tag{11.153}$$

This is plotted in figure 11.4.

Problem 11.24. Consider the *exterior* Dirichlet-Laplace problem for $\psi(r, \theta)$: with $r > a$, solve

$$\triangle \psi = 0\,, \qquad u(a, \theta) = f(\theta)\,. \tag{11.154}$$

To whatever extent you can, find an external analog of Poisson's formula (11.116).

Problem 11.25. Consider the *internal* Neumann-Laplace problem for $u(r,\theta)$: with $r > a$, solve

$$\triangle\psi = 0\,, \qquad u_r(a,\theta) = f(\theta)\,. \tag{11.155}$$

Before attempting solution, show that $f(\theta)$ must satisfy a solvability solution. Given that f satisfies the solvability condition, show that to within an arbitrary constant

$$\psi(r,\theta) = -a\oint f(\theta')\ln\left(1 - 2\frac{r}{a}\cos(\theta-\theta') + \frac{r^2}{a^2}\right)\frac{\mathrm{d}\theta'}{2\pi} \tag{11.156}$$

Lecture 12

The wave equation in higher dimensions: $\phi_{tt} = c^2 \triangle \phi$

12.1 Acoustics

Linearize

$$\rho(\boldsymbol{u}_t + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}) + \boldsymbol{\nabla} p = 0\,, \tag{12.1}$$

$$\rho_t + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0\,, \tag{12.2}$$

$$p = k\rho^\gamma\,, \tag{12.3}$$

about the base state

$$\boldsymbol{u} = 0\,, \qquad \rho = \rho_0\,, \qquad p = p_0\,. \tag{12.4}$$

The linearized equations are

$$\rho_0 \boldsymbol{u}_{1t} + \boldsymbol{\nabla} p_1 = 0\,, \tag{12.5}$$

$$\rho_{1t} + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_1 = 0\,, \tag{12.6}$$

$$p_1 = c^2 \rho_1\,, \tag{12.7}$$

where the sound speed is

$$c^2 = k\gamma\rho_0^{\gamma-1}\,. \tag{12.8}$$

With some simple eliminations

$$p_{1tt} = c^2 \triangle p_1\,. \tag{12.9}$$

We can represent the velocity as

$$\boldsymbol{u}_1 = \boldsymbol{\nabla}\phi\,, \tag{12.10}$$

where ϕ is the velocity potential. (This involves an assumption that there is no vorticity at $t = 0$.) Using ϕ, the momentum equation is

$$\boldsymbol{\nabla}\left(\phi_t + p_1/\rho_0\right) = 0\,. \tag{12.11}$$

Removing the $\boldsymbol{\nabla}$ introduces a function of t as a constant of integration. Requiring that the disturbance is zero at great distances from the origin implies that this function is zero. Hence

$$\phi_t + p_1/\rho_0 = 0\,. \tag{12.12}$$

The potential also satisfies the wave equation

$$\phi_{tt} = c^2 \triangle \phi\,. \tag{12.13}$$

The speed of sound in air is about 330 $\mathrm{m\,s^{-1}}$; in the ocean the sound speed is $\sim$ 1500 $\mathrm{m\,s^{-1}}$. Young human ears can hear frequencies between about 20 and 20,000 Hertz; the corresponding wavelengths are roughly between 15m and 1.5cm.

12.2 Spherical waves in $d = 3$

Looking for spherically symmetric solutions of the 3D wave equation we are confronted with

$$\phi_{tt} = c^2 \left(\phi_{rr} + \frac{2}{r}\phi_r \right) . \tag{12.14}$$

Multiplying this equation by r, we observe that it is equivalent to

$$\partial_t^2 \left(r\phi \right) = c^2 \partial_r^2 \left(r\phi \right) , \tag{12.15}$$

We recognize the 1D wave equation and so we can invoke D'Alembert's 1D solution and write down the general solution of (12.14)

$$\phi = \frac{a(r-ct)}{r} + \frac{b(r-ct)}{r} . \tag{12.16}$$

We've used a remarkable identity for the spherically symmetric part of the 3D Laplacian:

$$\partial_r^2 + \frac{2}{r}\partial_r = r^{-1}\partial_r^2\, r\,. \tag{12.17}$$

Please don't take this for granted: there is no equivalent result for the cylindrically symmetric part of the 2D Laplacian.

12.2.1 The causal Green's function

The term

$$\frac{a(r-ct)}{r} \tag{12.18}$$

in the general solution (12.16) is an outwardly propagating spherical wave. Thus it seems that

$$g(r,t) \stackrel{\text{def}}{=} \frac{\delta(r-ct)}{4\pi c r} , \tag{12.19}$$

is the spherically expanding disturbance created by an impulsive point disturbance at the origin e.g. sound generated by an explosion at $(r, t) = (0, 0)$. (The normalization factor $4\pi c$ is explained below.)

Now we can build a more general solution of the 3D wave equation by superposing these impulsive disturbances:

$$\phi(\boldsymbol{r}, t) = \int \frac{\delta(|\boldsymbol{r} - \boldsymbol{r}'| - ct)}{4\pi c|\boldsymbol{r} - \boldsymbol{r}'|} f(\boldsymbol{r}') \,\mathrm{d}V_{\boldsymbol{r}'} \,. \tag{12.20}$$

This construction satisfies 3D the wave equation. But what are the initial conditions? To answer this question we begin by rewriting (12.20) more expansively. First, we put the origin of the coordinate system at the field point $\boldsymbol{r}$, and then we use a spherical coordinate system with $\mathrm{d}V_{\boldsymbol{r}'} = \mathrm{d}\Omega r'^2 \mathrm{d}r'$ where Ω denotes the angular variables e.g., $\Omega = (\lambda\,, \theta)$ and $\mathrm{d}\Omega = \mathrm{d}\lambda \sin\theta \mathrm{d}\theta$ if you prefer. After these machinations

$$\phi(\boldsymbol{0}, t) = \int \mathrm{d}\Omega \int_0^\infty \mathrm{d}r' r'^2 \,\frac{\delta(r' - ct)}{4\pi c r'} f(r', \Omega) \,, \tag{12.21}$$

$$= \frac{t}{4\pi} \int f(ct, \Omega) \,\mathrm{d}\Omega \,. \tag{12.22}$$

We define the "spherical mean operator" as

$$\mathcal{S}^{ct}[f] \stackrel{\text{def}}{=} \frac{1}{4\pi} \int f(ct, \Omega) \,\mathrm{d}\Omega \,. \tag{12.23}$$

In other words $\mathcal{S}^{ct} f$ is the mean of $f(\boldsymbol{r})$ averaged over a sphere of radius ct. Note that the normalization in (12.23) is correct: $\mathcal{S}^{ct}[1] = 1$ — (12.23) is written as an angular average, not an area average. Using the spherical mean notation, we write (12.22) as

$$\phi(\boldsymbol{0}, t) = t\, \mathcal{S}^{ct}[f] \,. \tag{12.24}$$

Considering $t \to 0$ we see that

$$\lim_{ct \to 0} \mathcal{S}^{ct}[f] = f(\boldsymbol{0}) \,, \tag{12.25}$$

and therefore

$$\lim_{ct \to 0} \phi(\boldsymbol{0}, t) = 0 \,. \tag{12.26}$$

On the other hand the time derivative of (12.24) is

$$\phi_t = \mathcal{S}^{ct}[f] + ct\mathcal{S}^{ct}[f_r] \tag{12.27}$$

and therefore

$$\lim_{ct \to 0} \phi_t(\boldsymbol{0}, t) = f(\boldsymbol{0}) \,. \tag{12.28}$$

Thus ϕ constructed in (12.20) satisfies the 3D wave equation

$$\phi_{tt} = c^2 \triangle \phi \,, \tag{12.29}$$

with the initial conditions

$$\phi(\boldsymbol{r}, 0) = 0 \,, \qquad \text{and} \qquad \phi_t(\boldsymbol{r}, 0) = f(\boldsymbol{r}) \,. \tag{12.30}$$

General initial conditions

How do we solve (12.29) equation with general initial conditions

$$\phi(\boldsymbol{r},0) = g(\boldsymbol{r})\,, \qquad \text{and} \qquad \phi_t(\boldsymbol{r},0) = f(\boldsymbol{r})\,? \tag{12.31}$$

No problem:

$$\phi(\boldsymbol{r},t) = \partial_t \int \frac{\delta(|\boldsymbol{r}-\boldsymbol{r}'| - ct)}{4\pi c|\boldsymbol{r}-\boldsymbol{r}'|} g(\boldsymbol{r}')\,\mathrm{d}V_{\boldsymbol{r}'} + \int \frac{\delta(|\boldsymbol{r}-\boldsymbol{r}'| - ct)}{4\pi c|\boldsymbol{r}-\boldsymbol{r}'|} f(\boldsymbol{r}')\,\mathrm{d}V_{\boldsymbol{r}'}\,. \tag{12.32}$$

(This is a problem for the student.)

12.2.2 Kirchoff's solution of the IVP using spherical means

If one wants to use (12.32) to solve a problem then the spherical mean notation is probably more convenient. In that case it is necessary to use the spherical mean centered on a general field point $\boldsymbol{r}$. Here is the definition

$$\mathcal{S}_{\boldsymbol{r}}^{ct}[g] = \frac{1}{4\pi}\int_0^{\pi}\!\!\int_0^{2\pi} f(x + ct\sin\theta\cos\lambda, y + ct\sin\theta\sin\lambda, z + ct\cos\theta)\,\mathrm{d}\lambda\sin\theta\mathrm{d}\theta \tag{12.33}$$

With this notation, (12.32) is equivalent to

$$\phi(\boldsymbol{r},t) = \partial_t\, t\mathcal{S}_{\boldsymbol{r}}^{ct}[g] + t\mathcal{S}_{\boldsymbol{r}}^{ct}[f]\,, \tag{12.34}$$

$$= \mathcal{S}_{\boldsymbol{r}}^{ct}[g] + ct\mathcal{S}_{\boldsymbol{r}}^{ct}[\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\nabla} g] + t\mathcal{S}_{\boldsymbol{r}}^{ct}[f]\,. \tag{12.35}$$

This is *Kirchoff's formula*, which can be considered the 3D analog of D'Alembert's solution to the 1D wave equation.

Huygen's principle

Huygen's principle is a verbalization of (12.20): solutions of the 3D wave equation can be constructed by superposing spherically expanding waves. In particular, a spherical wave is produced by an initial point disturbance and a more general initial disturbance, concentrated on a surface, is propagated by constructing spheres of radius ct about every point on the initial surface and finding the envelope of those spheres.

The balloon burst problem

Drop all the subscript 1's on perturbation fields so $p_1 \mapsto p$, and so on.

Suppose that at $t = 0$ pressure anomaly inside a spherical balloon of radius a is a constant p_* and the pressure anomaly outside is 0. Using the notation

$$p = \psi(r,t)p_*\,, \tag{12.36}$$

we have the 3D wave equation

$$\psi_{tt} = c^2 \triangle \psi \tag{12.37}$$

with the initial condition

$$\psi(r,0) = \begin{cases} 1, & \text{if } 0 < r < a; \\ 0, & \text{if } a < r. \end{cases} \tag{12.38}$$

The general spherical wave solution is

$$\psi = \frac{f(r-ct)}{r} + \frac{g(r+ct)}{r}, \tag{12.39}$$

and to satisfy the initial condition we must have

$$f(r) + g(r) = \begin{cases} r, & \text{if } r < a; \\ 0, & \text{if } r > a; \end{cases} \tag{12.40}$$

Note too that $p_t \propto \boldsymbol{\nabla\cdot u}$ and thus if $\boldsymbol{u} = 0$ at $t = 0$ then so is p_t. Thus we also have

$$f'(r) = g'(r). \tag{12.41}$$

In fact, looking at the characteristic diagram in figure 12.1, we expect that $\psi = 1$ and $\psi_t = 0$ throughout the blue region — in this region the gas is undisturbed by the rupture of the enclosure at $r = a$. We can satisfy all these conditions if we simply take

$$f(s) = g(s) = \frac{s}{2}, \tag{12.42}$$

then (12.39) becomes

$$\psi = \frac{r-ct}{2r} + \frac{r+ct}{2r}, \tag{12.43}$$

$$= 1. \tag{12.44}$$

This is a valid solution in the blue region and it certainly satisfies the initial conditions in (12.40) and (12.41). Note that $f(s)$ is defined in (12.42) over the interval $-a < s < a$ i.e., into the region with $r < 0$ in figure 12.1.

The green region of figure 12.1 is ventilated only by the characteristics associated with $f(r - ct)$ — this is signal that is expanding radially outwards from the explosion. Thus in the green region of figure 12.1

$$\psi = \frac{r-ct}{2r}. \tag{12.45}$$

A snapshot of the pressure distribution is remarkable: there is a negative pressure anomaly in the region $ct - a < r < ct$.

We used some inspired guesswork above, so it is sporting to check that our solution conserves mass. The density anomaly is $\rho = p/c^2$, so once $t > a/c$, the total mass in the expanding wave is

$$m = \frac{4\pi p_*}{c^2} \int_{ct-a}^{ct+a} \frac{r-ct}{2r} r^2 \mathrm{d}r, \tag{12.46}$$

$$= \frac{2\pi p_*}{c^2} \left[\tfrac{1}{3} r^3 - ct\tfrac{1}{2} r^2\right]_{ct-a}^{ct+a}, \tag{12.47}$$

$$= \frac{4\pi a^3 p_*}{c^2}. \tag{12.48}$$

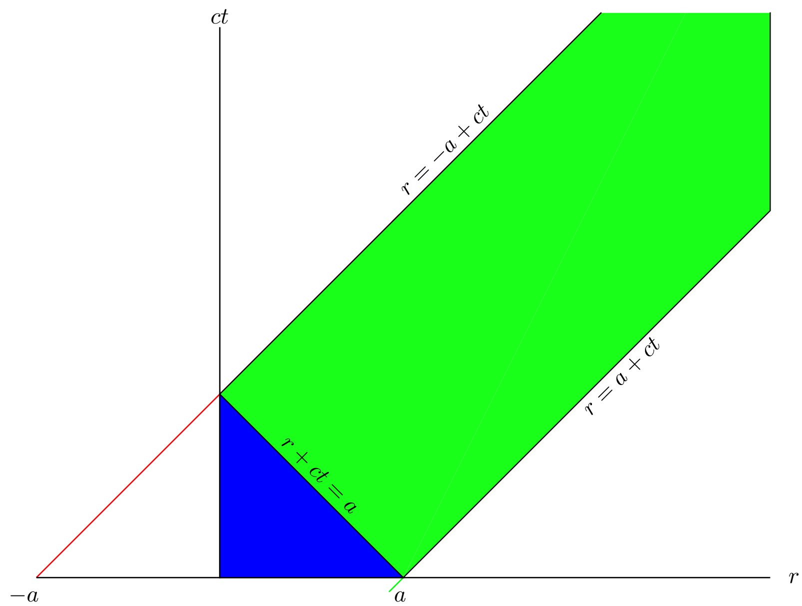


Figure 12.1: Characteristic diagram for the balloon burst problem. In the blue region $\psi = 1$ and in the unshaded regions $\psi = 0$.

12.3 Descent to $d = 2$

To solve the 2D wave equation we look for 3D solutions that are independent of z! This is called the method of descent because we go down from $d = 3$ to $d = 2$. Thus to solve the 2D wave equation with initial conditions $\phi(x, y, 0) = g(x, y)$ and $\phi_t(x, y) = f(x, y)$ we go to Kirchoff's formula (12.35) and take f and g to independent of z. We can then rewrite the $d = 3$ spherical average in a form that is easier to think about in $d = 2$.

Now return to consideration of the spherical average of a function $f(x, y)$. To perform the spherical average about the field point $\boldsymbol{r}$ we can write

$$\boldsymbol{r}' = \boldsymbol{r} + ct\,\boldsymbol{n} \tag{12.49}$$

where $\boldsymbol{r}'$ is a point on the averaging sphere and $\boldsymbol{n}$ is a unit vector. We can then use a spherical coordinate system centered on $\boldsymbol{r}$ and write

$$\boldsymbol{n} = (\sin\theta\cos\lambda, \sin\theta\sin\lambda, \cos\theta)\,. \tag{12.50}$$

Thus the spherical mean of $f(x', y')$ is

$$\mathcal{S}_{\boldsymbol{r}}^{ct}[f] = \frac{1}{4\pi}\oint\int_0^\pi f\left(x + ct\sin\theta\cos\lambda, y + ct\sin\theta\sin\lambda\right)\sin\theta \mathrm{d}\theta\,\mathrm{d}\lambda\,. \tag{12.51}$$

Change integration variables from λ and θ to

$$\xi \stackrel{\text{def}}{=} x + ct\sin\theta\cos\lambda\,, \qquad \text{and} \qquad \eta \stackrel{\text{def}}{=} y + ct\sin\theta\sin\lambda\,, \tag{12.52}$$

and note that

$$\sin\theta \mathrm{d}\theta\, \mathrm{d}\lambda = \sin\theta \left|\frac{\partial(\lambda,\theta)}{\partial(\xi,\eta)}\right| \mathrm{d}\xi\mathrm{d}\eta\,, \tag{12.53}$$

$$= \frac{\mathrm{d}\xi\mathrm{d}\eta}{(ct)^2\cos\theta}\,, \tag{12.54}$$

$$= \frac{\mathrm{d}\xi\mathrm{d}\eta}{ct\,\sqrt{(ct)^2-(\xi-x)^2-(\eta-y)^2}}\,. \tag{12.55}$$

(See problem 12.7 for a more geometric version of these manipulations.)

Now substitute (12.55) into (12.51). We must be careful because averaging over the sphere in (12.51) covers the disc $(\xi-x)^2+(\eta-y)^2<(ct)^2$ twice — so we must include a factor of 2. Thus the spherical mean in (12.51) becomes

$$\mathcal{S}_{\boldsymbol{r}}^{ct}[f] = \underbrace{\frac{1}{2\pi ct}\iint f(\xi,\eta)\frac{H[(ct)^2-(\xi-x)^2-(\eta-y)^2]}{\sqrt{(ct)^2-(\xi-x)^2-(\eta-y)^2}}\,\mathrm{d}\xi\mathrm{d}\eta}_{\stackrel{\text{def}}{=}\mathcal{W}_{\boldsymbol{r}}^{ct}[f]}\,, \tag{12.56}$$

where H is the Heaviside step function and $\mathcal{W}_{\boldsymbol{r}}^{ct}[f]$ is the *weighted* 2D average over a disc of radius ct centered on $\boldsymbol{r}=(x,y)$. Using $\mathcal{W}_{\boldsymbol{r}}^{ct}$, the 2D version of Kirchoff's formula is

$$\phi = t\mathcal{W}_{\boldsymbol{r}}^{ct}[f] + \partial_t t\mathcal{W}_{\boldsymbol{r}}^{ct}[g]\,. \tag{12.57}$$

Due diligence requires us to check that

$$\mathcal{W}_{\boldsymbol{r}}^{ct}[1] = \frac{1}{2\pi ct}\iint \frac{H[(ct)^2-\xi^2-\eta^2]}{\sqrt{(ct)^2-\xi^2-\eta^2}}\,\mathrm{d}\xi\mathrm{d}\eta\,, \tag{12.58}$$

$$= \frac{1}{2\pi}\oint \mathrm{d}\theta \int_0^1 \frac{s\,\mathrm{d}s}{\sqrt{1-s^2}}\,, \tag{12.59}$$

$$= 1\,, \tag{12.60}$$

as it must be. The average W is over the *interior* of the circle of radius ct centered on $\boldsymbol{r}$ — not just the edge as it was in $d=3$

Even if f and g have compact support, ϕ will remain non-zero at the field point $\boldsymbol{r}$ at arbitrarily large t. This is because the initial conditions correspond in $d=3$ to a prism extending along the z-axis. Thus in $d=2$ Huygen's principle is not valid. Does Huygen's principle apply in $d=1$? (No.)

12.4 Inhomogeneous wave equation: Duhamel again

Consider waves forced by a source

$$\phi_{tt} - c^2\triangle\phi = s(\boldsymbol{r},t)\,, \tag{12.61}$$

with initial conditions

$$\phi(\boldsymbol{r},0) = \phi_t(\boldsymbol{r},0) = 0\,. \tag{12.62}$$

According to our trusted philosophy — tested on the diffusion equation — we must first solve the problem

$$g_{tt} - c^2 \triangle g = \delta(\boldsymbol{r})\delta(t)\,, \tag{12.63}$$

with no disturbance before the $\delta(\boldsymbol{r})\delta(t)$ source acts: $g(\boldsymbol{r}, t<0) = 0$. Once we possess $g(\boldsymbol{r}, t)$, the solution of (12.61) is

$$\phi(\boldsymbol{r}, t) = \int_0^t \iiint g(\boldsymbol{r}-\boldsymbol{r}', t-t') s(\boldsymbol{r}', t')\, \mathrm{d}\boldsymbol{r}'\, \mathrm{d}t'\,. \tag{12.64}$$

As an exercise, you are asked to show that

$$g(\boldsymbol{r}, t) = \frac{\delta(|\boldsymbol{r}| - ct)}{4\pi c|\boldsymbol{r}|} H(t)\,. \tag{12.65}$$

(Hint: the initial value problem solved in section 12.2 is equivalent to the source $s(\boldsymbol{r}, t) = \phi_t(\boldsymbol{r}, 0)\delta(t)$ in (12.61).) With (12.65) in hand, we have

$$\phi(\boldsymbol{r}, t) = \frac{1}{4\pi c} \int_0^t \iiint s(\boldsymbol{r}', t') \frac{\delta\left(|\boldsymbol{r}-\boldsymbol{r}'| - c(t-t')\right)}{4\pi c|\boldsymbol{r}-\boldsymbol{r}'|}\, \mathrm{d}\boldsymbol{r}'\mathrm{d}t'\,. \tag{12.66}$$

Using the δ-function in (12.66), to perform the t'-integral we have

$$\phi(\boldsymbol{r}, t) = \frac{1}{4\pi} \iiint \frac{s(\boldsymbol{r}', t - c^{-1}|\boldsymbol{r}-\boldsymbol{r}'|)}{|\boldsymbol{r}-\boldsymbol{r}'|}\, \mathrm{d}\boldsymbol{r}'\,. \tag{12.67}$$

In (12.67),

$$s(\boldsymbol{r}', t - c^{-1}|\boldsymbol{r}-\boldsymbol{r}'|) = \text{the ``retarded source''}. \tag{12.68}$$

There is delay time

$$\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c} \tag{12.69}$$

between when the signal leaves the source point $\boldsymbol{r}'$ and when it reaches the field point $\boldsymbol{r}$. If the source wasn't retarded we would be solving the Poisson equation $-c^2\triangle\phi = s$ and the effect of s is felt immediately over all of space.

12.5 Radiation from a moving source

We consider the 2D wave equation

$$\left(\partial_t^2 - c^2\partial_x^2 - c^2\partial_y^2\right)\Phi = \mathrm{e}^{\epsilon t} S(x-ut, y)\,, \tag{12.70}$$

where S is the source moving with speed u. We use a slow switch-on starting at $t = -\infty$ to make sense of some singular terms: ultimately we take $\epsilon \to 0$. The emission,

$$W \stackrel{\text{def}}{=} \iint \Phi_t S\, \mathrm{d}x\mathrm{d}y\,, \tag{12.71}$$

is of prime interest.

The quasisteady solution

We look for a solution which is quasi-steady in the frame of the source:

$$\Phi = e^{\epsilon t}\phi(\tilde{x}, y)\,, \tag{12.72}$$

where

$$\tilde{x} \stackrel{\text{def}}{=} x - ut\,. \tag{12.73}$$

With the change of frame, the problem is

$$\left[(\epsilon - u\partial_{\tilde{x}})^2 - c^2\partial_{\tilde{x}}^2 - c^2\partial_y^2\right]\phi = S\,. \tag{12.74}$$

Using the Fourier transform

$$\hat{\phi} = \iint e^{-ik\tilde{x}-ily}\phi(\tilde{x}, y)\,d\tilde{x}dy \tag{12.75}$$

we obtain

$$\left[c^2k^2 + c^2l^2 - (ku + i\epsilon)^2\right]\hat{\phi} = \hat{S}\,. \tag{12.76}$$

equivalently

$$\hat{\phi} = \frac{1}{2}\frac{1}{uk + i\epsilon}\left[\frac{1}{c\kappa - uk - i\epsilon} - \frac{1}{c\kappa + uk + i\epsilon}\right]\hat{S}\,, \tag{12.77}$$

where $\kappa \stackrel{\text{def}}{=} \sqrt{k^2 + l^2}$.

The emission is

$$W = e^{\epsilon t}\iint S\,(\epsilon - u\partial_{\tilde{x}})\,\phi\,d\tilde{x}dy\,, \tag{12.78}$$

$$= e^{\epsilon t}\iint \hat{S}^*(\epsilon - iuk)\hat{\phi}\,\frac{dk\,dl}{(2\pi)^2}\,. \tag{12.79}$$

Using (12.77), the emission becomes

$$W = -\frac{i}{2}e^{\epsilon t}\iint\left[\frac{1}{c\kappa - uk - i\epsilon} - \frac{1}{c\kappa + uk + i\epsilon}\right]\left|\hat{S}\right|^2\frac{dk\,dl}{(2\pi)^2}\,, \tag{12.80}$$

$$= -\frac{i}{2}e^{\epsilon t}\iint\left[\frac{c\kappa - uk + i\epsilon}{(c\kappa - uk)^2 + \epsilon^2} - \frac{c\kappa + uk - i\epsilon}{(c\kappa + uk)^2 + \epsilon^2}\right]\left|\hat{S}\right|^2\frac{dk\,dl}{(2\pi)^2}\,, \tag{12.81}$$

$$= \frac{1}{2}e^{\epsilon t}\iint\left[\frac{\epsilon}{(c\kappa - uk)^2 + \epsilon^2} - \frac{\epsilon}{(c\kappa + uk)^2 + \epsilon^2}\right]\left|\hat{S}\right|^2\frac{dk\,dl}{(2\pi)^2}\,. \tag{12.82}$$

Now take $\epsilon \to 0$ using

$$\lim_{\epsilon\to 0}\frac{\epsilon}{(c\kappa \pm uk)^2 + \epsilon^2} = \pi\delta\left[c\kappa \pm uk\right]\,. \tag{12.83}$$

This expresses the emission as integrals along the curves

$$c\kappa \pm uk = 0\,, \qquad \text{or} \qquad \sqrt{u^2 - c^2}\,k = \pm c\,l\,. \tag{12.84}$$

The δ-functions are activated only if $u > c$ i.e., only if the source is "supersonic".

Let's use the δ-functions to do the k-integral:

$$W = \frac{1}{8\pi}\frac{u}{c^2 - u^2}\int\left|\hat{S}\left(\frac{cl}{\sqrt{u^2 - c^2}}, l\right)\right|^2 dl \tag{12.85}$$

12.6 Problems

Problem 12.1. Carefully explain how and why (12.32) satisfies the initial condition $\phi(\boldsymbol{r}, 0) = g$.

Problem 12.2. Formulate the balloon burst problem in terms of the velocity potential ϕ (rather than pressure) and evaluate the spherical means in Kirchoff's solution to determine both $\phi(\boldsymbol{r}, t)$ and pressure. Show that your result for pressure agrees with the solution in the notes. Find the radial velocity, u, and show that when the wave arrives a radius $r > a$, at time $ct = r - a$, the radial velocity u jumps suddenly from 0 to $p_* a/2cr$. Calculate the total mass flux through a sphere of radius $r > a$

$$q = 4\pi r^2 u(r, t) \tag{12.86}$$

and show that eventually all of the excess mass initially within the balloon passes through this sphere.

Problem 12.3. A gas with sound speed c is enclosed in a rigid sphere with radius a. Show that the frequencies purely radial oscillation are $c\xi_n/a$ where ξ_1, $\xi_2 \cdots$ are the positive roots of $\tan\xi = \xi$.

Problem 12.4. Consider an initial ($t = 0$) acoustic disturbance that is non-zero only inside a sphere of radius a centered on the origin. At what points in space-time is the disturbance subsequently ($t > 0$) zero?

Problem 12.5. Solve the 3D wave equation $\phi_{tt} = c^2 \triangle \phi$ with the initial conditions

$$(a): \quad \phi(\boldsymbol{r}, 0) = 0\,, \qquad \text{and} \qquad \phi_t(\boldsymbol{r}, 0) = \frac{1}{a^2 + r^2}\,, \tag{12.87}$$

and with

$$(b): \quad \phi(\boldsymbol{r}, 0) = \frac{1}{a^2 + r^2}\,, \qquad \text{and} \qquad \phi_t(\boldsymbol{r}, 0) = 0\,. \tag{12.88}$$

Problem 12.6. Consider a sphere of radius a with thermal diffusivity κ and acoustic wave speed c. There is a small explosion at the center. Standing on the surface, does one first feel the thermal signal or the acoustic signal? Discuss this in qualitative terms.

Problem 12.7. Consider a surface $z = h(x, y)$ above the (x, y)-plane. Denoting an element of area in this surface by $\mathrm{d}S$ show geometrically that

$$\mathrm{d}S = \sqrt{1 + (h_x)^2 + (h_y)^2}\, \mathrm{d}x\mathrm{d}y\,, \tag{12.89}$$

where $\mathrm{d}x\mathrm{d}y$ is the area element in the (x, y)-plane below $\mathrm{d}S$. Evaluate the factor $\sqrt{1 + (h_x)^2 + (h_y)^2}$ in the special case

$$h = \sqrt{a^2 - x^2 - y^2}\,. \tag{12.90}$$

Lecture 13

Classification of second-order pdes

13.1 Preliminary examples

Let's start with the simplest second-order PDE for an unknown function $u(x, y)$:

This is a *hyperbolic* PDE.

$$u_{xy} = 0\,. \tag{13.1}$$

The solution is

$$u = a(x) + b(y)\,, \tag{13.2}$$

where a and b are arbitrary functions of x and y. Boundary or initial conditions are required to determine a and b. The example

$$u_{xx} + u_{xy} - 6u_{yy} = 0 \tag{13.3}$$

is only a little more difficult: substituting

$$u = U(\lambda x + y) \tag{13.4}$$

we find

$$(\lambda^2 + \lambda - 6)\, U' = 0\,, \qquad \Rightarrow \qquad \lambda = -3\,, \text{or } 2. \tag{13.5}$$

Hence the solution is

$$u = a(-3x + y) + b(2x + y)\,, \tag{13.6}$$

where a and b are arbitrary functions.

These simple examples suggest that the solution of second-order PDEs involve two arbitrary functions. This should remind you of D'Alembert's solution of the wave equation

$$\zeta_{tt} - c^2 \zeta_{xx} = 0\,, \tag{13.7}$$

which begins by observing that a general solution is

$$\zeta = L(x + ct) + R(x - ct)\,, \tag{13.8}$$

and then proceeds to determine the arbitrary functions R and L by applying initial conditions.

13.2 Second-order pdes with variable coefficients

In the remainder of this lecture we will make more systematic arguments towards the simplification and classification of PDEs of the form

$$\mathcal{L}u = \phi(\zeta_x, u_y, u, x, y)\,. \tag{13.9}$$

Above

$$\mathcal{L} \stackrel{\text{def}}{=} a\partial_x^2 + 2b\partial_x\partial_y + c\partial_y^2\,, \tag{13.10}$$

with a, b and c functions only of x and y. $\mathcal{L}$ is a linear second-order differential operator. Although a, b and c are not constants we can still develop a useful simplification of the differential operator $\mathcal{L}$.

Let's change variables from x and y in (13.9) to $\xi(x,y)$ and $\eta(x,y)$. The function $\xi(x,y)$ and $\eta(x,y)$ must provide a good coordinate system. This means that

$$J \stackrel{\text{def}}{=} \xi_x\eta_y - \xi_y\eta_x\,, \tag{13.11}$$

$$\neq 0\,. \tag{13.12}$$

We now regard the solution u of (13.9) as a function of ξ and η and use the chain rule

$$u_x = \xi_x u_\xi + \eta_x u_\eta \tag{13.13}$$

and

$$\zeta_{xx} = \xi_x^2 u_{\xi\xi} + 2\xi_x\eta_x u_{\xi\eta} + \eta_x^2 u_{\eta\eta} + \xi_{xx}u_\xi + \eta_{xx}u_\eta\,, \tag{13.14}$$

and so on. Thus in terms of ξ and η, the PDE (13.9) becomes

$$\begin{aligned}\big(a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2\big)u_{\xi\xi}\\ + 2\big(a\xi_x\eta_x + b\xi_x\eta_y + b\xi_y\eta_x + c\xi_y\eta_y\big)u_{\xi\eta}\\ + \big(a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2\big)u_{\eta\eta} = \psi\,,\end{aligned} \tag{13.15}$$

where ψ collects all lower order derivatives:

$$\psi \stackrel{\text{def}}{=} \phi(\zeta_x, u_y, u, x, y) - (\mathcal{L}\xi)u_\xi - (\mathcal{L}\eta)u_\eta\,. \tag{13.16}$$

The idea is to choose ξ and η so that the left of (13.15) is simplified. For example, we can make the coefficient of $u_{\xi\xi}$ zero by choosing ξ so that

$$a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0\,. \tag{13.17}$$

This is a quadratic equation for ξ_y/ξ_y and the problem is simple when the *discriminant* of the quadratic form in (13.17)

$$b^2 - ac \tag{13.18}$$

is either positive, zero or negative *everywhere* in the domain of interest. We'll discuss the three cases — positive, zero or negative — in turn. But first note that the sign of (13.18) cannot be changed by changing coordinates from (x,y)

to (ξ,η). Denoting the coefficients of $u_{\xi\xi}$, $u_{\xi\eta}$ and $u_{\eta\eta}$ in (13.15) by α, 2β and γ, one can show by direct calculation that

$$b^2 - ac = (\beta^2 - \alpha\gamma)J^2\,, \tag{13.19}$$

where J is the Jacobian in (13.12). Thus the sign of $b^2 - ac$ is the same as that of $\beta^2 - \alpha\gamma$.

The hyperbolic case: $b^2 - ac > 0$

The PDE (13.1) is exemplary. In general if $b^2 - ac > 0$ then we can set the coefficients of $u_{\xi\xi}$ and $u_{\eta\eta}$ in (13.15) to zero. Thus we can determine $\xi(x,y)$ via

$$a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0\,, \qquad \Rightarrow \qquad \frac{\xi_x}{\xi_y} = \frac{-b + \sqrt{b^2 - ac}}{a}\,, \tag{13.20}$$

and $\eta(x,y)$ by

$$a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0\,, \qquad \Rightarrow \qquad \frac{\eta_x}{\eta_y} = \frac{-b - \sqrt{b^2 - ac}}{a}\,, \tag{13.21}$$

We must solve the first-order PDEs above. With this choice of ξ and η the PDE (13.15) reduces to

$$u_{\xi\eta} = \psi/2\beta\,. \tag{13.22}$$

The identity (13.19) assures us that $\beta \neq 0$ — so the division by 2β in (13.22) is hunky dory. The curves defined by $\xi(x,y) = c_1$ and $\eta(x,y) = c_2$ are *characteristics*.

Example: Reduce

$$u_{xx} - x^2 u_{yy} = 0 \tag{13.23}$$

to the canonical form in (13.22).

In this example $a = 1$, $b = 0$ and $c = -x^2$ and $b^2 - ac = x^2$. To beat the PDE into the canonical form (13.22) we require that

$$\xi_x^2 - x^2\xi_y^2 = 0\,, \qquad \text{or} \qquad (\xi_x - x\xi_y)(\xi_x + x\xi_y) = 0\,, \tag{13.24}$$

and the same for η. The simplest solutions are

$$\xi = y + \tfrac{1}{2}x^2\,, \qquad \text{and} \qquad \eta = y - \tfrac{1}{2}x^2\,, \tag{13.25}$$

so that

$$x = \sqrt{\xi - \eta}\,, \qquad \text{and} \qquad y = \tfrac{1}{2}(\xi + \eta)\,. \tag{13.26}$$

Now change variables to ξ and η. The transformed equation is in (13.15) and (13.16). Noting that

$$\mathcal{L}\xi = x = \sqrt{\xi - \eta}\,, \qquad \mathcal{L}\eta = -x = -\sqrt{\xi - \eta}\,, \tag{13.27}$$

$$\beta = \xi_x\eta_x - x^2\xi_y\eta_y = -2x^2 = -2(\xi - \eta)\,, \tag{13.28}$$

we find

$$u_{\xi\eta} = \frac{u_\xi - u_\eta}{4(\xi - \eta)}\,. \tag{13.29}$$

The parabolic case: $b^2 - ac = 0$

In this case the roots of the quadratic equation (13.17) are equal. We define ξ as in the previous hyperbolic case and then take η as any other function with $J \neq 0$. Thus, as in the previous case, we have $\alpha = 0$ and we are delighted to observe that from (13.19) β is also zero! Thus, dividing by γ, the parabolic canonical form is

$$u_{\eta\eta} = \psi/\gamma \,. \tag{13.30}$$

Example: Reduce

$$u_{xx} + 2u_{xy} + u_{yy} = 0 \tag{13.31}$$

to canonical form.

In this example

$$\xi_x^2 + 2\xi_x\xi_y + \xi_y^2 = (\xi_x + \xi_y)^2 = 0 \,, \tag{13.32}$$

and therefore ξ is an arbitrary function of $x - y$. We take

$$\xi = x - y \,, \qquad \text{and} \qquad \eta = x + y \,. \tag{13.33}$$

It is easy to check that $\alpha = \beta = 0$ and $\gamma = \eta_x^2 + 2\eta_x\eta_y + \eta_y^2 = 4$. Hence the transformed equation is

$$4u_{\eta\eta} = 0 \tag{13.34}$$

with solution $u = f(\xi) + \eta g(\xi)$ with f and g arbitrary functions.

Example: Reduce

$$\frac{x}{y}u_{xx} + 2u_{xy} + \frac{y}{x}u_{yy} = 0 \tag{13.35}$$

to the canonical form in (13.30).

In this case $ac - b^2 = 0$ and

$$\frac{x}{y}\xi_x^2 + 2\xi_x\xi_y + \frac{y}{x}\xi_y^2 = 0 \,, \qquad \text{or} \qquad x\xi_x + y\xi_y = 0 \,. \tag{13.36}$$

Thus ξ is an arbitrary function of x/y and we make some simplest choices

$$\xi = \frac{x}{y} \,, \qquad \eta = xy \,. \tag{13.37}$$

We choose η above because it looks pretty and leads to an easy solution for (x, y) in terms (ξ, η):

$$x = \sqrt{\xi\eta} \,, \qquad y = \sqrt{\frac{\eta}{\xi}} \,. \tag{13.38}$$

As expected, the middle coefficient on the left of (13.15) is zero (check by substitution). Thus the first two coefficients on the left of (13.15) are zero and the third coefficient is

$$\frac{x}{y}\eta_x^2 + 2\eta_x\eta_y + \xi_y\eta_x + \frac{y}{x}\eta_y^2 = 4\eta \,. \tag{13.39}$$

We can get the lower order derivatives from (13.16):

$$\phi = 0 \,, \qquad \mathcal{L}\xi = 0 \,, \qquad \mathcal{L}\eta = 2 \,. \tag{13.40}$$

Hence the canonized equation is

$$u_{\eta\eta} + \frac{u_\eta}{2\eta} = 0 \,, \tag{13.41}$$

with solution

$$u = p(\xi)\sqrt{\eta} + q(\xi) \,, \tag{13.42}$$

where p and q are arbitrary functions.

The elliptic case: $b^2 - ac < 0$

This is the same as the elliptic case except that the roots of

$$a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0 \tag{13.43}$$

are complex conjugates and thus $\eta = \xi^*$. Proceeding as in the hyperbolic case, we find

$$2\beta u_{\xi\xi^*} = \psi\,. \tag{13.44}$$

To write the equation in terms of real variables, suppose

$$\xi = \mu + \mathrm{i}\nu \tag{13.45}$$

and then

$$\mu = \tfrac{1}{2}\xi + \tfrac{1}{2}\xi^*\,, \qquad \nu = -\tfrac{1}{2}\mathrm{i}\xi + \tfrac{1}{2}\mathrm{i}\xi^* \tag{13.46}$$

$$\partial_\xi = \tfrac{1}{2}\partial_\mu - \tfrac{1}{2}\mathrm{i}\partial_\nu\,, \qquad \partial_{\xi^*} = \tfrac{1}{2}\partial_\mu + \tfrac{1}{2}\partial_\nu\,, \tag{13.47}$$

$$\text{and} \qquad \partial_\xi\partial_{\xi^*} = \tfrac{1}{4}\big(\partial_\mu^2 + \partial_\nu^2\big)\,. \tag{13.48}$$

Thus the elliptic canonical form is

$$u_{\mu\mu} + u_{\nu\nu} = 2\psi/\beta\,, \tag{13.49}$$

with ψ in (13.16).

Example: Reduce

$$u_{xx} + x^2 u_{yy} = 0 \tag{13.50}$$

to the canonical form in (13.49).

In this example $a = 1$, $b = 0$ and $c = x^2$ and $b^2 - ac = -x^2$. To beat the PDE into the canonical form we require that

$$\xi_x^2 + x^2\xi_y^2 = 0\,, \qquad \text{or} \qquad (\xi_x - \mathrm{i}x\xi_y)(\xi_x + \mathrm{i}x\xi_y) = 0\,, \tag{13.51}$$

and the same for η. The simplest solutions are

$$\xi = \mathrm{i}y + \tfrac{1}{2}x^2\,, \qquad \text{and} \qquad \eta = -\mathrm{i}y + \tfrac{1}{2}x^2\,, \tag{13.52}$$

so that

$$\mu = \tfrac{1}{2}x^2\,, \qquad \text{and} \qquad \nu = y\,. \tag{13.53}$$

Changing variables to μ and ν we find we find

$$u_{\mu\mu} + u_{\nu\nu} + \tfrac{1}{2}u_\mu = 0\,, \tag{13.54}$$

Style points for getting rid of the first-derivative term with a further change of variable $u = \mu^{-1/4}v$, resulting in

$$v_{\mu\mu} + v_{\nu\nu} + \frac{3v}{16\mu^2} = 0\,. \tag{13.55}$$

13.3 The wave equation with non-uniform c

We consider the wave equation

$$\varrho\zeta_{tt} + T\zeta_{xx} = f\,, \tag{13.56}$$

where the tension T is constant but the density ϱ is non-uniform. Hence the wave speed, $c = \sqrt{T/\varrho}$, varies with x. We apply forcing $f(x,t)$ on the right and we consider segment $0 < x < \ell$ with initial conditions

$$\zeta(x,0) = \zeta_0(x)\,, \qquad \text{and} \qquad \zeta_t(x,0) = v_0(x)\,. \tag{13.57}$$

and boundary conditions

$$\zeta(0,t) = a_0(t)\,, \qquad \text{and} \qquad \zeta(\ell,t) = a_\ell(t)\,. \tag{13.58}$$

Uniqueness

First we use energy conservation to show that the PDE posed above has a unique solution. Recall the energy equation

$$\mathcal{E}_t + \mathcal{J}_x = \zeta_t f\,, \tag{13.59}$$

where the energy density and flux are

$$\mathcal{E} = \tfrac{1}{2}\varrho\zeta_t^2 + \tfrac{1}{2}T\zeta_x^2\,, \qquad \text{and} \qquad \mathcal{J} = -T\zeta_t\zeta_x\,. \tag{13.60}$$

Integrating (13.59) over the rectangular region

$$0 < x < \ell\,, \qquad \text{and} \qquad 0 < t < t_* \tag{13.61}$$

we obtain

$$\int_0^\ell \mathcal{E}(x,t_*)\,\mathrm{d}x = \int_0^\ell \mathcal{E}(x,0)\,\mathrm{d}x + \int_0^{t_*} \mathcal{J}(0,t)\,\mathrm{d}t - \int_0^{t_*} \mathcal{J}(\ell,t)\,\mathrm{d}t + \iint f\zeta_t\,\mathrm{d}x\mathrm{d}t\,. \tag{13.62}$$

This says that the energy at time t_* is equal to the intial energy, plus whatever has come or gone through the boundaries at $x = 0$ and ℓ, plus whatever has been injected by the forcing $f(x,t)$.

Now suppose we have two solutions $\zeta_1(x,t)$ and $\zeta_2(x,t)$ of the problem in (13.56) through (13.58). Consider the difference $v = \zeta_1 - \zeta_2$. Because the PDE and boundary-initial conditions are linear $v(x,t)$ satisfies (13.56) with $f = 0$ the boundary initial conditions with $\zeta_0 = v_0 = a_0 = a_\ell = 0$. Hence the analog of (13.62) for $v(x,t)$ is

$$\int_0^\ell \tfrac{1}{2}\varrho v_t^2 + \tfrac{1}{2}T v_x^2\,\mathrm{d}x\bigg|_{@t_*} = 0\,. \tag{13.63}$$

We are assuming that $T > 0$ and $\rho(x) > 0$ and thus the integrand above is non-negative. Thus $v(x,t)$ must be zero.

Reduction to canonical form

Because $c = \sqrt{T/\varrho(x)}$ is not constant the wave equation

$$\zeta_{tt} - c^2 \zeta_{xx} = 0 \,, \tag{13.64}$$

is not in canonical form. To bring the equation to canonical form we introduce a new coordinates $\xi = \xi(x)$, so that the wave equation becomes

$$\zeta_{tt} - (c\xi_x)^2 \zeta_{\xi\xi} - c^2 \xi_{xx} \zeta_\xi = 0 \,. \tag{13.65}$$

Choosing $\xi(x)$ so that

$$c\xi_x = 1 \,, \qquad \text{or} \qquad \xi(x) \stackrel{\text{def}}{=} \int^x \frac{\mathrm{d}x'}{c(x')} \,, \tag{13.66}$$

the wave equation becomes

$$\zeta_{tt} - \zeta_{\xi\xi} + \frac{c_\xi}{c} \zeta_\xi = 0 \,. \tag{13.67}$$

$$c^2 \xi_{xx} = -c_x = -\frac{c_\xi}{c}$$

Although we have achieved canonical form, we might be dissatisfied with (13.67): the first-derivative term makes it difficult to derive the energy conservation law. For example, if we multiply (13.67) by ζ_t the term $a\zeta_\xi \zeta_t$ is nothing but trouble. This motivates a further transformation check

$$\zeta(\xi, t) = f(\xi) v(\xi, t) \,, \tag{13.68}$$

resulting in

$$v_{tt} - v_{\xi\xi} - \left(2\frac{f_\xi}{f} - \frac{c_\xi}{c} \right) v_\xi - \left(\frac{f_{\xi\xi}}{f} - \frac{c_\xi}{c}\frac{f_\xi}{f} \right) v = 0 \,. \tag{13.69}$$

$$\frac{c_\xi}{c}\frac{(\sqrt{c})_\xi}{\sqrt{c}} - \frac{(\sqrt{c})_{\xi\xi}}{\sqrt{c}} = \frac{3}{4}\left(\frac{c_\xi}{c}\right)^2 - \frac{1}{2}\frac{c_{\xi\xi}}{c} \,, = \sqrt{c}\,\partial_\xi^2 \frac{1}{\sqrt{c}} \,.$$

We kill the v_ξ-term by choosing $f = \sqrt{c}$. In terms of v, the PDE is

$$v_{tt} - v_{\xi\xi} + \sqrt{c} \left(\frac{1}{\sqrt{c}} \right)_{\xi\xi} v = 0 \,. \tag{13.70}$$

The v-equation above, with an undifferentiated term, is simpler than the ζ-equation in (13.67).

Exercise: Obtain an energy conservation for the wave equation with non-uniform c.

Characteristics with non-uniform speed c

The characteristics of (13.64) are the curves defined

$$t - \xi(x) = c_- \,, \qquad \text{and} \qquad t + \xi(x) = c_+ \tag{13.71}$$

where c_+ and c_- are constants. Figure 13.1 shows the characteristics passing through a spacetime point (x_*, t_*). There is a quasi-triangular region $\mathcal{D}$ enclosed by the characteristics and the x-axis. In the case of constant c we showed that $\mathcal{D}$ was the domain of dependence of (x_*, t_*) i.e., the solution at (x_*, t_*) is determined by inital conditions on the segment between x^+ and x_- on the x-axis and forcing $f(x,t)$ inside $\mathcal{D}$.

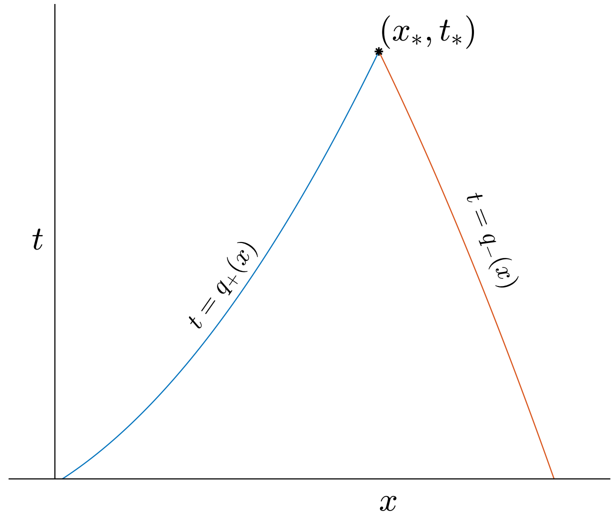


Figure 13.1

13.4 Cauchy data

Cauchy data on the y-axis

Suppose that we specify the solution of

$$a u_{xx} + 2b u_{xy} + c u_{yy} = \phi(u_x, u_y, u, x, y)\,. \tag{13.72}$$

on the y-axis:

$$u(0, y) = f(y) \tag{13.73}$$

where $f(y)$ is a known function of y. Taking y-derivatives of (13.73) we can obtain derivatives along the y-axis

$$u_y(0, y) = f'(y)\,, \qquad u_{yy}(0, y) = f''\,, \tag{13.74}$$

and so on. There is no way of getting information "off the y-axis" unless we also specify

Specifying both u and u_x on the y-axis is *Cauchy data.*

$$u_x(0, y) = g(y)\,. \tag{13.75}$$

Again we can take derivatives along the axis to obtain

$$u_{xy}(0, y) = g'\,, \qquad u_{xyy}(0, y) = g''\,, \tag{13.76}$$

and so on. We can also use the PDE (13.72) to obtain

$$u_{xx}(0, y) = \frac{1}{a}\left[\phi - 2b u_{xy} - c u_{yy}\right]_{@x=0} \tag{13.77}$$

This is OK provided that $a(0, y)$ has no zeros in the region of interest. Now we can take y-derivatives of (13.77) and, to obtain $u_{xxx}(0, y)$, we can take x-derivatives of the PDE (13.9) — and so on.

We have assumed:

(a) specified Cauchy data $f(y)$ and $g(y)$ on the line $x = 0$;

(b) suitable differentiability of f, g, a, b and c;

(c) $a(0, y) \neq 0$.

If all conditions above are met then we can obtain *all* derivatives of u on the y-axis. Thus we can move u off of the y-axis and into the (x, y)-plane:

$$u(x,y) = u(0,y) + xu_x(0,y) + \tfrac{1}{2}x^2u_{xx}(0,y) + \tfrac{1}{6}x^3u_{xxx}(0,y) + \cdots \qquad (13.78)$$

Perhaps the series above does not converge? There is a result, the Cauchy-Kowalewski Theorem, that assures us that the series above determines a unique solution in some neighbourhood of the y-axis.

Cauchy data on a curve

Now consider the problem with Cauchy data specified on a curve $\mathcal{C}$ in the (x, y)-plane. This means that we know both u and the unit normal derivative $u_n = \boldsymbol{n} \cdot \boldsymbol{\nabla} u$ on $\mathcal{C}$. We suppose that $\mathcal{C}$ is specified by

$\boldsymbol{n}$ is the normal vector to $\mathcal{C}$ i.e.,

$$\boldsymbol{n} = \boldsymbol{\nabla}\xi/|\boldsymbol{\nabla}\xi|\,.$$

$$\xi(x, y) = 0 \qquad (13.79)$$

where $\xi(x, y)$ is some function. We assume that

$$|\boldsymbol{\nabla}\xi| \neq 0 \text{ on } \mathcal{C}. \qquad (13.80)$$

Thus $\boldsymbol{n}$ is well defined and ξ changes sign as one passes through $\mathcal{C}$. Knowing u on $\mathcal{C}$ we can calculate the derivative of u along $\mathcal{C}$. Because we also know the normal derivative u_n we can obtain ζ_x and u_y, or indeed any directional derivative of u, on $\mathcal{C}$. To determine higher derivatives normal to $\mathcal{C}$ we must use the PDE (13.9).

13.5 Problems

Problem 13.1. Find the general solution of the PDEs

$$3u_{xx} + 7u_{xy} + 2u_{yy} = 0\,, \qquad 4v_{xx} + 2v_{xy} + v_{yy} = 0\,, \qquad w_{xx} - 6w_{xy} + 9w_{yy} = 0\,,$$

in terms of "arbitrary functions".

Problem 13.2. Transform the Klein-Gordon equation,

$$u_{tt} - c^2u_{xx} + \sigma^2u = 0\,, \qquad (13.81)$$

into the form

$$u_{\xi\eta} = \left(\frac{\sigma}{2c}\right)^2 u\,. \qquad (13.82)$$

Problem 13.3. Transform

$$v_{tt} + 3v_{xt} + 2v_{xx} + v_t - v_x = 0\,, \tag{13.83}$$

first into $v_{\xi\eta} + 2v_\xi + 3v_\eta = 0$, and then into $w_{\xi\eta} = 6w$.

Problem 13.4. Reduce

$$2u_{xx} - 5u_{xy} + 2u_{yy} + \zeta_x + u_y = 0 \tag{13.84}$$

to the canonical form in (13.22).

Problem 13.5. *(i)* Consider Tricomi's equation

$$u_{xx} - xu_{yy} = 0\,, \tag{13.85}$$

in the half-plane $x > 0$. With $\nu = \nu(x)$, transform Tricomi's equation to

$$u_{\nu\nu} - u_{yy} + \frac{u_\nu}{3\nu} = 0\,. \tag{13.86}$$

Bring (13.86) to the canonical form in (13.22). *(ii)* Consider Tricomi's equation in the half plane $x < 0$. transform the equation to the canonical form in (13.49).

Lecture 14

Partial difference equations

14.1 The θ-transform

Let us consider a infinite (in both directions) row of tanks. Each tank has volume V m^3 and we label them $n = 0, \pm 1, \pm 2, \cdots$. There is a flux Qm^{3}s^{-1} going to the left and the same flux to the right; each tank has two outlets and two inlets. Then the equation for tracer conservation in this system is

$$\dot{C}_n = \alpha \left[C_{n-1} - 2C_n + C_{n+1} \right] , \quad \text{with} \quad \alpha \overset{\text{def}}{=} Q/V \, . \tag{14.1}$$

The initial condition for might be $\mathcal{C}_n(0) = \delta_{n0}$; that is, tank 0 contains all of the dissolved chemical and the others are unpolluted.

Probabilists recognize this as a 'continuous time random walk'. You can think of each tank as a site occupied by a number of random walkers; there is a constant probability per unit time, α, of hopping to the right and the same probability of hopping to the left. This is also an ingredient in Turing's model of morphogenesis in which cells exchange morphogens with their nearest neighbours.

Equation 14.1 is an infinite set of differential–difference equations. Solving it looks tough, but help arrives from an unexpected direction. Introduce the function

If $z \overset{\text{def}}{=} e^{i\kappa}$ then

$$F(\kappa, t) \overset{\text{def}}{=} \sum_{n=-\infty}^{\infty} C_n(t) z^n \, .$$

This is the "z-transform".

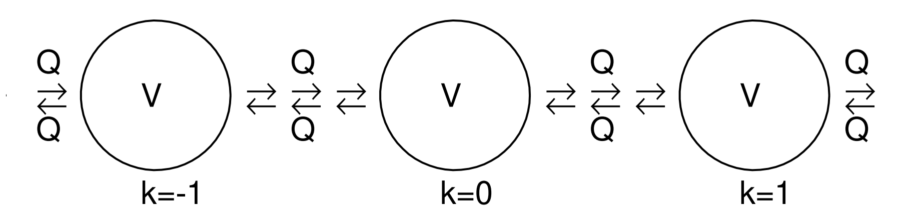


Figure 14.1: `FourSer77.eps`

$$F(\kappa,t) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} C_n(t)e^{\mathrm{i}n\kappa}. \tag{14.2}$$

Given the answer to our problem, $C_n(t)$, we can obtain $F(\kappa,t)$ by using the Fourier series in (14.2). But the inverse is also true: given $F(\kappa,t)$ we can get the answer to our problem by evaluating the integral

$$C_n(t) = \int_{-\pi}^{\pi} F(\kappa,t)e^{-\mathrm{i}n\kappa}\,\frac{\mathrm{d}\kappa}{2\pi}\,. \tag{14.3}$$

This is a very significant observation because it is easy to show from (**??**) that

$$F_t = -2\alpha(1-\cos\kappa)F, \quad \Rightarrow \quad F(\kappa,t) = F(\kappa,0)\mathrm{e}^{-2(1-\cos\kappa)\alpha t}\,. \tag{14.4}$$

Next, since we now possess $F(\kappa,t)$, we use (14.3) to obtain an integral representation of the solution of (14.1):

$$C_n(t) = \int_{-\pi}^{\pi} F(\kappa,0)e^{-\mathrm{i}n\kappa-2(1-\cos\kappa)\alpha t}\,\frac{\mathrm{d}\kappa}{2\pi}\,. \tag{14.5}$$

Do not be too impressed with this yet — the integral in (14.5) is not a transparent representation of the solution. However having an integral is major progress because it is easier to numerically evaluate integrals than to time step differential equations. And there are powerful analytic methods for approximately evaluating integrals.

Take a simple initial condition, namely $\mathcal{C}_n(0) = \delta_{n0}$, so that $F(\kappa,0) = 1$. With this choice we can use symmetry arguments to reduce the solution in (14.4) to the form

$$C_n(t) = \mathrm{e}^{-2\alpha t}\int_0^{\pi} \mathrm{e}^{2\alpha t\cos\kappa}\cos n\kappa\,\frac{\mathrm{d}\kappa}{\pi}\,. \tag{14.6}$$

We try to evaluate (14.6) by looking up the integral in thick books like Gradshteyn & Ryzhik or Abramowitz & Stegun. In this problem our efforts are crowned with success because after some excavation we find the following result:

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos n\theta\,\mathrm{d}\theta\,, \tag{14.7}$$

where $I_n(z)$ is the modified Bessel function of integer order. So, now we get

$$C_n(t) = e^{-2\alpha t}I_n(2\alpha t). \tag{14.8}$$

This is progress; for instance, `matlab` has the routine `besseli` which calculates $I_n(z)$. Moreover, there are many results known about I_n so that we can make a pretty good qualitative visualization of the solution even without turning to numerical evaluation.

In figure **??** you can see that at large times the concentration is a slowly changing function of n and this suggests that we make a 'continuum' approximation by regarding n as a continuously changing variable, analogous to position, so that

$$C(n+1,t) = C(n,t) + C_n(n,t) + (1/2)C_{nn}(n,t) + \cdots \tag{14.9}$$

Thus on the right hand side of (??)

$$C_{n-1} - 2C_n + C_{n+1} \approx C_{nn}. \tag{14.10}$$

At large times, when the solution is a slowly changing function of n, the differential–difference equation (??) is approximated by the diffusion equation

$$C_t = \alpha C_{nn}\,. \tag{14.11}$$

It seems plausible that the initial condition which corresponds to releasing all of the chemical at $n = 0$ is probably

$$C(0,t) = \delta(n). \tag{14.12}$$

But we know the solution of this intial value problem is our friend the Gaussian

$$C_n(t) \approx \frac{e^{-n^2/4\alpha t}}{2\sqrt{\pi\alpha t}}. \tag{14.13}$$

Unlike our earlier exact result in (14.8) the approximation above is very transparent — it is very easy to visualize the solution without numerical evaluation. Partly, this is because the answer is in terms of elementary functions, but the main simpification is the discovery of the similarity form.

14.2 Mach bands

The physical basis for vision is that retinal cells "fire" when stimulated by light. A single cell, excised from the retina of a horseshoe crab, fires at a rate R (spikes per second, traveling down the axon) which is proportional to the logarithm of the number of incident photons:

$$R = R_0 \frac{\ln I}{\ln I_0}\,. \tag{14.14}$$

Here I_0 is some standard illumination (photons per second) which elicits R_0 spikes per second. To avoid lots of logarithms we write

$$R = kL\,, \qquad L \overset{\text{def}}{=} \ln I\,. \tag{14.15}$$

Another experimental fact is *lateral inhibition*: the response of cell n is reduced by stimulation of the neighbouring cells $n \pm 1$. Thus the model of a one-dimensional retina is

$$R_n = kI_n - \alpha \underbrace{\left[R_{n-1} + \alpha R_{n+1}\right]}_{\text{inhibition}}\,. \tag{14.16}$$

Now let's work out a few solutions.

Example 1: unifrom illumination

Consider unifrom illumination, $I_n = I$. Evidently

$$R = k\frac{L}{1+2\alpha}\,. \tag{14.17}$$

Since $\alpha > 0$ the response of a cell in the array is less than the repsonse of an isolated cell.

Example 2: the Green's function

Suppose we stimulate just the cell at $n = 0$:

$$G_n = k\delta_{n0} - \alpha\left[G_{n-1} + \alpha G_{n+1}\right]\,. \tag{14.18}$$

To solve this system we again consider the transform

$$\mathcal{G}(\kappa) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} G_n \mathrm{e}^{\mathrm{i}n\kappa}\,, \tag{14.19}$$

and we quickly find

$$\mathcal{G}(\kappa) = \frac{k}{1+2\alpha\cos\kappa}\,. \tag{14.20}$$

We could now recover G_n by evaluating the integrals

$$G_n = \oint \frac{\mathrm{e}^{-\mathrm{i}n\kappa}}{1+2\alpha\cos\kappa}\frac{\mathrm{d}\kappa}{2\pi}\,. \tag{14.21}$$

This is a standard exercise in residue calculus. (Or see problem 14.5)) We summarize the result:

$$\mathcal{G} = \frac{k}{\sqrt{1-4\alpha^2}} \sum_{n=-\infty}^{\infty} (-\beta)^{-|n|}\mathrm{e}^{\mathrm{i}n\kappa}\,, \tag{14.22}$$

where

$$\beta \stackrel{\text{def}}{=} \sqrt{\frac{1}{4\alpha^2} - 1} + \frac{1}{2\alpha}\,. \tag{14.23}$$

We assume $0 < \alpha < 1/2$ — else it is lateral excitation rather than inhibition. (Why?)

The response at $n = 0$ is

$$G_0 = \frac{k}{\sqrt{1-4\alpha^2}} \tag{14.24}$$

— this is always greater than the response of an isolated cell. The stimulated cell at $n = 0$ inhibits its neighbours at $n = \pm 1$. This in turn lowers the inhibition at $n = 0$, and also at $n = \pm 2$ and so on. In the trade this is known as *disinhibtion*.

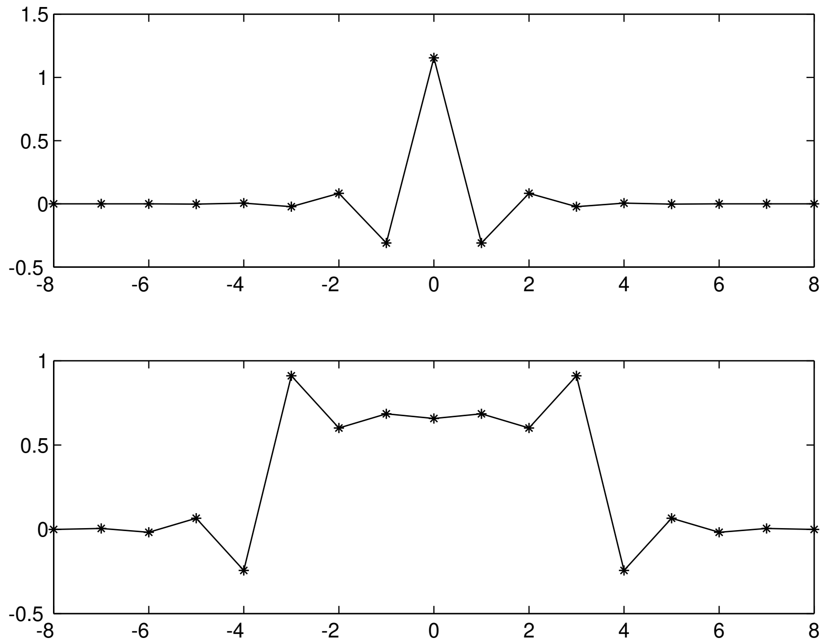


Figure 14.2: Top panel: the Green's function (14.22) with $\alpha = 1/4$. The response at $n = 0$ is about 15% greater than the response of an isolated cell. Lower panel: the response of the array if the seven central cells, $n = [-3, -2 \ldots 2, 3]$ are uniformly excited. The cells at the edge ($n = \pm 4$ and $n = \pm 3$) have an enhanced response. This is how your visual system detects 'edges'. `MachBand.eps`

The response of the array is shown in the upper panel figure 14.2. Notice we can't have a negative response so the solution we have just found should be superposed on the response to a uniform illumination L. Thus the actual firing rate is

$$R_n = \frac{kL}{1+2\alpha} + \frac{kA}{\sqrt{1-4\alpha^2}} \left(\frac{1}{\beta}\right)^{|n|} . \tag{14.25}$$

Example 3: edges

If you look closely at the border between a uniformly light region and uniformly dark region you'll notice that the edge is enhanced — there is a bright bar running along the border. This is a Mach band. The phenomenon was observed by impressionists who wondered if it was necessary to represent the band in their paintings. The physicist Mach said: "Of course not — just paint realistically and the observer's nervous system will supply the band." This advice was ignored by Degas, Monet and others who routinely overcompensate for the Mach band in their paintings.

14.3 Problems

Problem 14.1. Formulate a version of (14.1) whose continuum approximation is the advection equation

$$C_t + C_n \approx 0\,. \tag{14.26}$$

Solve the discrete system and comapre your answer with the relevant solution of (14.26)

Problem 14.2. Suppose that t is large in (14.6). Use Laplace's method to asymptotically evaluate the integral and show that the answer agrees with the similarity solution in (14.13).

Problem 14.3. Consider an infinite set of coupled oscillators

$$\ddot{\theta}_n = \theta_{n-1} - 2\theta_n + \theta_{n+1}; \qquad \theta_n(0) = \delta_{n0}, \quad \dot{\theta}_n(0) = 0\,.$$

Find a solution of the system above in terms of the Bessel functions:

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos[z\sin\theta - n\theta]\mathrm{d}\theta = \frac{\mathrm{i}^{-n}}{\pi}\int_0^{\pi} e^{\mathrm{i}z\cos\theta}\cos n\theta\,\mathrm{d}\theta\,, \qquad \text{AS §9.1.21}\,.$$

Give a simple expression for the decay of θ_0 at large times. Use MATLAB to plot the displacements as a function of n at fixed times such as $t = 1$, $t = 2$ etcetera.

Problem 14.4. The impulse response of coupled oscillators in the previous problem is the solution of

$$\ddot{G}_n = G_{n+1} - 2G_n + G_{n+1} + \delta_{n0}\delta(t)\,.$$

Use the Fourier series method to find a compact integral representation of the solution. Can the integrals be evaluated in terms of Bessel functions?

Problem 14.5. Fill in all the steps between (14.20) and (14.23).