Dynamic Enthalpy, Conservative Temperature, and the Seawater Boussinesq Approximation

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ABSTRACT

A new seawater Boussinesq system is introduced, and it is shown that this approximation to the equations of motion of a compressible binary solution has an energy conservation law that is a consistent approximation to the Bernoulli equation of the full system. The seawater Boussinesq approximation simplifies the mass conservation equation to $\nabla \cdot \mathbf{u} = 0$, employs the nonlinear equation of state of seawater to obtain the buoyancy force, and uses the conservative temperature introduced by McDougall as a thermal variable. The conserved energy consists of the kinetic energy plus the Boussinesq dynamic enthalpy $h^{\ddagger}$, which is the integral of the buoyancy with respect to geopotential height Z at a fixed conservative temperature and salinity. In the Boussinesq approximation, the full specific enthalpy h is the sum of four terms: McDougall's potential enthalpy, minus the geopotential $g_0 Z$, plus the Boussinesq dynamic enthalpy $h^{\ddagger}$, and plus the dynamic pressure. The seawater Boussinesq approximation removes the large and dynamically inert contributions to h, and it reveals the important conversions between kinetic energy and $h^{\ddagger}$.

1. Introduction

This note demonstrates that the Boussinesq approximation used in physical oceanography has an energy conservation law that consistently approximates energy conservation in the full equations of motion. The Boussinesq approximation employed by physical oceanographers has three main ingredients:

1) the exact density $\rho(\mathbf{x}, t)$ in the inertial terms of the momentum equation is replaced by a constant reference density ρ_0,
2) the mass conservation equation is approximated by $\nabla \cdot \mathbf{u} = 0$, and
3) the full equation of state (EOS) relates the buoyancy of seawater to temperature, salinity, and an approximate pressure $P_0 - \rho_0 g_0 Z$, where $Z(\mathbf{x})$ is the geopotential height (i.e., the gravitational–centrifugal potential divided by the standard gravity at mean sea level, g_0).

Because of the third point, this "seawater Boussinesq approximation" is more complicated than the standard Boussinesq approximation in which thermodynamic relations are simplified by linearizing the EOS (Spiegel and Veronis 1960). The extra complexity of the seawater Boussinesq approximation is required to capture oceanographic processes, such as cabbeling and the thermobaric instability (McDougall 1987a).

Dewar et al. (1998) argue that because of the approximation discussed in point 3 there is a spurious pressure gradient, which in western boundary currents leads to velocity errors as large as several centimeters per second. To avoid this quantitative and systematic error, they recommend that ocean circulation models might use the full pressure in the EOS. However, to do so while continuing to use $\nabla \cdot \mathbf{u} = 0$ means that the approximation does not conserve energy. Lack of energy conservation is a qualitative failure of an approximation, and thus we insist that Boussinesq models should make the approximation discussed in point 3.

In an interesting discussion of the energetics of thermobaric instability, Ingersoll (2005) uses an oceanographic anelastic approximation in which basic state quantities, such as ρ_0, are functions of z rather than constants. There are several reasons for preferring the seawater Boussinesq approximation to Ingersoll's anelastic approximation. First, the density of the ocean is close to constant, and thus most ocean models and theories use $\nabla \cdot \mathbf{u} = 0$ rather than the anelastic mass

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conservation equation. Second, to obtain an anelastic energy conservation law, Ingersoll restricts attention to a linearized EOS. Thus, without further development the anelastic model does not contain the principal non-linearity of the EOS, namely, the quadratic dependence of buoyancy on temperature. Consequently, cabbeling is not captured by the anelastic model.

McDougall (2003) and Ingersoll (2005) emphasize the central importance of enthalpy as an oceanographic variable. Nycander (2009a, manuscript submitted to *J. Phys. Oceanogr.*, hereafter NY09) has recently proposed a new definition of oceanographic neutral surfaces as being orthogonal to an enthalpic gradient. Enthalpy also proves to be the essential thermodynamic potential of this paper.

Section 2 presents the seawater Boussinesq equations and shows that this approximation has a consistent energy conservation law for the quantity $\frac{1}{2}|\mathbf{u}|^2 + h^\ddagger$, where $h^\ddagger$ is the Boussinesq dynamic enthalpy. Section 3 discusses the exact equations of motion for a compressible binary solution; and this review highlights the enthalpy h, the conservative temperature Θ, and near conservation of the Bernoulli density $\frac{1}{2}|\mathbf{U}|^2 + g_0 Z + h$ ($\mathbf{U}$ is the exact compressible velocity). Section 4 shows that the Boussinesq energy density $\frac{1}{2}|\mathbf{u}|^2 + h^\ddagger$ is the small part of $\frac{1}{2}|\mathbf{U}|^2 + g_0 Z + h$ that participates in mechanical (as opposed to thermodynamic) processes. The appendix presents the Boussinesq approximation to the internal energy of seawater.

2. The seawater Boussinesq approximation

To formulate the seawater Boussinesq approximation, the density of seawater is written as

$$\rho = \frac{\rho_0}{1 + g_0^{-1} b}, \qquad (1)$$

where ρ_0 is a constant reference density and b is the buoyancy.[1] In terms of the specific volume $\upsilon \equiv 1/\rho$, the buoyancy is $b = g_0(\rho_0 \upsilon - 1)$. Corresponding to (1), the absolute pressure is represented as

$$P = P_0 - g_0 \rho_0 Z + \rho_0 p, \qquad (2)$$

where $P_0 = 101\,325$ Pa is the pressure at sea level of 1 atm. The dynamic part of the pressure is $p(\mathbf{x}, t)$ in (2).

Throughout this work, the subscript 0 indicates a constant reference quantity evaluated at convenient reference values. For example, if $T_0 = 24°\text{C}$, $P_0 = 101\,325$ Pa,

and $S_0 = 35$ on the practical salinity scale, then, using the United Nations Educational, Scientific and Cultural Organization (UNESCO) EOS 80 polynomial,

$$\rho_0 = \text{EOS}_{80}(S_0, T_0, P_0) = 1023.7\,\text{kg m}^{-3}. \qquad (3)$$

The basic conditions for the validity of the Boussinesq approximation are

$$g_0^{-1} b \ll 1 \quad \text{and} \quad p \ll \rho_0^{-1} P_0 - g_0 Z. \qquad (4)$$

The first inequality says that there are only small fractional changes in density about the reference value ρ_0. Extreme examples, such as

$$\text{EOS}_{80}(0, T_0, P_0) = 997.3\,\text{kg m}^{-3},$$

$$\text{EOS}_{80}(S_0, -1°\text{C}, P_0) = 1028.2\,\text{kg m}^{-3}, \quad \text{and}$$

$$\text{EOS}_{80}(S_0, T_0, 500P_0) = 1044.5\,\text{kg m}^{-3}$$

indicate that the parameter

$$\epsilon \equiv \max_{\forall \mathbf{x}, t} \left| \frac{b}{g_0} \right| \qquad (5)$$

is not likely to be greater than 0.03. The second inequality in (4) says that the nonhydrostatic dynamic pressure variations $\rho_0 p(\mathbf{x}, t)$ are much less than the pressure variations resulting from the hydrostatic background. This requirement is key in simplifying the thermodynamic relations—the seawater Boussinesq approximation replaces P by $P_0 - g_0 \rho_0 Z(\mathbf{x})$ in all thermodynamic relations. Here, $\rho_0 p$ can be estimated as the range of pressure variations at a fixed depth, which are due to internal gravity waves, convective plumes, and eddies, etc. A large mesoscale eddy has a $\rho_0 p \sim 10^4$ Pa, which is equivalent to the variation of the hydrostatic background pressure over a distance of only 1 m.

Apart from small deviations owing to the uneven distribution of the earth's mass, the geopotential height $Z(\mathbf{x})$ is almost equal to the geometric height z and ∇Z is almost equal to the unit vector $\hat{\mathbf{z}}$. The distinction between z and $Z(\mathbf{x})$ is maintained because it brings conceptual clarity to the following discussion: in the seawater Boussinesq approximation, depth has both a geometric and a thermodynamic role. Thermodynamically, Z is a surrogate for the background hydrostatic pressure $P_0 - \rho_0 g_0 Z(\mathbf{x})$ and this is usefully distinguished with Z.

Making the approximations outlined in the introduction, the seawater Boussinesq equations are

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[1] The conventional definition is $\rho = \rho_0(1 - g_0^{-1} b)$. There are small differences between the two bs. The definition of (1) leads to a cleaner expansion in section 4.

$$\frac{D\mathbf{u}}{Dt} + \mathbf{f} \times \mathbf{u} + \nabla p = b\nabla Z + \mathring{\mathbf{u}}, \qquad (6)$$

$$\nabla \cdot \mathbf{u} = 0, \qquad (7)$$

$$\frac{D\Theta}{Dt} = \mathring{\Theta}, \quad \text{and} \qquad (8)$$

$$\frac{DS}{Dt} = \mathring{S}. \qquad (9)$$

The quantities $\mathring{\mathbf{u}}$, $\mathring{\Theta}$, and $\mathring{S}$ on the right are diabatic processes and external forcings whose exact form is unimportant for this discussion. Here, S is the salinity and Θ is the "conservative temperature" defined by McDougall (2003); the decision to frame the Boussinesq approximation in terms of conservative temperature (rather than potential temperature θ) is discussed further in section 3. On the right of (6), $b\nabla Z$ is the buoyant acceleration.

The system (6) through (9) is closed with an EOS expressing the buoyancy $b(\mathbf{x}, t)$ in terms of the conservative temperature $\Theta(\mathbf{x}, t)$, the salinity $S(\mathbf{x}, t)$, and the background hydrostatic pressure $P_0 - \rho_0 gZ$. This EOS is written as

$$b = \tilde{b}(\Theta, Z, S). \qquad (10)$$

The undecorated b on the left of (10) denotes the buoyancy $b(\mathbf{x}, t)$ at a point in space–time. On the right, $\tilde{b}$ denotes a thermodynamic function with three arguments as indicated. This tilde notation distinguishes thermodynamic functions, such as $\tilde{b}(\Theta, Z, S)$, from fields, such as $b(\mathbf{x}, t)$. So, for instance,

$$\frac{Db}{Dt} = \mathring{\Theta}\frac{\partial \tilde{b}}{\partial \Theta} + \frac{DZ}{Dt}\frac{\partial \tilde{b}}{\partial Z} + \mathring{S}\frac{\partial \tilde{b}}{\partial S}. \qquad (11)$$

Derivatives with respect to the coordinate z are denoted by a subscript and derivatives with respect to the thermodynamic variable Z are denoted by quotients of partials:

$$b_z = \Theta_z \frac{\partial \tilde{b}}{\partial \Theta} + Z_z \frac{\partial \tilde{b}}{\partial Z} + S_z \frac{\partial \tilde{b}}{\partial S}. \qquad (12)$$

To obtain a Boussinesq energy conservation equation, it is crucial to introduce the thermodynamic function

$$\tilde{h}^{\ddagger}(\Theta, Z, S) \equiv \int_Z^0 \tilde{b}(\Theta, Z', S)\, dZ' \qquad (13)$$

and the associated field $h^{\ddagger}(\mathbf{x}, t)$. The constant of integration in this definition ensures that $\tilde{h}^{\ddagger}(\Theta, 0, S) = 0$.

Following a 2005 personal communication from W. R. Young, Vallis (2006) introduces $\tilde{h}^{\ddagger}$, denoting it by Π, in a section on Boussinesq energetics. For reasons that

emerge in section 4 of this study, it is appropriate to refer to $h^{\ddagger}$ in the present Boussinesq context, as the dynamic enthalpy, or more explicitly as the Boussinesq dynamic enthalpy. In section 4, we introduce an exact dynamic enthalpy for a fully compressible fluid; $h^{\ddagger}$ enters the Bernoulli equation in much the same way as does this exact dynamic enthalpy.

The material derivative of the Boussinesq dynamic enthalpy is

$$\frac{Dh^{\ddagger}}{Dt} = \mathring{\Theta}\frac{\partial \tilde{h}^{\ddagger}}{\partial \Theta} - \frac{DZ}{Dt}b + \mathring{S}\frac{\partial \tilde{h}^{\ddagger}}{\partial S}, \qquad (14)$$

and taking $\mathbf{u}$ times the momentum equation,

$$\frac{1}{2}\frac{D|\mathbf{u}|^2}{Dt} + \nabla \cdot (p\mathbf{u}) = \frac{DZ}{Dt}b + \mathbf{u} \cdot \mathring{\mathbf{u}}. \qquad (15)$$

Eliminating $(DZ/Dt)b$ between (14) and (15) gives the energy conservation law

$$\frac{D}{Dt}\left(\frac{1}{2}|\mathbf{u}|^2 + h^{\ddagger}\right) + \nabla \cdot (\mathbf{u}p) = \mathring{\Theta}\frac{\partial \tilde{h}^{\ddagger}}{\partial \Theta} + \mathring{S}\frac{\partial \tilde{h}^{\ddagger}}{\partial S} + \mathbf{u} \cdot \mathring{\mathbf{u}}. \qquad (16)$$

Using $\nabla \cdot \mathbf{u} = 0$, one can rewrite (16) as

$$\frac{D}{Dt}\left(\frac{1}{2}|\mathbf{u}|^2 + h^{\ddagger} + p\right) = p_t + \mathring{\Theta}\frac{\partial \tilde{h}^{\ddagger}}{\partial \Theta} + \mathring{S}\frac{\partial \tilde{h}^{\ddagger}}{\partial S} + \mathbf{u} \cdot \mathring{\mathbf{u}}. \qquad (17)$$

This form emphasizes that $\frac{1}{2}|\mathbf{u}|^2 + h^{\ddagger} + p$ can be regarded as a type of Bernoulli density; the term p_t on the right means that the $\frac{1}{2}|\mathbf{u}|^2 + h^{\ddagger} + p$ does not satisfy an exact conservation equation. Nonconservation of the Bernoulli density resulting from the unsteady pressure term is a well-known issue with the full equations of motion [see the discussion in section 3 and in McDougall (2003)]. The relation between $\frac{1}{2}|\mathbf{u}|^2 + h^{\ddagger} + p$ and the exact Bernoulli density is clarified in section 4.

Notice that in the EOS (10), pressure is approximated by the hydrostatic background $P_0 - g_0\rho_0 Z$. If one uses the total pressure in the EOS, then the resulting system does not conserve energy. In some ocean circulation models the velocity field is incompressible, and the complete EOS, with pressure $P_0 - \rho_0 g_0 Z$, is used to calculate buoyancy. These models have a consistent Boussinesq energy conservation law.

3. The equations of motion of a compressible binary solution

Turning now to the fundamental basis of the mechanical energy conservation law in (16), we consider

the exact equations for the fluid mechanics of a compressible binary solution:

$$\frac{D\mathbf{U}}{Dt} + \mathbf{f} \times \mathbf{U} + v\nabla P + g_0 \nabla Z = \mathring{\mathbf{U}}, \tag{18}$$

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{U} = 0, \tag{19}$$

$$\frac{De}{Dt} + vP\nabla \cdot \mathbf{U} = \mathring{e}, \tag{20}$$

$$\frac{D\eta}{Dt} = \mathring{\eta}, \quad \text{and} \tag{21}$$

$$\frac{DS}{Dt} = \mathring{S} \tag{22}$$

(Davis 1994; Salmon 1998; Vallis 2006). In the above equations, $v \equiv \rho^{-1}$ is the specific volume, $e(\mathbf{x}, t)$ is the internal energy, $P(\mathbf{x}, t)$ is the total pressure in (2), and $\mathbf{U}(\mathbf{x}, t)$ is the exact compressible velocity. The specific entropy is $\eta(\mathbf{x}, t)$ and the salinity is $S(\mathbf{x}, t)$. As in (6) through (9), we avoid explicit expressions for the diabatic terms by using the notation $\mathring{e}$, etc.

The system is completed by a sixth equation, which is a thermodynamic relation specifying the specific internal energy e as a function of the specific entropy η, the specific volume $v \equiv \rho^{-1}$, and the salinity S as

$$e = \tilde{e}(\eta, v, S). \tag{23}$$

Complete thermodynamic information can be derived from the function $\tilde{e}$ and the thermodynamic law

$$d\tilde{e} = \tilde{T} \, d\eta - \tilde{P} \, dv + \tilde{\mu} \, dS \tag{24}$$

(e.g., Callen 1985). Thus, from the relation above one has three more thermodynamic functions of η, v, and S:

$$\tilde{T} \equiv \frac{\partial \tilde{e}}{\partial \eta}, \quad \tilde{P} \equiv -\frac{\partial \tilde{e}}{\partial v}, \quad \tilde{\mu} \equiv \frac{\partial \tilde{e}}{\partial S}. \tag{25}$$

A seawater EOS is obtained by eliminating η between $\tilde{T}(\eta, v, S)$ and $\tilde{P}(\eta, v, S)$ to obtain $v = \rho^{-1}$, as a function of T, P, and S.

Making the replacement $d \rightarrow D/Dt$ in (24) and using (19) through (22), we see that consistency requires that the diabatic terms satisfy the condition

$$\mathring{\eta} = T^{-1}(\mathring{e} - \mu\mathring{S}). \tag{26}$$

The diabatic fluxes of internal energy, entropy, and salt satisfy the relation above (see appendix A of Davis 1994).

The equations above are not in the most convenient form for the physical oceanography of seawater. Fofonoff (1962) and McDougall (2003) show that it is better to use the enthalpy

$$\tilde{h}(\eta, P, S) \equiv \tilde{e} + P\tilde{v} \tag{27}$$

as a thermodynamic potential. A main advantage of this formulation is that mixing at constant pressure conserves enthalpy.

As indicated in (27), $\tilde{h}$ is naturally a thermodynamic function of the two adiabatically conservative variables η and S and the pressure P. Thus,

$$d\tilde{h} = \tilde{T} \, d\eta + \tilde{v} \, dP + \tilde{\mu} \, dS, \quad \text{so that} \tag{28}$$

$$\tilde{T} = \frac{\partial \tilde{h}}{\partial \eta}, \quad \tilde{v} = \frac{\partial \tilde{h}}{\partial P}, \quad \tilde{\mu} = \frac{\partial \tilde{h}}{\partial S}. \tag{29}$$

Combining the internal energy Eq. (20) with the thermodynamic law (24) and the definition (27), one finds that

$$\frac{Dh}{Dt} = v\frac{DP}{Dt} + \mathring{e}. \tag{30}$$

Combining (30) with the conservation law for kinetic energy obtained by taking $\mathbf{U} \cdot$ (18), one obtains this Bernoulli equation:

$$\frac{D}{Dt}\left(\frac{1}{2}|\mathbf{U}|^2 + g_0 Z + h\right) = vP_t + \mathring{e} + \mathbf{U} \cdot \mathring{\mathbf{U}}. \tag{31}$$

In many ways the Bernoulli equation above is the most useful statement of physical oceanographic energy conservation. However, a crippling disadvantage of the Bernoulli energy density

$$\mathcal{B} \equiv \frac{1}{2}|\mathbf{U}|^2 + g_0 Z + h \tag{32}$$

is that the kinetic energy density $\frac{1}{2}|\mathbf{U}|^2$ is many orders of magnitude smaller than $g_0 Z$ and h, that is, the dynamically important part of $\mathcal{B}$ is concealed beneath much larger terms.

In the next section, we show that the Boussinesq energy conservation law, (16) or (17), should be regarded as a simplification of (31) that reveals the dynamically important part of $\mathcal{B}$. The first step in picking apart $\mathcal{B}$ is to decompose the enthalpy h. Thus, integrating the middle relation in (29) from P to P_0

$$\tilde{h}(\eta, P, S) = \tilde{h}(\eta, P_0, S) + \int_{P_0}^{P} \tilde{v}(\eta, P', S) \, dP'. \tag{33}$$

McDougall's conservative temperature $\tilde{\Theta}$ is defined by

$$c_P^0 \tilde{\Theta}(\eta, S) \equiv \tilde{h}(\eta, P_0, S), \tag{34}$$

where $c_P^0 = 3991.867\ 957\ 119\ 63\ldots$ J kg^{-1} K^{-1} is close to the average value of the specific heat capacity at the sea surface of today's ocean. Thus, $c_P^0 \Theta$ is the potential enthalpy, and it is convenient to define the dynamic enthalpy as

$$\tilde{h}^{\dagger}(\eta, P, S) \equiv \int_{P_0}^{P} \tilde{v}(\eta, P', S)\, dP'. \tag{35}$$

The total enthalpy on the left of (33) is the sum of the potential enthalpy $c_P^0 \Theta$ and the dynamic enthalpy $h^{\dagger}$.

The material derivative of the dynamic enthalpy $h^{\dagger}$ is

$$\frac{Dh^{\dagger}}{Dt} = \frac{\partial \tilde{h}^{\dagger}}{\partial \eta}\mathring{\eta} + v\frac{DP}{Dt} + \frac{\partial \tilde{h}^{\dagger}}{\partial S}\mathring{S}. \tag{36}$$

Subtracting the expression above from Dh/Dt in (30) one finds

$$c_P^0 \frac{D\Theta}{Dt} = \mathring{e} - \frac{\partial \tilde{h}^{\dagger}}{\partial \eta}\mathring{\eta} - \frac{\partial \tilde{h}^{\dagger}}{\partial S}\mathring{S}. \tag{37}$$

All the results in this section are exact consequences of the equations of compressible fluid mechanics and thermodynamics. One exact result that we have not bothered to write is a close relative to (31); this is the conservation equation for the total energy $\mathcal{E} \equiv \frac{1}{2}|\mathbf{U}|^2 + g_0 Z + e$. In section 4, we turn to Boussinesq energetics and it is $\mathcal{B}$, rather than $\mathcal{E}$, that dominates this discussion. Appendix A provides the Boussinesq version of internal energy.

4. The seawater Boussinesq approximation again

The seawater Boussinesq approximation is obtained with a Helmholtz decomposition of the total velocity $\mathbf{U}$ as

$$\mathbf{U} = \mathbf{u} + \nabla \chi, \tag{38}$$

where $\mathbf{u}$ is the solenoidal or incompressible part of $\mathbf{U}$ and χ is a scalar potential that generates the small compressible part of $\mathbf{U}$. The seawater Boussinesq approximation is obtained with the approximation $\mathbf{U} \approx \mathbf{u}$. In the momentum equation (18), the pressure force is approximated as

$$v\nabla P \approx -g_0 \nabla Z + \nabla p - b\nabla Z, \tag{39}$$

which involves neglecting a small term $b\nabla p/g_0$. The Boussinesq mass conservation equation $\nabla \cdot \mathbf{u} = 0$ is

obtained by replacing ρ by ρ_0 in the mass conservation (19). If required, the small divergence of $\mathbf{U}$ can be diagnosed from the higher-order terms in the mass conservation equation as

$$\nabla \cdot \mathbf{U} = \nabla^2 \chi \approx \frac{1}{g_0}\frac{Db}{Dt}. \tag{40}$$

After using $\mathbf{U} \approx \mathbf{u}$, the Boussinesq salinity equation (9) corresponds closely to exact salinity conservation in (22). The easiest path to the Boussinesq heat equation (8) is to follow McDougall (2003) and neglect the final two terms on the right of (37). This approximation produces the Boussinesq temperature equation (8), with

$$\mathring{\Theta} \equiv \frac{\mathring{e}}{c_P^0}. \tag{41}$$

McDougall provides a full discussion of the small errors introduced by this approximation, and he also emphasizes the important conceptual advantages of Θ over potential temperature.

The most puzzling issue presented by the approximation outlined above is the physical interpretation of the Boussinesq energy conservation law (16) and the role of the thermodynamic function $\tilde{h}^{\ddagger}(\Theta, Z, S)$ defined in (13). To clarify this, we apply the Boussinesq approximation to the dynamic enthalpy $\tilde{h}^{\dagger}(\eta, P, S)$. Replacing η by the equivalent variable Θ, the definition of $\tilde{h}^{\dagger}$ in (35) simplifies to

$$\tilde{h}^{\dagger}(\Theta, P, S) = \rho_0^{-1}\int_{P_0}^{P} 1 + g_0^{-1}\tilde{b}(\Theta, P', S)\, dP', \tag{42}$$

$$\approx \rho_0^{-1}(P - P_0) + \tilde{h}^{\ddagger}(\Theta, Z, S), \tag{43}$$

where $\tilde{h}^{\ddagger}$ is the Boussinesq dynamic enthalpy from section 2. Because of the definition of b in (1), Eq. (42) is exact. To evaluate the integral of the large constant 1 on the right of (42), one must use the exact pressure in (2). However, to evaluate the integral of the $O(\epsilon)$ term $g_0^{-1}\tilde{b}$, $O(\epsilon^2)$ accuracy is preserved with $Z \approx (P_0 - P)/(g_0\rho_0)$.

At a field point, (43) can be written as

$$h^{\dagger}(\mathbf{x}, t) + g_0 Z(\mathbf{x}) \approx h^{\ddagger}(\mathbf{x}, t) + p(\mathbf{x}, t). \tag{44}$$

McDougall (2003) remarks that $c_P^0 \Theta$ and $h^{\dagger}$ have the same numerical order of magnitude. Moreover, the two terms on the left of (44) almost cancel, so that $h^{\ddagger}$ and p are much smaller than $h^{\dagger}$, $g_0 Z$, and $c_P^0 \Theta$.

We summarize our progress to this point by saying that within the seawater Boussinesq approximation the

full enthalpy h plus the gravitational potential energy $g_0 Z$ is decomposed into three terms:

$$h + g_0 Z \approx c_P^0 \Theta + h^\ddagger + p. \qquad (45)$$

Thus the difference between the Bernoulli density $\mathcal{B} = \frac{1}{2}|\mathbf{U}|^2 + g_0 Z + h$ and the potential enthalpy $c_P^0 \Theta$ can be written as

$$\mathcal{B} - c_P^0 \Theta \approx \frac{1}{2}|\mathbf{u}|^2 + p + h^\ddagger. \qquad (46)$$

In (45) and (46) the terms $\mathcal{B}$, $c_P^0 \Theta$, and $g_0 Z$ are much larger than $p + \tilde{h}^\ddagger$ and $\frac{1}{2}|\mathbf{u}|^2$. But it is only the combination $\frac{1}{2}|\mathbf{u}|^2 + p + \tilde{h}^\ddagger$ that matters for dynamics.

The Boussinesq equation (14) for $Dh^\ddagger/Dt$ can be re-derived by substituting the approximation (44) into the expression for the exact material derivative[2] of $h^\dagger$ in (36). Neglecting the $O(\epsilon^2)$ contribution bDp/Dt, the middle term on the right of (36) is

$$v \frac{DP}{Dt} \approx -g_0 \frac{DZ}{Dt} - b \frac{DZ}{Dt} + \frac{Dp}{Dt}. \qquad (47)$$

The terms $-g_0 DZ/Dt$ and Dp/Dt above then cancel against terms on the left of (36) arising from $p - g_0 Z$ in (44). Thus, in the Boussinesq limit (36) collapses to (14). This alternative derivation shows that the Boussinesq dynamic enthalpy $h^\ddagger$ is that small part of the total enthalpy that couples to the momentum equations via the buoyancy flux $(DZ/Dt)b$. Also, the Boussinesq approximation satisfies the enthalpy equation very accurately; using the conservative temperature as the thermal variable captures the leading-order terms, and the Boussinesq enthalpy equation (14) mops up the $O(\epsilon)$ terms.

Finally, to obtain the Boussinesq version of the Bernoulli equation, one substitutes the approximation (46) into the exact Bernoulli equation (31) to obtain

$$\frac{D}{Dt}\left[\frac{1}{2}|\mathbf{u}|^2 + p + h^\ddagger\right] = p_t + \frac{\partial \tilde{h}^\ddagger}{\partial \Theta}\mathring{\Theta} + \frac{\partial \tilde{h}^\ddagger}{\partial S}\mathring{S} + \mathbf{u} \cdot \mathring{\mathbf{u}}. \qquad (48)$$

The result above is Eq. (17), rewritten here for emphasis.

5. Discussion and conclusions

Because ocean currents are much slower than molecular velocities and the speed of sound, ocean kinetic and gravitational energies are negligible relative to internal energy. The seawater Boussinesq approximation of this paper accounts for the large molecular energies

via conservation of conservative temperature in (8) and also for the much smaller energy of ocean currents via the Boussinesq energy in (16) and (17). Small transformations of energy between kinetic, gravitational, and internal are dynamically crucial in processes such as thermobaric convection and for the "epsilon theorems" discussed by Paparella and Young (2002), McIntyre (2009), and Nycander (2009). The Boussinesq dynamic enthalpy $h^\ddagger$ consistently accounts for the gravitational and internal energies. The buoyancy flux $(DZ/Dt)b$, in (14) and (15), is the agent that results in conversion between $\frac{1}{2}|\mathbf{u}|^2$ and $h^\ddagger$.

The Boussinesq dynamic enthalpy $h^\ddagger$ and the full dynamic enthalpy $h^\dagger$ both figure prominently in NY09's reexamination of the "neutral surface" concept (McDougall 1987b; Jackett and McDougall 1997). In Nycander's view, it is enthalpy rather than buoyancy that defines the local planes along which seawater mixes most readily.

The internal energy $e^\ddagger$ and the divergence $\nabla \cdot \mathbf{U}$ play secondary roles in the Boussinesq approximation, because both quantities can be diagnosed from within the Boussinesq approximation. But aside from a reassuring verification of the consistency of the approximate energetics, there is not usually a compelling reason for such a close examination of the entrails. Indeed, $h^\ddagger$ is the central thermodynamic function of the Boussinesq equations of motion. The enthalpy is not altered by the pressure work and $\nabla \cdot \mathbf{U}$ is immaterial to the enthalpy changes encoded in $h^\ddagger$.

In addition to the enthalpy, the Bernoulli function $\mathcal{B}$ also plays a central role in these considerations. McDougall (2003) discusses the virtues of the Bernoulli density $\mathcal{B}$ versus those of the potential enthalpy $c_P^0 \Theta$. The analysis of section 4 shows that the seawater Boussinesq approximation provides access to the information in both $\mathcal{B}$ and in $c_P^0 \Theta$. This is accomplished by using what amounts to an almost exact equation for $c_P^0 \Theta$ (as explained by McDougall 2003) and separately dealing with the dynamically important difference $\mathcal{B} - c_P^0 \Theta$. The beauty of the seawater Boussinesq approximation, as developed in section 2, is that the approximation displays the crucial dynamical role of tiny energies, such as $\frac{1}{2}|\mathbf{u}|^2 + h^\ddagger$, while shielding the user from the details of section 4. Critical to that shielding is the introduction of the Boussinesq dynamic enthalpy $h^\ddagger$. The origin of $h^\ddagger$ is mysterious from within the Boussinesq framework, but it becomes clear once the full thermodynamics are considered, as done in section 4.

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[2] One can replace η in (36) by the equivalent variable Θ.

thorough and constructive reviews of this paper. WRY is supported by NSF OCE07–26320.

APPENDIX

Boussinesq Internal Energy

In this appendix we return to the Boussinesq equations of section 2 and obtain the connection between $h^{\ddagger}$ and internal energy. The first step is simplifying the definition of enthalpy in (27). In the Boussinesq limit,

$$vP \approx \frac{P_0}{\rho_0} - g_0 Z + \left(\frac{P_0}{\rho_0 g_0} - Z\right) b + p, \quad \text{(A1)}$$

where the $O(\epsilon^2)$ term pb/g_0 is neglected. Putting (A1) into $e = h - Pv$, and using the decomposition of h in (45), we have

$$e \approx c_P^0 \Theta - \frac{P_0}{\rho_0}\left(1 + \frac{b}{g_0}\right) + \underbrace{h^{\ddagger} + Zb}_{\equiv e^{\ddagger}}. \quad \text{(A2)}$$

In other words, given the equation of state $\tilde{b}(\Theta, Z, S)$ from section 2, one defines the Boussinesq dynamic enthalpy $h^{\ddagger}$ via (13), and then the Boussinesq internal energy $e^{\ddagger}$ is defined as the thermodynamic function

$$\tilde{e}^{\ddagger}(\Theta, Z, S) \equiv \underbrace{\int_Z^0 \tilde{b}(\Theta, Z', S)\, dZ'}_{\equiv h^{\ddagger}} + Z\tilde{b}(\Theta, Z, S). \quad \text{(A3)}$$

Recalling the expression for $Dh^{\ddagger}/Dt$ in (14), it follows that

$$\frac{De^{\ddagger}}{Dt} = \left(\mathring{\Theta}\frac{\partial \tilde{h}^{\ddagger}}{\partial \Theta} + \mathring{S}\frac{\partial \tilde{h}^{\ddagger}}{\partial S}\right) + Z\frac{Db}{Dt}. \quad \text{(A4)}$$

With the Boussinesq approximation for $\nabla \cdot \mathbf{U}$ in (40), the final term on the right-hand side can be written as

$-\rho_0 g_0 Z\nabla \cdot \mathbf{U}$ and recognized as a Boussinesq expression for pressure work (see also NY09).

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