Semicompressible Ocean Dynamics

W. K. DEWAR AND J. SCHOONOVER

Department of Earth, Ocean and Atmospheric Science, Florida State University, Tallahassee, Florida

T. J. MCDOUGALL

School of Mathematics and Statistics, University of New South Wales, Sydney, New South Wales, Australia

W. R. YOUNG

Scripps Institution of Oceanography, University of California, San Diego, La Jolla, California

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ABSTRACT

The equations of motion are reexamined with the objective of improving upon the Boussinesq approximation. The authors derive new equations that conserve energy, filter out sound waves, are more accurate than the Boussinesq set, and are computationally competitive with them. The new equations are partly enabled by exploiting a reversible exchange between internal and gravitational potential fluid energy. To improve upon these equations appears to require the inclusion of acoustics, at which point one should use full Navier–Stokes. This study recommends the new sets for testing in general circulation modeling.

1. Introduction

The workhorse equations of ocean theory and modeling are the Boussinesq equations, characterized by a nondivergent velocity field and a constant density except where acted on by gravity. Virtually all ocean numerical models employ the Boussinesq equations because, although they filter out sound waves (which decreases their computational load), they are an accurate approximation to the Navier–Stokes equations and include most recognized ocean dynamics. A consistent form of these equations for a linear equation of state (EOS) has been known since 1960 (Spiegel and Veronis 1960). Accurate ocean modeling, however, requires that a realistic, nonlinear equation of state be employed. It has only been in the last few years that a consistent form for the energetics equation of seawater Boussinesq fluid has been derived (Young 2010). One of the novel aspects of this development was the identification of dynamic enthalpy as a Boussinesq potential energy density. Dynamic enthalpy $\hat{h}^{\dagger}$ is given by

$$\hat{h}^{\dagger}(S_A, \Theta, P_*) = \frac{1}{\rho_o g}\int_{P_o}^{P_*} \hat{b}(S_A, \Theta, \widetilde{P_*})\, d\widetilde{P_*}, \qquad (1)$$

where the integration of the buoyancy $\hat{b}$ involved "freezing" the Conservative Temperature Θ and the Absolute Salinity S_A at their in situ values at static pressure $P_*(z)$:

$$P_* = P_o - \rho_o g z, \qquad (2)$$

where $P_o = 101\,325$ Pa is a standard atmospheric pressure at sea level. The quantity ρ_o in (1) and (2) is the reference density used to replace full density everywhere except where it multiplies gravity g. Buoyancy is given by

$$b(x, y, z, t) = \hat{b}(S_A, \Theta, P_*) = -g\frac{[\rho(S_A, \Theta, P_*) - \rho_o]}{\rho_o}. \qquad (3)$$

Note that our notation differs from that used in Young (2010), who represented pressure in (1) with the pressure-like variable Z, where Z has units of length. We retain an explicit static pressure here for reasons that will become

Corresponding author address: W. K. Dewar, Dept. of Earth, Ocean and Atmospheric Science, Florida State University, P.O. Box 3064520, Tallahassee, FL 32309.
E-mail: wdewar@fsu.edu

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clear. This gives rise to the multiplicative factor appearing in front of the integral in (1). We also adopt the convention that "hats" on variables identify them as functions of the thermodynamic variables S_A, Θ, P_*, while the same variables without the hat implies they are considered as functions of position and time, that is, as field variables.

For all its advantages, the Boussinesq approximation still involves errors, one of which was investigated by Dewar, et al. (1998; DHMY hereinafter). They pointed out that using (2) introduced systematic errors in density which when integrated over depths of O(1km) could produce pressure gradient errors that, when expressed in terms of geostrophic velocity were up to $0.05\,\mathrm{m\,s^{-1}}$. The source of the error was the finite compressibility of seawater, which is neglected in the Boussinesq set. The regions most prone to this were western boundary currents and, since they are centrally involved in meridional ocean heat transport, it was argued to be worthwhile to correct this error. DHMY therefore recommended that density be computed using the full (static plus dynamic) pressure. Young (2010) shows that this is not energetically consistent, which is a strong motivation to revisit how this error can be remedied. There are other well-known shortcomings of the Boussinesq set, such as difficulty in directly computing thermosteric sea level variations that one would also hope to improve.

The purpose of this short note is to describe and rationalize two new sets of equations, the types I and II semicompressible sets, with the potential to replace the Boussinesq equation in ocean modeling. These sets address several of the shortcomings of the Boussinesq set, including the pressure gradient error of DHMY, while retaining the valuable feature of filtering out sound waves. The computational load of the new sets compares quite well with the Boussinesq equations while offering greater accuracy.

Similar issues have been examined in the atmospheric sciences and astrophysics, where compressibility, as measured by the Mach number (i.e., fluid speed divided by the sound speed), is much greater. A set of equations like our type I equations below has been derived for atmospheric settings by Durran (2008) and Klein and Pauluis (2012). Discussion of the anelastic approximation, somewhat like our type II equations, and its extensions to moist atmospheres appears in Pauluis (2008). Methods for uniting Boussinesq and non-Boussinesq (including fully compressible) approaches under a common numerical umbrella are discussed in Smolarkiewicz et al. (2014), again with atmospheric applications in mind. We differ from these efforts in the scalings used to derive the equations and in the use of a realistic seawater equation of state. The latter leads us to a different expression for the potential energy (i.e., dynamic enthalpy) than appears in those papers, although the same functional form was discussed in Pauluis (2008).

The error noted in DHMY has also attracted the attention of others in the ocean modeling community, most notably Dukowicz (2001) and Shchepetkin and McWilliams (2011), who have adopted different solution strategies. They recommend modifying standard Boussinesq models by using a "stiffened" equation of state that accounts for compressibility by a normalizing function depending only on pressure. The needed modifications to existing Boussinesq models are minor. Shchepetkin and McWilliams (2011) provide a thorough discussion of the techniques and ramifications of the use of this method and provide a practical stiffened equation of state. Other related issues received a full discussion in Griffies and Adcroft (2008). Inasmuch as the field is in an exploratory state with these improvements to Boussinesq, all of these approaches can be thought of as competitors from which modeling experience will eventually crown a winner.

In the following, we rationalize the form of the new equations by analyzing the stratified Navier–Stokes equations. A leading-order balance between internal and gravitational potential energy variations associated with the reversible effects of compression is identified that renders the kinetic energy equation similar to that of a Boussinesq fluid. The construction of the semicompressible equations builds on that cancellation. We also demonstrate that they have well-behaved energetics, a prerequisite for an acceptable equation set emphasized by Young (2010).

2. Analysis of the stratified Navier–Stokes equations

The stratified Navier–Stokes momentum, mass, state, tracer, and fundamental thermodynamic relation (Gibbs relation) are

$$\rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u}) = -\nabla p - [\rho - \rho_r(z)]g\mathbf{k} + \rho\mathbf{u}^o\,, \tag{4a}$$

$$\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0\,, \tag{4b}$$

$$\rho = \hat{\rho}(S_A, \Theta, P)\,, \tag{4c}$$

$$\frac{d}{dt}\Theta = \Theta^o\,, \tag{4d}$$

$$\frac{d}{dt}S = S^o\,, \quad \text{and} \tag{4e}$$

$$dE = -Pd\left(\frac{1}{\rho}\right) + \mu dS + T\,d\eta\,, \tag{4f}$$

where subscript t denotes a partial time derivative, d/dt is the material derivative, E is the internal energy, $2\boldsymbol{\Omega}$ is the Coriolis vector, and boldface characters are vectors. The quantity η denotes specific entropy, S_A is the Absolute Salinity on the reference-composition salinity scale (Millero et al. 2008), and μ is the relative chemical potential of seawater. The quantity Θ is Conservative Temperature (McDougall 2003; IOC et al. 2010), and T is absolute in situ temperature in degrees Kelvin. We differentiate in notation between absolute pressure P and dynamic pressure p, such that

$$P = P_o + P_r(z) + p = P_* + p. \tag{5}$$

The quantity $P_r(z)$ is related to the quantity $\rho_r(z)$ in (4a) by

$$\frac{\partial}{\partial z} P_r(z) = -g\rho_r(z), \tag{6}$$

where $\rho_r(z)$ is a reference density profile. The lone constraint on ρ_r is that it need be a function only of z. Here, we are more general than the Boussinesq approximation, where $\rho_r(z) = \rho_o$ and $P_r(z) = -\rho_o gz$. Young (2010) uses $\rho_o = 1023.7\,\mathrm{kg\,m^{-3}}$; ocean models usually use $\rho_o = 1035.0\,\mathrm{kg\,m^{-3}}$ (Griffies 2004). The quantity P_* denotes the static pressure caused by a mean atmospheric load and a seawater density profile $\rho_r(z)$. Although not important, for simplicity, we omit viscosity and the diffusion of tracers, represented in (4) by the symbols decorated with the superscript ^{o}.

A kinetic energy equation is formed from (4) by dotting the momentum equations with velocity, yielding

$$\rho(K_t + \mathbf{u}\cdot\nabla K) = -\mathbf{u}\cdot\nabla p - gw(\rho - \rho_r), \tag{7}$$

where $K = (\mathbf{u}\cdot\mathbf{u})/2$ is the kinetic energy density, and w is the vertical velocity.

Manipulating the right-hand side of (7) leads to

$$-\mathbf{u}\cdot\nabla p - wg(\rho - \rho_r) = -\nabla\cdot p\mathbf{u} - \frac{p}{\rho}\frac{d\rho}{dt} - gw(\rho - \rho_r). \tag{8}$$

Density is now expanded as

$$\hat\rho(S_A, \Theta, P) = \hat\rho(S_A, \Theta, P_* + p) = \hat\rho_B(S_A, \Theta, P_*) + \frac{\partial}{\partial p}\hat\rho_B(S_A, \Theta, P_*)p + O(p^2). \tag{9}$$

The notation $\hat\rho_B$ (the subscript B implying Boussinesq) will denote the density estimate obtained by the use of static pressure in the equation of state rather than total pressure (see also Klein and Pauluis 2012; Durran 2008). Equation (9) introduces the first approximation of the analysis. The error in (9) can be estimated by comparing the next order term in the Taylor expansion for density to the linear term:

$$\frac{\left(\frac{\partial^2}{\partial p^2}\hat\rho\right)p^2}{\left(\frac{\partial}{\partial p}\hat\rho\right)p} = \frac{1}{c_s}\left(\frac{\partial}{\partial p}c_s\right)\rho_o g N_s \sim O\left(\frac{\mathrm{s}}{1500\,\mathrm{m}} \times \frac{40\,\mathrm{m}}{\mathrm{s}\,\rho_o g\,500\,\mathrm{m}}\rho_o g\,1\,\mathrm{m}\right) \sim 5\times10^{-5}, \tag{10}$$

where the pressure perturbation is approximated by the surface contribution, N_s is the free-surface height, and $\rho_p = c_s^{-2}$ is the inverse of sound speed (in seawater) squared. The ratio in (10) is quite small so the $O(p^2)$ contribution will be neglected.

Substituting (9) into (8) and considering density as a thermodynamic variable while continuing to ignore all nonconservative terms,

$$-\nabla\cdot p\mathbf{u} - \underline{\frac{p}{\rho}\frac{d\hat\rho}{dt}} - wg(\rho - \rho_r) = -\nabla\cdot p\mathbf{u} - \frac{p}{\rho}\frac{\partial}{\partial p}\hat\rho\frac{dP}{dt} - wg(\rho_B - \rho_r) - \frac{wgp}{c_s^2}, \tag{11}$$

where the chain rule has been used on the underlined term, and $\partial/\partial p\,(\hat\rho_B) = 1/c_s^2$ has been employed. The total pressure P in (11) can now be broken into its static and dynamic parts:

$$-\nabla\cdot p\mathbf{u} - \frac{p}{\rho}\frac{1}{c_s^2}\frac{dP}{dt} - wg(\rho_B - \rho_r) - wg\frac{p}{c_s^2} = -\nabla\cdot p\mathbf{u} + \underline{\frac{\rho_r}{\rho}\frac{gwp}{c_s^2}} - wg(\rho_B - \rho_r) - \underline{\frac{wgp}{c_s^2}} + O(p^2), \tag{12}$$

where $d/dt\,P_* = \rho_r gw$ has been used, and the $O(p^2)$ denotes the order of the approximation. Note that upon using $\rho = \rho_r + \rho'$, the denominator of the first underlined term in (12) shows that at leading order the two underlined terms in (12) cancel, with the result

$$-\nabla\cdot p\mathbf{u} + \underline{\frac{p}{\rho}\frac{1}{c_s^2}\rho_r gw} - wg(\rho_B - \rho_r) - \underline{\frac{wgp}{c_s^2}} + O(p^2) = -\nabla\cdot p\mathbf{u} - \underline{\underline{wg(\rho_B - \rho_r) - \frac{pg\rho' w}{\rho c_s^2}}} + O(p^2, \rho'^2). \tag{13}$$

The $O(\rho'^2)$ denotes the order of the approximation. Comparing the double underlined terms on the right-hand side of (13),

$$\frac{p\rho' gw}{c_s^2\rho_r wg\rho'} = \frac{gN_s}{c_s^2} = O\left[\frac{10\,\mathrm{m}}{\mathrm{s}^2}\frac{1\,\mathrm{m}}{(1500\,\mathrm{m\,s^{-1}})^2}\right] \sim 5\times 10^{-6}, \tag{14}$$

which we again classify as negligible and neglect the second double underlined term from here on.

Even though compressibility has been allowed in these equations, the kinetic energy equation to a high degree of accuracy becomes

$$\rho(K_t + \mathbf{u}\cdot\nabla K) = -\nabla\cdot p\mathbf{u} - gw(\rho_B - \rho_r), \tag{15}$$

which, because of the appearance of Boussinesq density on the right-hand side, closely resembles the kinetic energy of the classical Boussinesq set. Indeed, employing the usual approximation that density on the left-hand side of (15) is the constant ρ_o, the Boussinesq kinetic energy equation is obtained. This occurs because much of the divergence is completely reversible and simply trades energy between potential and internal.

Tracing the analysis back to momentum, it is seen that the vertical momentum equation to an order consistent with (15) is

$$\rho(w_t + \mathbf{u}\cdot\nabla w + 2\boldsymbol{\Omega}\times\mathbf{u}\cdot\mathbf{k}) = -p_z - g(\rho_B - \rho_r) - \frac{gp}{c_s^2}, \tag{16}$$

which involves a Boussinesq-like form of buoyancy on the right-hand side along with a correction of that buoyancy because of compression. Although gp/c_s^2 is a small correction relative to the other terms in (16), it is responsible for the pressure gradient error noted in DHMY.

To illustrate, consider a fluid with a free-surface variation of $N_s = 1\,\mathrm{m}$ over a lateral separation of 50 km, which is typical of the Gulf Stream and western boundary currents generally. Even if the Conservative Temperature and Absolute Salinity surfaces are perfectly level, subsurface lateral density anomalies ρ' develop because of compression in combination with the sea surface elevation:

$$\rho' \sim p'\frac{\partial}{\partial p}\rho \sim g\rho_o N_s\frac{\partial}{\partial p}\rho = \frac{g\rho_o N_s}{c_s^2}.$$

Integrating these anomalies hydrostatically generates a pressure anomaly over the 50-km separation of

$$g\int_{2000\mathrm{m}}^{0}\frac{p'(x,y,z)}{c_s^2}dz \sim \frac{g\rho_o g N_s H}{c_s^2} \sim \frac{10\,\mathrm{m}}{\mathrm{s}^2}\rho_o\frac{10\,\mathrm{m}\;\;1\,\mathrm{m}}{\mathrm{s}^2}\frac{2000\,\mathrm{m}}{2.25\times 10^6\,\mathrm{m^2\,s^{-2}}} = \rho_o 10^{-1}\,\mathrm{m^2\,s^{-2}} \tag{17}$$

at a depth of 2 km. This pressure gradient expressed in terms of an equivalent geostrophic flow is

$$\delta v = \frac{1}{\rho_o f}\frac{\delta p}{\delta x} \sim \frac{\mathrm{s}}{10^{-4}}\frac{\rho_o 1\,\mathrm{m}^2}{\mathrm{s}^2}\frac{1}{\rho_o 5\times 10^4\,\mathrm{m}} \sim .02\,\mathrm{m\,s^{-1}}.$$

The gradient arises entirely from non-Boussinesq effects and is of vexingly large amplitude, as emphasized in DHMY. It is seen, however, that allowing for the non-Boussinesq correction to density does not substantively change the kinetic energy equation from its Boussinesq form. This provides hope that suitable equations accounting for this issue and simultaneously filtering acoustics while conserving energy can be found.

We now consider the characteristics of an enhanced equation set that can remove the DHMY error in an acceptable manner. Further examination of the above analysis shows that for the needed cancellation between internal and potential energy to occur, it is necessary that the velocity field be divergent. The density variable included in the new equations must then be richer than a constant to allow for this, as well as being an accurate approximation to full density. We also require that the new equation set obey energy conservation. Therefore, as seen from (15), the density variable of the new set must satisfy a conservation equation. Last, we would like to build in as physically realistic fluid compression as possible while simultaneously filtering out sound waves.

There appear to be two choices for a density variable that meet the above constraints, while providing an accurate estimate of the real density.

a. *Type I semicompressible equations*

The first such density variable is the Boussinesq density ρ_B. Requiring that ρ_B satisfy a conservation equation implies

$$\nabla\cdot\mathbf{u} = \frac{-1}{\rho_B}\frac{d}{dt}\rho_B = \frac{w\rho_r g}{c_s^2\rho_B}, \tag{18}$$

again ignoring diabatic effects. The overall accuracy of (18) lies essentially in the smallness of the right-hand side compared to the individual components of the left-hand side; this ratio scales as

$$\frac{gH}{c_s^2} = \frac{10\,\mathrm{m}\cdot 5\times 10^3\,\mathrm{m}}{\mathrm{s}^2(1500\,\mathrm{m\,s^{-1}})^2} \sim 2\times 10^{-2}, \tag{19}$$

which is the usual scaling argument in support of the Boussinesq approximation, the shortcomings of which are emphasized in DHMY. When compared to the full mass conservation equation

$$\nabla\cdot\mathbf{u} = \frac{-1}{\rho}\frac{d}{dt}\rho = \frac{w\rho_r g}{\rho c_s^2} - \underline{\frac{1}{\rho c_s^2}\frac{d}{dt}p}\,, \tag{20}$$

the most problematic error is seen to come from the underlined term. Comparing the quantities on the right-hand side,

$$\frac{w\rho_r g}{\rho c_s^2}\frac{\rho c_s^2}{\frac{dp}{dt}} \sim \frac{w\rho_o g}{\rho_o g\frac{d}{dt}N_s} \sim O(1),$$

shows them to be comparable, so the elimination of the underlined term is ad hoc. It is essential to remove dynamic pressure, however, in order to filter out sound waves. In this sense, (18) is as good as can be achieved in a model without acoustics while retaining a locally accurate density variable.

In the type I equations, (18) takes the place of the Boussinesq continuity equation and ρ_B systematically replaces density. The kinetic energy equation formed from the momentum equations and using (18) becomes

$$\rho_B(K_t + \mathbf{u}\cdot\nabla K) = -\nabla\cdot(p\mathbf{u}) + \underline{\frac{pw\rho_r g}{c_s^2\rho_B}} - gw(\rho_B - \rho_r) - \underline{\frac{pwg}{c_s^2}}\,, \tag{21}$$

where writing $\rho_B = \rho_r + \rho'$ it is seen that the two underlined terms cancel to the leading order as desired. However, to get perfect energy conservation from the equations we are developing, the cancellation must be complete. Therefore, it is necessary to modify the vertical momentum equation to the form

$$\rho_B(w_r + \mathbf{u}\cdot\nabla w + 2\boldsymbol{\Omega}\times\mathbf{u}\cdot\mathbf{k}) = -\frac{\partial}{\partial z}p - g(\rho_B - \rho_r) - \underline{\frac{gp\rho_r}{c_s^2\rho_B}}\,, \tag{22}$$

where the density ratio ρ_r/ρ_B in the underlined term has been included. This ratio is very close to unity everywhere, thus introducing a modification to the momentum equation consistent with the errors within the analysis (recall that the remnant of the two underlined terms was shown to be negligible in the discussion of the Navier–Stokes equations). This modification to the momentum equation is ad hoc, but in its introduction of a small error in the equations, it is like the replacement of full density by the constant ρ_o in the Boussinesq equations.

Using Boussinesq density consistently along with (18) and (20) then yields the energy equation

$$[\rho_B(K + h_{\mathrm{I}}^{\dagger})]_t + \nabla\cdot\mathbf{u}[\rho_B(K + h_{\mathrm{I}}^{\dagger}) + p] = 0, \tag{23}$$

where $h_{\mathrm{I}}^{\dagger}$ plays the role of the potential energy density and is given by the dynamic enthalpy-like variable

$$h_{\mathrm{I}}^{\dagger} = -\int_{P_o}^{P_*}\frac{(\rho_B - \rho_r)}{\rho_B\rho_r}d\tilde{P}, \tag{24}$$

which differs slightly from (1). Again, diabatic terms have been neglected.

To summarize, the type I semicompressible equations, now including all diabatic terms, are

$$\rho_B(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u}) = -\nabla p - \left[\rho_B - \rho_r(z) + \underline{\frac{p\rho_r}{c_s^2\rho_B}}\right]g\mathbf{k} + \rho_B\mathbf{u}^o\,, \tag{25a}$$

$$\frac{d}{dt}\rho_B + \rho_B\nabla\cdot\mathbf{u} = 0, \text{ or equivalently, } \nabla\cdot\mathbf{u} = \frac{wg\rho_r}{c_s^2\rho_B} + R, \tag{25b}$$

$$\rho_B = \hat{\rho}(S_{\mathrm{A}}, \Theta, P_*), \tag{25c}$$

$$\frac{d}{dt}S_{\mathrm{A}} = S_A^o, \quad \text{and} \tag{25d}$$

$$\frac{d}{dt}\Theta = \Theta^o\,. \tag{25e}$$

These satisfy the energy equation

$$[\rho_B(K + h_{\mathrm{I}}^{\dagger})]_t + \nabla\cdot\{\mathbf{u}[\rho_B(K + h_{\mathrm{I}}^{\dagger}) + p\mathbf{u}]\} = pR + \rho_B\mathbf{u}\cdot\mathbf{u}^o. \tag{26}$$

The nonconservative terms R in the velocity divergence and energy equations take the form recommended by McDougall and Garrett (1992), involving equation of state nonlinearities and molecular effects only. The underlined term in the momentum [(25a)] addresses the error identified by DHMY. Equations (25) eliminate sound waves because density is evaluated using static pressure.

Klein and Pauluis (2012) proposed the same approach providing examples using a moist atmosphere; our contribution beyond their paper is in the scaling justification of the equations for seawater. Our form of energy, involving dynamic enthalpy, differs in detail from theirs. The thermodynamic consequences, if any, of this distinction are currently under investigation. We also emphasize that these equations are the same as the pseudocompressible equations of Klein and Pauluis (2012) that have seen use in the atmospheric and astrophysical communities. Although (25) could be referred to as the pseudocompressible equations, we continue with the type I designation in this paper to emphasize here the derivation of two separate sets of semicompressible equations. We do hope pointing out the correspondence between our type I set and the pseudocompressible set will aid in cross-disciplinary interactions.

The structure of (25) is very similar to the classic seawater Boussinesq equations, consisting of five prognostic equations and two diagnostic equations. They require no new computations relative to the Boussinesq equations. They have a straightforward hydrostatic limit and, in the case of nonhydrostatic dynamics, involve the solution of an elliptic equation. Representing (25) with the convenient notation

$$\mathbf{u}_t = \mathbf{U} - \frac{\nabla p}{\rho_B}, \tag{27}$$

the elliptic equation is

$$\nabla \cdot \frac{\nabla p}{\rho_B} = \nabla \cdot \mathbf{U} - \frac{(\rho_B W - p_z) g \rho_r}{c_s^2 \rho_B^2} \underline{+ \frac{w \rho_r g}{\rho_B c_s^2} \left[\left(\frac{2}{c_s} \frac{\partial c_s}{\partial \Theta} + \frac{1}{\rho_B} \frac{\partial \rho_B}{\partial \Theta} \right) \frac{\partial}{\partial t} \Theta + \left(\frac{2}{c_s} \frac{\partial c_s}{\partial S} + \frac{1}{\rho_B} \frac{\partial \rho_B}{\partial S} \right) \frac{\partial}{\partial t} S \right] - \frac{\partial}{\partial t} R} . \tag{28}$$

The quantity W denotes the vertical member of (27). The Boussinesq density on the left-hand side and the underlined terms on the right-hand side are absent from the Boussinesq elliptic operator.

b. Type II semicompressible equations

We wish to capture the fluid divergence as accurately as possible. The full statement of adiabatic divergence from mass conservation is [ignoring for a moment the R term that we will reinstate in (35)]

$$\begin{aligned} \nabla \cdot \mathbf{u} &= \frac{-1}{\rho c_s^2} \left(\frac{d}{dt} P^* + \frac{d}{dt} p \right) + O(p^2) \\ &= \frac{-1}{\rho c_s^2} \left(-w g \rho_r + \frac{\partial}{\partial t} p + u \frac{\partial}{\partial x} p + v \frac{\partial}{\partial y} p + w \frac{\partial}{\partial z} p \right), \end{aligned} \tag{29}$$

where we have decomposed total pressure into its static and dynamic contributions. Involving the dynamic pressure in the divergence equation in general leads to sound waves. However, isolating the hydrostatic part of the dynamic pressure, (29) can be written as

$$\begin{aligned} \nabla \cdot \mathbf{u} = \frac{-1}{\rho c_s^2} \bigg[& -w g \rho_r + \frac{\partial}{\partial t} p + u \frac{\partial}{\partial x} p + v \frac{\partial}{\partial y} p \\ & - w g (\rho_B - \rho_r) + w \frac{\partial}{\partial z} p_n \bigg], \end{aligned} \tag{30}$$

where p_n denotes the nonhydrostatic contribution to pressure. Clearly, vertical advection of the hydrostatic pressure can be safely incorporated into the divergence without exciting sound. The other contributions from the material derivative of dynamic pressure must not appear as they involve the free surface and interior pressure anomalies that can move at c_s. The truncated form of (30) that emerges is

$$\nabla \cdot \mathbf{u} = \frac{wg}{c_s^2}, \tag{31}$$

where we have recognized the cancellation in (30) and restricted the density in the denominator to the Boussinesq density. This is perhaps the most accurate statement of fluid divergence that can be made that stops short of allowing sound waves. It is different than the divergence obtained from strictly working with Boussinesq density [see (18)] by the multiplicative factor the type I equations used in vertical momentum. The type II equations explore (31) as the statement of fluid divergence.

We need a density variable ρ_{II} to solve (4b) in a manner equivalent to (31), that is,

$$\frac{wg}{c_{s\mathrm{II}}^2} = \frac{-1}{\rho_{\mathrm{II}}} (\rho_{\mathrm{II}t} + u \rho_{\mathrm{II}tx} + v \rho_{\mathrm{II}y} + w \rho_{\mathrm{II}z}). \tag{32}$$

This requires the density variable depend only on the vertical coordinate and satisfy

$$\frac{-g}{c_{s\mathrm{II}}^2} = \ln(\rho_{\mathrm{II}})_z, \tag{33}$$

whose solution is

$$\rho_{\mathrm{II}} = \rho_{\mathrm{II}}(0) \exp\left(- \int_0^z g / c_{s\mathrm{II}}^2 \, dz \right), \tag{34}$$

where $\rho_{\mathrm{II}}(0)$ is a constant of integration. In (34), the sound speed can be a function of depth, but cannot depend on horizontal coordinates. It is thus necessary to select a reference sound speed profile and choose a surface density. With this, the reference density profile can be calculated and used in the type II set. Because of the form of (31), it is not necessary to modify the buoyancy term in the vertical momentum equation to achieve energy conservation. However, it is necessary to use the preselected sound speed profile wherever sound speed appears, as opposed to having it be determined by the local Boussinesq density profile as in the type I set.

In summary, the type II semicompressible equations are

$$\rho_{\mathrm{II}}(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u}) = -\nabla p - \left(\rho_B - \rho_{\mathrm{II}} + \underline{\frac{p}{c_{s\mathrm{II}}^2}}\right)g\mathbf{k} + \rho_{\mathrm{II}}\mathbf{u}^o, \tag{35a}$$

$$\frac{d}{dt}\rho_{\mathrm{II}} + \rho_{\mathrm{II}}\nabla\cdot\mathbf{u} = 0, \text{ or equivalently,}$$

$$\nabla\cdot\mathbf{u} = \frac{wg}{c_{s\mathrm{II}}^2} + R, \tag{35b}$$

$$\rho_B = \hat{\rho}(S_{\mathrm{A}}, \Theta, P_*), \tag{35c}$$

$$\frac{d}{dt}S_{\mathrm{A}} = S_{\mathrm{A}}^o, \quad \text{and} \tag{35d}$$

$$\frac{d}{dt}\Theta = \Theta^o, \tag{35e}$$

with (34) defining the reference density profile. The underlined quantity in (35a) corrects for the DHMY error. The nonconservative effects in the velocity divergence equation are again taken from McDougall and Garrett (1992). These equations satisfy the energy constraint

$$[\rho_{\mathrm{II}}(K + h_{\mathrm{II}}^{\dagger})]_t + \nabla\cdot\mathbf{u}[\rho_{\mathrm{II}}(K + h_{\mathrm{II}}^{\dagger}) + p] = pR + \rho_{\mathrm{II}}\mathbf{u}\cdot\mathbf{u}^o, \tag{36}$$

where the role of potential energy density is played by

$$h_{\mathrm{II}}^{\dagger} = -\int_{P_o}^{P_*}\frac{(\rho_B - \rho_{\mathrm{II}})}{\rho_{\mathrm{II}}^2}\,dP. \tag{37}$$

In the use of a z-dependent density profile, the type II set might appear to resemble the anelastic approximation of Ingersoll (2005), but there are several differences. First, the density profile is not open for selection, rather the sound speed profile is. Second, the full seawater equation of state is used, as opposed to the linear EOS appearing in the anelastic set. Last, the anelastic equations do not include the correction of the error in DHMY. We also differ from the set derived in Pauluis (2008) in the exploitation of a reference sound speed profile, a seawater equation of state, and the effective form of our potential energy [(37)].

3. Discussion

The above equation sets move beyond the Boussinesq approximation in two main ways. The most obvious is in the explicit inclusion of compression, which shows in the divergence equation and appears there and elsewhere in the form of the sound speed. The second way is in the recognition of the effects of nonconstant density when computing fluid acceleration. The interesting fact emerging from the above is that these two features are linked. It is not possible to introduce one without the other, while guaranteeing that the equations have a well-behaved energy equation. It is also interesting that the effort needed for the solution of these equation sets is quite comparable to that needed for the classical Boussinesq equations. While the formulation of the problem is more tedious, execution times should not suffer greatly for the additional accuracy.

It is potentially useful that the velocity divergence does not vanish. This moves the system away from volume conservation and more in line with mass conservation. The type I equations are decidedly superior to type II in this regard. They conserve the Boussinesq mass represented by the Boussinesq density, which itself is an excellent approximation to the full density everywhere. Conservation of mass, rather than volume, leads to desirable properties like the direct computation of sea level.

The latter requires accurate velocity divergence, of course, and we should point out this is still the most problematic characteristic of the equations. As emphasized in (20), it is necessary to exclude the dynamic pressure from divergence to filter sound waves, but the scale size of this quantity is the same as the term that is retained. Equivalently, we only capture part of the divergence in our equations. On the other hand, to improve on the semicompressible sets apparently necessarily involves acoustic modes, at which point one should move to the full Navier–Stokes equations.

4. Summary

An analysis of the equations of motion argues that the Boussinesq equations can be extended to include the leading-order effects of compression. For fluids with small Rossby number, like the oceans, the error so removed from the equations is sizeable. The modifications needed to implement this correction in most OGCMs are nontrivial, but appear not to involve any more laborious computation than that needed for the

Boussinesq set. The reason for the well-behaved nature of these equations with respect to energy conservation is that the associated variation in fluid internal energy is provided at leading order by a gravitational potential energy variation.

We have proposed two sets of equations that meet our criteria of improved accuracy over the Boussinesq equations and consistent energetics. They are developed in different ways and have different strengths and weaknesses. Probably the more powerful of the two is the type I set. This much more completely allows for variable density (e.g., the presence of ρ_B on the left-hand side of the momentum equations) and accurate sound speed computations (computed from the Boussinesq density profile). Its drawbacks are the small but artificial modification of one of the coefficients in the vertical momentum equation and an incomplete, if nonzero, statement of fluid divergence (Klein and Pauluis 2012).

The type II set does not involve density in nearly as complete a fashion, but in its recognition of nonconstant reference densities improves on Boussinesq. It has a slightly more accurate statement of fluid divergence than the type I set, although this is a minimal distinction. Probably its greatest strength will prove to be in its computational efficiency. The use of a specified sound speed profile and a horizontally uniform reference density profile should lead to a smaller computational burden and hence faster execution. Of course, this comes at the cost of decreased overall accuracy.

Both sets correct the pressure gradient error identified by DHMY and both conserve energy.

It is possible that these equations will lend themselves to theoretical analysis, but probably they will prove most useful in numerical applications. We recommend that the types I and II sets be tested in ocean circulation settings, from which we can assess the full impact of their increased accuracy.

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