Topographic Rectification of Tidal Currents[1]

W. R. YOUNG

University of California, San Diego and *Marine Physical Laboratory, Scripps Institution of Oceanography, La Jolla, CA 92093*

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ABSTRACT

The rectification of oscillatory tidal currents on the sloping sides of a *low* submarine bank is discussed using the moment method. This method has been previously used in shear dispersion studies where it is used to analyze the advection–diffusion equation. In the present problem it is applied to the barotropic potential vorticity equation linearized about an oscillatory, spatially uniform tidal velocity. To apply the method it is necessary to assume that the topography produces only a small change in depth. The method economically provides the most important qualitative properties (*e.g.*, transport, location and width) of the time averaged current.

These results are obtained without making an harmonic truncation. They can then be used to assess the accuracy of the harmonic truncation approximation used by other authors. It is shown that harmonic truncation correctly predicts the transport and location of the rectified current when the bank is low. However if the width of the bank is much less than a tidal excursion distance, harmonic truncation may give a very mistaken impression of the width of the rectified current.

Finally, lateral vorticity diffusion is included in the moment calculation. It is shown that this dissipative process does not change the transport or location of the rectified current. It does however increase its width.

1. Introduction

In a recent article, Loder (1980) extended Huthnance's (1973) study of tidal rectification by considering the important effects of interaction between the mean current and tidal current. Robinson (1981) discussed the physical process responsible for rectification. The configuration studied by these authors is shown schematically in Fig. 1. The submarine bank creates relative vorticity through vortex stretching. If there is some dissipation, then this vorticity can result in a rectified current along the bank (into the page in Fig. 1).

Huthnance (1973) and Loder (1980) solved the vertically integrated momentum equation using an harmonic truncation. The present note complements these earlier studies by assessing the accuracy of the harmonic truncation in the case where the bank produces only a small total change in depth. To accomplish this, integral moments of the vorticity distribution are calculated without discarding any harmonics. These results can be compared with integral moments calculated from the approximate, harmonically truncated solution. This comparison shows that the harmonic truncation, carefully interpreted, is reliable.

The moment method used here is also of interest in its own right. In problems where the topography is low it has three advantages over harmonic truncation. First, it provides the integral moments of the vorticity distribution exactly. These integral moments are easily related to important qualitative properties of the along-bank current distribution. For example, in the present note, the transport, location and width of the rectified current are calculated exactly. Higher moments, such as the skewness, can be calculated in principle although the algebra becomes very involved. Second, the moment method also shows that the first few moments of the vorticity distribution depend only on the first few moments of the bottom slope distribution. Thus, to apply the moment method it is not necessary to assume a particular form for the topography. Third, it is straightforward to include a number of additional effects (such as lateral vorticity diffusion) which are difficult to discuss using other methods.

The greatest weakness of the moment method is that it can only treat slight variations in depth. Thus, referring to Fig. 1, it is necessary to assume that

$$H_0 \gg \eta. \qquad (1.1)$$

The assumption (1.1) allows two simplifications. First, the tidal velocity over the bank is approximately independent of x:

$$U = U_0 \cos\omega t, \quad v = V_0 \cos(\omega t + \alpha). \qquad (1.2)$$

Second, the barotropic potential vorticity (Pedlosky, 1979):

$$q \equiv (f + \zeta)/(H_0 - \eta),$$

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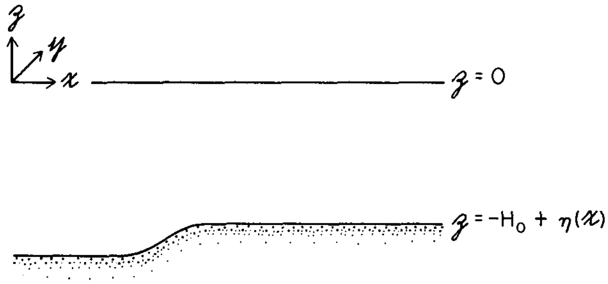

FIG. 1. Definition of axes.

can be simplified to

$$q = (f/H_0) + (\zeta + b)/H_0, \tag{1.3a}$$

$$b \equiv (f/H_0)\eta. \tag{1.3b}$$

One does not have to make the very restrictive assumption (1.1) to apply the harmonic truncation method.

With (1.3a) the linearized barotropic potential vorticity equation is

$$\left(\frac{\partial}{\partial t} + U_0 \cos\omega t \frac{\partial}{\partial x}\right)(\zeta + b) = -\delta\zeta + (\delta/f)V_0 \cos(\omega t + \alpha)b_x + \nu\nabla^2\zeta. \tag{1.4}$$

Eq. (1.4) is conservation of potential vorticity linearized about an oscillatory tidal flow. Dissipation is provided by a combination of bottom drag ($\delta\zeta$) and lateral diffusion of relative vorticity ($\nu\nabla^2\zeta$). Differential bottom friction (due to changes in depth) also results in a direct production of relative vorticity; this is the origin of the second term on the rhs of (1.4). The derivation of this equation in the context of tidal rectification has been discussed in detail by Zimmerman (1978, 1980). Zimmerman solves (1.4) using a Fourier transform and so relates the spectral properties of the residual currents to the spectral properties of the topography. One can also discuss a deterministic topography such as that in Fig. 1 using the Fourier transform method (Zimmerman, 1981; Loder, 1981). It seems, however, that the moment method is simpler.

It is convenient to nondimensionalize (1.4) using the following scalings (asterisks temporarily denotes a nondimensional variable)[2]:

$$(x, y) = (U_0/\omega)(x_*, y_*), \tag{1.5a}$$

$$t = (1/\omega)t_*, \tag{1.5b}$$

$$(\delta, f) = \omega(\delta_*, f_*), \tag{1.5c}$$

$$\nu = (U_0^2/\omega)\nu_*, \tag{1.5d}$$

$$b = Bb_*, \tag{1.5e}$$

$$\zeta = B\zeta_*. \tag{1.5f}$$

In (1.5e)–(1.5f) B is the total jump in b across the bank; thus from (1.3b):

$$B = (f/H_0)\,[\text{change in depth}].$$

One can now rewrite (1.4) in terms of nondimensional variables. It is convenient to group all the terms containing ζ on the lhs of the equation and place all the other terms (which may be thought of as source terms for ζ) on the rhs. Thus, dropping the subscript asterisks and denoting derivatives by subscripts:

$$\zeta_t + \cos t\,\zeta_x + \delta\zeta - \nu\zeta_{xx} = -s\cos t + \lambda s\cos(t + \alpha), \tag{1.6}$$

where

$$s = b_x = (f/H_0)\,[\text{bottom slope}], \tag{1.7a}$$

$$\lambda = (\delta/f)(V_0/U_0). \tag{1.7b}$$

The two source terms on the rhs of (1.6) have different physical origins (Huthnance, 1973). The first is related to the production of relative vorticity by topographic vortex stretching. The second arises from differential bottom friction (*e.g.*, bottom drag is stronger where the water is shallower). One can however rewrite the equation as

$$\zeta_t + \cos t\,\zeta_x + \delta\zeta - \nu\zeta_{xx} = -\rho\cos(t + \vartheta)s, \tag{1.8a}$$

$$\rho = [1 - 2\lambda\cos\alpha + \lambda^2]^{1/2}, \tag{1.8b}$$

$$\vartheta = \tan^{-1}\left[\frac{\lambda\sin(\alpha)}{\lambda\cos(\alpha) - 1}\right], \tag{1.8c}$$

so that the two source terms are combined into one.

Loder (1980), argued that over the sloping sides of Georges Bank, the topographic stretching term dominates relative-vorticity production. This corresponds to the limit

$$\lambda \to 0, \quad \rho \to 1 \quad \text{and} \quad \vartheta \to 0, \tag{1.9}$$

in (1.8a) Loder (1980) suggested that the frictional production mechanism is more important in the shallow central region of Georges Bank. The frictional mechanism becomes dominant when

$$\lambda \to \infty, \quad \rho \to \lambda, \quad \vartheta \to \alpha. \tag{1.10}$$

2. The moment method

The moment method has been widely employed in studies of shear dispersion (*e.g.*, Aris, 1956; Saffman, 1962; Young *et al.*, 1982). In the present problem it economically provides the most important properties of the solution of (1.8).

We begin by introducing the notation

[2] The notation is standard: f is twice the local vertical rotation of the earth and ζ is the relative vorticity.

$$\langle f\rangle = \int_{-\infty}^{\infty} f dx. \qquad (2.1)$$

The moment method relies on the fact that the moments of the vorticity distribution, $\langle x^n\zeta\rangle$, can be calculated from a closed hierarchy. Thus, from (1.8a),

$$\langle\zeta\rangle_t + \delta\langle\zeta\rangle = -\rho\cos(t+\vartheta), \qquad (2.2a)$$

$$\langle x\zeta\rangle_t + \delta\langle x\zeta\rangle = \cos t\langle\zeta\rangle, \qquad (2.2b)$$

$$\langle x^2\zeta\rangle_t + \delta\langle x^2\zeta\rangle = 2\cos t\langle x\zeta\rangle + 2\nu\langle\zeta\rangle - \rho\cos(t+\vartheta)\langle x^2 s\rangle, \qquad (2.2c)$$

$$\langle x^3\zeta\rangle_t + \delta\langle x^3\zeta\rangle = 3\cos t\langle x^2\zeta\rangle + 6\nu\langle x\zeta\rangle - \rho\cos(t+\vartheta)\langle x^3 s\rangle, \qquad (2.2d)$$

and so on for the higher moments. Note that because of (1.5e),

$$\langle s\rangle = 1, \qquad (2.3a)$$

and we are free to place the origin of the coordinate system so that

$$\langle xs\rangle = 0. \qquad (2.3b)$$

One can now solve (2.2). The moments of the vorticity distribution are related to the moments of the velocity distribution using

$$\zeta = v_x, \qquad (2.4a)$$

$$\zeta(x=\pm\infty) = 0. \qquad (2.4b)$$

Integrating by parts then:

$$\langle x\zeta\rangle = -\langle v\rangle, \qquad (2.5a)$$

$$\langle x^2\zeta\rangle = -2\langle xv\rangle, \qquad (2.5b)$$

$$\langle x^3\zeta\rangle = -3\langle x^2 v\rangle. \qquad (2.5c)$$

The solution of (2.2a) is

$$\langle\zeta\rangle = -(1+\delta^2)^{-1/2}\rho\cos(t+\vartheta-\varphi_1), \qquad (2.6a)$$

$$\varphi_1 = \tan^{-1}(\delta^{-1}). \qquad (2.6b)$$

Since

$$\langle\zeta\rangle = v(x=+\infty) - v(x=-\infty),$$

this result shows that there is a jump in v across the bank. In dimensional variables this jump is proportional to (BU_0/ω). If the linearization about the tidal velocity in (1)–(2) is to be valid, then

$$(\text{jump in } v)/V_0 \ll 1,$$

or

$$B/\omega \ll 1. \qquad (2.7)$$

Substituting (2.6a) into (2.2b) gives an equation for $\langle x\zeta\rangle$. The solution is

$$\langle x\zeta\rangle = -\tfrac{1}{2}\rho\delta^{-1}(1+\delta^2)^{-1/2}\cos(\vartheta-\varphi_1) - \tfrac{1}{2}\rho[(1+\delta^2)(4+\delta^2)]^{-1/2} \times \cos(2t+\vartheta-\varphi_1-\varphi_2), \qquad (2.8a)$$

where

$$\varphi_2 = \tan^{-1}(2\delta^{-1}). \qquad (2.8b)$$

Time averaging (2.8a) and using (2.5a) gives

$$\langle\bar v\rangle = \tfrac{1}{2}\rho\delta^{-1}(1+\delta^2)^{-1/2}\cos(\vartheta-\varphi_1), \qquad (2.9a)$$

$$= \tfrac{1}{2}(1+\delta^2)^{-1}[1-\lambda(\cos\alpha+\delta^{-1}\sin\alpha)], \qquad (2.9b)$$

where the definition of ϑ, (1.8c), was used to go from (2.9a) to (2.96). (Hereafter the overbar will denote a time average). Eq. (2.9a) shows clearly that the combination of bottom friction and slope produces a rectified current.

If one takes the limit $\delta \to 0$ with (V_0/U_0f) fixed then (2.9b) reduces to

$$\langle\bar v\rangle = \tfrac{1}{2}[1-(V_0/U_0f)\sin\alpha]; \qquad (2.10a)$$

or, in dimensional variables,

$$\langle\bar v\rangle = \tfrac{1}{2}(BU_0^2/\omega^2)[1-(V_0\omega/U_0f)\sin\alpha]. \qquad (2.10b)$$

The above expression for $\int_{-\infty}^{\infty} v dx$ has also been obtained, without harmonic truncation and without the assumption $\eta \ll H_0$ by Huthnance (1973). Note that without the assumption $\eta \ll H_0$, $\langle v\rangle$ cannot be interpreted as a transport.

One can now calculate $\langle x^2\zeta\rangle$ by solving (2.2c) with $\langle x\zeta\rangle$ in (2.8a) and $\langle\zeta\rangle$ in (2.6a). The most important point is that

$$\langle x^2\bar\zeta\rangle = -2\langle x\bar v\rangle = 0, \qquad (2.11)$$

so that the center of mass of the rectified jet coincides with the center of mass of the slope distribution.

Now to get some idea of the width of the rectified jet it is necessary to calculate $\langle x^3\zeta\rangle = -3\langle x^2\bar v\rangle$ from (2.2d). The result of the calculation is

$$\langle x^2\bar v\rangle/\langle\bar v\rangle = \langle x^2 s\rangle + (1+\delta^2)^{-1} \times\left[1+\frac{\sin\varphi_2\cos(\vartheta-2\varphi_1-\varphi_2)}{4\sin\varphi_1\cos(\vartheta-\varphi_1)}\right] + 3\left(\frac{\nu}{\delta}\right)\left[1+\frac{\cos(\vartheta-3\varphi_1)}{3\cos(\vartheta-\varphi_1)}\right]. \qquad (2.12)$$

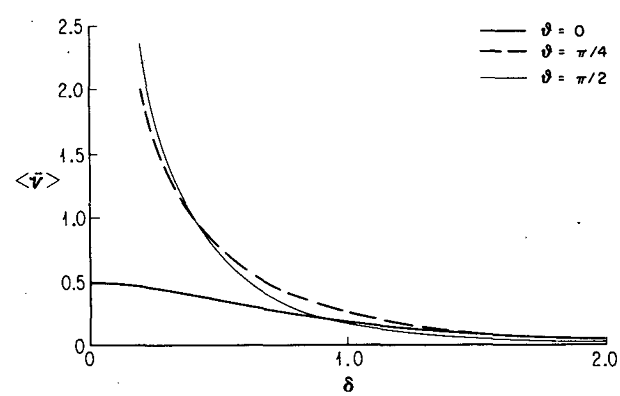

FIG. 2. Velocity $\langle\bar v\rangle$ as a function of δ for various values of ϑ. In dimensional units $\langle\bar v\rangle$ is measured in Bu_0^2/ω^2.

One can, in principle, compute more moments of $\bar{v}$. I suspect however, that the point of diminishing returns has been reached. Eqs. (2.9), (2.11) and (2.12) are expressions for the transport, location and width of the rectified jet in the limit where the topography is low. In the next section these results are compared with the corresponding expressions obtained using harmonic truncation.

Finally, to conclude this section, suppose that an explicit expression for $\bar{v}(x)$ is required. The simplest suggestion is to fit a Gaussian:

$$\bar{v} = Ve^{-x^2/2l^2}, \tag{2.13}$$

and adjust V and l so that $\langle\bar{v}\rangle$ and $\langle x^2\bar{v}\rangle$ agree with (2.9) and (2.12). One finds:

$$l^2 = \langle x^2\bar{v}\rangle/\langle\bar{v}\rangle, \tag{2.14a}$$

$$V = \langle\bar{v}\rangle/(2\pi)^{1/2}l. \tag{2.14b}$$

It is not necessary to introduce the Gaussian form (2.13) to define l^2 by (2.14a).

In Fig. 3, l in (2.14a) is plotted as a function of δ for various values of ϑ. In this figure it is assumed that $\langle x^2s\rangle = \nu = 0$; from (2.12) it is straightforward to include the effects of finite slope width and lateral diffusivity.

3. Comparison of the exact results with those obtained by harmonic truncation

The harmonic truncation method is summarized in Appendix A.

Comparing (A9) with (2.9) we see that the harmonic truncation method provides the integrated Eulerian transport of the rectified jet correctly. The moment method shows that this important property is independent of lateral diffusivity [*i.e.*, ν does not appear in the exact result (2.9a)].

Similarly, comparing (2.12) and (A10), both methods predict that the center of the jet coincides with the center of the slope distribution. Note that once again this property of the time-averaged flow is independent of ν.

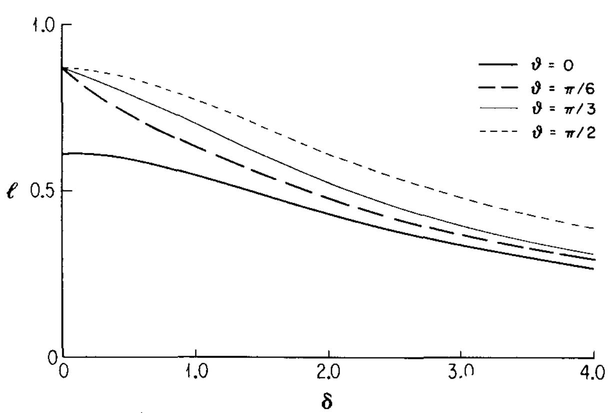

FIG. 3. Jet width l defined in (2.14a) as a function of δ for various values of ϑ. In this figure $\langle x^2s\rangle = \nu = 0$ and in dimensional units l is measured in $U_0\omega$.

Straightforward scale analysis shows that harmonic truncation is justified when the length scale of the slope is much greater than a tidal excursion distance. Loder (1980), Huthnance (1981), Robinson (1981) and Zimmerman (1981) have all considered the opposite extreme where the change in depth is abrupt. The first two authors solved the problem without harmonic truncation but in both cases it is difficult to extract the qualitative properties of the current distribution from the mathematics. Robinson (1981) solved the problem numerically using a Lagrangian approach. While this approach provides physical insight into the rectification mechanism, it is again difficult without further numerical work to obtain the Eulerian current. Loder (1980) solved the problem using harmonic truncation. The solution he found has non-zero residual vorticity more than a particle excursion distance from the source. If $\nu = 0$ (as it was in Loder's calculation) then this is impossible, and so the harmonic truncation is clearly unreliable if the change in depth is abrupt.

Because the change in depth is not necessarily gradual the harmonic truncation is approximate, and this becomes apparent when one compares the exact width of the jet, l as given by (2.14) and (2.16a), with the approximation l_H in (A12). To make this comparison take $\nu = 0$ and $\langle x^2s\rangle = 0$. In dimensional terms the latter condition means that the tidal excursion distance is much greater than the slope width. Note that in the opposite limit, $\langle x^2s\rangle \gg 1$, the two methods agree:

$$l_H^2 = l^2 = \langle x^2s\rangle.$$

Now we have:

$$l^2/l_H^2 = 1 + \frac{\sin\varphi_2\cos(\vartheta - 2\varphi_1 - \varphi_2)}{4\sin\varphi_1\cos(\vartheta - \varphi_1)}. \tag{3.1}$$

First suppose $\vartheta = 0$ so that the Coriolis mechanism is dominant. In this case, using (2.6b) and (2.8b), (3.1) reduces to

$$l^2/l_H^2 = 1 + \frac{(\delta^2 - 5)}{2(\delta^2 + 4)}. \tag{3.2}$$

This result shows that harmonic truncation overestimates the width of the jet by ~60% when δ is small. On the other hand when δ is large the width of the jet is underestimated by 20%. Note that finite slope width reduces the relative error so that the above figures are a worst-case estimate.

Now suppose $\vartheta = \pi/2$ so that the frictional mechanism is dominant. In this case (3.1) reduces to

$$l^2/l_H^2 = 1 + \frac{2\delta^2 - 1}{\delta^2 + 4}. \tag{3.3}$$

Once again the width of the jet is overestimated when

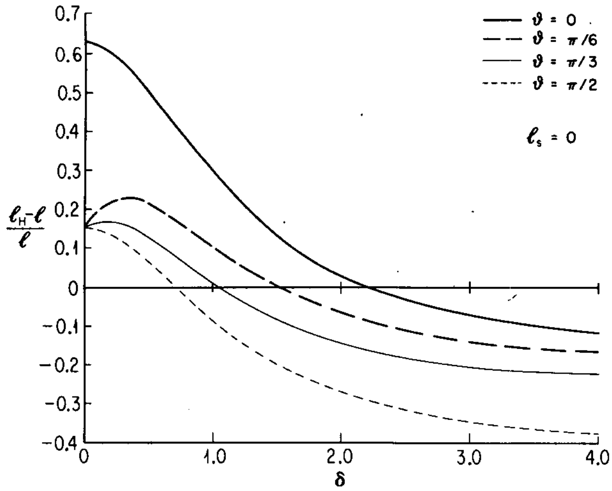

FIG. 4. The relative error $(l_H/l) - 1$ as a function of δ for various values of ϑ. When δ is small, harmonic truncation overestimates l, while when δ is large, l is underestimated.

δ is small, although only by 15%. On the other hand when δ is large the width of the jet is seriously underestimated.

These conclusions are summarized in Fig. 4 where the relative error:

$$e = (l_h/l) - 1 \tag{3.4}$$

is plotted as a function of δ for various values of ϑ.

Once again I emphasize that this is a worst-case estimate of the errors associated with harmonic truncation. Moderate values of slope width

$$l_s^2 = \langle x^2 s \rangle / \langle s \rangle, \tag{3.5}$$

can significantly reduce e. For example in Fig. 5, I

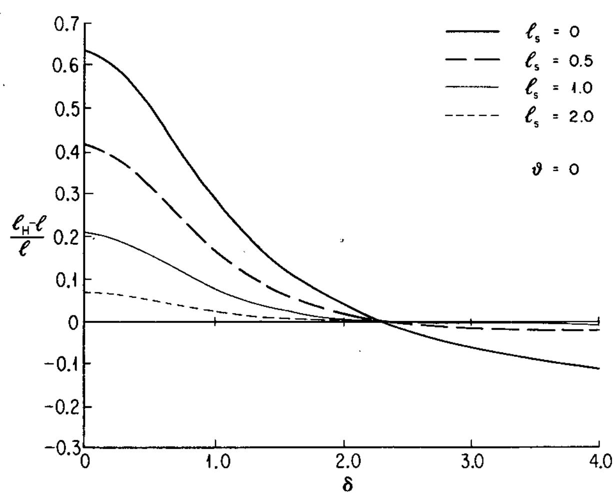

FIG. 5. The relative error as a function of δ for various values of l_s with ϑ fixed at 0. Note how the error in slope width is reduced as l_s on increases.

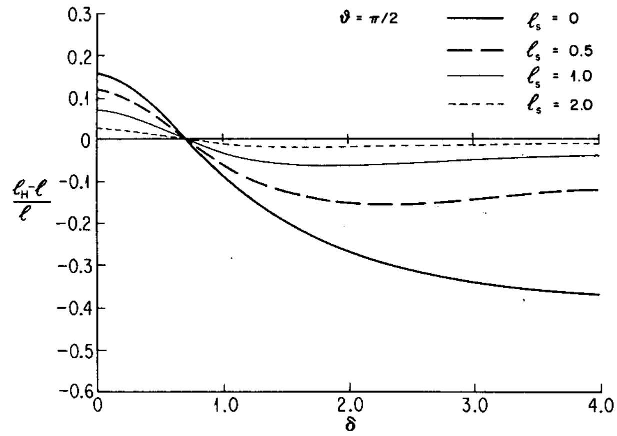

FIG. 6. The relative error as a function of δ for various values of l_s with ϑ fixed at $\pi/2$.

plot e as a function of δ for increasing values of l_s with ϑ fixed at 0. In Fig. 6, ϑ is fixed at $\pi/2$ and the reduction in e with increasing l_s is even more marked.

4. Conclusion

Subject to the approximation (1.1), the moment method can be used to calculate the most important qualitative properties of the rectified current. The approximation (1.1) is very restrictive however, and so harmonic truncation is still the only analytic method available in general. Comparison of the two methods in Section 3 indicates that harmonic truncation is reliable provided the slope width is greater than about one tidal excursion distance (this conclusion depends on δ and ϑ, see Figs. 5 and 6).

Once (1.1) has been accepted however, the moment method has a number of advantages over harmonic truncation. First, it provides exact results. Second, one does not assume a particular form for the slope distribution; all that is required are the first few moments of s. Third, one can include a number of effects that complicate harmonic truncation. For instance, in this article lateral diffusivity of the relative vorticity has been included. It is straightforward to include other effects such as two-dimensional topography, lateral shear in the tidal current and additional temporal harmonics (including a steady flow over the bank).

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APPENDIX

The Harmonic Truncation Method

For completeness, in this Appendix I shall summarize the approximate solution of (1.8a) using harmonic truncation. I assume that $\nu = 0$.

Substituting

$$\zeta = A(x) + B(x)\cos t + C(x)\sin t \qquad \text{(A1)}$$

into (1.8a) and discarding all the higher harmonics gives

$$\tfrac{1}{2}B' + \delta A = 0, \qquad \text{(A2)}$$

$$C + A' + \delta B = -\rho\cos(\vartheta)s, \qquad \text{(A3)}$$

$$-B + \delta C = \rho\sin(\vartheta)s, \qquad \text{(A4)}$$

where the prime denotes differentiation with respect to x.

From (A2)–(A4) one obtains:

$$B'' - 2(1+\delta^2)B = 2\rho(1+\delta^2)^{1/2}\cos(\vartheta - \varphi_1)s, \qquad \text{(A5)}$$

by eliminating A and C. Eq. (A5) is a second-order differential equation for B. The slope distribution $s(x)$ enters as a forcing term. Rather than choose particular forms for s and solve (A5), I shall calculate the moments of B. From (A1) and (A2) it follows that

$$\langle\bar{v}\rangle = -\frac{1}{2\delta}\langle B\rangle, \qquad \text{(A6)}$$

$$\langle\overline{xv}\rangle = -\frac{1}{2\delta}\langle xB\rangle, \qquad \text{(A7)}$$

$$\langle x^2\bar{v}\rangle = -\frac{1}{2\delta}\langle x^2 B\rangle, \qquad \text{(A8)}$$

so that we can compare the exact moments of $\bar{v}$, calculated in Section 2, with the approximate moments of $\bar{v}$ calculated from (A5) and (A6)–(A8).

Integrating (A5) from $+\infty$ to $-\infty$ and using (A6) gives

$$\langle\bar{v}\rangle = \tfrac{1}{2}\rho\delta^{-1}(1+\delta^2)^{-1/2}\cos(\vartheta - \varphi_1). \qquad \text{(A9)}$$

Now multiply (A5) by x and integrate to obtain

$$\langle\overline{xv}\rangle = 0. \qquad \text{(A10)}$$

Finally multiply (A5) by x^2 and integrate to obtain

$$\langle x^2\bar{v}\rangle = (1+\delta^2)^{-1}[\langle\bar{v}\rangle + \tfrac{1}{2}\rho\delta^1(1+\delta^2)^{1/2} \times \cos(\vartheta - \varphi_1)\langle x^2 s\rangle]. \qquad \text{(A11)}$$

From (A9) and (A11) it follows that the width of the jet is

$$l_H{}^2 \equiv \langle x^2\bar{v}\rangle/\langle\bar{v}\rangle, \quad = \langle x^2 s\rangle + (1+\delta^2)^{-1}. \qquad \text{(A12)}$$

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