SIO203CMAE294C, Spring 2010, Mid-term

9:30am to 10:50am, closed book

Turn the page!

Problem 1

Find the solution, and the domain of definition, of the PDE

$$u_x - x u_y = 0\,, \qquad \text{with the requirement that } u(x, x^2/2) = x.$$

Please show the domain of definition in a sketch.

Problem 2

Consider steady-state, thermal diffusion in an infinite three-dimensional medium. The temperature $u(r)$ is determined by

$$\kappa \nabla^2 u = -q \begin{cases} 1\,, & \text{if } r < a; \\ 0\,, & \text{if } r > a, \end{cases} \tag{1}$$

where the source strength q is a constant. The temperature far from the spherical source is $\lim_{r\to\infty} u(r) = u_\infty$. *(i)* Evaluate the integral over all space of the source on the right hand side of (1). *(ii)* Find the central temperature $u(0)$.

Hint: Gauss's divergence theorem is

$$\int_V \boldsymbol{\nabla \cdot f}\, \mathrm{d}^3 v = \int_{\partial V} \boldsymbol{f \cdot n}\, \mathrm{d}^2 s\,.$$

Above ∂V is the surface of the volume V, and $\boldsymbol{n}$ is the outward unit normal.

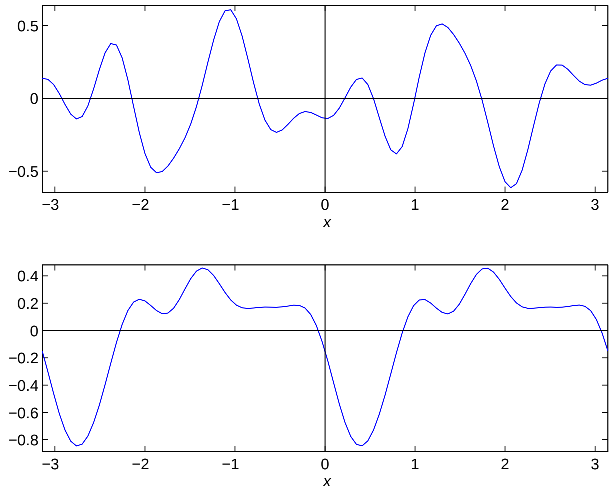

Figure 1: Figure for problem **3**.

Problem 3

The figure above shows two functions of x on the fundamental interval $-\pi < x < \pi$. Here are four possible Fourier series representations of these two functions:

$$
\begin{aligned}
f_1(x) &= a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + a_4 \cos 4x + \cdots \\
f_2(x) &= b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + \cdots \\
f_3(x) &= a_1 \cos x + a_3 \cos 3x + a_5 \cos 5x + \cdots \\
&\qquad + b_1 \sin x + b_3 \sin 3x + b_5 \sin 5x + \cdots \\
f_4(x) &= a_2 \cos 2x + a_4 \cos 4x + a_6 \cos 6x + \cdots \\
&\qquad + b_2 \sin 2x + b_4 \sin 4x + b_6 \sin 6x + \cdots
\end{aligned}
$$

(i) Which representation might apply to the function shown in the top panel of the figure? *(ii)* Which representation might apply to the function shown in the bottom panel of the figure? Lucky guesses don't count: explain your reasoning in twenty or thirty words.

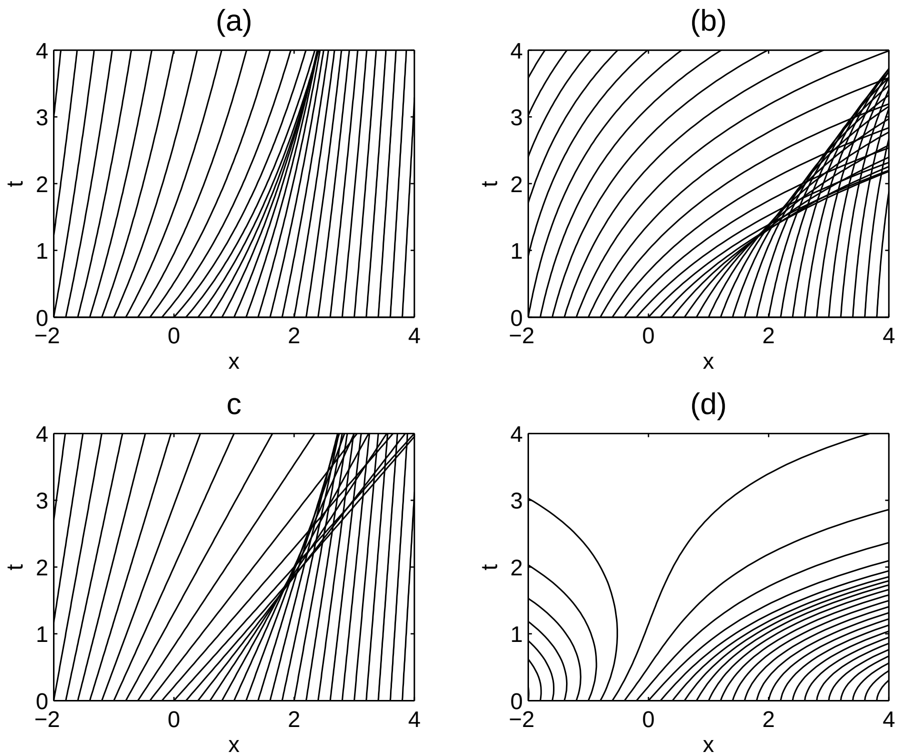

Figure 2: Match the characteristic diagram to the PDE's in **problem 4**.

Problem 4

Here are four PDE's

$$(1) \quad u_{1t} + u_1 u_{1x} = x\,, \qquad (2)\; u_{2t} + u_2 u_{2x} = \alpha u_2\,,$$

$$(3) \quad u_{3t} + u_3 u_{3x} = -\alpha u_3^2\,, \qquad (4) \quad u_{4t} + u_4 u_{4x} = 0\,.$$

In PDE's (2) and (3), α is a positive constant. Figure 2 shows four characteristic diagrams; the initial condition is

$$u_n(x,0) = \frac{1}{1+x^2}$$

in every case. Match the diagram with the PDE. Lucky guesses don't count, so explain your reasoning in thirty words or less.

Problem 5

Consider the PDE

$$u_t = (\beta|x|u_x)_x \,, \qquad u(x,0) = \delta(x)\,.$$

(i) What are the dimensions of β and the dimensions of $u(x,t)$? *(ii)* Determine

$$m(t) \equiv \int_{-\infty}^{\infty} u(x,t)\,\mathrm{d}x\,.$$

(iii) The PDE might have a similarity solution of the form

$$u(x,t) = t^a U\left(\frac{|x|}{\beta t^b}\right)\,. \tag{2}$$

Find the exponents a and b that are consistent with the PDE *and* with the initial condition. *(iv)* Find this similarity solution.