Deep-Sea Research, 1962, Vol. 9, pp. 185 to 197. Pergamon Press Ltd. Printed in Great Britain.

The effect of the earth's rotation on the propagation of ocean waves over long distances

G. E. BACKUS*

(*Received* 5 *April* 1962)

Abstract—The theory of group velocity is generalized to waves on curved surfaces, and applied to calculating the effect of the earth's rotation on the trajectory and wave vector of a wave packet whose waves have lengths much less than the radius of the earth and periods much less than one day. A geometrical description of the perturbed trajectory is given. If Ω is the angular velocity of the earth, g gravity, h the ocean depth, and k and ω the wave number and frequency, the effects are of the order of $\Omega\,(h/g)^{\frac{1}{2}}$ and independent of ω for shallow water waves; they are of the order of Ω/ω for deep water waves; and they are largest for waves with $kh = 1{\cdot}878973$.

1. INTRODUCTION

IT IS the purpose of the present paper to discuss the effect of the earth's rotation on the propagation of small-amplitude ocean waves over long distances. The problem is to discuss adequately the joint effect of the sphericity of the ocean surface and rotation when the vertical acceleration of the fluid cannot be neglected.

2. THE LOCAL DISPERSION RELATION

Since we are interested only in waves much shorter than a, the radius of the earth, we can treat the waves' local behaviour as if the earth were plane. Therefore, let $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ be unit vectors in a Cartesian coordinate system, $\hat{\mathbf{z}}$ pointing up. Let x, y, z be the corresponding Cartesian coordinates, z vanishing at the surface of the undisturbed ocean. Let $u\hat{\mathbf{x}} + v\hat{\mathbf{y}} + w\hat{\mathbf{z}}$ be the velocity of the water, p its pressure, and ρ its density. Let p_1 be $p\rho^{-1} + gz$. Let $\mathbf{\Omega}$ be the angular velocity of the earth's rotation, and $\mathbf{f} = 2\mathbf{\Omega}$. Then, ignoring the earth's sphericity, the local equations of motion are these :

$$\left.\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0,\\
\frac{\partial u}{\partial t} - f_z\, v + f_y\, w &= -\frac{\partial p_1}{\partial x},\\
f_z\, u + \frac{\partial v}{\partial t} - f_x\, w &= -\frac{\partial p_1}{\partial y},\\
-f_y\, u + f_x\, v + \frac{\partial w}{\partial t} &= -\frac{\partial p_1}{\partial z}.
\end{aligned}\right\} \tag{1}$$

*Institute of Geophysics, University of California, La Jolla.

In equations (1) terms of second order in the velocity have been neglected. The linearized boundary conditions are

$$\frac{\partial p_1}{\partial t} = gw \ at \ z = 0, \tag{2}$$

and $w = 0$ at $z = -h$, the ocean bottom

We seek solutions whose time and space dependence is $F(z)\, e^{-i\omega t + ilx}$; that is, we consider plane waves, and orient our coordinate system so that $\partial/\partial y = 0$. Then equations (1) become

$$\left.\begin{aligned} ilu + Dw &= 0, \\ -i\omega u - f_z v + f_y w &= -ilp_1, \\ f_z u - i\omega v - f_x w &= 0, \\ -f_y u + f_x v - i\omega w &= -Dp_1. \end{aligned}\right\} \tag{3}$$

Here D is d/dz. From the first three of equations (3),

$$\left.\begin{aligned} u &= \frac{i}{l} Dw, \\ v &= \frac{if_x}{\omega} w + \frac{f_z}{\omega l} Dw, \\ p_1 &= \left(\frac{f_x f_z + i\omega f_y}{\omega l}\right) w + \frac{i}{\omega l^2}(\omega^2 - f_z^2)\, Dw. \end{aligned}\right\} \tag{4}$$

Then from the fourth of equations (3),

$$(\omega^2 - f_z^2)\, D^2 w - 2ilf_x f_z\, Dw - l^2(\omega^2 - f_x^2)\, w = 0. \tag{5}$$

Thus, since $w(-h) = 0$,

$$w = A\left[e^{n_+(z+h)} - e^{n_-(z+h)}\right] e^{ilx - i\omega t}$$

where A is an undetermined constant and

$$\frac{n_\pm}{l} = \frac{if_x f_z \pm \omega(\omega^2 - f_x^2 - f_z^2)^{\frac{1}{2}}}{\omega^2 - f_z^2}. \tag{6}$$

Then the boundary condition at $z = 0$ implies

$$\frac{\omega(\omega^2 - f_x^2 - f_z^2)^{\frac{1}{2}}}{gl - \omega f_y} = \tanh\left[\frac{hl\omega(\omega^2 - f_x^2 - f_z^2)^{\frac{1}{2}}}{\omega^2 - f_z^2}\right]. \tag{7}$$

Equation (7) is the dispersion relation for surface waves travelling in the x direction on the plane surface of a steadily rotating ocean of depth h. For any fixed l, there are infinitely many real values of ω which satisfy equation (7) and the inequality $\omega^2 < f_x^2 + f_z^2$. These values of ω have a point of accumulation at $\omega = f_z$. For each of them, n_+ and n_- are pure imaginary, so the amplitude of the disturbance does not die out with depth, no matter how small l is. As Ω approaches zero, these waves approach steady flows, so they are not the analogues in a rotating fluid of the ordinary water waves. They deserve more attention, but are not relevant to the subject of the present paper.

For any fixed l, there is precisely one positive ω which satisfies equation (7) and the inequality $\omega^2 > f_x^2 + f_z^2$. This gives the analogue of ordinary water waves in a rotating fluid. Henceforth, we shall consider only waves whose frequencies are much larger than $(f_x^2 + f_y^2 + f_z^2)^{\frac{1}{2}}$.

When $(f_x^2 + f_y^2 + f_z^2) \ll \omega^2$, equation (7) can be solved explicitly for ω. The result is

$$\omega = (gl \tanh hl)^{\frac{1}{2}} - \frac{f_y}{2} \tanh (lh) + 0\,(f^2).$$

If the wave has propagation vector (wave vector) $l\hat{x} + m\hat{y}$, with $m \neq 0$, this equation is

$$\omega = g^{\frac{1}{2}} (l^2 + m^2)^{\frac{1}{4}} \tanh^{\frac{1}{2}} h\,(l^2 + m^2)^{\frac{1}{2}} + \\ + \left(\frac{m\Omega_x - l\Omega_y}{(l^2 + m^2)^{\frac{1}{2}}}\right) \tanh h\,(l^2 + m^2)^{\frac{1}{2}} + O\,(\Omega^2). \tag{8}$$

3. THE GROUP VELOCITY ON A CURVED SURFACE

The extension of the ideas of group velocity to short waves on a curved surface is closely connected with HAMILTON's ideas about the ray approximation in optics. Concerning the extent to which the question has been settled by previous workers, for example ECKART (1948, 1960), LANDAU and LIFSCHITZ (1951, 1959), LIGHTHILL (1955, 1960), URSELL (1960), and WHITHAM (1955, 1961a, b), the author feels it appropriate to quote WHITHAM (1961a) : 'It is not always clear how much is implicit in the above references.'

The central problem in a discussion of group velocity is to define it. Should the statement that energy propagates with the group velocity be regarded as a theorem or as the definition of group velocity? The latter point of view is the more physical one, but it leads to the complication that group velocity must be derived anew for each kind of wave, and the fact that group velocity is $d\omega/dk$ for sound waves, Rayleigh waves, and ocean waves seems almost a coincidence. JEFFREYS (1956, p. 512) and WHITHAM (1961a) suggest that the group velocity of waves with wave vector $\boldsymbol{k}$ in a dispersed wave train be defined kinematically as the velocity at which an observer must move if he is always to find around him waves of the given wave vector $\boldsymbol{k}$. From this definition, the general validity of the expression $d\omega/dk$ for the group velocity in a homogeneous medium follows immediately. The fact that energy propagates with the group velocity becomes a theorem, proofs of which are given by JEFFREYS (1956, p. 514) and WHITHAM (1961a). If this kinematic definition of group velocity is adopted, the group velocity is used to find the *location* of the waves with wave vector $\boldsymbol{k}$ at each instance, while the fact that energy propagates with the group velocity is used to find the *amplitude* of the waves with wave vector $\boldsymbol{k}$ at each instant. In the present paper, we discuss only the location, not the amplitude, of waves propagating on a rotating earth, so we shall have no further concern with the energy theorem.

The difficulty with Jeffreys' definition of group velocity is that it cannot be used to discuss wave packets which change their wave vector as they move; in fact, by JEFFREYS' definition such a situation can never arise. Consequently, JEFFREYS' discussion of wave packets cannot be extended to inhomogeneous media. LANDAU and LIFSCHITZ (1951) and WHITHAM (1961a) have overcome this difficulty by showing that for a dispersed wave train in an inhomogeneous dispersive medium, if waves of wave vector $\boldsymbol{k}$ are known to be in a given small region at time t_0, then at any later time there is a small region in which the wave vector of the waves is determined, although it may not be $\boldsymbol{k}$. The position of this patch as a function of time is determined by its initial position and the initial value of $\boldsymbol{k}$, and is independent of amplitude (in the linear approximation) and of the initial character of the wave field outside the initial patch. The velocity of this patch of wave vector information is defined to be the group velocity. With this definition of group velocity, the distribution of wave vectors with position in a dispersed wave train at any instant can be calculated from the group velocity and the initial distribution of wave vectors, even in an inhomogeneous, flat medium.

We propose to generalize LANDAU and LIFSCHITZ's and WHITHAM's idea slightly, so as to be able to use it on curved surfaces. Specifically, we need a technique applicable to surface ocean waves on a rotating earth. The need for dealing with curved surfaces is obvious if we want to discuss transoceanic propagation. The need for dealing with inhomogeneities arises because the vertical component of the earth's angular velocity varies with position. If we could assume that the trajectories of wave groups (that is, patches of wave-vector information) were great circles, the results of the preceding section would suffice. As we shall see, the trajectories of wave groups on a rotating earth deviate from great circles to first order in the angular velocity of the earth. Thus the wave vector produced at any instant at a receiving station by a distant source will differ from that on a non-rotating earth by two terms of first order in the earth's angular velocity; one term arises directly from the theory of waves on a flat ocean, discussed in section 2, and the other arises from the failure of wave-group trajectories to be great circles.

Consider a two-dimensional surface whose points are described by (not necessarily orthogonal) coordinates q^1, q^2. (The remarks which follow can be extended immediately to n-dimensional manifolds). Let x^1, x^2 be local Cartesian coordinates in the plane tangent to the surface at a point P. Let $\hat{x}_1$, $\hat{x}_2$ be unit vectors in the directions of increase of x^1 and x^2. We assume that the surface is able to support waves which are short compared to its radii of curvature and which near P have the form $A \exp(-i\omega t + ik_1 x^1 + ik_2 x^2)$. Here A is a slowly varying amplitude ($|\nabla A| \ll |\boldsymbol{k}A|$, $|\partial A/\partial t| \ll \omega\,|A|$). The frequency ω is a known function of the wave vector $\boldsymbol{k} = k_1\,\hat{x}_1 + k_2\,\hat{x}_2$, which function may also vary slowly with position and time :

$$\omega = F(q^1, q^2, t, k_1, k_2) \tag{9}$$

$$\text{and } |\nabla F| \ll |\boldsymbol{k}F| \text{ and } |\partial F/\partial t| \ll \omega\,|F|.$$

A wave train whose spatial extent is of the order of the radii of curvature of the surface, or whose history carries it over a region of such extent, must consist of local sinusoidal pieces which fit properly together over the large region. Thus we assume that the wave can be represented in the form

$$A(q^1, q^2, t)\, e^{i\cdot S(q^1, q^2, t)}$$

where A is again a slowly varying amplitude factor, and S is the phase of the wave train as a function of position and time. Then near position q^1, q^2 the wave has angular frequency

$$\omega = -\frac{\partial S}{\partial t}$$

and the wave vector's components are

$$k_i = \frac{\partial S}{\partial x^i} = \frac{\partial S}{\partial q^j}\frac{\partial q^j}{\partial x^i}.$$

Here we use the summation convention. The numbers $p_1 = \partial S/\partial q^1$, $p_2 = \partial S/\partial q^2$ may be regarded as the covariant components of the local wave vector in the coordinate system q^1, q^2, since the relations connecting the k's and p's are

$$k_i = \frac{\partial q^j}{\partial x^i} p_j, \quad p_i = \frac{\partial x^j}{\partial q^i} k_j.$$

The frequency ω can be expressed as a function of p_1, p_2 instead of k_1, k_2 :

$$\omega = H(q^1, q^2, t, p_1, p_2) \tag{10}$$

where $H(q^1, q^2, t, p_1, p_2) = F(q^1, q^2, t, k_1, k_2)$.

Equation (10) can be regarded as a first-order partial differential equation for the phase S :

$$\frac{\partial S}{\partial t} + H\left(q^1, q^2, t, \frac{\partial S}{\partial q^1}, \frac{\partial S}{\partial q^2}\right) = 0. \tag{11}$$

The theory of the solution of (11) leads to the idea of group velocity quite independently of any energy or amplitude considerations. The relevant theorem is Pfaff's uniqueness theorem (COURANT and HILBERT, 1937) : suppose that S satisfies equation (11). Suppose that at time zero at position q_0^1, q_0^2 the values of S and $\partial S/\partial q^1$ and $\partial S/\partial q^2$ are known to be S^0 and p_1^0, p_2^0. Solve the following ordinary differential equations (the characteristic equations of (11)) subject to the initial conditions $q^i(0) = q^i$, $p_i(0) = p_i^0$, $S(0) = S^0$:

$$\begin{aligned} \frac{dp_i}{dt} &= -\frac{\partial H(q, t, p)}{\partial q^i} \\ \frac{dq^i}{dt} &= \frac{\partial H(q, t, p)}{\partial p_i} \\ \frac{dS}{dt} &= p_i \frac{\partial H(q, t, p)}{\partial p} - H(q, t, p). \end{aligned} \tag{12}$$

Then at time t at position $q^1(t)$, $q^2(t)$ the values of S and $\partial S/\partial q^1$, $\partial S/\partial q^2$ are $S(t)$ and $p_1(t)$, $p_2(t)$.

The physical content of this theorem can be pictured as follows : if the phase is known everywhere at time zero, it can be found everywhere at time t. But to find the phase and its gradient, S and $\partial S/\partial q^i$, at time t at the position $q^1(t)$, $q^2(t)$ it is not necessary to know $S(q^1, q^2, 0)$ for all q^1, q^2. All that is required is $S(q_0^1, q_0^2, 0)$ and $S\,\partial(q_0^1, q_0^2, 0)/\partial q^1$ and $\partial S(q_0^1, q_0^2, 0)/\partial q^2$. The wave vector $(\partial S/\partial q^1, \partial S/\partial q^2)$ and the phase S propagate their values along the track $q^1(t)$, $q^2(t)$ as if they were the momentum and action of a particle whose position and momentum at time zero were $(q^1{}_0, q^2{}_0)$, $(p^0{}_1, p^0{}_2)$ and whose Hamiltonian was the $H(q, t, p)$ of equation (10). This ' particle ' is a wave group or wave packet in the sense that it represents a local train of waves whose behaviour can be calculated without reference to the other waves which may be present. For such a wave packet, as is well known from Hamiltonian mechanics (GOLDSTEIN, 1951), $d\omega/dt = dH/dt = \partial H/\partial t$, so if H does not depend explicitly on the time, the frequency of the waves in the packet does not change as the packet propagates. If $\partial H/\partial q^1 = \partial H/\partial q^2 = 0$, then $dp_i/dt = 0$; the covariant components of the wave vector do not change as the wave propagates. Then in case the surface is flat the other two equations of motion are $dx^i/dt = \partial\omega/\partial k_i$. The usual theory of group velocity in a flat space appears as a special case of the Hamiltonian theory.

WHITHAM's (1961b) discussion of group velocity in flat spaces can be read verbatim as a proof of PFAFF's uniqueness theorem in curved spaces. The argument is this : suppose $S(q^1, q^2, t)$ satisfies equation (11). Then the function $p_1(q^1, q^2, t) = \partial S(q^1, q^2, t)/\partial q^i$ satisfy equations obtained by differentiating (11) with respect to q^i :

$$\frac{\partial p_i}{\partial t} = -\frac{\partial}{\partial q^i}\left[H(q^1, q^2, t, p_1, (q^1, q^2, t), p_2(q^1, q^2, t))\right].$$

Carrying out the differentiation on the right,

$$\frac{\partial p_i}{\partial t} = -\frac{\partial H}{\partial q^i} - \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial q^i}.$$

But $\partial p_j/\partial q^i = \partial p_i/\partial q^j$ so

$$\frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial p^j}\frac{\partial p_i}{\partial q^j} = -\frac{\partial H}{\partial q^i}.$$

Now consider a point $q^1(t)$, $q^2(t)$ which at time t moves with velocity

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}(q^1(t), q^2(t), t, p_1(q^1(t), q^2(t), t), p_2(q^1(t), q^2(t), t)).$$

The motion of this point can be computed, since the functions $S(q^1, q^2, t)$ and $p_i(q^1, q^2, t)$ are known. The rate at which $p_i(q^1(t), q^2(t), t)$ changes with time is

$$\frac{dp_i}{dt} = \frac{\partial p_i}{\partial t} + \frac{\partial p_i}{\partial q^j}\frac{dq^j}{dt} = \frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial p_j}\frac{\partial p_i}{\partial q^j} = -\frac{\partial H}{\partial q^i}.$$

Then the time rate of change of $S(q^1(t), q^2(t), t)$ is

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q^i}\frac{dq^i}{dt} = p_i\frac{\partial H}{\partial p_i} - H.$$

These remarks prove Pfaff's uniqueness theorem.

4. CLASSICAL PERTURBATION THEORY

Suppose that a particle has Hamiltonian $H = H_0(q^i, p_i) + H_1(q^i, p_i)$, where H_1 represents a small perturbation on the motion. Given the initial values $q^i(t_0)$, $p_i(t_0)$, what will be $q^i(t)$ and $p_i(t)$? Let $q_0{}^i(t)$, $p_i{}^0(t)$ be the solutions of

$$\frac{dq_0^i}{dt} = \frac{\partial H_0}{\partial p_i}, \quad \frac{dp_i^0}{dt} = -\frac{\partial H_0}{\partial q^i}$$

which satisfy $q_0^i(t_0) = q^i(t_0)$, $p_i^0(t) = p_i(t_0)$. Let the actual motion be $q^i(t) = q_0^i(t) + q_1^i(t)$, $p_i(t) = p_i^0(t) + p_i^1(t)$ where q_1^i and p_i^1 are small perturbations. Then the initial conditions on p_i^1 and q_1^i are that they must vanish at $t = t_0$. The exact equations of motion are

$$\frac{dq_0^i}{dt} + \frac{dq_1^i}{dt} = \frac{\partial}{\partial p_i} H_0(q_0^i + q_1^i, p_i^0 + p_i^1) + \frac{\partial}{\partial p_i} H_1(q_0^i + q_1^i, p_i^0 + p_i^1),$$

$$\frac{dp_i^0}{dt} + \frac{dp_i^1}{dt} = -\frac{\partial}{\partial q^i} H_0(q_0^i + q_1^i, p_i^0 + p_i^1) - \frac{\partial}{\partial q^i} H_1(q_0^i + q_1^i, p_i^0 + p_i^1).$$

If we neglect second order terms in the small perturbations q_1^i, p_i^1, the equations for these perturbations are

$$\begin{aligned} \frac{dq_1^i}{dt} &= \left(q_1^j \frac{\partial}{\partial q_j} + p_j^1 \frac{\partial}{\partial p_j}\right) \frac{\partial H_0}{\partial p_i}(q_0^i, p_i^0) + \frac{\partial}{\partial p_i} H_1(q_0^i, p_i^0), \\ \frac{dp_i^1}{dt} &= -\left(q_1^j \frac{\partial}{\partial q_j} + p_j^1 \frac{\partial}{\partial p_j}\right) \frac{\partial H_0}{\partial q^i}(q_0^i, p_i^0) - \frac{\partial}{\partial q^i} H_1(q_0^i, p_i^0). \end{aligned} \tag{13}$$

Thus the perturbations q_1^i, p_i^1 can be found by solving an inhomogeneous linear system of ordinary differential equations subject to the initial condition that q_1^i and p_i^1 vanish at $t = t_0$. The coefficients in the equations for q_1^i and p_i^1 are determined by the unperturbed solutions $q_0^i(t)$ and $p_i^0(t)$.

The technique described above has been studied in great detail in celestial mechanics, and it is known that if H_0 generates a periodic motion, the perturbation theory will often generate secular terms which grow linearly with time. Eventually these secular terms will become large enough to invalidate the perturbation theory, and then more sophisticated techniques, such as the use of action-angle variables, are required. In the present problem, it will turn out that before the secular terms become large enough to invalidate the perturbation theory the wave packet will have run into a continent.

5. A WAVE PACKET ON THE SURFACE OF THE EARTH

Let $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ be unit vectors of a Cartesian coordinate system fixed in space. Let θ and ϕ be coaltitude and longitude on a non-rotating sphere with radius a, with surface gravity g, and covered by an ocean of uniform depth h. Let $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\Phi}}$ be unit vectors in the directions of increasing θ and ϕ on the surface of the sphere. A wave packet with wave vector $k_\theta \hat{\boldsymbol{\theta}} + k_\phi \hat{\boldsymbol{\Phi}}$ has covariant wave numbers

$$p_\theta = ak_\theta, \quad p_\phi = a \sin\theta\, k_\phi, \quad \text{for } \nabla = \frac{\hat{\boldsymbol{\theta}}}{a}\frac{\partial}{\partial\theta} + \frac{\hat{\boldsymbol{\Phi}}}{a\sin\theta}\frac{\partial}{\partial\phi}.$$

The wave packet moves like a particle with the Hamiltonian

$$H_0(\theta, \phi, p_\theta, p_\phi) = \left(\frac{g}{a}\right)^{\frac{1}{2}} (p_\theta^2 + p_\phi^2 \operatorname{cosec}^2\theta)^{\frac{1}{4}} \tanh^{\frac{1}{2}}\left[\frac{h}{a}(p_\theta^2 + p_\phi^2 \operatorname{cosec}^2\theta)^{\frac{1}{2}}\right]. \tag{14}$$

In this case we know the solution of the Hamiltonian equations of motion : the packet moves around a great circle with the group velocity

$$G(k) = \frac{(gk \tanh hk)^{\frac{1}{2}}}{2k}\left(1 + \frac{2kh}{\sinh 2kh}\right). \tag{15}$$

During the motion, the magnitude, k, of the wave vector $\boldsymbol{k}$ remains constant, while the direction of $\boldsymbol{k}$ is always tangent to the path of motion. We introduce a coordinate system appropriate to this particular wave packet. The z axis is chosen normal to the plane of the great circle traversed by the packet, and the origin of time is the instant when the packet has zero longitude. Then the motion of the packet is described by the following equations :

$$\theta = 0, \quad \phi = \frac{G}{a}t, \quad p_\theta = 0, \quad p_\phi = ak. \tag{16}$$

Now suppose the earth rotates with angular velocity $\boldsymbol{\Omega}$, much less than the angular frequency, ω, of the waves in the packet. Then, to first order in Ω/ω, the Hamiltonian describing the packet's motion is $H_0 + H_1$, where H_1 is the second term in equation (8) :

$$H_1 = \frac{\Omega_\theta p_\phi \operatorname{cosec}\theta - \Omega_\phi p_\theta}{(p_\theta^2 + p_\phi^2 \operatorname{cosec}^2\theta)^{\frac{1}{2}}} \tanh\left[\frac{h}{a}(p_\theta^2 + p_\phi^2 \operatorname{cosec}^2\theta)^{\frac{1}{2}}\right].$$

Without loss of generality, we can take $\boldsymbol{\Omega} = \hat{\boldsymbol{x}}\Omega\sin\Theta + \hat{\boldsymbol{z}}\Omega\cos\Theta$. Then

$$H_1 = \Omega\left[\frac{p_\theta \sin\Theta\sin\phi + p_\phi(\sin\Theta\cot\theta\cos\phi - \cos\Theta)}{(p_\theta^2 + p_\phi^2\operatorname{cosec}^2\theta)^{\frac{1}{2}}}\right] \tanh\left[\frac{h}{a}(p_\theta^2 + p_\phi^2\operatorname{cosec}^2\theta)^{\frac{1}{2}}\right]. \tag{17}$$

Suppose that the packet starts at time t_0 at colatitude $\theta_0 = \pi/2$, longitude $\phi_0 = Gt_0/a$, with $p_\theta = 0$, $p_\phi = ak$. If $\boldsymbol{\Omega} = \boldsymbol{0}$, the packet will travel around the equator of the coordinate system introduced to describe its motion, in the manner exhibited by equations (16). What effect will the earth's rotation have on the motion and wave vector of the packet ?

To apply the perturbation theory to this problem, we must evaluate the second partial derivatives of H_0 and the first partial derivatives of H_1 with respect to θ, ϕ, p_θ, p_ϕ on the unperturbed path (16), and then use these coefficients in equations (13). The calculations are omitted. The results are these :

$$\frac{\partial H_1}{\partial\theta} = -\Omega\sin\Theta\tanh kh\cos\left(\frac{G}{a}t\right),$$

$$\frac{\partial H_1}{\partial p_\theta} = \frac{\Omega\sin\Theta\tanh kh}{ak}\sin\left(\frac{G}{a}t\right),$$

$$\frac{\partial H_1}{\partial\phi} = 0,$$

$$\frac{\partial H_1}{\partial p_\phi} = -\frac{h\Omega\cos\Theta\operatorname{sech}^2 kh}{a}.$$

All second derivatives of H_0 vanish on the path (16), except that

$$a^2 k^2 \frac{\partial^2 H_0}{\partial p_\theta^2} = \frac{\partial^2 H_0}{\partial \theta^2} = kG(k)$$

and

$$a^2 k^2 \frac{\partial^2 H_0}{\partial p_\phi^2} = k^2 \frac{dG(k)}{dk}.$$

Then equations (13) become

$$\begin{aligned} \frac{dp_\phi^1}{dt} &= 0, \\ \frac{d\phi_1}{dt} - \frac{G'(k)}{a^2} p_\phi^1 &= -\frac{G}{a}\eta\,; \end{aligned} \tag{18}$$

and

$$\begin{aligned} \frac{dp_\theta^1}{dt} + kG\theta_1 &= \epsilon kG \cos\left(\frac{G}{a}t\right), \\ \frac{d\theta_1}{dt} - \frac{G}{a^2 k} p_\theta^1 &= \frac{G}{a}\epsilon \sin\left(\frac{G}{a}t\right). \end{aligned} \tag{19}$$

Here the small dimensionless constants ϵ and η are

$$\begin{aligned} \epsilon &= \frac{\Omega \sin\Theta \tanh kh}{kG}, \\ \eta &= \frac{h\,\Omega \cos\Theta \operatorname{sech}^2 kh.}{G} \end{aligned} \tag{20}$$

The initial conditions are that θ_1, p_θ^1, ϕ_1, p_ϕ^1 should vanish at time t_0. The solutions of equations (18) and (19) satisfying these initial conditions are

$$\begin{aligned} p_\phi^1(t) &= 0, \\ \phi_1(t) &= -\eta\frac{G}{a}(t - t_0), \\ p_\theta^1(t) &= ak\epsilon\left(\frac{G}{a}t - \phi_0\right)\cos\left(\frac{G}{a}t\right), \\ \theta_1(t) &= \epsilon\left(\frac{G}{a}t - \phi_0\right)\sin\left(\frac{G}{a}t\right). \end{aligned}$$

It is convenient to give p_ϕ^1, ϕ_1, p_θ^1, θ_1 as functions not of t but of the longitude ϕ of the unperturbed wave packet. Since $\phi = Gt/a$, we have

$$\begin{aligned} p_\phi^1(\phi) &= 0, \\ \phi_1(\phi) &= -\eta(\phi - \phi_0), \end{aligned} \tag{21}$$

and

$$\begin{aligned} p_\theta^1(\phi) &= ak\epsilon(\phi - \phi_0)\cos\phi, \\ \theta_1(\phi) &= \epsilon(\phi - \phi_0)\sin\phi. \end{aligned} \tag{22}$$

These equations do contain secular terms, so they are not valid for all time.

They are valid as long as the magnitudes of ϕ_1, θ_1, p_θ^1/ak are all much less than π. Therefore they are certainly valid as long as

$$|\phi - \phi_0| \ll \pi/\eta \quad \text{and} \quad |\phi - \phi_0| \ll \pi/\epsilon. \tag{23}$$

To estimate ϵ and η we write

$$\epsilon = 2\Omega \left(\frac{h}{g}\right)^{\frac{1}{2}} (\sin \Theta)\, E(kh),$$

$$\eta = 2\Omega \left(\frac{h}{g}\right)^{\frac{1}{2}} (\cos \Theta)\, N(kh)$$

$$\text{where } E(x) = \left(\frac{\tanh x}{x}\right)^{\frac{1}{2}} \left(1 + \frac{2x}{\sinh 2x}\right)^{-1}$$

$$\text{and } N(x) = \left(\frac{x}{\tanh x}\right)^{\frac{1}{2}} \left(1 + \frac{2x}{\sinh 2x}\right)^{-1} \operatorname{sech}^2 x.$$

The function $N(x)$ decreases monotonically from its value at $x = o$, which is $\frac{1}{2}$. We conclude that for all kh, $\eta \leqslant \Omega\, (h/g)^{\frac{1}{2}} \cos \Theta$. The function $E(x)$ has a single maximum at $x = \xi$, with

$$\begin{aligned} \xi &= 1{\cdot}878973, \\ E(\xi) &= 0{\cdot}606316. \end{aligned} \tag{24}$$

Therefore $\epsilon \leqslant 1{\cdot}213\, \Omega\, (h/g)^{\frac{1}{2}} \sin \Theta$ for waves of any wavelength.

Thus the growth of the secular terms does not invalidate our solution as long as

$$|\phi - \phi_0| \ll \frac{\pi}{1{\cdot}213\, \Omega} \left(\frac{g}{h}\right)^{\frac{1}{2}}.$$

For the earth, with $\Omega = 7{\cdot}25 \times 10^{-5}\,\text{sec}^{-1}$, $g = 980\,\text{cm/sec}^2$, and $h = 4$ km, the solution is valid as long as $|\phi - \phi_0| \ll (277)\,(2\pi)$; that is, it is valid as long as the number of times the packet has circled the earth is much less than 277. This upper limit is larger for a shallower ocean.

6. DISCUSSION OF THE RESULTS

The geometrical interpretation of equations (21) and (22) is as follows: rotation slightly changes the group velocity of the packet from G to $G(1 - \eta)$. Furthermore, the path traversed by the packet is no longer a great circle. The packet travels with speed $G(1 - \eta)$ around the circle in which the earth's surface intersects a plane rotating about the axis of $\mathbf{\Omega}$ at angular velocity $-\epsilon G a^{-1} \operatorname{cosec} \Theta$ radians/sec. This plane is the plane $z = o$ at time t_0, and rotates through the angle $-2\pi\epsilon \operatorname{cosec} \Theta$ radians for each circuit of the packet around the earth. Define the instantaneous orbital plane of the packet as the plane through the centre of the earth which is tangent to the trajectory of the particle at the position of the particle. Then the instantaneous orbital plane rotates with angular velocity $-\epsilon G a^{-1} \operatorname{cosec} \Theta$ about the axis of $\mathbf{\Omega}$.

The wave vector $\boldsymbol{k}$ is not tangent to the path of the packet. The angle it makes with the direction of motion of the packet (positive if counterclockwise when viewed from above) is

$$\frac{p_\theta^1}{ak} - \frac{d\theta_1}{d\phi} = -\epsilon \sin \phi. \tag{25}$$

Thus as the packet travels, the wave vector $\boldsymbol{k}$ oscillates across the direction of travel with amplitude ϵ, making one oscillation for each circuit of the packet around the earth. The magnitude of $\boldsymbol{k}$ is constant during the motion to first order in Ω/ω.

The deviation of the trajectory from the expected great circle and the deviation of the wave vector's direction from that of the trajectory are proportional to ϵ, while the deviation of the speed of the packet from G is proportional to η. In the short wave limit, $kh \to \infty$,

$$\epsilon = \frac{2\Omega \sin \Theta}{(gk)^{\frac{1}{2}}} = \frac{2\Omega}{\omega} \sin \Theta$$

and

$$\eta = 0.$$

Thus the magnitude of the group velocity of deep water waves is unaffected by rotation, but these waves do deviate from the expected great circle trajectory by amounts proportional to their periods. In the long wave limit, $\epsilon = \Omega\,(h/g)^{\frac{1}{2}} \sin \Theta$, $\eta = \Omega\,(h/g)^{\frac{1}{2}} \cos \Theta$, so rotation produces deviations in both directions and group speed which are independent of the period of the waves. The waves for which ϵ is the largest are those which have $kh = 1{\cdot}878973$ (see equation (24)). In an ocean 4 km deep these waves would have a length of 13·6 km and a period of 94 seconds.

Taking $h = 4$ km on the earth, we have $|\epsilon| \leqslant 1{\cdot}81 \times 10^{-3}\,|\sin \Theta|$ and $|\eta| \leqslant 1{\cdot}49 \times 10^{-3}\,|\cos \Theta|$. For ocean waves with a period of 12 seconds, $\epsilon = 2{\cdot}77 \times 10^{-4} \sin \Theta$ and $\eta = 1{\cdot}19 \times 10^{-102} \cos \Theta$. The effect of the earth's rotation on ocean waves is very small.

It should be noted that so far the discussion applies only to oceans whose depths are independent of position or much greater than the wave-length of the waves ($kh \gg 1$). If kh is of the order of 1 or less and h varies with position, the waves will 'feel the bottom' and refraction will occur. The trajectories of refracted wave packets on a non-rotating earth are computed by solving Hamilton's equations of motion with the Hamiltonian (14), h being now an explicit function of θ and ϕ. The discussion then proceeds as in section 5, except that these refracted trajectories must be used in place of the trajectories (16). If the variation of h with position is a small perturbation, then the small additional perturbation due to rotation is correctly given by equations (21) and (22), at least to first order in the two perturbations.

ECKART (1950) has suggested computing wave refraction in flat spaces by analogy with particle mechanics, but has not used the Hamiltonian formulation, in which the wave vector is computed directly as the canonical momentum.

7. LOCATION OF WAVE SOURCES

Suppose that at time $t_0 + T$ an observer measures the wave vector of a wave passing him. Suppose he knows the time t_0 at which this packet originated. Then he can calculate the position at which the packet originated. How large will be his error if he neglects the earth's rotation in this calculation ?

In the coordinate system appropriate to the wave packet, as defined in section 5, let the position of the observer be ϕ_M, θ_M, while the wave vector of the packet has covariant components $p_{\phi M}$, $p_{\theta M}$. $|\theta_M| \ll 1$, $p_{\phi M} \approx ak$, and $|p_{\theta M}/ak| \ll 1$. The great circle trajectory which the observer, neglecting rotation, attributes to the packet is

$$\theta = \frac{\pi}{2} + \theta_M \cos(\phi - \phi_M) + \frac{p_{\theta M}}{ak} \sin(\phi - \phi_M),$$

$$\phi = \phi_0 + \frac{G}{a}(t - t_0);$$

here terms of second order in θ_M and $p_{\phi M}$ have been neglected. The observer gives the position of the packet at time t_0 as

$$\phi_0' = \phi_M - \frac{G}{a} T,$$

$$\theta_0' = \frac{\pi}{2} + \theta_M \cos(\phi_0' - \phi_M) + \frac{p_{\theta M}}{ak} \sin(\phi_0' - \phi_M).$$

The correct trajectory of the packet is given by equations (21) and (22). Therefore the differences between the true coordinates of the packet at time t_0 and those calculated by the observer who neglects rotation are

$$\phi_0' - \phi_0 = -\eta(\phi_M - \phi_0),$$

and

$$\theta_0' - \theta_0 = \epsilon(\phi_M - \phi_0) \sin \phi_0, \tag{26}$$

with $\theta_0 = \pi/2$. An observer facing the arriving packet will calculate for it a position of origin which is too distant by the amount $a\eta(\phi_M - \phi_0)$ km, and too far left by the amount $\epsilon(\phi_M - \phi_0) \sin \phi_0$ radians.

MUNK, SNODGRASS, MILLER and BARBER (1962) have kindly communicated to the author that some of their observations of the arrival at La Jolla of waves generated by distant storms appear to put the source as much as 0·1 radians left of the storm, as seen by an observer facing the storm. They think this effect may be produced by local refraction. The results of the present paper establish that it is not produced by the rotation of the earth. Rotation would produce an error in the observed direction, but smaller by one or two orders of magnitude, depending on the periods of the waves.

REFERENCES

COURANT, R. and HILBERT, D. (1937) *Methoden der Mathematischen Physik*, Vol. II, pp. 63–95. Julius Springer, Berlin.

ECKART, C. (1948) Approximate solutions of one-dimensional wave equations. *Rev. Mod. Phys.*, **20**, 399–417.

ECKART, C. (1950) The ray-particle analogy. *J. Mar. Res.*, **9**, 139–144.

ECKART, C. (1960) *Hydrodynamics of Oceans and Atmospheres*, Chapter XI. Pergamon Press, New York.

GOLDSTEIN, H. (1951) *Classical Mechanics*, p. 220. Addison-Wesley, Cambridge, Mass.

JEFFREYS, H. and B. S. (1956) *Methods of Mathematical Physics*. Cambridge University Press (third edition).

LANDAU, L. and LIFSCHITZ, E. (1951) *The Classical Theory of Fields*, p. 138. Addison-Wesley, Cambridge, Mass.
LANDAU, L. and LIFSCHITZ, E. (1959) *Fluid Mechanics*, § 66 and § 67. Addison-Wesley, Reading, Mass.
LIGHTHILL, M. J. and WHITHAM, G. B. (1955) On kinematic waves. I. Flood movement in long rivers. *Proc. Roy. Soc. Lond.*, (A), **229**, 281–316.
LIGHTHILL, M. J. (1960) Studies on magnetohydrodynamic waves and other anisotropic wave motions. *Phil. Trans. Roy. Soc. Lond.*, (A), **252**, 397–430.
MUNK, W., SNODGRASS, F., MILLER, G. and BARBER, N. F. (1962) Directional recording of swell from distant storm. *Phil. Trans. Roy. Soc. Lond.*, (A), (in press).
URSELL, F. (1960) On Kelvin's ship-wave pattern. *J. Fluid Mech.*, **9**, 333–346.
WHITHAM, G. B. (1955) See LIGHTHILL and WHITHAM (1955).
WHITHAM, G. B. (1961a) A note on group velocity. *J. Fluid Mech.*, **9**, 347–352.
WHITHAM, G. B. (1961b) Group velocity and energy propagation for three-dimensional waves. *Comm. Pure Appl. Math.*, **14**, 675–691.