SIO203CMAE294C, Spring 2018, Mid-term

1:00pm to 2:30pm, open notes

Problem 1

Solve the PDE

$$u_x + u_y = 0\,, \qquad \text{with data} \qquad u(x,-x) = \mathrm{e}^x\,. \tag{1}$$

Problem 2

Here are four complex Fourier series

$$S_1(x) = \frac{\sinh \alpha\pi}{\pi} \sum_{m-=\infty}^{\infty} \frac{(-1)^m \,\mathrm{i}m\,\mathrm{e}^{\mathrm{i}mx}}{\alpha^2+m^2}\,, \tag{2}$$

$$S_2(x) = \frac{\sinh \alpha\pi}{\pi} \sum_{m-=\infty}^{\infty} \frac{(-1)^m \,m\,\mathrm{e}^{\mathrm{i}mx}}{\alpha^2+m^2}\,, \tag{3}$$

$$S_3(x) = \frac{\sinh \pi\alpha}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^m \,\mathrm{i}\alpha\,\mathrm{e}^{\mathrm{i}mx}}{\alpha^2+m^2}\,, \tag{4}$$

$$S_4(x) = \frac{\sinh \pi\alpha}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^m \,\alpha\,\mathrm{e}^{\mathrm{i}mx}}{\alpha^2+m^2}\,. \tag{5}$$

On the interval $-\pi < x < \pi$, which series is equal to $\cosh \alpha x$ and which is equal to $\sinh \alpha x$? Lucky guesses don't count so explain your reasoning (thirty words or less). Considering the end point $x = \pi$, what does the $\cosh \alpha x$ series converge to, and what does the $\sinh \alpha x$ series converge to?

Problem 3

Solve the PDE

$$u_t + \ln(u)\, u_x = 0\,, \qquad \text{with inital condition} \qquad u(x,0) = \mathrm{e}^x\,. \tag{6}$$

Problem 4

Consider the PDE

$$u_t = \kappa u_{xxx} + \frac{\delta(x)}{t^{1/2}}\,, \qquad \text{with initial condition } u(x,0) = 0. \tag{7}$$

(i) Assuming that $\lim_{x\to\pm\infty} u(x,t) = 0$, and without solving the PDE, show that

$$\int_{-\infty}^{\infty} u(x,t)\,\mathrm{d}x = 2\sqrt{t}\,. \tag{8}$$

(ii) Find the values of p and q in

$$u(x,t) = t^p U(\eta) \qquad \text{and} \qquad \eta \stackrel{\text{def}}{=} x/(\kappa t)^q\,, \tag{9}$$

which make this a possible similarity solution of (7). *(iii)* Find the ODE that $U(\eta)$ must satisfy. (There is no need to solve the ODE.)

Problem 5

Consider an age-stratified population, with histogram $h(a,t)$ satisfying

$$h_t + h_a = -\mathrm{e}^{-t} h\,. \tag{10}$$

The initial condition is

$$h(a,0) = N\mathrm{e}^{-a}\,, \tag{11}$$

and the birth rate $h(0,t)$ is adjusted so that the population is constant:

$$\forall t \geq 0: \qquad N = \int_0^{\infty} h(a,t)\,\mathrm{d}a\,. \tag{12}$$

Solve the PDE and exhibit $h(a,t)$.