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Elliptical vortices in shallow water

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A compact rotating shallow mass of fluid with a free boundary is released on a horizontal plane. Initially it is supposed that horizontal sections through the mass are elliptical, vertical sections are parabolic and the two velocities are linear functions of the horizontal coordinates. Thus four numbers suffice to describe the configuration of the fluid and another four the velocity field. The subsequent motion preserves this simple structure, and so the shallow-water equations reduce to ordinary differential equations in time for the eight parameters required to specify the initial condition.

This eighth-order system can be reduced to quadratures using the well-known invariants of the shallow-water equations. There are five such integrals: volume, energy, enstrophy, angular momentum, and a fifth, which lacks a familiar name. The remaining three degrees of freedom can be related to the *shape* (i.e. eccentricity), horizontal *size* (i.e. radius of gyration) and *orientation* of the mass. In general, all of these are periodic functions of time but with different characteristic timescales. Simplest are size changes, which occur at the inertial frequency f. Shape changes are superinertial, while orientation changes may be subinertial or superinertial.

This solution is used to discuss the unsteady motion of a non-axisymmetric Gulf Stream ring. We argue that size and shape changes excite internal gravity waves in the underlying fluid, while orientation changes generate Rossby waves. While this wave radiation decreases the energy of the ring, and may alter the angular momentum, it cannot lead to a state of no motion because the volume and enstrophy are unaffected.

1. Introduction

This article discusses the motion of a compact rotating mass of fluid whose boundary is free. It is assumed that the aspect ratio and initial conditions are such that the shallow-water equations describe the evolution of the liquid.

The geophysical motivation of this study is the dynamics of warm-core rings. These clockwise eddies of warm Sargasso Sea water are formed when a northward meander of the Gulf Stream becomes so extreme that it herniates, detaches from the stream and subsequently evolves as an isolated structure (Fofonoff 1981). Seen from above on a satellite photograph, they are roughly elliptical with a radius of about 100 km (Brown *et al.* 1984; Evans *et al.* 1984; Cushman-Roisin, Heil & Nof (1985). Traditional hydrography shows that the warm water is bounded below at about 500 m by a sharp thermocline. The reduced gravity at this interface is about 1 cm s^{-2} (Csanady 1979).

The model outlined in the first paragraph idealizes the structure of these rings by supposing that the boundary between the warm Sargasso Sea water and the underlying shelf water is a sharp interface that is free to move. The lower layer is magined to be infinitely deep, and consequently is motionless. Additionally the ıpatial gradient of the planetary vorticity field (the β-effect) is ignored. Finally, lissipation and forcing are neglected. All of these assumptions are unreasonable, and so the primary geophysical motivation for this study is an investigation of the effects of strong departures from axisymmetry. These may be due to accidental events at the birth of the ring or perhaps strong shearing events in the ambient fluid. Once present, they may be as dynamically important as the various processes that the present model ignores. Finally, the analysis here may serve as the basis of an expansion that includes these neglected processes as perturbations. For example, Flierl (1984*a*, *b*), Killworth (1983) and Nof (1981, 1983) have all used the axisymmetric solution as the basis for an expansion that includes the β-effect as a perturbation. An analogous calculation might use elliptical vortices as a starting point.

To obtain simplification, it is necessary to suppose that the initial condition is special (see (3.1) below). Consequently, it is shown that the shallow-water equations can be reduced to an eighth-order set of ordinary differential equations in time. Then the various integrals of the motion (e.g. volume, energy, angular momentum and enstrophy) are used to further reduce and simplify this eighth-order system.

The reduction of the shallow-water equations in (3.1) has a long history. Perhaps Goldsbrough (1930) was the first to observe that the shallow-water equations simplify if the velocities are linear functions and the depth is a quadratic function of the coordinates. But the nineteenth-century theory of self-gravitating masses (e.g. Lamb 1932; Chandrasekhar 1969) is based on analogous polynomial solutions. Goldsbrough's observation is the basis of Ball's (1965) and Thacker's (1981) investigations of tidal oscillations in elliptical basins whose depth profile is parabolic. More recently, and in the context of warm-core rings, Cushman-Roisin *et al.* (1985) and the present author rediscovered these polynomial solutions. Cushman-Roisin *et al.* found an elegant particular solution: a uniformly rotating ellipse with constant shape (i.e. eccentricity) and size (i.e. area), which has been christened the 'rodon'.

The present study is directed at understanding the qualitative properties of polynomial solutions. In particular, a systematic application of Ball's (1963) general theorems reduces the evolution equations to quadratures. Thus it is shown that the uniformly rotating ellipse described by Cushman-Roisin *et al.* (1985) requires certain special relations amongst the constants of motion. In general, when these do not hold, an elliptical vortex does not rotate uniformly without change of shape and size. But the motion is still sufficiently simple as to permit complete qualitative understanding without resort to linearization about axisymmetry or extensive numerical calculation.

One important issue that is ignored is the stability of these solutions. However, in one very special case the solution reduces to the geostrophically balanced, zero-potential-vorticity profile whose instability has been demonstrated by Griffiths, Killworth and Stern (1982). On the other hand, Killworth (1983) has shown that the axisymmetric special case is stable. Thus both cases are possible, and a thorough study of the ageostrophic stability of the intermediate elliptical vortices is of great interest. This stability calculation is surprisingly tractable (B. Cushman-Roisin and P. Ripa, personal communication) because, as in the theory of self-gravitating masses, the linearized perturbation equations also have polynomial solutions.

2. The shallow-water equations and some general results

The shallow water equations are

$$\frac{\mathrm{D}u}{\mathrm{D}t}-fv=-gh_x,\quad \frac{\mathrm{D}v}{\mathrm{D}t}+fu=-gh_y,\quad \frac{\mathrm{D}h}{\mathrm{D}t}+h(u_x+v_y)=0, \tag{2.1}$$

where h is the depth of the fluid. The velocities u and v are independent of z, and f is the Coriolis parameter. g is the gravitational acceleration, which in the geophysical context may be a reduced gravity. It is assumed that the mass of fluid is finite and its boundary is free. If $y = L(x,t)$ denotes this free, moving boundary then the boundary condition is

$$h(x, L(x,t), t) = 0, \quad v = L_t + uL_x. \tag{2.2}$$

The remainder of this section summarizes some general results due to Ball (1963), which form the basis of the analysis in §3.

The most obvious conserved quantity is the volume of fluid

$$Q \equiv \int h\,\mathrm{d}A \tag{2.3}$$

Additional conserved quantities are the energy

$$E \equiv \int \{\tfrac{1}{2}h(u^2+v^2)+\tfrac{1}{2}gh^2\}\,\mathrm{d}A \tag{2.4}$$

and the angular momentum

$$J \equiv \int h\{(xv-yu)+\tfrac{1}{2}f(x^2+y^2)\}\,\mathrm{d}A \tag{2.5}$$

The potential vorticity

$$q \equiv (f+v_x-u_y)/h \tag{2.6}$$

is conserved following particles:

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0 \tag{2.7}$$

This in turn implies conservation of the generalized enstrophy

$$Z \equiv \int hF(q)\,\mathrm{d}A, \tag{2.8}$$

where F is an arbitrary function.

Ball's first theorem concerns the motion of the centre of mass

$$(X, Y)\,Q \equiv \int (x,y)\,h\,\mathrm{d}A, \tag{2.9}$$

where Q is defined in (2.3). From (2.1), it follows that

$$\frac{\mathrm{d}^2X}{\mathrm{d}t^2}-f\frac{\mathrm{d}Y}{\mathrm{d}t}=0,\quad \frac{\mathrm{d}^2Y}{\mathrm{d}t^2}+f\frac{\mathrm{d}X}{\mathrm{d}t}=0 \tag{2.10}$$

and so the centre of gravity executes inertial oscillations. Further, Ball demonstrates that the additional motions relative to this centre are independent of the motion of the centre.

Ball's second theorem concerns the expansion and contraction of the fluid about this centre. Thus, introducing the moment of inertia

$$I \equiv \int h\{(x-X)^2+(y-Y)^2\}\,\mathrm{d}A, \tag{2.11}$$

he shows it satisfies a remarkably simple equation:

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2}+f^2 I = 2(2E+fJ). \tag{2.12}$$

The above equation can be integrated once:

$$\left\{\left(\frac{\mathrm{d}I}{\mathrm{d}t}\right)^2+f^2I^2\right\}-4(2E+fJ)\,I = -C, \tag{2.13}$$

where C is another constant of the motion. To summarize: the constants of motion are Q, E, J, Z and C. These may not be all independent: for instance, it is shown in §3 that in an axisymmetric distension (defined below after (3.3)) Z and C are functions of Q, E and J. In fact, Ball established that in general

$$\beta \equiv 9\pi(C-2J^2)/8gQ^3 \tag{2.14}$$

is greater than one. If β is precisely one then the motion must be axisymmetric (a pure distension in Ball's terminology).

To the extent that the geometry of the mass is characterized by (X, Y) and I, one has a simple picture of how an arbitrary initial condition evolves. However, the preceding general results give no information on how departures from axisymmetry affect the evolution of the system. I have been unable to find any simple general results that supplement (2.9) and (2.11). Instead, a particular solution that illustrates these issues is presented.

3. A special solution of the shallow-water equations

Substitution of

$$u = ax+by, \quad v = cx+dy, \quad h = k-\tfrac{1}{2}lx^2-mxy-\tfrac{1}{2}ny^2 \tag{3.1}$$

into (2.1) gives ordinary differential equations that determine the time evolution of the eight functions a, b, c, d, k, l, m, n. In (3.1) the coordinates (x, y) are relative to the moving centre of mass (see (2.9)). Because the internal motions of the fluid are unaffected by the motion of the centre of mass, there is no loss of generality in this.

The resulting ordinary differential equations are

$$\left.\begin{aligned} \dot a &= -a^2-bc-fc+gl,\\ \dot b &= -ab-bd-fd+gm,\\ \dot c &= -dc-ac-fa+gm,\\ \dot d &= -d^2-bc-fb-gn \end{aligned}\right\} \tag{3.2}$$

and

$$\begin{aligned} \dot k &= -(a+d)\,k,\\ \dot l &= -3al-2cm-dl,\\ \dot m &= -2(a+d)-bl-cn,\\ \dot n &= -3dn-2bm-an. \end{aligned} \tag{3.3}$$

In the geophysical context g is the reduced gravity between the warm water mass and the resting infinitely deep lower layer (typically $1\ \mathrm{cm\ s^{-2}}$). f is the Coriolis parameter, which is about $10^{-4}\,\mathrm{s}^{-1}$. One point that is easy to overlook is that the kinematic boundary condition (2.2) is identically satisfied by the above solution.

The most recent investigation of (3.2) and (3.3) is that of Cushman-Roisin *et al.* (1985). They note that there are two steady solutions:

(i) a one-dimensional parallel flow, i.e. $(\partial/\partial x)(u,v,h) = 0$;
(ii) an axisymmetric vortex, i.e. $(\partial/\partial\phi)(u,v,h) = 0$.

In both these cases four of the eight variables are identically zero. They then proceed to linearize (3.2) and (3.3) by assuming that departures from the axisymmetric solution with constant radius are small. Because observed Gulf Stream rings have modest eccentricities (perhaps 0.5–0.8), this is not unreasonable. From a geophysical perspective the most important result to emerge from this analysis is the existence of a low frequency (i.e. subinertial) mode. This consists of a slow clockwise rotation of the ellipse without change of form. Cushman-Roisin *et al.* speculate that this accounts for the observed rotation, at about 9° per day, of elliptical rings.

Besides these linearized solutions they also discuss two exact nonlinear solutions of (3.2) and (3.3). The first is axisymmetric and is Ball's (1963) 'distension'. This is simply a radial pulsation of a circular vortex at the inertial frequency. The second solution is more interesting and is essentially a nonlinear analogue of the subinertial mode described above: an elliptical vortex (with arbitrary eccentricity) rotates uniformly clockwise without change of form. The frequency of rotation goes to zero as the eccentricity approaches one.

The present analysis aims to reduce (3.2) and (3.3) to quadratures. This clarifies the conditions under which the special nonlinear solutions discussed above emerge from arbitrary initial conditions. Additionally it is shown how the values of the various constants of motion determine observable quantities such as the rate of rotation (in general non-uniform) of the ring.

To simplify (3.2) and (3.3), we exploit the general results in §2. Thus we know the time evolution of Q, E, J, Z, I and C. The first step is to express these six variables in terms of a, b, c, d, k, l, m, n. This is simple in principle, but the algebra is quite involved. Because the results in §2 provide six variables, while (3.1) introduces eight, we need two additional variables, which are defined in (3.6) and (3.8).

The simplest two relations are

$$Q = \pi k^2/(ln-m^2)^{\frac{1}{2}}, \tag{3.4a}$$

$$Z = (c-b+f)/k, \tag{3.4b}$$

and it should be noted that for the special solution (3.1) the infinite family of invariants in (2.7) reduces to constancy of Z in (3.4*b*). To prove this, one simplifies the integral in (2.7) by first using the principal-axis system in which m in (3.1) is zero. Rescaling the coordinate axes so that the region of integration is circular shows that the arbitrary function F affects only the value of a dimensionless constant of proportionality.

For reference the following geometric results should be noted:

$$Q = \tfrac{1}{2}kA, \quad A^4 = (4\pi Q)^2/(ln-m^2), \quad I = A^3(n+l)/24\pi^2, \tag{3.5}$$

where A is the area of the elliptical base and I is the moment of inertia defined in (2.10). For an axisymmetric vortex Q, A and I are not independent. Rather any one of these three can be calculated from the other two. Thus, introducing

$$\alpha \equiv 3\pi I/AQ, \tag{3.6}$$

one can show that α is one for an axisymmetric state, and is otherwise greater than one. α will be referred to as the 'shape parameter'. In fact, one can show that

$$r \equiv \frac{\text{minor axis}}{\text{major axis}} = [\alpha-(\alpha^2-1)^{\frac{1}{2}}]^2, \tag{3.7a}$$

$$\alpha = \tfrac{1}{2}[r^{\frac{1}{2}}+r^{-\frac{1}{2}}]. \tag{3.7b}$$

(It should be noted that r is less than one, so (3.7 b) implies a unique value of α for each value of r). α is one of the two new variables that supplement those from §2. The other is the angle

$$\mu \equiv \tan^{-1}\left(\frac{2m}{l-n}\right); \tag{3.8}$$

$\frac{1}{2}\mu$ is the angle through which the coordinate system in (3.1) must be rotated so that $m = 0$, i.e. the angle at which the principal axes of the elliptical base are inclined to the initial coordinate system.

In summary, a, b, c, d, k, l, m, n are replaced by $Q, E, J, Z, I, C, \alpha, \mu$. Some intermediate steps are given in Appendix A. The final result is

$$\left(\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}\right)^2 + V(\alpha;\beta,j,z) = 0, \tag{3.9a}$$

$$V = -\alpha^2\{(\alpha^2-1)(\beta-\alpha)-z^2(j-\alpha)^2\}, \tag{3.9b}$$

$$\beta = \frac{9\pi(C-2J^2)}{8gQ^3}, \quad z^2 = \frac{QZ^2}{4\pi g}, \quad j = \frac{3\pi J}{Q^2 Z}, \tag{3.9c–e}$$

so that α is governed by a simple nonlinear equation with a non-uniformly progressing time τ, defined in (3.10) below. The function V depends on the three non-dimensional constants β, z^2 and j. As was mentioned after (2.13), β is always greater than or equal to one. If β is equal to one then the mass of fluid is axisymmetric. The non-dimensional angular momentum j has also been defined so that $j = 1$ for an axisymmetric state.

The qualitative analysis of (3.9) in §4 hinges on the observation that α in (3.9) is restricted to the interval in which V is negative. For instance, if $z^2 = 0$ then this interval is $(1, \beta)$, so that the shape changes periodically from circular ($\alpha = 1$) to the maximum eccentricity possible ($\alpha = \beta$). In §4 the various inequalities that constrain the initial specification of α, β, j and z are discussed in detail.

In (3.9) the time variable has also been changed:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{fI_0}{I} = \frac{f}{i}, \quad i \equiv \frac{I}{I_0}, \tag{3.10}$$

where I_0 is a reference moment of inertia:

$$I_0 \equiv \frac{4}{3f}\left(\frac{gQ^3}{\pi}\right)^{\frac{1}{2}}. \tag{3.11}$$

The evolution of the non-dimensional moment of inertia i is obtained by non-dimensionalizing and rearranging (2.12) and (2.13):

$$\left(\frac{\mathrm{d}i}{\mathrm{d}t}\right)^2 = f^2\{8ei-\beta-(i-jz)^2\}, \tag{3.12}$$

where

$$e \equiv E/f^2 I_0 \tag{3.13}$$

is a non-dimensional energy.

Finally, μ is calculated from

$$\frac{\mathrm{d}\mu}{\mathrm{d}\tau} = -f\frac{\mathrm{d}t}{\mathrm{d}\tau} + \alpha z \frac{j\alpha - 1}{\alpha^2 - 1} \tag{3.14}$$

Thus, to summarize, instead of the eight functions of time in (3.2) and (3.3), we have five constants, Q, Z, J, E and C, and three functions of time, i, α and μ. The dimensionless moment of inertia i can be obtained quite easily from (3.12). The evolution of α is determined by (3.9), and, while it is not possible to solve this in closed form, it is easy to understand it qualitatively. Finally, the angle μ is obtained from (3.14). The structure of this set is disappointingly simple. Although (3.2), (3.3) is an eighth-order system, the various integrals in §2 constrain the motion to be regular.

In §4 various limiting and special cases of (3.9*a*), (3.12) and (3.14) are discussed.

4. A qualitative analysis of the motion

To systematically discuss the dependence of the dynamics on the various parameters introduced in § 3, it is helpful to distinguish between variations in *size* of the ring and variations in *shape*. Because the radius of gyration $(I/Q)^{\frac{1}{2}}$ is proportional to $i^{\frac{1}{2}}$, variations in size are governed by (3.12). These necessarily occur at the inertial frequency (this is more easily seen from (2.12)).

Variations in shape are governed by (3.9). This is because vortices with the same value of α have the same ratio of major to minor axis, independent of the size of the eddy. It is shown below that shape changes are superinertial, i.e. the shape of the eddy changes more rapidly than its size.

4.1. *Solutions with permanent size*

These are solutions in which the various constants of the motion are related so that the only solution of (3.12) is

$$i = i_* \equiv (\beta + j^2 z^2)^{\frac{1}{2}}, \tag{4.1}$$

and this requires that

$$e = \tfrac{1}{4}[-jz + (\beta + j^2 z^2)^{\frac{1}{2}}] \tag{4.2}$$

When (4.2) is not satisfied the moment of inertia changes at the inertial frequency. It is still convenient to define i_* as in (4.1), and, in fact,

$$(i_*)^{-1} = \int_0^{2\pi} i^{-1}\,\mathrm{d}t/2\pi, \tag{4.3}$$

i.e. i_* is the harmonic mean of the changing moment of inertia.

It might appear that (4.2) is a very special condition. However, if it is not satisfied then the eddy should pulsate at the inertial frequency. While vigorous inertial motions do occur inside eddies, a gross pulsation at the inertial frequency is not a conspicuous feature of warm-core rings. Indeed, one is led to speculate that if the energy is greater than the right-hand side of (4.2) then the attendant inertial pulsations should excite inertial waves in the underlying cold fluid. This wave radiation would cease if it reduced the energy to the value in (4.2). An investigation of this radiation process is beyond the scope of the present paper, but this argument does suggest that solutions with permanent size are less special than they might first seem.

In any case, if (4.1) holds then τ is simply proportional to t:

$$\tau = ft(\beta + j^2 z^2)^{-\frac{1}{2}}, \tag{4.4}$$

and this will frequently be used to simplify subsequent results.

4.2. *Solutions with permanent shape*

The shape changes are determined by (3.9). Because $(\mathrm{d}\alpha/\mathrm{d}\tau)^2$ is positive, α is confined to the interval in which V is negative. For positive α (which is the only physically relevant case) this allowed interval is a subset of $(1, \beta)$ and will be denoted by (α_1, α_2):

$$V(\alpha_n; z^2, \beta, j) = 0, \quad n = 1 \text{ or } 2. \tag{4.5}$$

The above defines α_1 and α_2 implicitly as functions of (z^2, j, β).

Now, in general, α is a periodic function of τ that oscillates between the limits α_1 and α_2. To see this, we note that α cannot 'stop' in the middle of the interval (α_1, α_2) because $\mathrm{d}\alpha/\mathrm{d}\tau \neq 0$ there. Moreover, α cannot stop at the endpoints α_1 and α_2, because this would imply that $\mathrm{d}^2\alpha/\mathrm{d}\tau^2$ was discontinuous. (I am grateful to G. R. Flierl for this observation.) There are also the spurious steady solutions $\alpha = \alpha_1$ or α_2, discussed in Appendix A. Disregarding these, we conclude that the only way in which the shape of the vortex can be constant is that $\alpha_1 = \alpha_2$. In this case V has a double root at α_1 and is positive everywhere else. Thus the solution is trapped at α_1. These special solutions with constant shape are just the 'rodons' discovered by Cushman-Roisin *et al.* (1985). Their identification in this context makes it likely that they are the only solutions of (3.2) and (3.3) that can be expressed in terms of elementary functions. The improbable alternative is that (3.9) can be integrated exactly.

We now focus on the rodon solutions. The goal here is to locate the surface in (z^2, j, β)-space on which rodons reside. Thus in (3.9) these three parameters can be independently specified. However, it is clear that if $|j|$ becomes too large then eventually V has no positive zeros. Rodons are found at the critical values of j where the two zeros coalesce and further increases in $|j|$ result in the disappearance of the positive zeros of V.

The condition for a double zero is that V and $\partial V/\partial\alpha$ vanish at the same value of α. These relations can be rearranged to give z and j in terms of α and β:

$$z^2 = \frac{(3\alpha^2 - 2\alpha\beta - 1)^2}{4(\alpha^2-1)(\beta-\alpha)}, \quad j = \frac{\alpha^3 + \alpha - 2\beta}{3\alpha^2 - 2\alpha\beta - 1} \tag{4.7}$$

and these functions are contoured in figure 1. These figures are useful if one elects to describe the rodon solution using α and β as independent parameters.

Numerical elimination of α between z^2 and j, with β fixed, gives the curves in figure 2. These are useful if z^2 and β are used as the two parameters that describe the rodon. Thus, if z^2 and β are specified then there are two values of j at which solutions with permanent shape are found.

The first curve is

$$j_+(z^2, \beta) \to \begin{cases} \infty & \text{as} \quad z^2 \to 0, \\ \beta & \text{as} \quad z^2 \to \infty, \end{cases} \tag{4.8}$$

and on this curve

$$\beta > \alpha > \tfrac{1}{3}[\beta + (\beta^2+3)^{\frac{1}{2}}]. \tag{4.9}$$

The second curve is

$$j_-(z^2, \beta) \to \begin{cases} -\infty & \text{as} \quad z^2 \to 0, \\ 1 & \text{as} \quad z^2 \to \infty, \end{cases} \tag{4.10}$$

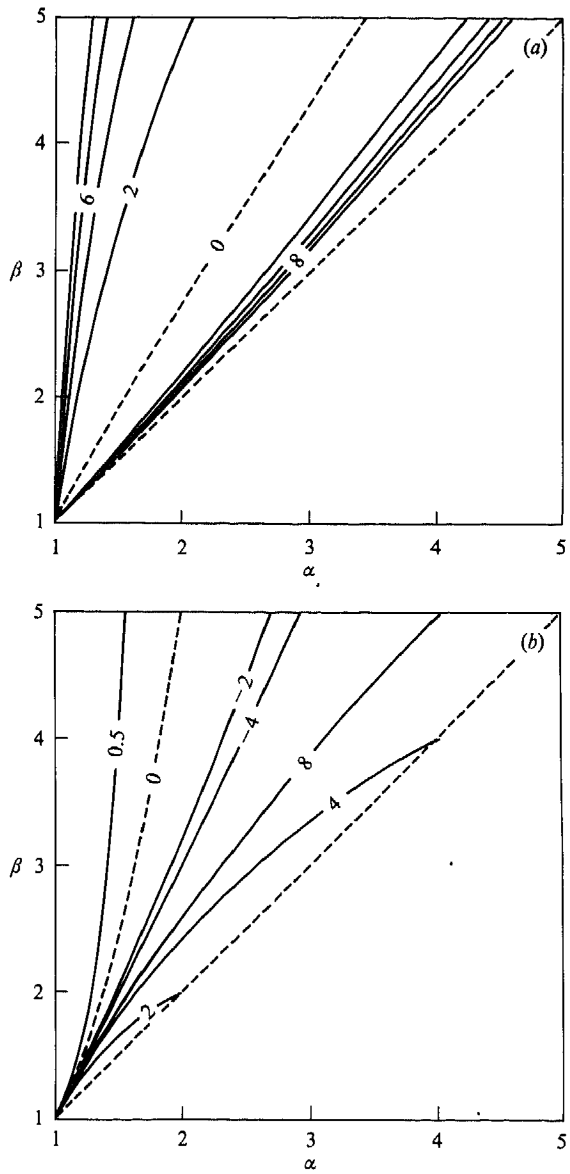

FIGURE 1. (*a*) The enstrophy z^2, defined in (3.9), as a function of α and β for a rodon. (*b*) The angular momentum j, defined in (3.9), as a function of α and β for a rodon. The rodon is a two-parameter family of solution. In this figure the two independent parameters have been taken as α (defined in (3.6) or (3.7)) and β (defined in (3.9)). Rodons with large α are very eccentric.

and here
$$\tfrac{1}{3}[\beta+(\beta^2+3)^{\frac{1}{2}}] > \alpha > 1. \tag{4.11}$$
Thus, on j_-, α is in the lower portion of the interval $(1, \beta)$, and so these shapes are more nearly circular than those on j_+.

One can also eliminate β between (4.7), and the result is
$$2z^2(\alpha j-1)(j-\alpha) = (\alpha^2-1)^2. \tag{4.12}$$
This shows that j cannot be in the interval (α^{-1}, α), and in fact it is straightforward to show that $j \geqslant \alpha$ on the upper curve in figure 2 and $j \leqslant \alpha^{-1}$ on the lower curve.

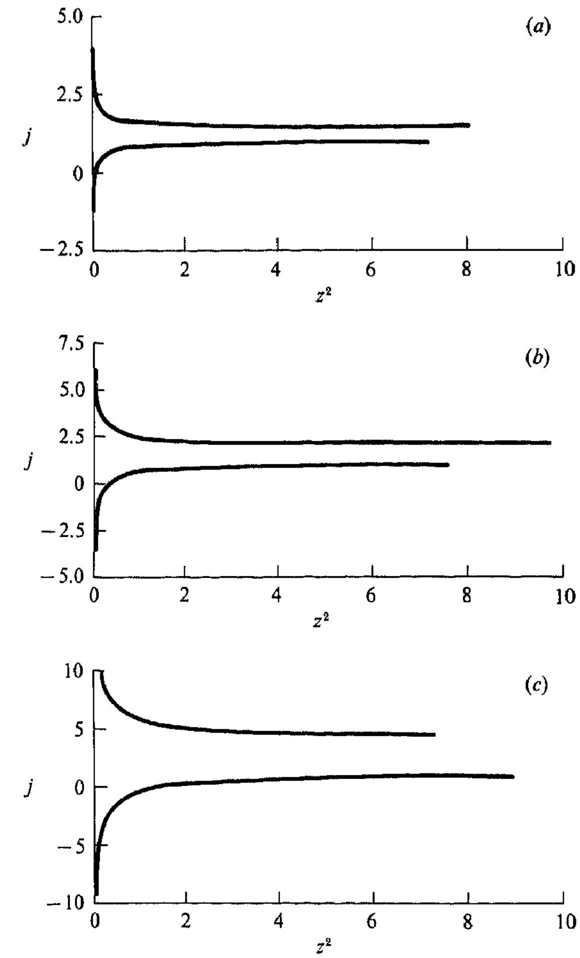

FIGURE 2. If z^2 and β are used as the two parameters defining a rodon then j^2 can be found from the graph above. (a) $\beta = \surd 2$, (b) $\beta = 2$, (c) $\beta = 4$. For given z^2 and β there are two possible rodons. The upper banch $j_+(z^2, \beta)$ are subinertial rodons, for which α is defined by (4.9). The lower branch $j_-(z^2, \beta)$ are superinertial rodons, for which α is limited by (4.11). An arbitrary initial condition (i.e. one that is not a rodon) must have $j_- \leqslant j \leqslant j_+$. Thus the curves above are a boundary in parameter space that encloses allowed values of j.

These curves are significant landmarks in parameter space. Consideration of the general problem (i.e. when the shape of the eddy changes) shows that, with β and z^2 prescribed, j must lie between j_+ and j_-. If j is outside this interval then V in (3.8) has no positive real zeros. This does not mean that the shallow-water equations have no solution with these values of j, but only that the solution is not of the very special form in (3.1).

It is interesting that rodons lie on the boundary of the allowed region of parameter space. It is tempting to speculate that there is a bifurcation to a different class of solutions when this boundary is crossed.

From the preceding remarks, it is clear that the rodons are a two-parameter family of solutions. The choice of parameters is of course arbitrary and different choices are convenient in different contexts. The parameters used by Cushman-Roisin *et al.* (1985) are essentially α and i_*. This is a particularly useful choice for comparison with observations. Various formulae that relate j, β, etc., to α and i_* are given in Appendix B.

The rodons can change both their size (3.12) and their orientation (3.14). Rewriting (3.14) in terms of t and using (4.12) gives

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = -f+fz\frac{\alpha}{i}\frac{(j\alpha-1)}{\alpha^2-1} \tag{4.13a}$$

$$= -f+\frac{f}{2z}\frac{\alpha}{i}\frac{\alpha^2-1}{j-\alpha}. \tag{4.13b}$$

Now on $j_-(z^2,\beta), j < \alpha^{-1}$, and the last term in (4.13a) is negative if z is positive. Thus on j_- the rotation rate is superinertial. On $j_+(z^2,\beta), j > \alpha$, and the last term in (4.13b) is positive if z is positive. Thus, on j_+ the rotation rate is subinertial. If z is negative then these conclusions are reversed, but in this case the vortex is inertially unstable.

It is difficult to compare the rotation rates in (4.13) with the expressions given by Cushman-Roisin *et al.* (1985). This is because they use i_* and α, rather than α and β, as independent variables. For more details see Appendix B.

4.3. *Near-permanent-form solutions*

In §4.2 solutions of permanent shape ($\mathrm{d}\alpha/\mathrm{d}\tau = 0$) were discussed. These solutions require certain relations between the constants of motion, e.g. (4.7). If these relations are only approximately satisfied then we expect that α will exhibit small harmonic oscillations about some mean value:

$$\alpha = \alpha_* + \epsilon, \quad \epsilon \ll 1, \tag{4.14}$$

In the above, α_* is defined to be the solution of

$$\partial V/\partial \alpha = 0,$$

and if ϵ is to be small then one must have

$$V(\alpha_*;\beta,j,z) = O(\epsilon^2) \ll 1$$

Thus, expanding V in a power series about α_*, (3.9) becomes

$$\left(\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau}\right)^2 + \frac{\epsilon^2}{2}\frac{\partial^2 V}{\partial \alpha^2} = -V, \tag{4.15}$$

where V and $\partial^2 V/\partial\alpha^2$ are evaluated at α_*. Direct calculation gives

$$\frac{\partial^2 V}{\partial \alpha^2} = -\frac{6V}{\alpha_*^2} + \alpha_*^2\{6\alpha_* - 2\beta + 2z^2\}, \tag{4.16}$$

and if ϵ is to be small then the first term on the right-hand side must be small relative to the second.

Thus (4.16) has harmonic solutions:

$$\epsilon = \epsilon_0 \cos\omega_\tau \tau, \quad \omega_\tau^2 = \alpha_*^2\{3\alpha_* - \beta + z^2\}, \quad \epsilon_0^2 = -V/\omega_\tau^2. \tag{4.17}$$

It is difficult to estimate the timescale of these small shape changes from this, because (4.15) uses τ as a time variable. However, assuming that the size is constant, one can use (4.4) to replace τ by t in (4.15):

$$\epsilon = \epsilon_0 \cos\omega t, \quad \omega^2 = \frac{f^2\alpha_*^2(3a_* - \beta + z^2)}{\beta + j^2 z^2}. \tag{4.18}$$

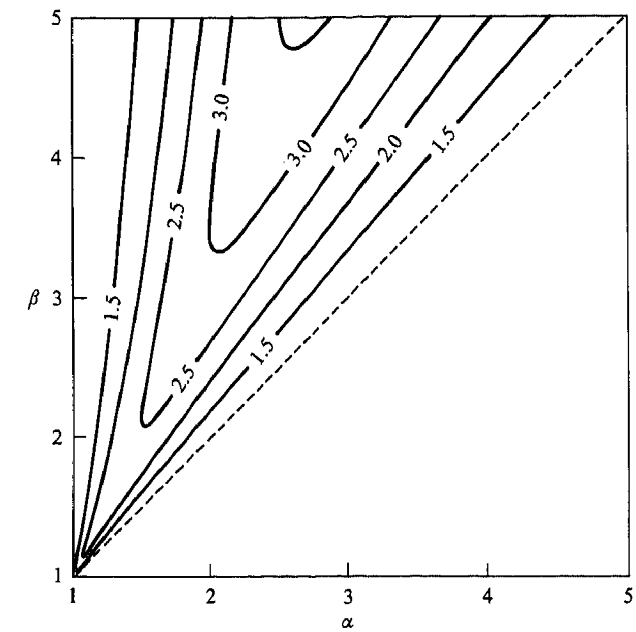

FIGURE 3. If a rodon is slightly perturbed then its shape changes periodically, e.g. (4.15). $(\omega/f)^2$ is plotted here, where ω is the frequency of these small oscillations. α and β are the two parameters that define the rodon. This function is always greater than one, so that shape changes are superinertial.

In figure 3, $(\omega/f)^2$ is contoured as a function of α_* and β. Because this function is always greater than one, shape changes are superinertial.

As an example, consider the special case when the shape is close to axisymmetric. Thus α, β and j are all close to 1 in (4.16), so

$$\omega^2 \approx \frac{f^2(2+z^2)}{1+z^2}. \tag{4.19}$$

For very large values of z^2 the shape changes occur at the inertial frequency. In all other cases they are faster.

Perturbations about an axisymmetric vortex are an important special case of (4.14), and it is worth while to return to (3.9) and analyse these in more detail. Thus, suppose

$$\left.\begin{aligned} \alpha &= 1+\gamma, \quad \gamma \ll 1, \\ \beta &= 1+\delta, \quad \delta \ll 1, \\ j &= 1+\rho, \quad \rho \ll 1, \end{aligned}\right\} \tag{4.20}$$

and substitute these into (3.8), neglecting cubic terms. The result is

$$(2+z^2)\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\right)^2 + (\gamma-\gamma_0)^2 = \gamma_1^2, \tag{4.21a}$$

$$\gamma_0 = (\delta+\rho z^2)/(2+z^2), \tag{4.21b}$$

$$\gamma_1^2 = \frac{\delta^2+2z^2\delta\rho-2z^2\rho^2}{(2+z^2)^2}. \tag{4.21c}$$

(In the notation of (4.13): $\alpha_* = 1+\gamma_0$ and $\epsilon = \gamma-\gamma_0$.) Thus the orbits in the $(\gamma, \mathrm{d}\gamma/\mathrm{d}\tau)$-plane are ellipses. From (4.21 *c*), it follows that ρ must satisfy the double inequality

$$\left.\begin{aligned} \rho_+ > \rho > \rho_-, \\ \rho_\pm = \tfrac{1}{2}\delta\{1 \pm [1+(2/z^2)]^{\frac{1}{2}}\}, \end{aligned}\right\} \quad (4.22)$$

because otherwise γ_1^2 is negative. This is the near-axisymmetric version of the inequality that restricts j to the interval (j_-, j_+). Note how as β approaches one $(\delta \to 0)$ the range of j becomes increasingly restricted. Numerical calculations in § 4.4 suggest that the above are likely to be excellent approximations when r in (3.7) is less than 2. This probably includes all the geophysically relevant cases.

4.4. *Some numerical examples*

To illustrate the preceding formalism, it may help to consider some specific examples, starting with the parameters which are most easily observed.

Consider a ring whose major axis is 300 km and whose minor axis is 150 km; the central depth is taken as 500 m and the reduced gravity as 2 cm s^{-2}.

If we assume that the ring is a rodon then the above are sufficient to determine all the various non-dimensional parameters. With $r = 2$ we calculate $\alpha = 1.06066$ from (3.7). It is significant that α is close to one even for a ring with such pronounced eccentricity. From (B 3) and (B 4), $R_\mathrm{d}^2 = 10^3$ km^2 and $R^2 = 11.25 \times 10^3$ km^2, so (B 2) gives

$$\left(\frac{R^2}{8R_\mathrm{d}^2}\right) = \left(\frac{i_*}{\alpha}\right)^2 = 1.40625.$$

Then, from (B 5),

$$\begin{aligned} \lambda_- &= 2.3949, & \beta_- &= 1.21034, \\ \lambda_+ &= 0.41754, & \beta_+ &= 1.08675. \end{aligned}$$

Now, given α and β, one can calculate z^2 and j from (4.7) (or equivalently from (B 6)). The result is

$$\begin{aligned} j_- &= 0.86629, & z_-^2 &= 0.49524, \\ j_+ &= 1.1543, & z_+^2 &= 0.371924. \end{aligned}$$

We see that $j_+ > \alpha$ and $j_- < \alpha^{-1}$. Thus there are two rodons compatible with the initial conditions. The subinertial mode is j_+, and, from (4.13) or (B 7),

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = -0.0772f,$$

while, for the superinertial mode j_-,

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = -1.3853f.$$

In the above example we noted that α was very close to 1 even though $r = 2$. In fact, with $r = 4$, $\alpha = 1.25$. These results suggest that the axisymmetric limit, summarized in (4.20)–(4.22), contains most of the geophysically interesting cases. Of course, β must also be close to one if these results are to be used. From (B 5) we see that, provided λ is not very large, β will be close to one when α is close to one. Now if $(R^2/8R_\mathrm{d}^2)$ is of order one then this is the case for both λ_+ and λ_-. In this case both the subinertial and superinertial mode are described by (4.20)–(4.22). However, if $R^2/8R_\mathrm{d}^2$ becomes too large then λ_- becomes large and β_- may be substantially different from one. To summarize, the subinertial mode j_+ is always described by

(4.20)–(4.22). The superinertial mode is described by this system only if $R^2/8R_d^2$ is not too large (i.e. not so large as to make β in (B 5) much greater than one). The above numerical example is a case in point. Here β_+ is 1.08 but β_- is 1.21, so at this rather modest value of $R^2/8R_d^2$ the expansion is starting to fail for the superinertial mode.

5. Conclusions and some speculations

Taking Goldsbrough's (1930) reduction (3.1) of the shallow-water equations we have emphasized how a systematic application of Ball's (1963) theorems reduces this eighth-order system to quadratures. Thus the eight time-dependent functions in (3.2) are replaced by five constants (Q, Z, E, J and C) and three time-dependent functions (I, α, μ). An important point is that the initial specification of $(a, b, \dots, n)$ implies certain constraints on the constants of motion. Thus they must satisfy $\beta > \alpha > 1$ (Ball 1963) and $j_+ \geqslant j \geqslant j_-$ (see figure 3). Solutions of the shallow-water equations that fail to satisfy these inequalities may exist, but they do not have the simple form posited in (3.1).

It was noted that the rodon solution of Cushman-Roisin *et al.* (1985) lies on the border of the allowed region in parameter space, i.e. $j = j_+(z^2, \beta)$ or $j_-(z^2, \beta)$. Further, if $z > 0$ then rodons on the j_+ branch rotate subinertially while those on the j_- branch have superinertial rotation rates.

After discussing solutions of permanent size or shape, we analysed the small oscillations that accompany a small perturbation of the constant shape solution. The most important point to emerge here is that shape changes are superinertial (figure 3). This perturbation analysis does not prove that shape changes are generally superinertial, but it is likely that this is the case.

Throughout this analysis it has been supposed that the fluid outside the elliptical eddy is unstratified and infinitely deep. Also the β-effect has been ignored. These idealizations ensure that the exterior fluid remains motionless and the eddy preserves its initial energy and angular momentum. However, if these assumptions are relaxed, the eddy will lose energy and alter its angular momentum, because its motion will generate both internal gravity waves and Rossby waves. The timescales suggest that shape and size changes will generate internal waves, while the subinertial orientation changes will generate Rossby waves.

However this wave radiation cannot completely spin-down the eddy. This is because the enstrophy Z remains invariant ((2.6) is unchanged if one includes the β-effect and pressure forces associated with lower-layer motion in (2.1)). It seems likely that the internal-wave radiation will alter E, J and C so that the solution has constant shape and size. Then Rossby-wave radiation, on a longer timescale, will 'axisymmetrize' the eddy. Throughout this process, Q and Z are unchanged. In the near-axisymmetric case the Rossby-wave radiation problem is difficult (see Flierl 1984*a*, *b*), and so analytic progress in the strongly non-axisymmetric problem posed here is unlikely. Thus, without a numerical model, these remarks remain plausible speculation.

Finally it is perhaps disappointing that the eighth-order system in (3.1) and (3.2) has such simple, regular behaviour. This is because the general arguments in §2 provide so many constants of the motion. One wonders if the regularity found here survives if the problem is changed slightly. For instance, Ball (1965) discussed the tidal modes of an elliptic paraboloid, but his analysis stopped short of the complete qualitative study given here. He noted, however, that the angular momentum of the liquid is no longer constant when it is confined in a basin whose cross-section is elliptical. Also the moment of inertia can be no longer calculated at the outset, and there is no longer an integral analogous to C in (2.13). This raises the possibility that the tidal motion in such a basin may be dynamically richer than the system studied here.

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Appendix A. Intermediate steps leading (3.8) *et seq.*

To change variables from a, b, c, d, k, l, m, n to $Q, E, J, Z, C, I, \alpha, \mu$, it is convenient to first introduce an intermediate set. Thus, instead of a, b, c, d we use

$$Z = (c-b+f)/k, \quad \delta = a+d, \quad \zeta^2 = (b+c)^2+(a-d)^2, \quad \phi = \tan^{-1}\left(\frac{b+c}{a-d}\right) \tag{A 1}$$

to describe the velocity field. Z is constant. δ is the rate of contraction or expansion of the area A. $\frac{1}{2}\phi$ is the angle through which the coordinate system in (3.1) must be rotated so that $b = -c$, i.e. so that the velocity strain tensor is antisymmetric. ζ is a measure of departures from axisymmetry: a circular vortex has $\zeta = 0$.

Next, instead of k, l, m, n, we use

$$Q = \pi k^2(ln-m^2)^{-\frac{1}{2}}, \quad \eta = l+n, \quad \xi^2 = (1-n)^2+4m^2, \quad \mu = \tan^{-1}\left(\frac{2m}{l-n}\right). \tag{A 2}$$

Q is the constant volume of the ring. $\frac{1}{2}\mu$ is the angle through which the coordinate system in (3.1) must be rotated so that $m = 0$. ξ is a measure of departures from axisymmetry: a circular vortex has $\xi = 0$. η is roughly an inverse measure of the diameter of the vortex.

Because Z and Q are constant, (3.2) and (3.3) are replaced by a sixth-order system:

$$\dot{\xi} = -2\delta\xi - \eta\zeta\cos(\mu-\phi), \tag{A 3a}$$

$$\dot{\eta} = -2\delta\eta - \xi\zeta\cos(\mu-\phi), \tag{A 3b}$$

$$\dot{\mu} = -f+\eta\frac{\zeta}{\xi}\sin(\mu-\phi)+\frac{2QZ}{A}, \tag{A 3c}$$

$$\dot{\zeta} = -\delta\zeta+g\xi\cos(\mu-\phi), \tag{A 3d}$$

$$\dot{\delta} = -\frac{1}{2}\left\{\delta^2+\zeta^2+f^2-\left(\frac{2QZ}{A}\right)^2-2g\eta\right\}, \tag{A 3e}$$

$$\dot{\phi} = -f+g\frac{\xi}{\zeta}\sin(\mu-\phi), \tag{A 3f}$$

and the geometric properties of the fluid are

$$A^4 = (8\pi Q)^2(\eta^2-\xi^2)^{-1}, \quad I = \frac{A^3}{24\pi^2\eta}. \tag{A 4}$$

The time derivatives of these, which can of course be deduced from (A 3), are:

$$\dot{A} = \delta A, \quad \dot{I} = \delta I - \frac{\xi\zeta A^3}{24\pi^2}\cos(\mu-\phi). \tag{A 5}$$

In terms of these new variables, the constants of motion from § 2 are

$$J = \frac{QZI}{A} + \frac{A^3\xi\zeta}{48\pi^2}\sin(\mu-\phi), \tag{A 6a}$$

$$E = g\frac{2Q^2}{3A} + \left[\delta^2 + \left(f - \frac{2QZ}{A}\right)^2 + \zeta^2\right]\frac{I}{8} - \frac{A^3\xi\zeta}{96\pi^2}\left[\delta\cos(\mu-\phi) + \left(f - \frac{2QZ}{A}\right)\sin(\mu-\phi)\right], \tag{A 6b}$$

$$C = 2J^2 + \left(\frac{Q}{3\pi}\right)^2 A^2(g\eta + \tfrac{1}{2}\zeta^2), \tag{A 6c}$$

and once again these relations can be verified directly from (A 3).

We now use (A 6) to simplify (A 3). It is convenient to introduce

$$\alpha \equiv \frac{3\pi I}{AQ} = \eta(\eta^2 - \xi^2)^{-\frac{1}{2}}, \tag{A 7}$$

and then from (A 4) and (A 6c):

$$\xi = \frac{8Q^3\alpha^2(\alpha^2-1)^{\frac{1}{2}}}{9\pi I^2}, \quad \eta = \frac{8Q^3\alpha^3}{9\pi I^2}, \quad \zeta^2 = \frac{16gQ^3}{9\pi}\left(\frac{\alpha}{I}\right)^2(\beta-\alpha), \tag{A 8}$$

where β is defined in (3.8). Using (A 8), we can write (A 6a) as

$$j = \alpha + z^{-1}[(\alpha^2-1)(\beta-\alpha)]^{\frac{1}{2}}\sin(\mu-\phi), \tag{A 9}$$

where j and z are defined in (3.8).

From (A 5), we can find an expression for the rate of change of α:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\tau} = -\alpha[(\alpha^2-1)(\beta-\alpha)]^{\frac{1}{2}}\cos(\mu-\phi), \tag{A 10}$$

where τ is the new time variable defined in (3.9). Then, eliminating the trigonometric functions between (A 9) and (A 10) gives (3.9a, b).

Using (A 8) to rewrite (A 3c) gives

$$\frac{\mathrm{d}\mu}{\mathrm{d}\tau} = -f\frac{\mathrm{d}t}{\mathrm{d}\tau} + \frac{\alpha z(j\alpha-1)}{\alpha^2-1}, \tag{A 11}$$

and (A 3f) becomes

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = -f\frac{\mathrm{d}t}{\mathrm{d}\tau} + \frac{\alpha z(j-\alpha)}{2(\beta-\alpha)}. \tag{A 12}$$

There is one subtle and confusing point associated with (3.9a). This is that the elimination of the trigonometric functions between (A 9) and (A 10) has introduced spurious, unphysical solutions. These are

$$\alpha = \alpha_* \quad \text{and} \quad V(\alpha_*) = 0, \quad \frac{\partial V(\alpha_*)}{\partial\alpha} \neq 0. \tag{A 13}$$

It is clear that $\alpha = \alpha_*$ is an exact, steady solution of (3.9a). But it is not an exact solution of (A 10)–(A 12). To see this, note that if $\mathrm{d}\alpha/\mathrm{d}\tau = 0$ in (A 10) then in general (unless $\alpha = 1$ or β) $\cos(\mu-\phi)$ must be zero. But, by direct substitution in (A 11) and (A 12), one finds that $(\mathrm{d}/\mathrm{d}\tau)(\mu-\phi) \neq 0$ when $\alpha = \alpha_*$ *unless* α_* is a double root of V. Thus, although $\alpha = \alpha_*$ is a mathematical solution of (3.9a), it is not a physical solution!

Analogous spurious solutions occur in elementary mechanics, and, just as they are disregarded there, they cause no concern here. Consider for instance the motion of a simple harmonic oscillator whose displacement is α. In suitable non-dimensional coordinates, Newton's equation of motion is

$$\ddot{\alpha}+\alpha = 0, \tag{A 14}$$

or, forming an energy equation by multiplying by $\dot{\alpha}$,

$$\tfrac{1}{2}\dot{\alpha}^2+\tfrac{1}{2}\alpha^2 = E. \tag{A 15}$$

Now $\alpha = \pm(2E)^{\frac{1}{2}}$ is an exact steady solution of (A 15) but clearly not of (A 14). The origin of this difficulty is that if $\dot{\alpha} = 0$ then (A 15) does not imply (A 14). Specifically, if (A 15) is differentiated then

$$\dot{\alpha}(\ddot{\alpha}+\alpha) = 0,$$

and the spurious solution satisfies the above because $\dot{\alpha} = 0$ rather than because of (A 14).

The spurious solutions of (3.9*a*) are entirely analogous to the above, but locating the zero divisor is more difficult.

Appendix B. α and i_* as independent variables

In §4 it was convenient to discuss constant-shape solutions using α and β as independent variables. Thus (4.7) expresses the angular momentum and the enstrophy of constant-shape solutions in terms of these. Likewise, (4.13) gives the rotation rate. These were the most convenient variables for theoretical analysis because they are directly related to constants of the motion.

However, it may also be convenient to use α and i_* as independent variables, for example, §4.4. i_* is defined in (4.1), and it can be verified from (3.12) that i_* is the harmonic mean of i.

First we use (4.7):

$$\begin{aligned} i_*^2 &= \beta+j^2z^2 \\ &= \left[4(\beta-\alpha)^2+\frac{(\alpha^2-1)^2}{4(\alpha^2-1)(\beta-\alpha)}\right]\alpha^2. \end{aligned} \tag{B 1}$$

Because the arithmetic mean is greater than the geometric mean, the above implies that

$$(i_*/\alpha)^2 \geqslant 1.$$

This can be related to other results in the literature if we express i_* and α in terms of dimensional variables. Using (3.5*a*), (3.6) and (3.10):

$$\left(\frac{i_*}{\alpha}\right)^2 = \left(\frac{R^2}{8R_{\mathrm{d}}}\right)^2 \geqslant 1, \tag{B 2}$$

where R is the geometric mean of the major and minor axes,

$$A = \pi R^2, \tag{B 3}$$

and R_{d} is the deformation radius based on the central depth,

$$R_{\mathrm{d}}^2 = gk/f^2. \tag{B 4}$$

Thus (B 2) generalizes Killworth's (1983) restriction on the minimum radius of an isolated eddy by including departures from axisymmetry (define R by (B 3)).

Now use (B 1) to express β in terms of α and i_*:

$$\left.\begin{aligned}\beta &= \alpha+\tfrac{1}{2}(\alpha^2-1)\,\lambda,\\ \lambda_{\mp} &= \left(\frac{i_*}{\alpha}\right)^2 \pm\left[\left(\frac{i_*}{\alpha}\right)^4-1\right]^{\frac{1}{2}}.\end{aligned}\right\} \tag{B 5}$$

The perverse sign convention above is so that the formulae agree with (4.8) and (4.10), i.e. λ_+ is for rodons on the j_+ branch. Together with (4.7), the above expresses z and j in terms of α and i_*:

$$j=\frac{\alpha-\lambda}{1-\alpha\lambda},\quad z^2=\frac{(1-\alpha\lambda)^2}{2\lambda}. \tag{B 6}$$

Finally, one can express the rotation rate in terms of α and i:

$$\frac{\mathrm{d}\mu}{\mathrm{d}t}=-f+\frac{f}{2}\frac{i_*}{i}\left\{\left[1-\left(\frac{\alpha}{i_*}\right)^2\right]^{\frac{1}{2}}\pm\left[1+\left(\frac{\alpha}{i_*}\right)^2\right]^{\frac{1}{2}}\right\}, \tag{B 7}$$

where the $+$ sign is for the subinertial modons (j_+) and the $-$ sign for the superinertial (j_-) branch. Using (B 2), we see that if the size is constant ($i=i_*$) then we recover the expression of Cushman-Roisin *et al.* (1985) for the rotation rate of the eddy.

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