Stressed horizontal convection

J. Hazewinkel\textsuperscript{1} F. Paparella\textsuperscript{2} and W.R. Young\textsuperscript{1}

\textsuperscript{1}Scripps Institution of Oceanography, La Jolla CA 92093-0213, USA
\textsuperscript{2}Department of Mathematics, University of Lecce, Italy

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Consider the problem of a Boussinesq fluid driven by applying both nonuniform temperature and stress at the top surface. On the other boundaries the conditions are thermally insulating and either no-slip or stress-free. The interesting case is when the direction of the steady applied surface stress opposes the sense of the buoyancy driven flow. We obtain two-dimensional numerical solutions showing a regime in which there is an upper cell with thermally indirect circulation (buoyant fluid is pushed downwards by the applied stress and heavy fluid is elevated), and a second deep cell with thermally direct circulation. In this two-cell regime the driving mechanisms are competitive in the sense that neither dominates the flow. A scaling argument shows that this balance requires that surface stress vary as the horizontal Rayleigh number to the three-fifths power.

1. Introduction

In 1908, Sandström began his work on horizontal convection with the remark: “The motive for these experiments was the following observation I made at the Bornö station in the Gullmarfjord on the west coast of Sweden. When the wind swept over the fjord, the water at the surface flowed in the direction of the wind. Yet, as soon as the wind ceased, it flowed back in the opposite direction.” For a complete English translation of Sandström’s paper, and a survey of the history of the horizontal-convection problem, see Kuhlbrodt (2008).

Horizontal convection is convection generated by imposing non-uniform buoyancy along a horizontal surface (Stern 1975). The problem has attracted considerable attention due to the suggestion of Munk & Wunsch (1998) that mechanical energy sources — such as the wind stress observed by Sandström (1908) — are necessary to sustain the ocean circulation. Recent work on horizontal convection is reviewed by Hughes & Griffiths (2008). Despite Sandström’s early recognition of the importance of surface wind stress, there has been only a little discussion of the interaction between imposed surface stress with surface buoyancy forcing within the context of horizontal convection. An early exception is the study of Beardsley & Festa (1972); more recently Ilicak & Vallis (2011) have examined the effect of an oscillatory stress on horizontal convection. In this paper we revisit the steady-stress problem formulated by Beardsley & Festa (1972) with greater computational resources, and with an improved understanding of horizontal convection.

The point of Munk & Wunsch (1998) regarding energy sources can be appreciated by considering the mechanical energy budget of horizontal convection (Paparella & Young 2002). Consider a three-dimensional rotating fluid in a rectangular box and represent the density as $\rho = \rho_0(1 - g^{-1}b)$, where $b$ is the “buoyancy”. The Boussinesq equations of
motion are then
\[ \frac{D\mathbf{u}}{Dt} + 2\Omega \times \mathbf{u} + \nabla p = b\dot{z} + \nu \nabla^2 \mathbf{u}, \]  
\[ \frac{Db}{Dt} = \kappa \nabla^2 b, \]  
\[ \nabla \cdot \mathbf{u} = 0. \]  
\[(1.1)\]  
\[(1.2)\]  
\[(1.3)\]

The kinematic viscosity is \( \nu \) and the thermal diffusivity is \( \kappa \). The boundary conditions on the velocity \( \mathbf{u} = (u, v, w) \) are \( \mathbf{u} \cdot \mathbf{n} = 0 \), where \( \mathbf{n} \) is the outward normal. The vertical coordinate is \( -H < z < 0 \). At the top surface, \( z = 0 \), the boundary conditions are \( b = b_s(x, y) \), and \( \nu(u_z, v_z) = \tau_s(x, y) \); \[(1.4)\]

The surface buoyancy \( b_s \) and surface stress \( \tau_s \) are prescribed with \( 0 \leq b_s(x, y) \leq b_{\text{max}} \) and \( 0 \leq |\tau_s(x, y)| \leq \tau_{\text{max}} \). There is no flux of heat through bottom, \( z = -H \), or through the sidewalls. The viscous boundary conditions on the bottom and sidewalls are some combination of no-slip and no-stress.

We denote the total volume and time average by angular brackets. Thus, for example, the mechanical energy dissipation is
\[ \varepsilon \overset{\text{def}}{=} \nu \langle \nabla u \cdot \nabla u + \nabla v \cdot \nabla v + \nabla w \cdot \nabla w \rangle. \]  
\[(1.5)\]

Taking the dot product of the momentum equation in (1.1) with \( \mathbf{u} \) and averaging over the volume, one has
\[ \varepsilon = \langle wb \rangle + H^{-1} \mathbf{u}_s \cdot \tau_s, \]  
\[(1.6)\]

where \( \mathbf{u}_s \overset{\text{def}}{=} \mathbf{u}(x, y, 0, t) \) is the surface velocity and the overbar denotes an average over \( x, y \) and \( t \). This shows that the viscous dissipation \( \varepsilon \) is balanced by the conversion of available potential energy\(^\dagger\) into kinetic energy via the correlation in \( \langle wb \rangle \) and by the stress work. In Sandström’s observation the wind is doing net positive work because the surface velocity is in the direction of the wind i.e., \( \mathbf{u}_s \cdot \tau_s > 0 \). However if the surface velocity is against the direction of the wind then the atmosphere is extracting energy from the ocean.

Taking the \( (x, y, t) \)-average of (1.2), and using the no-flux condition at \( z = -H \), one has
\[ \mathbf{w} \mathbf{b} - \kappa \mathbf{b}_z = 0. \]  
\[(1.7)\]

Thus there is no net vertical buoyancy flux through every level \( z = \text{constant} \). The “zero-flux” constraint (1.7) is a distinctive feature of horizontal convection, which remains true when the problem is enriched by the addition of the stress forcing \( \tau_s \). Another expression for the buoyancy flux \( \langle wb \rangle \) in (1.6) is obtained by averaging (1.7) over \( z \):
\[ \langle wb \rangle = \frac{\kappa \Delta \mathbf{b}}{H}, \]  
\[(1.8)\]

where \( \Delta \mathbf{b} \equiv \mathbf{b}(0) - \mathbf{b}(-H) \) is the difference between the horizontally averaged buoyancy at the top and bottom of the box.

\(^\dagger\) The role of available potential energy in horizontal convection has recently been examined by several authors (Tailleux 2009; Hughes et al. 2009; Winters & Young 2009). Relevant to this discussion is that \( \langle wb \rangle \) is the net rate of transfer between available potential energy and kinetic energy; thus (1.8) provides a restrictive bound on conversions between kinetic and available potential energy.
Eliminating the buoyancy flux $\langle wb \rangle$ between (1.6) and (1.8) one then has

$$H \varepsilon = \kappa \Delta \tilde{b} + SW,$$

(1.9)

where $SW \equiv \tau_s \cdot \mathbf{u}_s$ is the “stress work”. The left of (1.9) is positive definite. But, in general, the sign of the individual terms on the right is indefinite. For instance, if $SW > 0$, then it is impossible to discount the statically unstable possibility that $\Delta \tilde{b} \leq 0$.

Despite the motivation of Sandström’s original paper, most recent work on horizontal convection has taken $\tau_s = 0$ so that $\Delta \tilde{b}$ is positive, and the strength of the kinetic energy source is bounded because $\Delta \tilde{b} \leq \tilde{b}_{max}$. Thus one obtains the theorem of Parella & Young (2002), which provides a rigorous foundation for Sandström’s (1908) thermodynamic argument that nonuniform surface buoyancy alone cannot supply significant mechanical energy to the ocean circulation; see also McIntyre (2009) and Nycander (2010).

2. A model of stressed horizontal convection

For the numerical simulations we consider a two-dimensional non-rotating Boussinesq fluid in a rectangular box, where the motion is in the $(y, z)$-plane, with the vertical coordinate $-H < z < 0$, and horizontal coordinate $0 < y < L$. Following Beardsley & Festa (1972) and Rossby (1998), we consider horizontal convection in the streamfunction-
vorticity formulation with equations of motion

\[ \nabla^2 \psi_t + J(\psi, \nabla^2 \psi) = b_y + \nu \nabla^4 \psi, \quad (2.1) \]

\[ b_t + J(\psi, b) = \kappa \nabla^2 b. \quad (2.2) \]

Here \( J(p,q) = p_y q_z - p_z q_y \) is the Jacobian, the streamfunction is \( \psi(y, z, t) \), and the incompressible velocity is \( (v, w) = (-\psi_z, \psi_y) \). On the boundary of the domain \( \psi = 0 \).

Mechanical and buoyancy forcing is via the surface boundary conditions

\[ b(y, 0, t) = b_{\text{max}} \cos^2 \left( \frac{\pi y}{2L} \right), \quad \nu v_z(y, 0, t) = -\tau_{\text{max}} \sin \left( \frac{\pi y}{L} \right). \quad (2.3) \]

On the sidewalls and bottom there is no flux of buoyancy and no stress. In (2.3), \( \tau_{\text{max}} > 0 \) so that the surface stress drives a counterclockwise circulation in the box. This is opposite in direction to the clockwise circulation forced by the surface buoyancy. We refer to this sense of mechanical forcing as “thermally indirect”, meaning that buoyant fluid near \( y = 0 \) is being pushed downwards by the stress and the dense fluid at \( y = L \) is elevated.

The surface buoyancy decreases smoothly and monotonically from \( b = b_{\text{max}} \) at \( y = 0 \) to \( b = 0 \) at \( y = L \).

2.1. The control parameters

As a dimensionless measure of the strength the surface mechanical forcing we introduce the parameter

\[ S \overset{\text{def}}{=} \frac{\tau_{\text{max}} L^2}{\nu^2}. \quad (2.4) \]

The other three dimensionless parameters specifying this problem are familiar from earlier studies of unstressed horizontal convection. In the notation of Chiu-Webster et al. (2008), these are the aspect ratio, the horizontal Rayleigh number and the Prandtl number:

\[ A \overset{\text{def}}{=} \frac{H}{L}, \quad R \overset{\text{def}}{=} \frac{b_{\text{max}} L^3}{\nu^2}, \quad \sigma \overset{\text{def}}{=} \frac{\nu}{\kappa}. \quad (2.5) \]

2.2. Discussion of two representative solutions

Figure 1 shows snapshots of two numerical solutions. The steady flow in panel (a) is unstressed horizontal convection. The main features in the upper panel, such as the thin surface boundary layer and the almost unstratified abyss, are familiar from many earlier studies. In this solution the average bottom buoyancy is \( 0.12b_{\text{max}} \), which is considerably less than the mean surface buoyancy \( b_{\text{max}}/2 \).

Panel (b) shows the more complicated unsteady flow resulting from thermally indirect stress. It takes at least one vertical diffusion time, \( \kappa/H^2 \), to reach a statistically steady state. There is a shallow stress-driven counter-clockwise cell and a second deep clockwise cell. Animations show that the deep cell is associated with pulses of dense fluid that periodically fall along the right hand wall, beneath the densest part of the top surface. These pulses hit the bottom, turn the corner, and establish an unsteady bottom current flowing from \( y = L \) towards \( y = 0 \). The flow in the upper left quadrant of the domain is steady.

A two-celled circulation is also evident in the relatively low-Rayleigh number solutions shown in Figure 6 of Beardsley & Festa (1972). The two-cell circulation requires intermediate values of \( \tau_{\text{max}} \) so that the stress is strong enough to reverse the surface velocity, \( v_s = v(y, 0, t) \), but is not so strong as to overpower the buoyancy-driven circulation throughout the domain. A more quantitative estimate of the requisite \( \tau_{\text{max}} \) is given below in the discussion surrounding (3.5).
In Figure 1(b), \( v_s \) is in the same direction as the applied stress, so that the stress is doing positive work on the fluid. Using the surface stress in (2.3), the stress work in the power integral (1.9) is

\[
SW = -\tau_{\text{max}} v_s \sin \left( \pi y/L \right).
\]  

In Figure 1(b) the abyssal fluid is more buoyant than in panel (a): stress increases the abyssal buoyancy by pumping buoyant fluid downwards in the upper cell. In fact, the average bottom buoyancy is 0.55\( b_{\text{max}} \), so that in (1.9) \( \Delta \bar{b} = -0.05b_{\text{max}} \); because \( \Delta \bar{b} < 0 \) energy is being converted from kinetic to potential. Thus the positive SW balances the dissipation \( \varepsilon \), and SW also provides the net production of available potential energy required by \( \Delta \bar{b} < 0 \).

The solutions in Figure 1 show the unstressed case in panel (a), and in panel (b) a case with moderately strong stress in which there is a top-to-bottom inversion of the density. Figure 2(a), showing the horizontally averaged buoyancy \( \bar{b}(z) \), summarizes a suite of solutions in which \( \tau_{\text{max}} \) varies between the extremes shown in Figure 1. The abyssal buoyancy increases monotonically with \( \tau_{\text{max}} \); very small \( \tau_{\text{max}} \) produces small density inversions confined to the upper cell. At a particular value of \( \tau_{\text{max}} \) — which is less than the value in Figure 1(b) — the average bottom buoyancy is equal to \( b_{\text{max}}/2 \), so that \( \Delta \bar{b} = 0 \).

If \( \tau_{\text{max}} \) is increased past the value at which \( \Delta \bar{b} = 0 \), as it is in Figure 1(b), then there is a top-to-bottom density inversion, even though the squared buoyancy frequency,

\[
N^2 \equiv \bar{b}_z,
\]  

is negative only in the relatively small upper cell: see figure 2(b).

Increases in \( \tau_{\text{max}} \) lead to the point where the buoyant “blob” evident in the top left corner of Figure 1(b) is pushed down to the bottom. At this threshold, the lower thermally direct cell collapses and the circulation is thermally indirect everywhere i.e., the stress wins. Except at the top boundary, where a nonuniform buoyancy is prescribed, the buoyancy is homogenized to around 0.65\( b_{\text{max}} \). In this mechanically dominated regime the buoyancy is almost a passive scalar and the top boundary layer becomes very thin, so that it is difficult to reliably compute. Figure 1(b) shows the largest \( \tau_{\text{max}} \) at which we can afford the requisite numerical resolution. The stress work SW has an interesting dependence on \( \tau_{\text{max}} \). Small \( \tau_{\text{max}} \) does not manage to reverse the sign of \( v_s \) and therefore the surface flow is against the direction of the stress, so that \( SW < 0 \). That is, the first effect of thermally indirect surface stress is to make the circulation weaker. However increasing \( \tau_{\text{max}} \) eventually reverses \( v_s \) so that SW becomes positive. The transition can be identified precisely by the condition that \( SW = 0 \), which happens at a smaller value of \( \tau_{\text{max}} \) than the full-depth inversion signaled by \( \Delta \bar{b} = 0 \). In section 3 we use a scaling arguments to more quantitatively delineate the occurrence of the two transitions \( SW = 0 \) and \( \Delta \bar{b} = 0 \).

2.3. Remarks on the numerical solution

The system in (2.1) and (2.2) was solved using the same code as Paparella & Young (2002). The numerical representation of vorticity and temperature is a second-order finite difference in space with a staggered grid. The Jacobian terms are discretized with the Arakawa Jacobian formulation (Arakawa 1966) and the elliptic problem for the streamfunction with a given vorticity is solved with a multigrid method (Briggs 1987). The Laplacian terms are based on the DuFort-Frankel discretization (DuFort & Frankel 1953) which is a stable and relatively accurate scheme as long as the time step obeys the CFL-condition. In most presented simulations the CFL-number is less than 0.05. When the spatial resolution appears to be too coarse for the features in the simulated flow, runs
Figure 2. The left panel shows horizontally averaged buoyancy profiles, $\bar{b}(z)$ and the right panel the buoyancy frequency in (2.7). The Rayleigh number is $R = 64 \times 10^7$ with various values of $S$. The end points $S = 0$ and $9.6 \times 10^5$ are the solutions in Figure 1. The parameter in the legend is $S_\delta$ defined (3.5).

3. Scaling arguments

3.1. Rossby scaling

In the unstressed case, with $S = 0$, Rossby (1965)’s scaling argument provides a satisfactory condensation of all known numerical solutions e.g., Siggers et al. (2004); Chiu-Webster et al. (2008); Ilicak & Vallis (2011). The first step in Rossby’s argument is assuming that the typical vertical variation of buoyancy within the surface boundary layer is $b_{\text{max}}$. If the dominant boundary-layer balance in (2.1) is between $b_y$ and the vertical viscosity $\nu \psi_{zzzz}$, then one has

$$V = \frac{b_{\text{max}} \delta^3}{\nu L}, \quad (3.1)$$

where $\delta$ is the boundary layer thickness, and $V$ is the typical horizontal velocity in the boundary layer. A second scaling relation comes from a dominant balance in the buoyancy equation (2.2) between the advective terms, which scale as $V b_{\text{max}}/L$, and the vertical...
diffusion scaling as $\kappa b_{\text{max}}/\delta^2$. Thus one has

$$V = \frac{L\kappa}{\delta^2}. \quad (3.2)$$

From (3.1) and (3.2), Rossby (1965) obtains the boundary layer thickness

$$\delta = \frac{L}{R^{1/5}}, \quad (3.3)$$

and the boundary layer velocity

$$V = \frac{\kappa}{L} R^{2/5}. \quad (3.4)$$

It is instructive to estimate these Rossby scales using oceanographic parameter values (Stern 1975). Suppose that $b_{\text{max}} = 5 \times 10^{-2}$ m s$^{-2}$, corresponding to a temperature difference of 25K and a thermal expansion coefficient of $2 \times 10^{-4}$ K$^{-1}$. For the horizontal dimension, use the planetary scale $L = 10^7$ m, and molecular parameters $\kappa = 10^{-7}$ m$^2$ s$^{-1}$ and $\nu = 10\kappa$. Then $R = 5 \times 10^{22}$, and the Rossby boundary-layer scales are $\delta = 2.9$m and $V = 0.12$m s$^{-1}$. If instead one uses eddy diffusion and viscosity with $\nu = \kappa = 10^{-4}$m s$^{-2}$ then the Rayleigh number drops to $R = 5 \times 10^{27}$, and the implied Rossby scales are $\delta = 29$m and $V = 1.2$m s$^{-1}$. 

3.2. The effect of stress

In order for the surface stress to be competitive with the buoyancy forcing, one must have $\tau_{\text{max}} \sim \nu V/\delta$, so that the magnitude of the imposed surface stress is comparable to the stress across Rossby’s purely buoyancy boundary layer. In terms of control parameters, this condition is $S \sim \sigma^{-1} R^{3/5}$. This motivates the introduction of the non-dimensional parameter

$$S_\delta \overset{\text{def}}{=} \frac{\sigma S}{R^{3/5}}. \quad (3.5)$$

If $S_\delta \ll 1$ then the stress $\tau_{\text{max}}$ is only a weak perturbation of the buoyancy-driven boundary layer, while if $S_\delta \gg 1$ then the flow is strongly mechanically forced. Notice that the solutions summarized in Figure 2 have $S_\delta$ of order unity.

3.3. Static instability in the upper boundary layer

In the solutions shown in Figures 1 and 2, the stress is dragging dense fluid over light and inducing a statically inverted buoyancy field. This density inversion can remain stable if the local vertical Rayleigh number within Rossby’s purely buoyancy boundary layer is sufficiently small. One can estimate the relevant boundary-layer Rayleigh number as $b_{\text{max}} \delta^3/\nu \kappa \sim R^{2/5}$. This motivates the definition of a boundary-layer Rayleigh number

$$R_\delta \overset{\text{def}}{=} R^{2/5}. \quad (3.6)$$

If the mechanical stress is thermally indirect, with $S_\delta = 0(1)$, and $R$ is increased sufficiently then experience with Rayleigh-Bénard convection suggest that the static inversion should trigger convection once $R_\delta > 10^3$. The solutions at $R = 64 \times 10^7$ and $64 \times 10^8$ have $R_\delta = 3330$ and 8365 respectively, yet there is no indication of convection within the boundary layer in any of our simulations. Either $R_\delta = 8365$ is too small, or the shear across the boundary layer, $V/\delta \sim \kappa R^{3/5}/L^2$, is suppressing convection.

To assess the possible role of boundary-layer shear† we compare the shear time scale

† In a three-dimensional situation, convection rolls parallel to the $y$-axis, with overturning orthogonal to the plane of Figure 1, would be unaffected by the boundary-layer shear $v_z$, and are therefore the most likely mode of instability.
Figure 3. Buoyancy and streamfunction of three solutions with $R = 64 \times 10^7$, $\sigma = 10$ and $A = 4$. The parameter in (3.7) is $S_\delta / \sqrt{\sigma} = 0.39$, 0.65 and 1.05 in panels (a), (b) and (c) respectively.

Convection might occur if the parameter above is sufficiently small i.e., if parcels can fall through the boundary layer before being sheared into oblivion. On the other hand, to invert the boundary-layer buoyancy requires that $S_\delta$ be sufficiently large. With $\sigma = 1$ these two requirements cannot both be satisfied i.e., if the shear is strong enough to
invert the boundary-layer buoyancy, it is also strong enough to stabilize the inversion by preventing convective overturning in the plane of Figure 1.

These consideration led us to obtain solutions with $\sigma = 10$ so that $S_\delta$ can be significantly larger than one, while $S_\delta/\sqrt{\sigma}$ is less than one. Then, according to (3.7), the inverted boundary-layer buoyancy might result in convection within the boundary layer. Figure 3 shows the three such solutions, and indeed unsteady boundary-layer convection is evident in Figure 3(b).

It is remarkable that convection occurs only in the intermediate case in Figure 3(b): the other solutions in Figure 3 are steady. In panel (a) the stress is weak and the thermally indirect upper cell is undeveloped. Consequently the buoyancy inversion is not strong enough to result in boundary-layer convection. In panel (c) the strong stress produces a complex, but steady, pattern with two co-rotating cells within the boundary layer. Although the buoyancy inversion is strong, the flow in panel (c) is steady.

Animations of the solution in Figure 3(b) show that the thermally indirect flow in the upper cell produces an inversion which steadily becomes stronger till convective plumes suddenly appear and release the potential energy. Discharging the top-heavy inversion and quenching the convection requires several plumes to form and then fall through the boundary layer. As shown by the time series of potential energy in Figure 4 this process repeats cyclically so that there are epochs of convection followed by epochs during which the inversion is re-established: see Figure 5. Both stages are slow: a gradual build-up of the stress-driven inversion, followed by a slow discharge via a sequence of plumes.

4. Verification of the scaling

With the Rossby scaling relations we can collapse the results of computations in the range $64 \times 10^5 \leq R \leq 64 \times 10^8$ and $0 < S_\delta < 5.8$. In Figure 6(a) we confirm the scaling argument from section 3 by showing that the vertical coordinate $z/\delta$ collapses $\bar{b}(z)$ profiles at fixed values of $S_\delta$.

Figure 6(b) shows the horizontally averaged bottom buoyancy $\bar{b}(-H)$; the bottom buoyancy increases monotonically with increasing $S$ at fixed $R$. At fixed $S_\delta$, $\bar{b}(-H)$ decreases slowly with increasing $R$ and seems to approach a nonzero constant as $R \to \infty$; we have not been able to obtain a satisfactory scaling for the dependence of $\bar{b}(-H)/b_{\text{max}}$ on $R$. In figure 6(b) we see that $\Delta \bar{b} = 0.5b_{\text{max}}$ (i.e., $\Delta \bar{b} = 0$) when $S_\delta$ is in the range four to five (the value depends on $R$). In the sequence with $R = 64 \times 10^8$, the flow is weakly unsteady for all solutions with $S_\delta \geq 3$ and $\bar{b}(-H)$ is never larger than the mean top buoyancy $\bar{b}(0) = 0.5b_{\text{max}}$. This leads us to speculate that as $R \to \infty$, with $S_\delta$ fixed at
Figure 5. Panel (a) is a snapshot of the solution at the first peak of the potential-energy time series in Figure 4. Panel (b) is a snapshot at the minimum PE immediately after the snapshot in panel (a). The top right quarter of the domain in Figure 4 is shown, and the color scale has been changed to better show structure in the boundary layer. Eight plumes fall through the boundary layer between the two snapshots.

A largeish value such as five or six, that the bottom buoyancy \( \tilde{b}(-H) \) saturates at \( b_{max}/2 \), so that \( \Delta \tilde{b} = 0 \). In other words, at very high horizontal Rayleigh number it is impossible for surface stress to coerce a top-to-bottom density inversion. Via (1.8), this scenario also entails a shutdown of the conversion between available potential energy and kinetic energy.

As an index of the strength of horizontal convective heat transport, Paparella & Young (2002) introduced

\[
\Phi \equiv \frac{\langle \nabla b \cdot \nabla b \rangle}{\langle \nabla c \cdot \nabla c \rangle},
\]

where \( c(x) \) is the solution of the conduction problem \( \nabla^2 c = 0 \), with \( c \) satisfying the same boundary conditions as \( b \), i.e. \( c = b \) at the top surface and the normal derivative of \( c \) is zero at the other boundaries. Figure 6c shows \( \Phi \) for various \( R \) against \( S_\delta \). For the unsteady solutions, temporal pulsations in \( \Phi \) are removed by averaging over half a diffusive time. For all \( R \) there is an initial decrease in \( \Phi \) for increasing \( S_\delta \). At \( S_\delta \approx 2 \) there is a minimum in \( \Phi \) and as \( S_\delta \) increases past 2, the index \( \Phi \) increases. The behaviour observed in both \( \tilde{b}(-H) \) and \( \Phi \) is qualitatively similar to figures 7 and 9 of Beardsley & Festa (1972); those \( 32 \times 32 \) solutions were at \( R = 10^4 \) in a square container.

The effect of the surface stress, and the strongest evidence supporting the earlier scaling arguments, is in the surface velocity \( v_s \). In figure 6(d) we plot the scaled mean surface velocity \( R^{-2/5} \bar{v}_s \) against \( S_\delta \); there is good collapse of the data and \( \bar{v}_s \) reverses at \( S_\delta \approx 1 \).
for all $R$. The weakest horizontal convection, as indicated by the minimum in $\Phi$ shown in Figure 6c, is found at around $S_\delta \approx 2$ i.e., with thermally indirect $\bar{v}_s$.

5. Conclusion and discussion

The addition of steady, thermally indirect stress forcing to the problem of horizontal convection qualitatively changes the structure of the flow by inducing a two-celled circulation as in Figure 1(b). Rather small values of $\tau_{\text{max}}$ have a large effect. For example, using the numerical values leading to $R = 5 \times 10^{32}$ in the discussion after (3.4), one finds that $S_\delta = 1$ corresponds to $\tau_{\text{max}} = 4 \times 10^{-8} \text{m}^2\text{s}^{-2}$. The stress is then $\rho_0 \tau_{\text{max}} = 4 \times 10^{-5} \text{N m}^{-2}$, which is smaller than typical atmospheric values, 0.1 N m$^{-2}$, by a factor of over one thousand. If instead one uses eddy viscosity and diffusivity, so that $R = 5 \times 10^{27}$, then $S_\delta = 1$ corresponds to a stress $\rho_0 \tau_{\text{max}} = 4 \times 10^{-3} \text{N m}^{-2}$, which is still very significantly smaller than observed values.

Regarding the oceanographic implications of these results, the problem of horizontal convection is best thought of as an idealized and instructive thought experiment applying the “Sandström Ocean” rather than the “real ocean”. Like the real ocean, Sandström’s ocean is forced by stress and buoyancy only at the top surface. We have shown here that with steady stress forcing the Sandström Ocean exhibits the shallow wind-driven cell which is responsible for most of the heat transport in the real ocean (Talley 2003).

The numerical estimates made in this paper suppose that Rossby’s scaling remains
valid as $R \to \infty$ with $S_\delta$ fixed and order unity. The resulting very small values of stress $\tau_{\text{max}}$ and boundary layer thickness $\delta$ indicate that Sandström’s ocean is very different from the real ocean. This is the case even if one adopts substantial eddy viscosity and diffusivity (without specifying the energy source required to support this mixing). A related difference between the Sandström Ocean and the ocean is the relative lack of abyssal stratification in Figure 2(b). With applied surface stress, either steady or oscillatory, there is no energetic principle preventing the generation of shallow turbulence, vertical radiation of internal gravity waves, wave breaking, abyssal mixing and the development of deep stratification. Presumably this chain of events, which may require three-dimensional dynamics, will destroy Rossby scaling so that the boundary layer is realistically thick and the abyss is significantly stratified even in Sandström’s Ocean. We have shown that surface stress makes a considerable difference to the structure of the flow. But with $R = O(10^8)$ we have not been able to access this hypothetical regime in which Sandström’s ocean is self-mixing and the abyssal stratification is significant; see also Ilicak & Vallis (2011).

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