An Exact Thickness-Weighted Average Formulation of the Boussinesq Equations

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ABSTRACT

The author shows that a systematic application of thickness-weighted averaging to the Boussinesq equations of motion results in averaged equations of motion written entirely in terms of the thickness-weighted velocity; that is, the unweighted average velocity and the eddy-induced velocity do not appear in the averaged equations of motion. This thickness-weighted average (TWA) formulation is identical to the unaveraged equations, apart from eddy forcing by the divergence of three-dimensional Eliassen–Palm (EP) vectors in the two horizontal momentum equations. These EP vectors are second order in eddy amplitude and, moreover, the EP divergences can be expressed in terms of the eddy flux of the Rossby–Ertel potential vorticity derived from the TWA equations of motion. That is, there is a fully nonlinear and three-dimensional generalization of the one- and two-dimensional identities found by Taylor and Bretherton. The only assumption required to obtain this exact TWA formulation is that the buoyancy field is stacked vertically; that is, that the buoyancy frequency is never zero. Thus, the TWA formulation applies to nonrotating stably stratified turbulent flows, as well as to large-scale rapidly rotating flows. Though the TWA formulation is obtained by working on the equations of motion in buoyancy coordinates, the averaged equations of motion can then be translated into Cartesian coordinates, which is the most useful representation for many purposes.

1. Introduction

After averaging over 10-m scales, the stratification of the ocean is strongly statically stable and the circulation is nearly adiabatic. Physical oceanographers have therefore argued that mesoscale eddies mostly flux buoyancy and passive scalars along (but not through) mean buoyancy surfaces. This is equivalent to saying that the eddy transport of buoyancy is represented as an eddy-induced (or bolus) velocity (Rhines 1982; Gent and McWilliams 1990; Gent et al. 1995; McDougall and McIntosh 1996, 2001; Treguier et al. 1997; Griffies 1998; Greatbatch 1998; Plumb and Ferrari 2005). The sum of the eddy-induced velocity and the mean velocity is the residual velocity. It is the residual velocity that effectively advects large-scale tracers. A main preoccupation of ocean modelers in the 20 years since Gent and McWilliams (1990) has been devising and testing parameterizations expressing the eddy-induced velocity in terms of the large-scale density field (e.g., Killworth 1997; Visbeck et al. 1997; Aiki et al. 2004; Cessi 2008; Ferrari et al. 2010).

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An alternative to parameterization of the eddy-induced velocity is to formulate the large-scale ocean-circulation problem completely in terms of the residual velocity: that is, by formulating a residual-mean momentum equation. If one can use the residual velocity as a prognostic variable and abolish mention of the eddy-induced velocity and the mean velocity, then parameterization in the buoyancy equation is unnecessary. Instead, the parameterization problem is moved to the momentum equations, where it belongs.

This prospect motivated Ferreira and Marshall (2006) to pursue a formulation of the large-scale averaged equations of motion using the residual-mean velocity instead of the mean velocity and the eddy-induced velocity. These authors work in Cartesian coordinates using the transformed Eulerian mean (TEM) introduced by Andrews and McIntyre (1976) and the vector streamfunction of Treguier et al. (1997). To cast the equations of motion entirely in terms of the residual velocity, Ferreira and Marshall use a number of idealizations and approximations (such as small Rossby number) and parameterize eddies in the momentum equation as vertical viscosity (e.g., Rhines and Young 1982; Greatbatch and Lamb 1990; Greatbatch 1998). There are conceptual advantages to divorcing the momentum-equation parameterization problem from the approximations employed by Ferreira and Marshall to derive a residual-mean system. Finalizing the divorce by systematically deriving a totally residual-mean formulation of the Boussinesq primitive equations is the goal of this article.

The key step is averaging the equations of motion in buoyancy[1] coordinates, using an average weighted by the ''isopycnal thickness.'' We refer to this as the thickness-weighted average (TWA) formulation. The resulting exact description assumes neither small-isopycnal slope, rapid rotation, weak eddies, nor small diabatic effects. For example, the TWA formulation applies equally well to nonrotating fluids, provided only that the stratification is stable.

With hindsight, some of the ingredients in the TWA formulation (e.g., the definitions of $b^\#$ and $w^\#$ below) are already contained in de Szoeke and Bennett (1993), Smith (1999), and Greatbatch and McDougall (2003). A main point of de Szoeke and Bennett is that the Osborn–Cox relation between diabatic density flux and molecular dissipation actually provides the diapycnal (rather than vertical) flux of density (see also Winters and D'Asaro 1996). This is a second potent reason for using the TWA formulation.

In section 2, we review the kinematic problem of transforming from Cartesian coordinates (x, y, z, t) to buoyancy coordinates $(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t})$. In this framework the depth of a buoyancy surface, $z = \zeta(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t})$, is an independent variable and

$$\sigma \stackrel{\text{def}}{=} \zeta_{\tilde{b}} \qquad (1)$$

is the isopycnal ''thickness.'' Some new formulas providing the b-coordinate representation of gradient, divergence, and curl are obtained: (53) is particularly useful. In section 3, we review the thickness-weighted average, which is used to define the horizontal components of the residual velocity as

$$(\hat{u}, \hat{v}) \stackrel{\text{def}}{=} (\overline{\sigma u}, \overline{\sigma v})/\overline{\sigma} \qquad (2)$$

(Andrews 1983; de Szoeke and Bennett 1993). The overbar above denotes an ensemble average in buoyancy coordinates over realizations of the eddies. The third component of the three-dimensional incompressible residual velocity $\mathbf{u}^\#$ is not the thickness-weighted average $\hat{w}$: instead, using the standard Cartesian basis vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, the nondivergent residual velocity is $\mathbf{u}^\# = \hat{u}\mathbf{i} + \hat{v}\mathbf{j} + w^\#\mathbf{k}$; the vertical component $w^\#$ is defined in terms of the average depth of an isopycnal surface $\overline{\zeta}(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t})$ by (73). The ''averaging identities'' (72), (80), and (83) are key results in section 3.

Sections 5 and 6 turn to dynamics by starting with the hydrostatic equations of motion, written in b coordinates. After a thickness-weighted average, the equations of motion are transformed into Cartesian coordinates, (x, y, z, t). In the adiabatic case, this results in the Cartesian coordinate TWA system:

$$\hat{u}_t + \hat{u}\hat{u}_x + \hat{v}\hat{u}_y + w^\#\hat{u}_z - f\hat{v} + p^\#_x + \nabla\cdot\mathbf{E}^u = 0, \qquad (3)$$

$$\hat{v}_t + \hat{u}\hat{v}_x + \hat{v}\hat{v}_y + w^\#\hat{v}_z + f\hat{u} + p^\#_y + \nabla\cdot\mathbf{E}^v = 0, \qquad (4)$$

$$p^\#_z = b^\#, \qquad (5)$$

$$\hat{u}_x + \hat{v}_y + w^\#_z = 0, \qquad (6)$$

$$b^\#_t + \hat{u}b^\#_x + \hat{v}b^\#_y + w^\#b^\#_z = 0. \qquad (7)$$

The variables $p^\#$, $b^\#$, and $w^\#$ are defined in terms of the mean depth of buoyancy surface, $\overline{\zeta}(\tilde{x}, \tilde{y}, b, \tilde{t})$ [e.g., as in (59) and (73)]. The field $b^\#(x, y, z, t)$ is equal to the value of the buoyancy surface whose average depth is z.

The eddy forcing of the TWA system above is confined to the horizontal momentum equations and is via the divergence of the three-dimensional Eliassen–Palm (EP) vectors $\mathbf{E}^u$ and $\mathbf{E}^v$, defined in (124) and (125). These EP vectors are second-order in eddy amplitude and there is a three-dimensional generalization of Andrews's (1983) finite-amplitude zonal-mean EP theorem.

If the superscripts ^ and # are dropped, then, apart from the EP divergences $\nabla\cdot\mathbf{E}^u$ and $\nabla\cdot\mathbf{E}^v$, the TWA system (3)–(7) is identical to the primitive equations. Thus, the eddy parameterization problem devolves to relating the EP divergences to residual-mean quantities so that (3)–(7) is closed. Parameterization is not a main focus of this article. However, an important clue is provided by the relation between the divergence of the EP vectors and the eddy flux of the relevant form of Rossby–Ertel potential vorticity (PV), which is

$$\Pi^\# = \hat{u}_z b^\#_y - \hat{v}_z b^\#_x + (f + \hat{v}_x - \hat{u}_y)b^\#_z. \qquad (8)$$

Specifically, in the adiabatic case

$$\Pi^\#_t + \hat{u}\Pi^\#_x + \hat{v}\Pi^\#_y + w^\#\Pi^\#_z + \nabla\cdot\mathbf{F}^\# = 0, \qquad (9)$$

where the eddy PV flux is

[1] We use the Boussinesq approximation with a linear equation of state. The buoyancy b is defined in terms of the density ρ as $b \stackrel{\text{def}}{=} g(\rho_0 - \rho)/\rho_0$, where ρ_0 is the constant bulk density of the ocean. Thus, buoyancy coordinates are essentially the same as isopycnal coordinates.

$$\mathbf{F}^\# = (b_z^\#\mathbf{i} - b_x^\#\mathbf{k})\nabla\cdot\mathbf{E}^\upsilon - (b_z^\#\mathbf{j} - b_y^\#\mathbf{k})\nabla\cdot\mathbf{E}^u. \tag{10}$$

Notice that $\mathbf{F}^\#\cdot\nabla b^\# = 0$ so that the eddy PV flux $\mathbf{F}^\#$ lies in a $b^\#$ surface. Taking the dot product of $\mathbf{F}^\#$ with $\mathbf{i}$ and $\mathbf{j}$ expresses the EP divergences as components of the PV flux; thus one can write the horizontal momentum equations (3) and (4) as

$$\hat{u}_t + \hat{u}\hat{u}_x + \hat{\upsilon}\hat{u}_y + w^\#\hat{u}_z - f\hat{\upsilon} + p_x^\# - \mathbf{F}^\#\cdot\mathbf{j}/b_z^\# = 0 \tag{11}$$

and

$$\hat{\upsilon}_t + \hat{u}\hat{\upsilon}_x + \hat{\upsilon}\hat{\upsilon}_y + w^\#\hat{\upsilon}_z + f\hat{u} + p_y^\# + \mathbf{F}^\#\cdot\mathbf{i}/b_z^\# = 0. \tag{12}$$

The results in (10)–(12) provide a three-dimensional and fully nonlinear generalization of the identities discovered by Taylor (1915) and Bretherton (1966); for a historical review,[2] see Dritschel and McIntyre (2008).

Earlier three-dimensional generalizations of EP fluxes also introduce two vectors analogous to $\mathbf{E}^u$ and $\mathbf{E}^\upsilon$ above. These three-dimensional EP formulations include the quasigeostrophic approach of Plumb (1986), the thickness-weighted average approach of Lee and Leach (1996), and the TEM-based approach of Gent and McWilliams (1996). The system in (3)–(7) is simpler and more exact than these antecedents—simpler because in the TWA formulation there is only one velocity $\mathbf{u}^\#$. The main thrust of Gent and McWilliams (1996), Lee and Leach (1996), and Plumb and Ferrari (2005) is to advect the unweighted average velocity (i.e., $\overline{u}$) by the residual velocity $\mathbf{u}^\#$. By contrast, in (3)–(7) the residual velocity is advected by the residual velocity and the unweighted mean velocity does not appear.

2. Buoyancy coordinates: Kinematics

The main results in this work are obtained by transforming the equations of motion to buoyancy coordinates, averaging in buoyancy coordinates, and then moving back to Cartesian coordinates. An alternative formulation, avoiding the intermediate introduction of buoyancy coordinates, is provided by Jacobson and Aiki (2006).

Although the transformation of the equations of motion to buoyancy coordinates is standard (e.g., Starr 1945; de Szoeke and Bennett 1993; Griffies 2004), the TWA formulation in section 5 requires some results that go beyond the isopycnic formalism used by earlier authors. To systematically introduce this material, we begin by reviewing the transition from Cartesian coordinates to buoyancy coordinates. The key new result needed in section 5 is contained in the material surrounding Eqs. (52)–(54).

A point in space is located with $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the usual unit vectors aligned with right-handed Cartesian coordinates. Using this basis, the velocity of a fluid can be represented as

$$\mathbf{u} = u\mathbf{i} + \upsilon\mathbf{j} + w\mathbf{k}. \tag{13}$$

Within the Boussinesq approximation

$$\nabla\cdot\mathbf{u} = 0, \tag{14}$$

where $\nabla\cdot$ is the three-dimensional coordinate-invariant divergence operator.

It is convenient to write the density as $\rho = \rho_0(1 - g^{-1}b)$, where $b(\mathbf{x}, t)$ is the buoyancy. We suppose that b is almost materially conserved,

$$b_t + ub_x + \upsilon b_y + wb_z = \varpi. \tag{15}$$

The right of (15) represents small diabatic effects: for example, for diffusion, $\varpi = \kappa\nabla^2 b$. It is instructive to consider the coevolution of a passive scalar $c(\mathbf{x}, t)$ satisfying

$$c_t + uc_x + \upsilon c_y + wc_z = \gamma. \tag{16}$$

On the right of (16), γ denotes diabatic terms.

An essential assumption is that the buoyancy $b(\mathbf{x}, t)$ remains statically stable and "stacked"; that is, there is a monotonic relation between b and z. This assumption requires the "double averaging" procedure described by de Szoeke and Bennett (1993): the stacked field $b(\mathbf{x}, t)$ used here is obtained by first averaging the exact buoyancy field over scales of a few meters so that transient small-scale inversions are eliminated. In section 5, we further assume that after this averaging the dynamics is hydrostatic.

If there is a monotonic relation between b and z, then one can change coordinates from (x, y, z, t) to $(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t})$, where

$$\tilde{x} = x, \tag{17}$$

$$\tilde{y} = y, \tag{18}$$

$$\tilde{b} = b(x, y, z, t), \tag{19}$$

$$\tilde{t} = t. \tag{20}$$

[2] Dritschel and McIntyre (2008) refer to results like (10) as "Taylor identities." In my opinion, Bretherton's two-dimensional quasigeostrophic generalization deserves recognition alongside the one-dimensional identity of Taylor.

The superscript tilde distinguishes the coordinate labels $(\tilde{x}, \tilde{y}, \tilde{b})$ from fields in physical space. In particular, (19) identifies the particular buoyancy surface labeled by $\tilde{b}$. In the partial derivatives $\partial_{\tilde{x}}$, $\partial_{\tilde{y}}$, and $\partial_{\tilde{t}}$ below, the tilde reminds one that the derivative is "at constant b."

The notation $\tilde{b}$ helps one recognize that the scalar field $b(\mathbf{x}, t)$ is a physical quantity whose isopleths serve as coordinate surfaces. We will be using a curvilinear, nonorthogonal coordinate system $(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t})$, and $\tilde{b}$ surfaces happen to coincide with the physical isopycnals. Buoyancy surfaces are geometric objects existing independently of any coordinate system and, therefore, $\tilde{b}$ surfaces are quite different from surfaces of constant $\tilde{x}$ and $\tilde{y}$. In fact, buoyancy is being described with two different functional representations. One is the scalar field $b(\mathbf{x}, t)$ whose arguments are tied to the Cartesian coordinate system $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The other is a curvilinear representation, using a function $B(\tilde{x}, \tilde{y}, \tilde{b}, t)$ with the "trivial" form $B(\tilde{x}, \tilde{y}, \tilde{b}, t) = \tilde{b}$ (trivial mathematically though not conceptually).

The equations of motion are rewritten in terms of $(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t})$ using the rules

$$\partial_x = \partial_{\tilde{x}} + b_x \partial_{\tilde{b}}, \tag{21}$$

$$\partial_y = \partial_{\tilde{y}} + b_y \partial_{\tilde{b}}, \tag{22}$$

$$\partial_z = b_z \partial_{\tilde{b}}, \tag{23}$$

$$\partial_t = \partial_{\tilde{t}} + b_t \partial_{\tilde{b}}. \tag{24}$$

In buoyancy coordinates the depth of a buoyancy surface, $\zeta(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t})$, is an independent variable. The notation ζ distinguishes the function $\zeta(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t})$ from the value of the function at a particular point in density coordinates. Thus, we write

$$z = \zeta(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t}) \tag{25}$$

rather than $z = z(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t})$: in the latter expression one must hold in mind that z has a different meaning on the two sides of the equation and this is painful at around (60).

The Jacobian of the transformation from (x, y, z) to $(\tilde{x}, \tilde{y}, \tilde{b})$ is

$$\sigma(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t}) \overset{\text{def}}{=} \zeta_{\tilde{b}} \tag{26}$$

$$= 1/b_z, \tag{27}$$

where (27) is obtained by applying the differential operator in (23) to ζ. Thus, the element of volume is $d^3\mathbf{x} = dx\,dy\,dz = \sigma\,d\tilde{x}\,d\tilde{y}\,d\tilde{b}$. The assumption of a stacked buoyancy field ensures that the Jacobian σ is nonzero. We refer to σ as the thickness. The important relations,

$$\zeta_{\tilde{x}} = -\sigma b_x, \quad \zeta_{\tilde{y}} = -\sigma b_y, \quad \text{and} \quad \zeta_{\tilde{t}} = -\sigma b_t, \tag{28}$$

are obtained by applying the differential operators in (21)–(24) to ζ. Using (28), one can alternatively write the derivatives in (21)–(24) as

$$\partial_x = \partial_{\tilde{x}} - \zeta_{\tilde{x}} \sigma^{-1} \partial_{\tilde{b}}, \tag{29}$$

$$\partial_y = \partial_{\tilde{y}} - \zeta_{\tilde{y}} \sigma^{-1} \partial_{\tilde{b}}, \tag{30}$$

$$\partial_z = \sigma^{-1} \partial_{\tilde{b}}, \tag{31}$$

and

$$\partial_t = \partial_{\tilde{t}} - \zeta_{\tilde{t}} \sigma^{-1} \partial_{\tilde{b}}. \tag{32}$$

Isolating w from (15) and using (29)–(32), one has

$$w = \zeta_{\tilde{t}} + u\zeta_{\tilde{x}} + v\zeta_{\tilde{y}} + \varpi\zeta_{\tilde{b}}. \tag{33}$$

Using (29)–(33), the convective derivative,

$$\frac{D}{Dt} \overset{\text{def}}{=} \partial_t + u\partial_x + v\partial_y + w\partial_z, \tag{34}$$

is transformed to buoyancy coordinates as

$$\frac{D}{Dt} = \partial_{\tilde{t}} + u\partial_{\tilde{x}} + v\partial_{\tilde{y}} + \varpi\partial_{\tilde{b}}. \tag{35}$$

Thus, the passive scalar Eq. (16) becomes

$$c_{\tilde{t}} + uc_{\tilde{x}} + vc_{\tilde{y}} + \varpi c_{\tilde{b}} = \gamma \tag{36}$$

The diabatic term ϖ is equivalent to a velocity through buoyancy surfaces.

Taking a z derivative of (33), using $\nabla \cdot \mathbf{u} = 0$ and the rules in (29)–(32), we deduce that

$$\sigma_{\tilde{t}} + (\sigma u)_{\tilde{x}} + (\sigma v)_{\tilde{y}} + (\sigma\varpi)_{\tilde{b}} = 0. \tag{37}$$

The thickness equation (37) is equivalent to mass conservation in buoyancy coordinates.

a. *Basis vectors*

To this point, the development of buoyancy coordinates is broadly familiar to physical oceanographers and meteorologists (Starr 1945; de Szoeke and Bennett 1993; Griffies 2004). However, the full power of the buoyancy coordinates is not fully exploited unless one also understands how vectors and coordinate-invariant differential operators $\nabla\cdot$, $\nabla\times$, and the Laplacian ∇^2 are represented. To accomplish this we use the most elementary aspects of tensor analysis. Thus, we consider the nonorthogonal set of basis vectors

$$\mathbf{e}^1 \stackrel{\text{def}}{=} \mathbf{i}, \quad \mathbf{e}^2 \stackrel{\text{def}}{=} \mathbf{j}, \quad \mathbf{e}^3 \stackrel{\text{def}}{=} \nabla b \tag{38}$$

(e.g., Simmonds 1982). Above, $\mathbf{e}^1$ and $\mathbf{e}^2$ are the usual Cartesian unit vectors, while $\mathbf{e}^3$ is normal to a buoyancy surface. Notice that $(\mathbf{e}^1 \times \mathbf{e}^2) \cdot \mathbf{e}^3 = b_z = \sigma^{-1}$.

In parallel with $\mathbf{e}^j$ one can also introduce the dual basis vectors

$$\mathbf{e}_1 \stackrel{\text{def}}{=} \sigma \mathbf{e}^2 \times \mathbf{e}^3 = \mathbf{i} + \zeta_{\tilde{x}}\mathbf{k}, \tag{39}$$

$$\mathbf{e}_2 \stackrel{\text{def}}{=} \sigma \mathbf{e}^3 \times \mathbf{e}^1 = \mathbf{j} + \zeta_{\tilde{y}}\mathbf{k}, \tag{40}$$

$$\mathbf{e}_3 \stackrel{\text{def}}{=} \sigma \mathbf{e}^1 \times \mathbf{e}^2 = \sigma\mathbf{k}. \tag{41}$$

The vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are tangent to a buoyancy surface, and thus a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$ is a vector "lying in the buoyancy surface." The triple product of this basis set is $(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 = \sigma$, which is the reciprocal of the triple product $(\mathbf{e}^1 \times \mathbf{e}^2) \cdot \mathbf{e}^3$. The set $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is "bi-orthogonal" to $(\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3)$ in the sense that

$$\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j, \tag{42}$$

where δ_j^i is the Kronecker δ.

The differential operators $\partial_{\tilde{x}}$, $\partial_{\tilde{y}}$, and $\partial_{\tilde{b}}$ on the right of (29)–(32) can be written as directional derivatives along the $\mathbf{e}_j$-basis vectors:

$$\partial_{\tilde{x}} = \mathbf{e}_1 \cdot \nabla, \quad \partial_{\tilde{y}} = \mathbf{e}_2 \cdot \nabla, \quad \partial_{\tilde{b}} = \mathbf{e}_3 \cdot \nabla. \tag{43}$$

It turns out that the nonorthogonal set $\mathbf{e}_j$ provides the most useful b-coordinate basis for many purposes.

b. Three representations of a vector field

An arbitrary vector field, $\mathbf{q}(\mathbf{x}, t)$ for example, can be expanded in three different ways:

$$\mathbf{q} = q\mathbf{i} + r\mathbf{j} + s\mathbf{k}, \tag{44}$$

$$= q^1\mathbf{e}_1 + q^2\mathbf{e}_2 + q^3\mathbf{e}_3, \tag{45}$$

$$= q_1\mathbf{e}^1 + q_2\mathbf{e}^2 + q_3\mathbf{e}^3. \tag{46}$$

In tensor analysis, q^j are referred to as the contravariant components of $\mathbf{q}$ and q_j are the covariant components. The component of $\mathbf{q}$ along a basis vector is extracted as

$$q_j = \mathbf{e}_j \cdot \mathbf{q} \quad \text{and} \quad q^j = \mathbf{e}^j \cdot \mathbf{q}. \tag{47}$$

Thus, $\mathbf{q}$ can be written in terms of its Cartesian components q, r, and s as

$$\mathbf{q} = \underbrace{q}_{=q^1}\mathbf{e}_1 + \underbrace{r}_{=q^2}\mathbf{e}_2 + \underbrace{\sigma^{-1}(s - \zeta_{\tilde{x}}q - \zeta_{\tilde{y}}r)}_{=q^3}\mathbf{e}_3 \tag{48}$$

or as

$$\mathbf{q} = \underbrace{(q + s\zeta_{\tilde{x}})}_{=q_1}\mathbf{e}^1 + \underbrace{(r + s\zeta_{\tilde{y}})}_{=q_2}\mathbf{e}^2 + \underbrace{\sigma s}_{=q_3}\mathbf{e}^3. \tag{49}$$

An important result follows from the special case $\mathbf{q} = \mathbf{u}$: using the thickness equation (37), the contravariant representation of $\mathbf{u}$ is

$$\mathbf{u} = u\mathbf{e}_1 + v\mathbf{e}_2 + \sigma^{-1}(\zeta_{\tilde{t}} + \varpi\zeta_{\tilde{b}})\mathbf{e}_3. \tag{50}$$

The vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, defined in (39) and (40), are tangent to a buoyancy surface. Thus the first two terms on the right of (50) provide the part of $\mathbf{u}$ that "lies in a buoyancy surface." The functions u and v do double duty: u and v provide the components of $\mathbf{u}$ along the horizontal Cartesian directions $\mathbf{i}$ and $\mathbf{j}$ and also along the in-b-surface vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. If the flow is steady ($\zeta_{\tilde{t}} = 0$) and adiabatic ($\varpi = 0$), then the final term in (50) is zero and $\mathbf{u}$ lies in a buoyancy surface.

c. Gradient and divergence

A scalar field f can be written as either $f(x, y, z, t)$ or $f(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t})$. In Cartesian coordinates, the gradient is $\nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$. Using (38) and the definition of the basis $\mathbf{e}^j$ in (39)–(41), one has the natural covariant representation of the gradient

$$\nabla f(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t}) = f_{\tilde{x}}\nabla\tilde{x} + f_{\tilde{y}}\nabla\tilde{y} + f_{\tilde{b}}\nabla b,$$
$$= f_{\tilde{x}}\mathbf{e}^1 + f_{\tilde{y}}\mathbf{e}^2 + f_{\tilde{b}}\mathbf{e}^3. \tag{51}$$

Turning to the divergence, if a vector field $\boldsymbol{q}$ is presented in the $\mathbf{e}_j$ basis as

$$\mathbf{q} = q^1\mathbf{e}_1 + q^2\mathbf{e}_2 + q^3\mathbf{e}_3, \tag{52}$$

then the divergence is

$$\nabla \cdot \mathbf{q} = \sigma^{-1}(\sigma q^1)_{\tilde{x}} + \sigma^{-1}(\sigma q^2)_{\tilde{y}} + \sigma^{-1}(\sigma q^3)_{\tilde{b}}. \tag{53}$$

This very handy formula can be verified by noting that $\sigma\nabla \cdot \mathbf{e}_j = \nabla\sigma \cdot \mathbf{e}_j$ and applying standard vector identities to (52). It is instructive to calculate the divergence of $\mathbf{u}$ in (50) using (53) to recover (37).

Later we will crucially need the inverse of (53): the pattern $l_{\tilde{x}} + m_{\tilde{y}} + n_{\tilde{b}}$ signals the introduction[3] of a vector field $\sigma^{-1}(l\mathbf{e}_1 + m\mathbf{e}_2 + n\mathbf{e}_3)$ so that

$$l_{\tilde{x}} + m_{\tilde{y}} + n_{\tilde{b}} = \sigma\nabla\cdot\sigma^{-1}(l\mathbf{e}_1 + m\mathbf{e}_2 + n\mathbf{e}_3). \tag{54}$$

There are oversights in the oceanographic and meteorological literature made by claiming that $l_{\tilde{x}} + m_{\tilde{y}} + n_{\tilde{b}}$ is the divergence of a vector (l, m, n). This is dangerous because the basis in which the vector (l, m, n) is expressed is not stated (the Cartesian basis is implied) and because the various factors of σ in the correct expression (54) are easily overlooked. Further buoyancy coordinate relations, such as expressions for the curl and Laplacian, are in the appendix.

3. The kinematics of averaging

Although the thickness-weighted average is familiar, earlier works have not exhaustively exploited this procedure (Andrews 1983; Gent et al. 1995; Lee and Leach 1996; Treguier et al. 1997; Greatbatch and McDougall 2003). Thus, in this section we review the thickness-weighted average and obtain some new results needed in section 5.

The average of a field $\theta(\tilde{x},\tilde{y},\tilde{b},\tilde{t})$ is denoted by $\overline{\theta}(\tilde{x},\tilde{y},\tilde{b},\tilde{t})$. We insist that the average is a linear projection operator. This means that

$$\overline{\overline{\theta}} = \overline{\theta} \tag{55}$$

and

$$\overline{\overline{\theta}\phi} = \overline{\theta}\,\overline{\phi}. \tag{56}$$

We also require that the average commutes with derivatives with respect to $(\tilde{x},\tilde{y},\tilde{b},\tilde{t})$. For example,

$$\overline{\partial_{\tilde{x}}\theta} = \partial_{\tilde{x}}\overline{\theta} \quad \text{and} \quad \overline{\partial_{\tilde{t}}\theta} = \partial_{\tilde{t}}\overline{\theta}, \quad \text{etc.} \tag{57}$$

It is safest to think of this overbar as an ensemble average: space and time filters will usually only approximately satisfy the three essential conditions in (55)–(57) (Davis 1994).

The averaging operation introduced above is conducted in buoyancy coordinates. For example, to calculate the average of buoyancy $b(\boldsymbol{x}, t)$, we write buoyancy in buoyancy coordinates, as in (19), and therefore

$$\overline{b(\mathbf{x},t)} = \overline{\tilde{b}} = \tilde{b} = b(\mathbf{x},t). \tag{58}$$

Thus, buoyancy itself is unaffected by averaging. This emphasizes that the average of a field represented in buoyancy coordinates is not equal to the average of the same field represented in Cartesian coordinates (Smith 1999; Jacobson and Aiki 2006). A most important mean field in the TWA formulation is the mean depth of an isopycnal, $\overline{\zeta}(\tilde{x},\tilde{y},\tilde{b},\tilde{t})$, and $\overline{\sigma} = \overline{\zeta}_{\tilde{b}}$ is the mean thickness.

a. Returning to Cartesian coordinates

Although the average of θ is defined using the buoyancy coordinate representation of θ, given $\overline{\theta}(\tilde{x},\tilde{y},\tilde{b},t)$ one can return to the Cartesian representation. de Szoeke and Bennett (1993) make this transition by inverting the relation $z = \overline{\zeta}(\tilde{x},\tilde{y},\tilde{b},\tilde{t})$ to obtain a field $b = b^\sharp(x, y, z, t)$. In other words,

$$\tilde{b} = b^\sharp(x,y,\overline{\zeta}(\tilde{x},\tilde{y},\tilde{b},\tilde{t}),t) \tag{59}$$

and

$$z = \overline{\zeta}(\tilde{x},\tilde{y},b^\sharp(x,y,z,t),\tilde{t}). \tag{60}$$

It is $b^\sharp$ that serves as the buoyancy variable in the TWA formulation.

To understand $b^\sharp$, consider an Eulerian observer E at a fixed position $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Here E is always at the mean depth of some buoyancy surface, and from (60) that surface is $\tilde{b} = b^\sharp(x,y,z,t)$.

The analog of (27) is

$$\overline{\sigma} = \overline{\zeta}_{\tilde{b}} = 1/b^\sharp_z. \tag{61}$$

To prove (61), one simply takes the z derivative of (60). Likewise, one can verify that results such as (28) apply to averaged variables provided that ζ and σ are replaced by $\overline{\zeta}$ and $\overline{\sigma}$ and b is replaced by $b^\sharp$.

With the exception of the passive scalar equation (16), all important results from section 2 can be averaged simply by appropriately decorating the variables. That is, we are not troubled by eddy correlations until we consider the averaged passive-scalar equation in (89) below.

For example, the vectors $\overline{\mathbf{e}}_j$ are defined by averaging $\mathbf{e}_j$ in (39)–(41),

$$\overline{\mathbf{e}}_1 = \mathbf{i} + \overline{\zeta}_{\tilde{x}}\mathbf{k} = \mathbf{i} - b^\sharp_x\mathbf{k}/b^\sharp_z, \tag{62}$$

$$\overline{\mathbf{e}}_2 = \mathbf{j} + \overline{\zeta}_{\tilde{y}}\mathbf{k} = \mathbf{j} - b^\sharp_y\mathbf{k}/b^\sharp_z, \tag{63}$$

$$\overline{\mathbf{e}}_3 = \overline{\sigma}\mathbf{k} = \mathbf{k}/b^\sharp_z. \tag{64}$$

[3] The solution of the inverse problem is not unique: one can add an arbitrary solenoidal vector field to $\sigma^{-1}(l\mathbf{e}_1 + m\mathbf{e}_2 + n\mathbf{e}_3)$ without changing the divergence. Thus, (54) involves a gauge choice.

There are no eddy correlations introduced by averaging the $\mathbf{e}_1$ basis vectors in b coordinates. Note too that the vectors $\overline{\mathbf{e}}_1$ and $\overline{\mathbf{e}}_2$ in (62) and (63) are tangent to $b^\#$ surfaces; that is, after averaging $b^\#(\boldsymbol{x}, t)$ plays the role of $b(\boldsymbol{x}, t)$.

b. The thickness-weighted average

If $\theta(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t})$ is any field, then the thickness-weighted average of θ is

$$\hat{\theta} \overset{\text{def}}{=} \frac{\overline{\sigma\theta}}{\overline{\sigma}}. \tag{65}$$

For instance, the thickness-weighted average velocity components are

$$\overline{\sigma}\hat{u} = \overline{u\sigma}, \quad \text{and} \quad \overline{\sigma}\hat{v} = \overline{v\sigma}. \tag{66}$$

Following Andrews (1983), we refer to $\hat{u}$ and $\hat{v}$ as the residual velocities.

One must be sensitively aware that the thickness-weighted average caret does not satisfy property (57): that is, $\widehat{\partial_x u} \neq \partial_x \hat{u}$. Because $\hat{u}_x$ is ambiguous, we adopt the definition

$$\hat{u}_x \overset{\text{def}}{=} \partial_x \hat{u}, \quad \hat{v}_t \overset{\text{def}}{=} \partial_t \hat{v}, \quad \text{etc.} \tag{67}$$

That is, first take the thickness-weighted average and then the derivative.

The advantage of the thickness-weighted average is immediately clear if one averages (37) to obtain

$$\overline{\sigma}_{\tilde{t}} + (\hat{u}\overline{\sigma})_{\tilde{x}} + (\hat{v}\overline{\sigma})_{\tilde{y}} + (\hat{\varpi}\overline{\sigma})_{\tilde{b}} = 0. \tag{68}$$

There are no Reynolds eddy correlation terms in (68).

c. The thickness-weighted decomposition

Using the average, any field θ can be Reynolds decomposed as $\theta = \overline{\theta} + \theta'$. Indeed, the decomposition

$$\zeta = \overline{\zeta} + \zeta' \quad \text{and} \quad \sigma = \overline{\sigma} + \sigma' \tag{69}$$

is used throughout the TWA formulation. However, for all other variables the de Szoeke and Bennett (1993) thickness-weighted decomposition

$$\theta = \hat{\theta} + \theta'' \tag{70}$$

is more useful.

Equation (70) is a definition of the fluctuation θ''. As a consequence of (55)–(65), one has[4]

$$\overline{\sigma\theta''} = 0. \tag{71}$$

The decomposition (70), as well as the identity in (71), results in the key relation

$$\overline{\sigma\phi\theta} = \overline{\sigma}(\hat{\phi}\hat{\theta} + \widehat{\phi''\theta''}). \tag{72}$$

d. The three-dimensional residual velocity

In (66) we defined two components of the residual velocity. In analogy with (33), the third component is

$$w^\# \overset{\text{def}}{=} \overline{\zeta}_{\tilde{t}} + \hat{u}\overline{\zeta}_{\tilde{x}} + \hat{v}\overline{\zeta}_{\tilde{y}} + \hat{\varpi}\overline{\zeta}_{\tilde{b}} \tag{73}$$

(de Szoeke and Bennett 1993). Notice that $w^\# \neq \hat{w}$. In fact, $w^\#$ is not the average of any field.[5] Using $w^\#$, the three-dimensional residual velocity is

$$\mathbf{u}^\# \overset{\text{def}}{=} \hat{u}\mathbf{i} + \hat{v}\mathbf{j} + w^\#\mathbf{k} \tag{74}$$

$$= \hat{u}\overline{\mathbf{e}}_1 + \hat{v}\overline{\mathbf{e}}_2 + \overline{\sigma}^{-1}(\overline{\zeta}_{\tilde{t}} + \hat{\varpi}\overline{\zeta}_{\tilde{b}})\overline{\mathbf{e}}_3. \tag{75}$$

One can verify that $\nabla \cdot \mathbf{u}^\# = 0$ using either $\nabla\cdot$ in Cartesian coordinates or more readily with the buoyancy–coordinate formula in (53) (with $\sigma \to \overline{\sigma}$ and $\mathbf{e}_n \to \overline{\mathbf{e}}_n$).

Proceeding with this program, the convective derivative following the residual velocity $\mathbf{u}^\#$ is

$$\frac{D^\#}{Dt} \overset{\text{def}}{=} \partial_{\tilde{t}} + \hat{u}\partial_{\tilde{x}} + \hat{v}\partial_{\tilde{y}} + \hat{\varpi}\partial_{\tilde{b}} \tag{76}$$

$$= \partial_t + \hat{u}\partial_x + \hat{v}\partial_y + w^\#\partial_z. \tag{77}$$

The results above are analogous to the unaveraged convective derivative in (34) and (35).

To summarize, suppose one starts in z coordinates with $\boldsymbol{u}$ and b satisfying (14) and (15). One then transforms to b coordinates, takes the thickness-weighted average, and then moves back to z coordinates. When the dust settles, the variables in z coordinates are $\mathbf{u}^\#(x, y, z, t)$ and $b^\#(x, y, z, t)$, satisfying the analogs of (14) and (15): namely,

$$\nabla \cdot \mathbf{u}^\# = 0 \tag{78}$$

and

$$b_t^\# + \mathbf{u}^\# \cdot \nabla b^\# = \hat{\varpi}. \tag{79}$$

If the flow is adiabatic ($\hat{\varpi} = 0$) and steady ($\overline{\zeta}_t = 0$), then from (75) the residual velocity $\mathbf{u}^\#$ lies in a $b^\#$ surface.

[4] The unweighted average of θ'' is nonzero: $\overline{\sigma}\overline{\theta''} = -\overline{\sigma'\theta'}$.

[5] The superscript $\#$ flags nonmean fields, such as $b^\#$ and $\omega^\#$, that play in the mean-field equations.

Using the correct variables ($\hat{u}, \hat{v}, \omega^\#, b^\#, \hat{\varpi}, \overline{\zeta}, \overline{\mathbf{e}}_j$, and $\overline{\sigma}$), the TWA equations are identical in form to the unaveraged equations from section 2.

e. Residual-average identities

This identity,

$$\sigma\overline{\frac{D\theta}{Dt}} = \overline{\sigma}\left(\frac{D^\#\hat{\theta}}{Dt} + \nabla\cdot\mathbf{J}^\theta\right), \tag{80}$$

with the eddy flux of θ,

$$\mathbf{J}^\theta \overset{\text{def}}{=} \widehat{u''\theta''}\overline{\mathbf{e}}_1 + \widehat{v''\theta''}\overline{\mathbf{e}}_2 + \widehat{\varpi''\theta''}\overline{\mathbf{e}}_3, \tag{81}$$

is key in the TWA formulation.

The first step in proving (80) is to use the unaveraged thickness equation (37) to write

$$\sigma\frac{D\theta}{Dt} = (\sigma\theta)_{\tilde{t}} + (\sigma u\theta)_{\tilde{x}} + (\sigma v\theta)_{\tilde{y}} + (\sigma\varpi\theta)_{\tilde{b}}. \tag{82}$$

Averaging the expression above results in $(\overline{\sigma}\hat{\theta})_{\tilde{t}} + \cdots$ on the right. One uses (72) to handle the eddy correlations such as $\overline{\sigma u\theta}$ and (53) to recognize the divergence of the three-dimensional flux vector $\mathbf{J}^\theta$ in (81). Then, the averaged thickness equation (68) is used to maneuver $\overline{\sigma}$ back outside of the derivatives to finally obtain (80).

A second TWA identity comes from considering the divergence of a vector with contravariant expansion $\mathbf{q} = q^j\mathbf{e}_j$. Using the divergence formula in (53), one has

$$\overline{\sigma\nabla\cdot q} = \overline{\sigma}\nabla\cdot\widehat{q^j}\overline{\mathbf{e}}_j. \tag{83}$$

f. Comments on averaging vector fields in buoyancy coordinates

An unaveraged vector field can be represented in three equivalent forms, for example, as in the discussion surrounding (44)–(49). One might compute the thickness-weighted average of $\mathbf{q}$ using the representation (44) as simply

$$\hat{\mathbf{q}} = \hat{q}\mathbf{i} + \hat{r}\mathbf{j} + \hat{s}\mathbf{k}. \tag{84}$$

But, then $\nabla\cdot\mathbf{q} = 0$ does not guarantee that $\nabla\cdot\hat{\mathbf{q}} = 0$. This problem is acute when $\mathbf{q}$ is the velocity or a related field, such as the bolus velocity (e.g., see the discussion in section 10 of McDougall and McIntosh 2001).

In all respects, the contravariant representation

$$\mathbf{q} = q^j\mathbf{e}_j \tag{85}$$

is preferable. One cannot, of course, directly average (85) because the basis vectors $\mathbf{e}_j$ are fluctuating. However, with the representation (85), the TWA identity (83) shows that the vector

$$\mathbf{q}^\# \overset{\text{def}}{=} \widehat{q^j}\overline{\mathbf{e}}_j \tag{86}$$

maintains the zero-divergence property of the unaveraged $\mathbf{q}$.

These considerations are illustrated by an example drawn from McDougall and McIntosh (2001): the residual velocities defined in (66) can be broken apart as

$$(\hat{u},\hat{v}) = (\overline{u},\overline{v}) + \underbrace{\overline{\sigma}^{-1}(\overline{u'\sigma'},\overline{v'\sigma'})}_{\overset{\text{def}}{=}(u_B,v_B)}, \tag{87}$$

where u_B and v_B are the components of the vectorial bolus velocity,

$$\mathbf{u}_B \overset{\text{def}}{=} u_B\overline{\mathbf{e}}_1 + v_B\overline{\mathbf{e}}_2. \tag{88}$$

Following the discussion of McDougall and McIntosh (2001), $\mathbf{u}_B$ defined above is divergent and tangent to a $b^\#$ surface (i.e., $\mathbf{u}_B$ has no diapycnal component). However, the main thrust of this article is that the decomposition of the residual velocity $\mathbf{u}^\#$ into a mean part and a bolus term is unnecessary and even confusing. For example, although $\mathbf{u}^\#$ is nondivergent, $\mathbf{u}_B$ and $\mathbf{u}^\# - \mathbf{u}_B$ are both divergent. There is no clear advantage in using this decomposition of $\mathbf{u}^\#$: therefore we will have no more to do with $\mathbf{u}_B$.

g. The passive scalar

Applying (80) to the passive-scalar equation (16), one has

$$\frac{D^\#\hat{c}}{Dt} + \nabla\cdot\mathbf{J}^c = \hat{\gamma}, \tag{89}$$

where $\mathbf{J}^c$ is defined via (81). If the flow is adiabatic ($\varpi = 0$), then the passive-scalar eddy flux $\mathbf{J}^c$ is a linear combination of $\overline{\mathbf{e}}_1$ and $\overline{\mathbf{e}}_2$, and therefore the eddy flux $\mathbf{J}^c$ lies in a $b^\#$ surface. The averaged passive-scalar equation (89) is written in terms of the coordinate-independent differential operators $D^\#/Dt$ and $\nabla\cdot$, and (89) thus can easily be interpreted in either z or b coordinates.

One can show using earlier formulas that the passive-scalar variance $\widehat{c''^2}$ satisfies

$$\frac{1}{2}\frac{D^\#\widehat{c''^2}}{Dt} + \mathbf{J}^c\cdot\nabla\hat{c} + \nabla\cdot\mathbf{J}_3^c = \widehat{c''\gamma''}, \tag{90}$$

where the third-order flux is

$$\mathbf{J}_3^c \overset{\text{def}}{=} \widehat{u''\frac{1}{2}c''^2}\overline{\mathbf{e}}_1 + \widehat{v''\frac{1}{2}c''^2}\overline{\mathbf{e}}_2 + \widehat{\varpi''\frac{1}{2}c''^2}\overline{\mathbf{e}}_3. \tag{91}$$

Osborn–Cox arguments, based on the assumption of a balance between variance production by $\mathbf{J}^c\cdot\nabla\hat{c}$ and dissipation by $\widehat{c''\gamma''}$, indicate that $\mathbf{J}^c$ tends to be down $\nabla\hat{c}$.

h. Comments on the TWA passive-scalar equation

The TWA passive-scalar equation in (89) is not in the standard form found in earlier works (e.g., Gent et al. 1995; Treguier et al. 1997; Smith 1999; McDougall and McIntosh 2001) and in textbooks (Griffies 2004). To show the equivalence of (89) to the standard construction we limit attention to totally adiabatic flow (i.e., $\varpi = \gamma = 0$) and use the averaged thickness equation (68) to write (89) as

$$(\overline{\sigma}\hat{c})_{\tilde{t}} + (\overline{\sigma}\hat{u}\hat{c})_{\tilde{x}} + (\overline{\sigma}\hat{v}\hat{c})_{\tilde{y}} + (\overline{\sigma}\widehat{u''c''})_{\tilde{x}} + (\overline{\sigma}\widehat{v''c''})_{\tilde{y}} = 0. \tag{92}$$

The standard form introduces the horizontal velocity,

$$\hat{\mathbf{u}}_H \stackrel{\text{def}}{=} \hat{u}\mathbf{i} + \hat{v}\mathbf{j}; \tag{93}$$

the horizontal part of the eddy flux,

$$\boldsymbol{J}^c_H \stackrel{\text{def}}{=} \widehat{u''c''}\mathbf{i} + \widehat{v''c''}\mathbf{j}; \tag{94}$$

and a horizontal divergence operator, applied at constant b,

$$\boldsymbol{\nabla}_b \stackrel{\text{def}}{=} \mathbf{i}\partial_{\tilde{x}} + \mathbf{j}\partial_{\tilde{y}}. \tag{95}$$

Using these "horizontal variables," the averaged passive-scalar equation (92) is written in the form

$$(\overline{\sigma}\hat{c})_{\tilde{t}} + \boldsymbol{\nabla}_b \cdot (\overline{\sigma}\hat{\mathbf{u}}_H\hat{c} + \overline{\sigma}\mathbf{J}^c_H) = 0 \tag{96}$$

(e.g., Griffies 2004). Averaged tracer conservation in the form (96) has served as the basis of most previous papers on thickness-weighted averaging. But, the less familiar averaged conservation law in (89) proves to be crucial in formulating the TWA momentum equations (where $c = u$ and v). Thus, it is instructive to discuss the differences between (96) and (89). On one level these differences are notational, but notation is important.

Contemplating (96), Treguier et al. (1997) note the "curious point" that "vertical motion does not appear explicitly in the isopycnal formulation," yet advection by the horizontal compressible velocity $\hat{\mathbf{u}}_H$ "is equivalent to three-dimensional advection by a nondivergent velocity field in a z-coordinate model." One response to this remark is that, if one uses (89), then vertical advection does appear in the isopycnal formulation via the three-dimensional residual velocity $\mathbf{u}^\sharp$ and the three-dimensional eddy flux $\mathbf{J}^c$. Moreover, there are important advantages that might lead one to prefer the three-dimensional form in (89) over the equivalent horizontal form (96).

A main advantage is physical transparency: (96) entices one to conclude that the thickness-weighted tracer is advected by the horizontal velocity $\hat{\mathbf{u}}_H$ and that the eddy flux of thickness-weighted passive scalar is the horizontal vector $\mathbf{J}^c_H$, which would pierce sloping mean buoyancy surfaces. Both conclusions are, of course, incorrect for adiabatic flow. On the other hand, (89) correctly indicates that the TWA tracer is advected by the full three-dimensional residual velocity $\mathbf{u}^\sharp$ and that the relevant eddy flux is the three-dimensional in-$b^\sharp$-surface vector $\mathbf{J}^c$.

Given the differences between the three-dimensional vectors $\mathbf{u}^\sharp$ and $\mathbf{J}^c$ and the horizontal projections $\hat{\mathbf{u}}_H$ and $\mathbf{J}^c_H$, one might wonder how can (96) and (89) be equivalent? The point is that $\boldsymbol{\nabla}_b\cdot$ is not a true divergence: there is no analog of the Gauss theorem[6] that associates a divergence $\boldsymbol{\nabla}_b \cdot \mathbf{J}^c_H$ with the flux of the vector $\mathbf{J}^c_H$ through a control volume. One should recall that the coordinate-invariant differential operator $\boldsymbol{\nabla}\cdot$ in (89) is defined so that flux of a vector field through the surface of an infinitesimal control volume is equal to the product of the divergence and the volume enclosed. The shape of the enclosing surface is irrelevant (Morse and Feshbach 1953), and the definition of $\boldsymbol{\nabla}\cdot$ makes no reference to any coordinate systems; that is, $\boldsymbol{\nabla}\cdot$ in (89) is coordinate invariant. This is not the case for $\boldsymbol{\nabla}_b\cdot$ defined by (95). The utility of the divergence theorem is the main advantage of (89).

In defense of (96), one can argue that two-dimensional advection is simpler than three-dimensional advection and that reduction to two dimensions was the point of introducing b coordinates. But, in the TWA formulation b coordinates are only a bridge to the ultimate Cartesian coordinate version of the averaged equations. It is easier to cross the bridge from (89) than from (96) because the differential operators in (89) are coordinate invariant.

Finally, the TWA momentum equations in section 6 use the three-dimensional formalism in (89). In section 6, we construct three-dimensional Eliassen–Palm fluxes that, like $\mathbf{u}^\sharp$ and $\mathbf{J}^c$, are most naturally expanded in the basis $\overline{\mathbf{e}}_j$. Thus, a unified formulation encompassing both passive-scalar and momentum conservation hinges on (89).

4. Boundary conditions

Several authors have discussed the boundary conditions appropriate to TRM variables (Killworth 2001; McDougall and McIntosh 2001; Aiki and Yamagata

[6] In section 6.11.1, Griffies (2004) discusses the transformation of flux components and differential operators such as $\boldsymbol{\nabla}_b\cdot$ in generalized vertical coordinates. However, the transformations summarized by Griffies are more simply obtained by working with the basis $\mathbf{e}_j$ from the outset.

2006; Jacobson and Aiki 2006). The formulation in these earlier papers is framed using the quasi-Stokes streamfunction, which is not a variable used in the TWA formulation. However, the main conclusion is that the residual velocity should satisfy the same nonpenetration conditions as the unaveraged velocity[7]: namely,

$$\mathbf{u}^{\#}\cdot\mathbf{n} = 0, \tag{97}$$

where $\mathbf{n}$ is an outward normal to the boundary.

A subtlety is that the z-coordinate TWA equations are in the same domain as the unaveraged equations, even though the domain boundary is moving in b coordinates. This is illustrated with a simple kinematic example: consider the unaveraged velocity field $(u, v, w) = (0, \alpha\cos(x - t), 0)$ and suppose that the domain is $0 < z < 1$. The buoyancy field

$$b(x,y,z,t) = z + G[y + \alpha\sin(x - t)] \tag{98}$$

is a solution of the adiabatic version ($\varpi = 0$) of the buoyancy equation (15). It follows that the isopycnal depth is

$$\zeta(\tilde{x},\tilde{y},\tilde{b},\tilde{t}) = \tilde{b} - G[\tilde{y} + \alpha\sin(\tilde{x} - \tilde{t})], \tag{99}$$

provided that

$$G[\tilde{y} + \alpha\sin(\tilde{x} - \tilde{t})] < \tilde{b} < 1 + G[\tilde{y} + \alpha\sin(\tilde{x} - \tilde{t})]. \tag{100}$$

The wavy contours in Fig. 1 show the isopycnal depth ζ as a function of buoyancy and time at $(x, y) = 0$.

To calculate the average depth $\overline{\zeta}(\tilde{x},\tilde{y},\tilde{b},\tilde{t})$, one extends the definition of ζ beyond the range in (100) as shown in Fig. 1. This extension is the same as the prescription of Andrews (1983), based on the Lorenz convention, that buoyancy surfaces intersecting the boundary be continued "just under the surface."

From another perspective, one can imagine a "semi-Lagrangian" observer (SL) who sits at fixed horizontal position and moves vertically so as to remain on a target isopycnal. If SL never reaches the top or the bottom of the ocean, then SL collects an uninterrupted time series of depth $z_{\mathrm{SL}}(t)$; the time average of $z_{\mathrm{SL}}(t)$ is the average depth of the SL's target isopycnal. However, if SL's vertical motion takes him to either the top or the bottom of the ocean, then SL is stuck while the target isopycnal is unavailable. This is the "outcropping problem" illustrated in Fig. 1. The Lorenz convention demands that SL waits at the boundary and continues to record his constant depth until the target isopycnal reappears at SL's horizontal location. The average depth of the target isopycnal is computed using the entire time series $z_{\mathrm{SL}}(t)$, including the boundary waiting times during which $z_{\mathrm{SL}}(t)$ is constant.

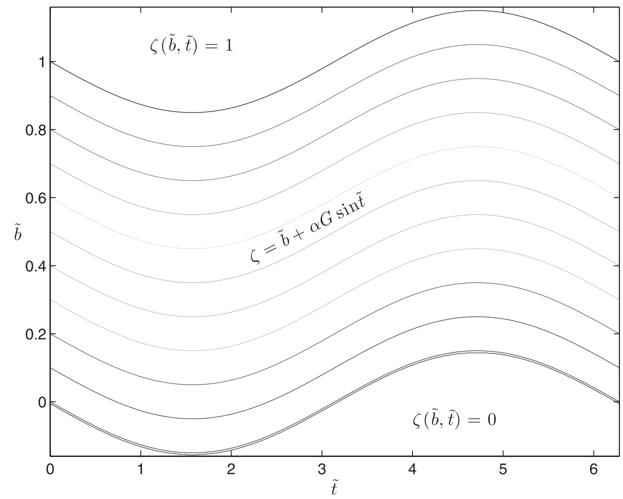

FIG. 1. The isopycnal depth $\zeta(\tilde{b},\tilde{t})$ in (99) at $(x, y) = 0$ as function of $\tilde{b}$ and $\tilde{t}$. In z coordinates the ocean depth is $0 < z < 1$ and ζ is extended with the constant value $\zeta = 1$ for isopycnals "above" the sea surface and $\zeta = 0$ for isopycnals "below" the bottom.

Using the extended ζ, one can compute the time average of $\zeta(\tilde{x},\tilde{y},\tilde{b},\tilde{t})$, that is, as a horizontal average through the field $\zeta(\tilde{b},\tilde{t})$ in Fig. 1. In this simple example $\overline{\zeta}$ can be obtained analytically. However, the expression is slightly complicated, and instead we show $\overline{\zeta}$ obtained by numerical integration in Fig. 2. Notice that $0 < \overline{\zeta} < 1$; that is, the mean depth is defined on the same interval as the unaveraged equations.

5. Dynamics in buoyancy coordinates

The Boussinesq primitive equations in z coordinates are

$$\frac{Du}{Dt} - fv + p_x = \mathcal{X}, \tag{101}$$

$$\frac{Dv}{Dt} + fu + p_y = \mathcal{Y}, \tag{102}$$

$$p_z = b, \tag{103}$$

$$u_x + v_y + w_z = 0, \tag{104}$$

$$\frac{Db}{Dt} = \varpi, \tag{105}$$

[7] We use the rigid-lid approximation so that the sea surface is $z = z_s$, where z_s is a constant. Thus, (97) is $\mathbf{w}^{\#}(x, y, z_s, t) = 0$.

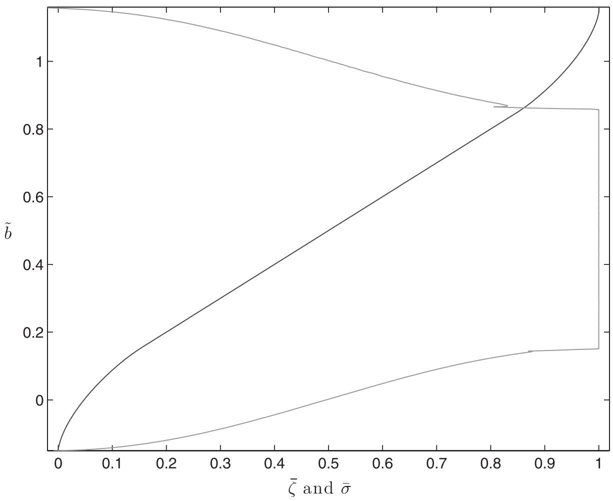

FIG. 2. The average isopycnal depth $\bar{\zeta}(\tilde{b})$ and the average thickness $\bar{\sigma} = \bar{\zeta}_{\tilde{b}}$ at $(x, y) = 0$ as function of $\tilde{b}$. The function $b^{\sharp}$ is the inverse of $\bar{\zeta}(\tilde{b})$ above and is defined on the original domain $0 < z < 1$. In the central part of the domain, $\alpha G < \tilde{b} < 1 - \alpha G$, the average depth is obtained from (99) as $\bar{\zeta} = \tilde{b}$, and therefore $\bar{\sigma} = 1$.

where the convective derivative D/Dt is defined in (34). In the horizontal momentum equations, $\mathcal{X}$ and $\mathcal{Y}$ denote adiabatic processes and body forces.

Now we write the equations of motion (101)–(105) in buoyancy coordinates: for example, using the b-coordinate representation of the convective derivative in (35). An important step is introduction of the Montgomery potential,

$$m(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t}) \overset{\text{def}}{=} p(x, y, \zeta(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t}), t) - \tilde{b}\zeta(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t}). \tag{106}$$

One can verify that $p_x = m_{\tilde{x}}$ etc. Then in b coordinates the equations of motion are

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv + m_{\tilde{x}} = \mathcal{X}, \tag{107}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + fu + m_{\tilde{y}} = \mathcal{Y}, \tag{108}$$

$$\zeta + m_{\tilde{b}} = 0, \tag{109}$$

$$\sigma_{\tilde{t}} + (\sigma u)_{\tilde{x}} + (\sigma v)_{\tilde{y}} + (\varpi\sigma)_{\tilde{b}} = 0, \tag{110}$$

where $\sigma \overset{\text{def}}{=} \zeta_{\tilde{b}} = -m_{\tilde{b}\tilde{b}}$. The convective derivative in b coordinates is given in (35).

a. Rossby–Ertel potential vorticity

One can write the horizontal momentum equations above as

$$u_{\tilde{t}} + \varpi u_{\tilde{b}} - \sigma v\Pi + \left(m + \frac{1}{2}u^2 + \frac{1}{2}v^2\right)_{\tilde{x}} = \mathcal{X} \tag{111}$$

and

$$v_{\tilde{t}} + \varpi v_{\tilde{b}} + \sigma u\Pi + \left(m + \frac{1}{2}u^2 + \frac{1}{2}v^2\right)_{\tilde{y}} = \mathcal{Y}, \tag{112}$$

where the Rossby–Ertel PV is

$$\Pi \overset{\text{def}}{=} \frac{f + v_{\tilde{x}} - u_{\tilde{y}}}{\sigma}. \tag{113}$$

Cross-differentiating to eliminate the Bernoulli terms, one obtains

$$(\sigma\Pi)_{\tilde{t}} + (\sigma u\Pi + \varpi v_{\tilde{b}} - \mathcal{Y})_{\tilde{x}} + (\sigma v\Pi - \varpi u_{\tilde{b}} + \mathcal{X})_{\tilde{y}} = 0; \tag{114}$$

the conservation law above is analogous to the "expanded" adiabatic passive scalar Eq. (92). The remarkable point is that (114) applies to PV even if the flow is diabatic.

b. The PV impermeability theorem

Haynes and McIntyre (1987, 1990) emphasize that a main advantage of (114) is that the PV impermeability theorem is immediate: at fixed $\tilde{x}$ and $\tilde{y}$ one can integrate (114) between $\tilde{b} = \tilde{b}_1$ and $\tilde{b} = \tilde{b}_2$ and obtain an expression for the rate of change of the total amount of PV substance in the layer $\tilde{b}_1 < \tilde{b} < \tilde{b}_2$. Since there are no $\tilde{b}$ derivatives in (114), the amount of PV substance in this buoyancy layer is not changed by flux through either bounding b surface.

Combining the layer-thickness equation (110) with (114) and using (53) to recognize a divergence, one obtains the PV conservation equation in the form

$$\frac{\mathrm{D}\Pi}{\mathrm{D}t} + \nabla \cdot \mathbf{\Gamma} = 0, \tag{115}$$

where the diabatic flux in (115) is

$$\mathbf{\Gamma} \overset{\text{def}}{=} -(\mathcal{X}\mathbf{i} + \mathcal{Y}\mathbf{j}) \times \nabla b - \varpi[\nabla \times (u\mathbf{i} + v\mathbf{j}) + \sigma^{-1}f\mathbf{e}_3] \tag{116}$$

(e.g., Haynes and McIntyre 1990); $\mathbf{\Gamma}$ can be expanded as

$$\sigma\mathbf{\Gamma} = -[(\mathcal{Y} - v_{\tilde{b}}\varpi)\mathbf{e}_1 - (\mathcal{X} - u_{\tilde{b}}\varpi)\mathbf{e}_2] - \sigma\varpi\Pi\mathbf{e}_3. \tag{117}$$

With (117) one readily finds $\mathbf{\Gamma} \cdot \nabla b = -\varpi\Pi$ so that $\mathbf{\Gamma}$ penetrates b surfaces. In their section 4, Haynes and McIntyre (1990) explain how this penetration is compatible with the PV impermeability theorem.

6. The TWA equations of motion

We now proceed with averaging the equations of motion in b coordinates. Following the discussion of kinematics in section 3, the average of the thickness equation (110) is (68). The average of the hydrostatic relation (109) is just $\overline{\zeta} = -\overline{m}_{\tilde{b}}$, with $\overline{\sigma} = \overline{\zeta}_{\tilde{b}} = -\overline{m}_{\tilde{b}\tilde{b}}$.

To average the horizontal momentum equations (107) and (108), one first multiplies by σ. The identity

$$\sigma m_{\tilde{x}} = -m_{\tilde{b}\tilde{b}} m_{\tilde{x}} \tag{118}$$

$$= (\zeta m_{\tilde{x}})_{\tilde{b}} + \left(\frac{1}{2}\zeta^2\right)_{\tilde{x}} \tag{119}$$

is key in dealing with the pressure gradient. Averaging (119) and using the mean hydrostatic relation, one has

$$\overline{\sigma m_{\tilde{x}}} = \overline{\sigma}\,\overline{m}_{\tilde{x}} + (\overline{\zeta' m'_{\tilde{x}}})_{\tilde{b}} + \left(\frac{1}{2}\overline{\zeta'^2}\right)_{\tilde{x}}. \tag{120}$$

Dividing (120) by $\overline{\sigma}$ and using (53) to recognize a divergence results in

$$\overline{\sigma}^{-1}\overline{\sigma m_{\tilde{x}}} = \overline{m}_{\tilde{x}} + \nabla\cdot\overline{\sigma}^{-1}\left(\frac{1}{2}\overline{\zeta'^2}\overline{\mathbf{e}}_1 + \overline{\zeta' m'_{\tilde{x}}}\overline{\mathbf{e}}_3\right). \tag{121}$$

The hydrostatic relation $\zeta + m_{\tilde{b}} = 0$ is used at several points in the manipulations above and is therefore essential to TWA.

The identity (121) and application of (80) to $\sigma Du/Dt$ and $\sigma Dv/Dt$ results in the TWA momentum equations,

$$\frac{D^\sharp \hat{u}}{Dt} - f\hat{v} + \overline{m}_{\tilde{x}} + \nabla\cdot\mathbf{E}^u = \hat{\mathcal{X}} \tag{122}$$

and

$$\frac{D^\sharp \hat{v}}{Dt} + f\hat{u} + \overline{m}_{\tilde{y}} + \nabla\cdot\mathbf{E}^v = \hat{\mathcal{Y}}. \tag{123}$$

The convective derivative $D^\sharp/Dt$ above is defined in (76), and the EP vectors $\mathbf{E}^u$ and $\mathbf{E}^v$ are

$$\mathbf{E}^u \stackrel{\text{def}}{=} \mathbf{J}^u + \overline{\sigma}^{-1}\left(\frac{1}{2}\overline{\zeta'^2}\overline{\mathbf{e}}_1 + \overline{\zeta' m'_{\tilde{x}}}\overline{\mathbf{e}}_3\right) \tag{124}$$

and

$$\mathbf{E}^v \stackrel{\text{def}}{=} \mathbf{J}^v + \overline{\sigma}^{-1}\left(\frac{1}{2}\overline{\zeta'^2}\overline{\mathbf{e}}_2 + \overline{\zeta' m'_{\tilde{y}}}\overline{\mathbf{e}}_3\right), \tag{125}$$

where $\mathbf{J}^u$ and $\mathbf{J}^v$ are defined via (81). In the adiabatic case (with $\varpi = 0$) the flux vectors $\mathbf{J}^u$ and $\mathbf{J}^v$ involve only $\overline{\mathbf{e}}_1$ and $\overline{\mathbf{e}}_2$, and therefore $\mathbf{J}^u$ and $\mathbf{J}^v$ lie in a $b^\sharp$ surface. Thus, on the right of (124) and (125) only the terms proportional to $\overline{\mathbf{e}}_3 = \overline{\sigma}\mathbf{k}$ transport momentum through $b^\sharp$ surfaces. This is the "inviscid pressure drag" identified by Rhines and Holland (1979), or "form drag."

In de Szoeke and Bennett (1993) and Greatbatch and McDougall (2003), the thickness-weighted velocity is advected by the thickness-weighted velocity, and therefore these are probably the closest antecedents of the thickness-weighted momentum equations (122) and (123). An advantage of the form in (122) and (123) is that the eddy forcing appears as the divergence of the three-dimensional Eliassen–Palm flux vectors $\mathbf{E}^u$ and $\mathbf{E}^v$.

a. The Rossby–Ertel PV equation

Following the same steps used to derive the unaveraged PV equation (115), one finds from the averaged momentum equations, (122) and (123), as well as from the averaged thickness equation (68), that

$$\frac{D^\sharp \Pi^\sharp}{Dt} + \nabla\cdot\mathbf{F}^\sharp + \nabla\cdot\mathbf{\Gamma}^\sharp = 0, \tag{126}$$

where

$$\Pi^\sharp \stackrel{\text{def}}{=} \frac{f + \hat{v}_{\tilde{x}} - \hat{u}_{\tilde{y}}}{\overline{\sigma}} \tag{127}$$

is a form of the Rossby–Ertel potential vorticity. In (126) the diabatic terms appear in

$$\overline{\sigma}\mathbf{\Gamma}^\sharp \stackrel{\text{def}}{=} -[(\hat{\mathcal{Y}} - \hat{v}_{\tilde{b}}\hat{\varpi})\overline{\mathbf{e}}_1 - (\hat{\mathcal{X}} - \hat{u}_{\tilde{b}}\hat{\varpi})\overline{\mathbf{e}}_2] - \overline{\sigma}\hat{\varpi}\Pi^\sharp\overline{\mathbf{e}}_3; \tag{128}$$

$\mathbf{\Gamma}^\sharp$ is the analog of the unaveraged $\mathbf{\Gamma}$ in (117). Also in (126)

$$\mathbf{F}^\sharp \stackrel{\text{def}}{=} \overline{\sigma}^{-1}(\nabla\cdot\mathbf{E}^v)\overline{\mathbf{e}}_1 - \overline{\sigma}^{-1}(\nabla\cdot\mathbf{E}^u)\overline{\mathbf{e}}_2 \tag{129}$$

is the eddy flux of $\Pi^\sharp$.

Taking the dot product of (129) with $\overline{\mathbf{e}}^1 = \mathbf{i}$ and $\overline{\mathbf{e}}^2 = \mathbf{j}$ expresses the EP divergences in terms of components of the PV eddy flux $\mathbf{F}^\sharp$. Thus, the TWA horizontal momentum equations can be written as

$$\hat{u}_{\tilde{t}} + \hat{\varpi}\hat{u}_{\tilde{b}} - \overline{\sigma}\hat{v}\Pi^\sharp + \left(\overline{m} + \frac{1}{2}\hat{u}^2 + \frac{1}{2}\hat{v}^2\right)_{\tilde{x}} = \hat{\mathcal{X}} + \overline{\sigma}\mathbf{j}\cdot\mathbf{F}^\sharp \tag{130}$$

and

$$\hat{v}_{\tilde{t}} + \hat{\varpi}\hat{v}_{\tilde{b}} - \overline{\sigma}\hat{u}\Pi^\sharp + \left(\overline{m} + \frac{1}{2}\hat{u}^2 + \frac{1}{2}\hat{v}^2\right)_{\tilde{y}} = \hat{\mathcal{Y}} - \overline{\sigma}\mathbf{i}\cdot\mathbf{F}^\sharp. \tag{131}$$

b. *Comments on the thickness-weighted Rossby–Ertel PV* $\hat{\Pi}$

Rather than $\Pi^\#$, Greatbatch (1998) and Smith (1999) use the thickness-weighted PV

$$\hat{\Pi} = \frac{f + \overline{v}_{\tilde{x}} - \overline{u}_{\tilde{y}}}{\overline{\sigma}}; \tag{132}$$

$\hat{\Pi}$ differs in the numerator from $\Pi^\#$ in (127). Using (80) and (83) to take the thickness-weighted average of the Rossby–Ertel PV equation (115), one obtains

$$\frac{D^\#\hat{\Pi}}{Dt} + \nabla\cdot\mathbf{J}^{\Pi} + \nabla\cdot\Upsilon = 0, \tag{133}$$

where the eddy flux $\mathbf{J}^{\Pi}$ is defined via (81) and the diabatic flux is

$$\overline{\sigma}\Upsilon \overset{\text{def}}{=} -[(\overline{\mathcal{Y}} - \overline{v_{\tilde{b}}\varpi})\overline{\mathbf{e}}_1 - (\overline{\mathcal{X}} - \overline{u_{\tilde{b}}\varpi})\overline{\mathbf{e}}_2] - \overline{\sigma}\widehat{\varpi\Pi}\overline{\mathbf{e}}_3. \tag{134}$$

Greatbatch (1998) and Smith (1999) show that $\hat{\Pi}$ and the fluctuation Π'' are involved in simple conservation laws. However, in the TWA formulation $\Pi^\#$ in (126) is the most useful expression of averaged PV conservation because the eddy flux $\mathbf{F}^\#$ in (129) is directly related to the Eliassen–Palm eddy forcing in the TWA momentum equations and $\Pi^\#$ contains the residual-mean velocities rather than the unweighted mean velocities in (132).

c. *The TWA formulation in z coordinates*

The TWA momentum equations (122) and (123) are written using the coordinate-invariant differential operators $D^\#/Dt$ and $\nabla\cdot$. Thus, with the residual convective derivative $D^\#/Dt$ given by (77), it is easy to rewrite (122) and (123) in Cartesian coordinates. An important step is taking the inverse of the definition of the Montgomery potential in (106) using variables appropriate to the averaged equations. This introduces the field $p^\#$ defined by

$$p^\#(x, y, z, t) \overset{\text{def}}{=} \overline{m}(\tilde{x}, \tilde{y}, b^\#(x, y, z, t), \tilde{t}) + zb^\#(x, y, z, t). \tag{135}$$

One can verify that $\overline{m}_{\tilde{x}} = p^\#_x$, $p^\#_z = b^\#$, etc. To translate (122) and (123) into z coordinates one replaces $\overline{m}_{\tilde{x}}$ and $\overline{m}_{\tilde{y}}$ by $p^\#_x$ and $p^\#_y$. The TWA equations of motion, written in Cartesian coordinates, are then

$$\frac{D^\#\hat{u}}{Dt} - f\hat{v} + p^\#_x + \nabla\cdot\mathbf{E}^u = \hat{\mathcal{X}}, \tag{136}$$

$$\frac{D^\#\hat{v}}{Dt} + f\hat{u} + p^\#_y + \nabla\cdot\mathbf{E}^v = \hat{\mathcal{Y}}, \tag{137}$$

$$p^\#_z = b^\#, \tag{138}$$

$$\hat{u}_x + \hat{v}_y + w^\#_z = 0, \tag{139}$$

$$\frac{D^\# b^\#}{Dt} = \hat{\varpi}. \tag{140}$$

The eddy forcing via $\nabla\cdot\mathbf{E}^u$ and $\nabla\cdot\mathbf{E}^v$ is confined to the horizontal components of the momentum balance in (136) and (137). Apart from these EP divergences, the TWA equations in (136)–(140) are identical in form to the unaveraged equations in (101)–(105).

7. Nonacceleration conditions

We now consider ‘‘nonacceleration conditions,’’ defined as (i) the system is adiabatic ($\varpi = \mathcal{X} = \mathcal{Y} = 0$); (ii) the flow is steady; and (iii) the EP divergences are zero,

$$\nabla\cdot\mathbf{E}^u = 0 \quad \text{and} \quad \nabla\cdot\mathbf{E}^v = 0. \tag{141}$$

With conditions (i) and (ii), the TWA thickness equation (68) is satisfied by the introduction of a ‘‘thickness streamfunction’’ $\Psi(\tilde{x}, \tilde{y}, \tilde{b})$ so that the in-$b^\#$-surface TWA velocity can be written as

$$\overline{\sigma}\mathbf{u}^\# = -\Psi_{\tilde{y}}\overline{\mathbf{e}}_1 + \Psi_{\tilde{x}}\overline{\mathbf{e}}_2. \tag{142}$$

Assumption (iii) implies that the horizontal momentum equations in (130) and (131) reduce to

$$-\Psi_{\tilde{x}}\Pi^\# + \left(\overline{m} + \frac{1}{2}\hat{u}^2 + \frac{1}{2}\hat{v}^2\right)_{\tilde{x}} = 0 \tag{143}$$

and

$$-\Psi_{\tilde{y}}\Pi^\# + \left(\overline{m} + \frac{1}{2}\hat{u}^2 + \frac{1}{2}\hat{v}^2\right)_{\tilde{y}} = 0. \tag{144}$$

That is, the TWA flow is balanced. It follows from the Taylor–Bretherton identity in (129) that under nonacceleration conditions $\mathbf{F}^\# = 0$ and $\Pi^\# = A(\Psi, \tilde{b})$, where A is some function. The latter conclusion is obtained by cross-differentiating (143) and (144) to eliminate the Bernoulli function or from the nonacceleration version of the PV equation in (126).

A partial converse is true: if $\mathbf{F}^\# = 0$, then condition (iii) in (141) follows from the Taylor–Bretherton identity (129), and the residual velocity $\mathbf{u}^\#$ is balanced. In other words, if $\mathbf{F}^\# = 0$, then nonacceleration conditions prevail and a steady adiabatic balanced residual flow $\mathbf{u}^\#$ coexists with a statistically stationary adiabatic eddy field.

A. Plumb (personal communication, 2011) has noted that the converse is not true: suppose that the PV flux is solenoidal with the form

$$\mathbf{F}^\sharp = \nabla \times (\phi \nabla b^\sharp) \tag{145}$$

$$= \sigma^{-1}(\phi_{\tilde{y}}\bar{\mathbf{e}}_1 - \phi_{\tilde{x}}\bar{\mathbf{e}}_2). \tag{146}$$

The PV flux above is in a $b^\sharp$ surface and has zero divergence. From the Taylor–Bretherton identity (129), the EP divergences are

$$\nabla \cdot \mathbf{E}^u = \phi_{\tilde{x}} \quad \text{and} \quad \nabla \cdot \mathbf{E}^v = \phi_{\tilde{y}}. \tag{147}$$

However, this nonzero EP eddy force can be absorbed into the gradient of the Bernoulli function, so a slightly modified version of the balance condition in (143) and (144) prevails:

$$-\Psi_{\tilde{x}}\Pi^\sharp + \left(\overline{m} + \phi + \frac{1}{2}\hat{u}^2 + \frac{1}{2}\hat{v}^2\right)_{\tilde{x}} = 0 \tag{148}$$

and

$$-\Psi_{\tilde{y}}\Pi^\sharp + \left(\overline{m} + \phi + \frac{1}{2}\hat{u}^2 + \frac{1}{2}\hat{v}^2\right)_{\tilde{y}} = 0. \tag{149}$$

The Plumb example[8] shows that, although the EP divergences are nonzero, the residual velocity in (142) remains in balance. Thus condition (iii) in (141) is sufficient but not necessary for balance.

The three-dimensional finite amplitude Eliassen–Palm relation developed here is also subject to a further caveat emphasized by Andrews (1983); unfortunately, we do not have here a nonacceleration theorem, analogous to small amplitude zonal-mean results that a steady adiabatic wave field must have $\mathbf{F}^\sharp = 0$ (e.g., Andrews and McIntyre 1976; Boyd 1976; Charney and Drazin 1961; Plumb 1986; Young and Rhines 1980).

8. Conclusions and discussion

The TWA formulation developed in this paper is a general and exact rewriting of the Boussinesq equations after averaging in buoyancy coordinates. In addition to hydrostatic balance, the TWA formulation requires nonzero b_z and the existence of an averaging operation with the three properties in (55)–(57). These are mild assumptions, and one might argue that TWA formulation does not take full advantage of additional simplifications that are appropriate in large-scale oceanography (e.g., geostrophic balance). Nonetheless, it is interesting to see how far one can proceed without making approximations. The TWA formulation provides a unified theoretical framework in which to diagnose eddy–mean flow interactions in ocean models and in which to pose the eddy parameterization problem. The main strength of TWA is that only the residual velocity $\mathbf{u}^\sharp$ features; that is, it is not necessary to separately consider the mean velocity and the eddy-induced velocity. This makes the TWA framework an attractive alternative to earlier formulations that use both mean and eddy-induced velocities as prognostic variables. In TWA, all tracers—including horizontal momentum—are subject to the thickness-weighted average and all tracers are advected by $\mathbf{u}^\sharp$.

In the TWA framework, the eddy parameterization problem is shifted to the horizontal momentum equations and devolves to expressing the EP divergences in terms of TWA fields. On large scales the dominant effect of eddies is vertical transmission of momentum by form drag, and vertical viscosity is the most obvious parameterization (e.g., Rhines and Young 1982; Greatbatch and Lamb 1990; Greatbatch 1998; Ferreira and Marshall 2006).

One might criticize the TWA formulation on the grounds that tracers other than buoyancy might be used to define a quasi-Lagrangian coordinate and a thickness-weighted average: for example, conservative temperature, salinity, or oxygen might serve instead. Different choices of vertical coordinate would lead to a quite different eddy-mean decomposition. Thus, it seems that the TWA formulation is not unique.

There are several responses to this criticism. First, buoyancy is the unique tracer that comes closest to satisfying the essential requirement that the Jacobian is nonzero: buoyancy is special because vertical inversions are dynamically eliminated and are never observed over distances more than a few meters. On the other hand, tracers such as salinity and oxygen have large-scale vertical inversions that are persistent features of the general circulation of the ocean. These inversions prohibit the use of these other tracers as generalized vertical coordinates.

Buoyancy is special again because one can argue that the small level of mechanical energy dissipation in the ocean implies that interior diapycnal diffusion is weak: below the mixed layer buoyancy surfaces are almost impermeable barriers or at least more impermeable than salinity or oxygen surfaces. This is the basis of the

[8] Plumb notes that a specific example of a PV flux with the structure in (146) is provided by a quasigeostrophic barotropic Rossby wave propagating along a zonal channel with streamfunction $\psi' = \sin y \cos(kx - \omega t)$. The cross-channel PV flux is zero, $\overline{v'q'} = 0$, but the along-channel flux is nonzero, $\overline{u'q'} \propto \sin 2y$. This zonal PV flux corresponds to an eddy force in the y-momentum equation, which can be absorbed into the pressure.

classical physical oceanographic argument that there is little or no mixing across buoyancy surfaces, coupled with significant tracer transport on buoyancy surfaces. The TWA formulation is a formal expression of this old intuition.

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APPENDIX

Further Buoyancy–Coordinate Relations

In this appendix we collect some formulas related to buoyancy coordinates. For example, the basis $\mathbf{e}^j$ is expressed in terms of $\mathbf{e}_j$ by

$$\mathbf{e}^1 = \mathbf{e}_1 - \zeta_{\tilde{x}}\sigma^{-1}\mathbf{e}_3, \tag{A1}$$

$$\mathbf{e}^2 = \mathbf{e}_2 - \zeta_{\tilde{y}}\sigma^{-1}\mathbf{e}_3, \tag{A2}$$

$$\mathbf{e}^3 = \underbrace{-\zeta_{\tilde{x}}\sigma^{-1}\mathbf{e}_1 - \zeta_{\tilde{y}}\sigma^{-1}\mathbf{e}_2 + \sigma^{-2}(1 + \zeta_{\tilde{x}}^2 + \zeta_{\tilde{y}}^2)\mathbf{e}_3}_{=\nabla b}. \tag{A3}$$

The inverse relation is

$$\mathbf{e}_1 = (1 + \zeta_{\tilde{x}}^2)\mathbf{e}^1 + \zeta_{\tilde{x}}\zeta_{\tilde{y}}\mathbf{e}^2 + \zeta_{\tilde{x}}\sigma\mathbf{e}^3, \tag{A4}$$

$$\mathbf{e}_2 = \zeta_{\tilde{x}}\zeta_{\tilde{y}}\mathbf{e}^1 + (1 + \zeta_{\tilde{y}}^2)\mathbf{e}^2 + \zeta_{\tilde{y}}\sigma\mathbf{e}^3, \tag{A5}$$

$$\mathbf{e}_3 = \sigma\zeta_{\tilde{x}}\mathbf{e}^1 + \sigma\zeta_{\tilde{y}}\mathbf{e}^2 + \sigma^2\mathbf{e}^3. \tag{A6}$$

a. The curl of a vector field

To express the curl of a vector field $\mathbf{q}$ in buoyancy coordinates, one starts with the covariant representation

$$\mathbf{q} = q_1\underbrace{\mathbf{e}^1}_{=\mathbf{i}} + q_2\underbrace{\mathbf{e}^2}_{=\mathbf{j}} + q_3\underbrace{\mathbf{e}^3}_{=\nabla b}. \tag{A7}$$

Then, since $\nabla \times \mathbf{e}^j = 0$, the curl of $\mathbf{q}$ is

$$\nabla \times \mathbf{q} = \nabla q_1 \times \mathbf{e}^1 + \nabla q_2 \times \mathbf{e}^2 + \nabla q_3 \times \mathbf{e}^3. \tag{A8}$$

Using (51), the gradients ∇q^j above are written in terms of the basis $\mathbf{e}^j$, resulting in the cross products $\mathbf{e}^i \times \mathbf{e}^j$. According to the definitions in (39)–(41), these cross products result in the dual basis $\mathbf{e}_j$ so that

$$\sigma\nabla \times \mathbf{q} = (q_{3\tilde{y}} - q_{2\tilde{b}})\mathbf{e}_1 + (q_{1\tilde{b}} - q_{3\tilde{x}})\mathbf{e}_2 + (q_{2\tilde{x}} - q_{1\tilde{y}})\mathbf{e}_3. \tag{A9}$$

Using the expression for the divergence in (53), it is easy to verify from (A9) that $\nabla \cdot \nabla \times \mathbf{q} = 0$.

An important application of (A9) is calculation of the vectorial vorticity featuring in the Rossby–Ertel PV. This one requires the curl of the horizontal velocity $\mathbf{u}_H = u\mathbf{e}^1 + v\mathbf{e}^2$, and from (A9) the curl is

$$\nabla \times \mathbf{u}_H = \sigma^{-1}[-v_{\tilde{b}}\mathbf{e}_1 + u_{\tilde{b}}\mathbf{e}_2 + (v_{\tilde{x}} - u_{\tilde{y}})\mathbf{e}_3] \tag{A10}$$

$$= -v_z\mathbf{i} + u_z\mathbf{j} + (v_x - u_y)\mathbf{k}. \tag{A11}$$

b. The Laplacian

We express the Laplacian of $c(\mathbf{x})$ in buoyancy coordinates by first writing $\nabla^2 c = \nabla \cdot \nabla c$ and then using the earlier results for gradient and divergence and (A1)–(A3). One finds

$$\sigma\nabla^2 c = (\zeta_{\tilde{b}}c_{\tilde{x}} - \zeta_{\tilde{x}}c_{\tilde{b}})_{\tilde{x}} + (\zeta_{\tilde{b}}c_{\tilde{y}} - \zeta_{\tilde{y}}c_{\tilde{b}})_{\tilde{y}} + [\sigma^{-1}(1 + \zeta_{\tilde{x}}^2 + \zeta_{\tilde{y}}^2)c_{\tilde{b}} - \zeta_{\tilde{x}}c_{\tilde{x}} - \zeta_{\tilde{y}}c_{\tilde{y}}]_{\tilde{b}}. \tag{A12}$$

A simple check is that $\nabla^2\zeta = 0$.

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