Problem 1

Find the solution, and the domain of definition, of the PDE

\[ u_x - xu_y = 0, \quad \text{with the requirement that } u(x, x^2/2) = x. \]

Please show the domain of definition in a sketch.

Problem 2

Consider steady-state, thermal diffusion in an infinite three-dimensional medium. The temperature \( u(r) \) is determined by

\[ \kappa \nabla^2 u = -q \begin{cases} 1, & \text{if } r < a; \\ 0, & \text{if } r > a, \end{cases} \tag{1} \]

where the source strength \( q \) is a constant. The temperature far from the spherical source is \( \lim_{r \to \infty} u(r) = u_\infty. \)  

(i) Evaluate the integral over all space of the source on the right hand side of (1).

(ii) Find the central temperature \( u(0). \)

Hint: Gauss’s divergence theorem is

\[ \int_V \nabla \cdot f \, d^3v = \int_{\partial V} f \cdot n \, d^2s. \]

Above \( \partial V \) is the surface of the volume \( V, \) and \( n \) is the outward unit normal.
Problem 3

The figure above shows two functions of $x$ on the fundamental interval $-\pi < x < \pi$. Here are four possible Fourier series representations of these two functions:

- $f_1(x) = a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + a_4 \cos 4x + \cdots$
- $f_2(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + \cdots$
- $f_3(x) = a_1 \cos x + a_3 \cos 3x + a_5 \cos 5x + \cdots$
  $\quad \quad \quad + b_1 \sin x + b_3 \sin 3x + b_5 \sin 5x + \cdots$
- $f_4(x) = a_2 \cos 2x + a_4 \cos 4x + a_6 \cos 6x + \cdots$
  $\quad \quad \quad + b_2 \sin 2x + b_4 \sin 4x + b_6 \sin 6x + \cdots$

(i) Which representation might apply to the function shown in the top panel of the figure? (ii) Which representation might apply to the function shown in the bottom panel of the figure? Lucky guesses don’t count: explain your reasoning in twenty or thirty words.
Problem 4

Here are four PDE’s

\begin{align*}
(1) \quad & u_{1t} + u_1 u_{1x} = x, \\
(2) \quad & u_{2t} + u_2 u_{2x} = \alpha u_2, \\
(3) \quad & u_{3t} + u_3 u_{3x} = -\alpha u_3^2, \\
(4) \quad & u_{4t} + u_4 u_{4x} = 0.
\end{align*}

In PDE’s (2) and (3), \( \alpha \) is a positive constant. Figure 2 shows four characteristic diagrams; the initial condition is

\[ u_n(x, 0) = \frac{1}{1 + x^2} \]

in every case. Match the diagram with the PDE. Lucky guesses don’t count, so explain your reasoning in thirty words or less.

Figure 2: Match the characteristic diagram to the PDE’s in problem 4.
Problem 5

Consider the PDE

\[ u_t = (\beta |x| u_x)_x, \quad u(x, 0) = \delta(x). \]

(i) What are the dimensions of \( \beta \) and the dimensions of \( u(x, t) \)? (ii) Determine

\[ m(t) \equiv \int_{-\infty}^{\infty} u(x, t) \, dx. \]

(iii) The PDE might have a similarity solution of the form

\[ u(x, t) = t^a U \left( \frac{|x|}{\beta t^b} \right). \quad (2) \]

Find the exponents \( a \) and \( b \) that are consistent with the PDE and with the initial condition. (iv) Find this similarity solution.