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On the transport of mass by time-varying ocean currents*

M. S. LONGUET-HIGGINS†

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Abstract—In order to calculate the mean mass flux past a given recording station it is necessary to know more than the mean velocity in a fixed, vertical section. One must add an additional term—the ' Stokes velocity ' which depends also on the time and distance scales of the fluctuating currents. In typical circumstances, where the fluctuations are larger than the mean current, the Stokes velocity may dominate the mass transport, and lead to the mass transport being opposite in direction to the mean current.

Some general expressions are given for the Stokes velocity, and these are studied in detail for the particular case of waves propagated along a sloping sea bed (double Kelvin waves). Such waves are always propagated with the shallower water to their right in the northern hemisphere. It is shown that in regions of small bottom gradient the Stokes velocity is in the same direction as the phase velocity, but in the region of large bottom gradient the sign of the Stokes velocity is reversed. The mean Stokes velocity is in the direction of wave propagation. However, the total transport (integrated with respect to the depth and width) is in the opposite direction.

1. INTRODUCTION

IN MOST parts of the ocean the fluctuations in the velocity at a fixed point are comparable with, or many times greater than, the mean velocity at that point. Now that such observations of current velocities are becoming increasingly available it may be opportune to draw attention to a paradoxical result often overlooked when interpreting such records, namely, that the mass transport past any fixed point does not depend solely on the mean velocity measured at that point, but depends besides on other properties of the field of motion.

To many people this appears at first unreasonable. But in the theory of surface waves the difference between the mean velocity at a given point and the *mass-transport* velocity—i.e. the mean velocity of a marked particle—has long been recognized. Essentially this difference is the same as the difference between the Eulerian mean and the Lagrangian mean velocity. If the mean greatly exceeds the fluctuations, the two are nearly equal. But in surface waves and also, as is now pointed out, in most oceanic currents, the two averages are quite different. They may easily be in opposite directions, perhaps leading to false conclusions as to the origins of water masses.

In the present paper we shall first discuss quite generally the relation between the Eulerian and the Lagrangian mean velocities, and show how the difference may be calculated. The results will then be applied in particular to barotropic motions in

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†Department of Oceanography, Oregon State University, Corvallis, Oregon 97331, U.S.A. Present addresses: National Institute of Oceanography, Wormley, Godalming, England and Dept. of Applied Mathematics, Silver Street, Cambridge, England.

the neighbourhood of a steep slope (double Kelvin waves). However, the analysis is applicable also to other kinds of oscillatory motion, particularly to tidal and inertial motions. A general discussion with recommendations will be found in Section 7.

2. THE STOKES RELATIONS

Suppose that the velocity field $\mathbf{u}$ is fluctuating with time in some way which, for the sake of argument, we may take at first to be periodic. We define $\mathbf{u}$ by the Eulerian system of coordinates $\mathbf{x} = (x, y, z)$ fixed in space. The time-average $\bar{\mathbf{u}}$ we take to be small compared to $\mathbf{u}$ but not in general zero. Now when a marked particle with posi-

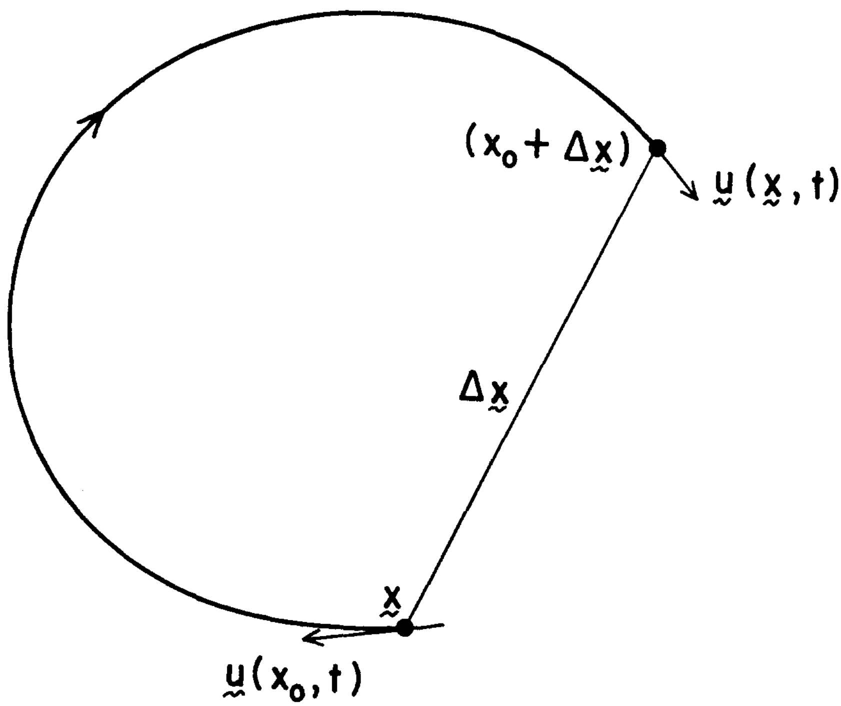


Fig. 1. The trajectory of a marked particle with initial position x_0.

tion $\mathbf{x}_0$ (Fig. 1) at time t_0 moves to a new position $\mathbf{x}_0 + \Delta\mathbf{x}$ at time t its velocity at the new position is not equal to the velocity $\mathbf{u}(\mathbf{x}_0, t)$ at $\mathbf{x}_0$ but to a slightly different velocity

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}_0, t) + \Delta\mathbf{x}\cdot\nabla\mathbf{u}(x_0, t) \tag{1}$$

depending on the space-gradient $\nabla\mathbf{u}$ of the velocity field. Equation (1) is of course correct only to order $\Delta\mathbf{x}$, quantities of higher order being neglected. Now if the particle oscillates in the neighbourhood of its initial position (Fig. 1), in such a way that $\Delta\mathbf{x}$ is still small compared to the local length-scale of the velocity field, we may write, to the same approximation,

$$\Delta\mathbf{x} = \int_{t_0}^{t} \mathbf{u}(\mathbf{x}_0, t)\,\mathrm{d}t. \tag{2}$$

Substituting into (1) and taking mean values over one or more wave cycles we obtain

$$\overline{\mathbf{u}(\mathbf{x}, t)} = \overline{\mathbf{u}(x_0, t)} + \overline{\int \mathbf{u}(\mathbf{x}_0, t)\, \mathrm{d}t \cdot \nabla \mathbf{u}(\mathbf{x}_0, t)}. \quad (3)$$

The left-hand side, which gives the mean velocity of a marked particle, is sometimes called the mass-transport velocity and may be denoted by the capital letter $\mathbf{U}$. Thus we have,

$$\mathbf{U} = \bar{\mathbf{u}} + \overline{\int \mathbf{u}\, \mathrm{d}t \,.\, \nabla \mathbf{u}}. \quad (4)$$

This is Stokes's formula for the mass-transport velocity in a water wave (STOKES, 1847) but it is clearly applicable in more general circumstances. It shows that even if the mean velocity at a fixed point (the Eulerian mean) is zero, the mass-transport velocity (the Lagrangian mean) is generally not zero. It will be convenient to refer to the difference between $\mathbf{U}$ and $\bar{\mathbf{u}}$ as the Stokes velocity $\mathbf{U}_s$. Thus

$$\mathbf{U}_s = \overline{\int \mathbf{u}\, \mathrm{d}t \cdot \nabla \mathbf{u}}. \quad (5)$$

The explicit application of equation (4) to progressive gravity waves in water of finite depth was given by STOKES (1847), and is partly reproduced by LAMB (1932) and others. It is found that the horizontal component of the mass-transport velocity is always forwards relative to the mean (that is, in the same direction as the phase velocity). The vertical component of the mass-transport velocity vanishes, as one would expect.

The paradoxical forward motion in a progressive gravity wave may also be explained as related to the mass carried forward by the wave crests plus the mass

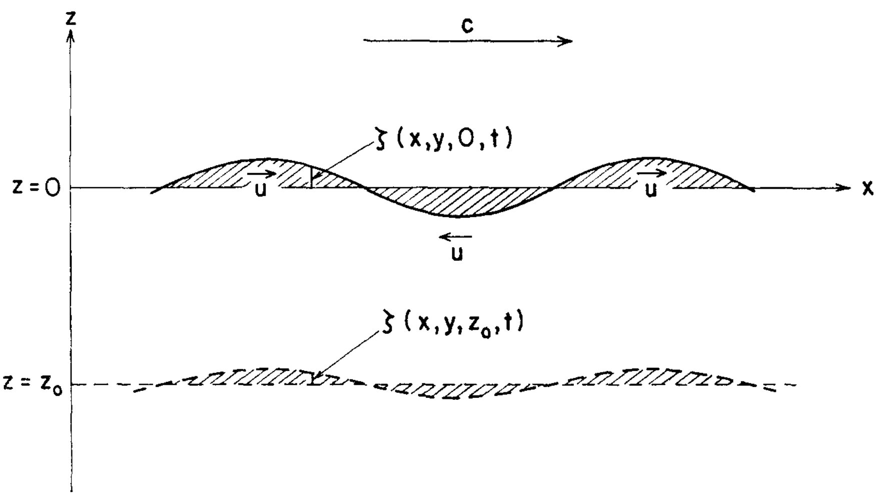


Fig. 2. Derivation of the *total* mass flux below a given level as being due to the positive mass carried by the wave crests minus the mass defect carried by the troughs; and a similar argument applied at a lower level within the fluid.

deficiency carried backward by the troughs—in fact to the mean value of $u\zeta$ (see Fig. 2). This gives in fact the *total* mass flux below the surface:

$$\overline{u\zeta} = \overline{u \int w\, \mathrm{d}t} = \int_{-h}^{0} U_s\, \mathrm{d}z = M(0), \quad (6)$$

say. But it should be emphasized that this mass flux is not all concentrated between the crests and the troughs of the waves. At any mean level z_0 within the fluid a similar surface $\zeta(x, y; z_0; t)$ may be drawn which moves, like the free surface, with the fluid. Applying the same argument to this surface as was applied to the free surface we see that the mass flux below the surface $z = z_0 + \zeta(x, y; z_0; t)$ is given by

$$\overline{(u\zeta)}_{z_0} = \int_{-h}^{z_0} U_s \, dz = M(z_0). \tag{7}$$

The distribution of momentum with depth can now be found by differentiating each element of (7):

$$U_s = \frac{dM}{dz_0} = \frac{\partial}{\partial z} \overline{(u\zeta)}_{z_0} = \frac{\partial}{\partial z} \overline{\left(u \int w \, dt\right)}. \tag{8}$$

In other words the forward momentum is distributed throughout the fluid, and is not simply located at the free surface. It is simple to show, by means of the equation of continuity, that the formulae (5) and (8) are equivalent. Thus we have

$$\begin{aligned} \frac{\partial}{\partial z} \overline{\left(u \int w \, dt\right)} &= \overline{u \int \frac{\partial w}{\partial z} \, dt} + \overline{\frac{\partial u}{\partial z} \int w \, dt} \\ &= \overline{u \int \left(-\frac{\partial u}{\partial x}\right) dt} + \overline{\int w \, dt \, \frac{\partial u}{\partial z}} \\ &= \overline{\int u \, dt \, \frac{\partial u}{\partial x}} + \overline{\int w \, dt \, \frac{\partial u}{\partial z}}. \end{aligned} \tag{9}$$

The last step follows from the fact that if A and B are any two periodic quantities with zero mean,

$$\overline{A \int B \, dt} + \overline{\int A \, dt \, B} = \frac{\partial}{\partial t} \overline{\int A \, dt \int B \, dt} = 0. \tag{10}$$

3. APPLICATION TO OCEAN CURRENTS

From equation (5) it can be seen that the Stokes velocity is generally of order $\overline{u^2}\,(T/L)$, where L and T denote typical scales of length and time of the current velocity $\mathbf{u}$. For example, if $\mathbf{u}$ is of order 20 cm/sec, while T and L are of order 1 day (10^5 sec) and 100 km (10^7 cm) respectively, then $\mathbf{U}_s$ is of order 4 cm/sec, which may be comparable with $\bar{\mathbf{u}}$.

Normally it will be necessary to take into account vertical gradients of velocity as well as horizontal gradients. For simplicity, however, we shall consider barotropic motions in which the velocity may be assumed uniform in any given vertical line. The currents will further be considered to be small perturbations on a state of rest, or at least of a state in which the mean velocity is small compared to the fluctuating component.

Then as basic equations, if we make the hydrostatic assumption, we have the equations of motion and continuity

$$\frac{D\mathbf{u}}{Dt} + \mathbf{f} \wedge \mathbf{u} = -g\nabla\zeta \tag{11}$$

and

$$\nabla \cdot (h\mathbf{u}) = -\frac{\partial \zeta}{\partial t} \tag{12}$$

where f, g, h and ζ denote twice the vertical component of the Earth's rotation vector, the acceleration of gravity, the mean depth of the fluid and the surface elevation, respectively. In evaluating the Stokes velocity $\mathbf{U}_s$ it is sufficient to use the linearized form of equation (11), since the right-hand side of equation (5) is bi-linear in $\mathbf{u}$. We shall be concerned with the two horizontal components of the Stokes velocity, namely

$$U_s = \overline{\int u \, dt \frac{\partial u}{\partial x}} + \overline{\int v \, dt \frac{\partial u}{\partial y}} \tag{13}$$

and

$$V_s = \overline{\int u \, dt \frac{\partial v}{\partial x}} + \overline{\int v \, dt \frac{\partial v}{\partial y}} . \tag{14}$$

It is well known that, for various given forms of the depth $h(x, y)$, equations (11) and (12) admit solutions in the form of waves, both of the first class (dominated by gravity) and of the second class (dominated by Coriolis forces). We shall therefore first derive some alternative forms of equations (13) and (14) appropriate to progressive wave motion.

Let C denote the phase velocity and let the x-axis be chosen in the direction of wave propagation. Then, using (10) we have

$$\overline{\int u \, dt \frac{\partial u}{\partial x}} = \overline{- u \int \frac{\partial u}{\partial x} dt} = \overline{u^2}/c \tag{15}$$

since u is a function of $(x - ct)$. Similarly

$$\overline{\int v \, dt \frac{\partial u}{\partial y}} = \overline{\int v \, dt \left(\frac{\partial v}{\partial x} - \omega\right)} = \overline{v^2}/c - \overline{\int v \, dt \, \omega} \tag{16}$$

where

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{17}$$

denotes the vorticity. Therefore altogether

$$U_s = (\overline{u^2} + \overline{v^2})/c - \overline{\int v \, dt \, \omega}. \tag{18}$$

It will be noted that the first term is always positive, that is to say in the direction of wave propagation. Incidentally we deduce that for irrotational waves, in which $\omega = 0$ and $\bar{u}$ also vanishes, the total mass transport M is given by

$$M = \int_{-h}^{0} U_s \, dz = \frac{2}{c} (K.E.) \tag{19}$$

where $K.E.$ denotes the density of the kinetic energy per unit horizontal area (cf. Starr, 1959).

An alternative form of equation (18) is

$$U_s = \overline{u^2}/c + \overline{\int v \, dt \frac{\partial u}{\partial y}} . \tag{20}$$

On the other hand for the y-component of the Stokes velocity we have

$$\overline{\int u \, dt \frac{\partial v}{\partial x}} = \overline{- u \int \frac{\partial v}{\partial x} dt} = \overline{uv}/c \tag{21}$$

and

$$\overline{\int v\,\mathrm{d}t\,\frac{\partial v}{\partial y}} = \overline{\int v\,\mathrm{d}t\left(\delta - \frac{\partial u}{\partial x}\right)} = \overline{\int v\,\mathrm{d}t\,\delta} - \overline{uv}/c \tag{22}$$

where

$$\delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{23}$$

denotes the divergence. Adding equations (20) and (21) we find

$$V_s = \overline{\int v\,\mathrm{d}t\,\delta}. \tag{24}$$

When δ vanishes, or is small, V_s is necessarily small also.

An alternative but useful expression for the Stokes velocity can be derived as follows. We have

$$\begin{aligned} U_s &= \overline{\int u\,\mathrm{d}t\,\frac{\partial u}{\partial x}} + \overline{\int v\,\mathrm{d}t\,\frac{\partial u}{\partial y}} \\ &= \overline{\int u\,\mathrm{d}t\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} + \overline{u\int \frac{\partial v}{\partial y}\,\mathrm{d}t + \frac{\partial u}{\partial y}\int v\,\mathrm{d}t} \\ &= \overline{\delta\int u\,\mathrm{d}t} + \overline{\frac{\partial}{\partial y}\left(u\int v\,\mathrm{d}t\right)}\,\cdot \end{aligned} \tag{25}$$

Now from the equation of continuity (12) we have

$$-h\delta = \frac{\partial \zeta}{\partial t} + \left(\frac{\partial h}{\partial x}u + \frac{\partial h}{\partial y}v\right) \tag{26}$$

and so on multiplying the above equation by h to get the vertically integrated mass flux in the x-direction we obtain

$$hU_s = \overline{u\zeta} + \frac{\partial h}{\partial x}\overline{u\int u\,\mathrm{d}t} + \frac{\partial h}{\partial y}\overline{u\int v\,\mathrm{d}t} + h\frac{\partial}{\partial y}\left(\overline{u\int v\,\mathrm{d}t}\right). \tag{27}$$

But $\overline{u\int u\,\mathrm{d}t}$ is identically zero, by setting $A = B = u$ in equation (10). Hence

$$hU_s = \overline{u\zeta} + \frac{\partial}{\partial y}\left(\overline{hu\int v\,\mathrm{d}t}\right) \tag{28}$$

and similarly

$$hV_s = \overline{v\zeta} - \frac{\partial}{\partial y}\left(\overline{hu\int v\,\mathrm{d}t}\right). \tag{29}$$

4. LINEAR BOTTOM TOPOGRAPHY

Of particular interest is the situation when the bottom topography is two-dimensional, and when the depth h is a function of one coordinate only; say the y-coordinate (see Fig. 3). We shall consider wave motions which propagate in the x-direction, i.e. such that

$$u, v, \zeta \propto \mathrm{e}^{i(kx-\sigma t)}. \tag{30}$$

On substituting ik and $-i\sigma$ for $\partial/\partial x$ and $\partial/\partial t$ respectively in the linearized form of equation (11) we obtain two simultaneous equations for u and v with solutions

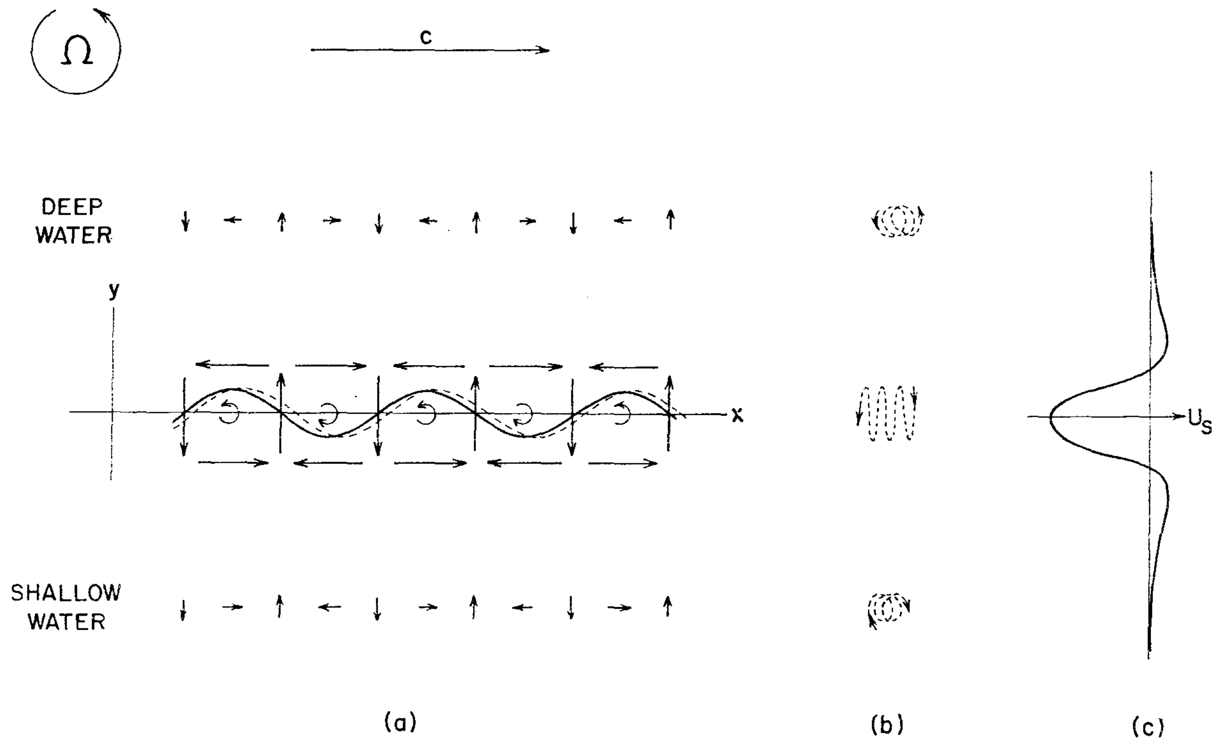


Fig. 3. A schematic picture of a double Kelvin wave. The bottom contours are parallel to the x-axis and the basic rotation Ω is anti-clockwise, as in the northern hemisphere. In Fig. 3a the full sine-wave represents a displaced line of particles. The broken sine wave represents the position of the same particles a short time later. The phase velocity c must be to the right. Figure 3b shows the particle orbits and Fig. 3c the Stokes velocity U_s.

$$\left.\begin{aligned} u &= \frac{-g}{f^2-\sigma^2}(f\zeta' + \sigma k\,\zeta) \\ v &= \frac{ig}{f^2-\sigma^2}(\sigma\zeta + fk\zeta) \end{aligned}\right\} \qquad (31)$$

where $'$ denotes $\partial/\partial y$. Substitution in the equation of continuity (12) then gives for ζ the second-order equation

$$(h\zeta')' = \left(\frac{f^2-\sigma^2}{g} + k^2 h - \frac{kf}{\sigma}h'\right)\zeta. \qquad (32)$$

If $h(y)$ is monotonically increasing with y, and tends to uniform values h_1 and h_2 as $y \to -\infty$ and $+\infty$ respectively then it can be shown that there exist waves, trapped in the transition region between $h = h_1$ and $h = h_2$, such that $\zeta \to 0$ as $y \to \pm\infty$. Such waves have been called double Kelvin waves (LONGUET-HIGGINS, 1968a, b).

These waves can only be propagated with the shallow water on their right, in the northern hemisphere. Thus if $h_1 < h_2$ the waves propagate in the positive x-direction: $\sigma/k = c > 0$.

A physical explanation of this type of wave motion can be given as follows, in the simplest case. Imagine a line of particles whose mean position is parallel to a contour of constant depth, as in Fig. 3. During the motion this line of particles has the form of a sine-wave. Now those vertical filaments of fluid that are displaced into deeper water become stretched, and so acquire additional vorticity. Hence, relative to the Earth, they have an anticlockwise spin, in the northern hemisphere. Likewise those filaments that are displaced into shallower water are contracted, and so relative to the Earth they develop a clockwise spin. The instantaneous velocity vectors are as shown in Fig. 3, the circulation being in cells of alternating sign. Clearly this results in a displacement of the original sine-wave such that the phase is always propagated towards the right.

The vertical displacement of the surface tends to counteract the stretching or contraction of the vertical filaments, but is never able to overcome it, in this type of motion. The energy of the motion is concentrated near the sloping region, where the restoring action of the vorticity is greatest. The amplitude of the particle motion falls off to each side of the sloping zone. In fact it decays exponentially at large distances, if the depth on each side tends to a uniform value.

It will be noted that the longitudinal component u of the particle velocity changes sign on the two sides of the sloping zone, whereas the transverse component v has always the same sign, at least in the lowest mode. Hence the orbital motion in the shallower water is in the opposite sense to that in the deeper water. On the shallower side the particle orbits are, to first order, circles or ellipses described in the clockwise sense, as in a Stokes wave lying on its side. On the deeper side the particle orbits are described in the anticlockwise sense, like a Stokes wave lying on its other side. Because the horizontal restoring mechanism only operates in the zone where the bottom gradient is appreciable, the wave energy tends to be greatest there. At large distances on either side of the sloping zone the orbital motion diminishes exponentially with $|y|$.

Consider now the mass-transport velocity in such a wave motion. On the shallow-water side, the orbits being clockwise, the particle motion is forward on the side of the orbit where the amplitude is greater, and backward where it is less. The contribution from the x-component of the velocity gradient is also positive. Hence the Stokes velocity is positive, i.e. in the direction of wave propagation. Exactly similar considerations show that the Stokes velocity is positive also on the deep-water side, at large distances from the sloping zone.

On the other hand it is easy to show that the total Stokes transport, integrated with respect to both depth and width, must be negative. For the total mass-flux M in the direction of wave propagation is given by

$$M = \int_{-\infty}^{\infty} hU_s \, \mathrm{d}y = \int_{-\infty}^{\infty} u\zeta \, \mathrm{d}y \tag{33}$$

from (28) since u and v vanish at the two limits.* Substituting for u from (31) and noting that

$$\int_{-\infty}^{\infty} \zeta' \zeta \, \mathrm{d}y = \left[\tfrac{1}{2} \zeta^2\right]_{-\infty}^{\infty} = 0 \tag{34}$$

we obtain

$$M = -\frac{g \, \sigma k}{f^2 - \sigma^2} \int_{-\infty}^{\infty} \zeta^2 \, \mathrm{d}y. \tag{35}$$

The integral is positive, in general. Then since $\sigma/k > 0$ and $\sigma^2 < f^2$ (LONGUET-HIGGINS, 1968b) it follows that

$$M < 0 \tag{36}$$

in other words, *the total Stokes flux is opposite to the direction of wave propagation.*

It follows that there must be a region, between the two extremes of deep and shallow water, where the Stokes flow is negative. This is indicated in Fig. 3 by the schematic profile of U_s shown on the right.

An explanation of the negative flow for points near the centre can also be given as follows. We have seen that the longitudinal component of the orbital velocity changes sign on some line y = constant in the sloping zone. In the neighbourhood of this line, the orbital motion, to first order, is purely transverse. But when a particle is on the shallow water side of its orbit, u is actually negative as can be seen from Fig. 3. Similarly when a particle is on the deep-water side, u is negative also. Hence on both sides of the orbit the particle receives a small negative displacement. Thus the Stokes velocity must be negative, at least at this point.

In the next Section we shall show how the profile of the mass-transport velocity may be calculated in detail.

5. THE MASS-TRANSPORT PROFILE

To calculate the mass-transport velocity as a function of position along the depth profile it is convenient first to choose units in which

$$m = 1, \quad f = 1 \tag{37}$$

and to write

$$\sigma = 1/\tau \tag{38}$$

so that τ is the wave period in pendulum days. Equations (31) and (32) then become

*Equation (33) may be compared with (6).

$$\left.\begin{aligned} u &= \frac{-g\tau}{\tau^2 - 1}(\tau\zeta' + \zeta) \\ v &= \frac{ig\tau}{\tau^2 - 1}(\zeta' + \tau\zeta) \end{aligned}\right\} \tag{39}$$

and

$$(h\zeta')' = \left(\frac{\tau^2 - 1}{g\tau^2} + h - \tau h'\right)\zeta \tag{40}$$

respectively.

From equation (20) we now have for the Stokes velocity

$$U_s = \frac{g^2\,\tau^3}{(\tau^2 - 1)^2}\left[\overline{(\tau\zeta' + \zeta)^2} + \overline{(\zeta' + \tau\zeta)\,(\tau\zeta'' + \zeta')}\right]. \tag{41}$$

Since $\zeta \propto e^{i(kx - \sigma t)}$ the mean values on the right-hand side may be replaced by the factor $\frac{1}{2}$, provided the exponential factors are removed. Thus we have simply

$$U_s = \frac{g^2\,\tau^3}{2\,(\tau^2 - 1)^2}\left[(\tau\zeta' + \zeta)^2 + (\zeta' + \tau\zeta)\,(\tau\zeta'' + \zeta')\right] \tag{42}$$

$$= \frac{g^2\,\tau^3}{2\,(\tau^2 - 1)^2}\left[\tau^2\,(\zeta^2 + \zeta\zeta'') + \tau\,(3\zeta\zeta' + \zeta'\zeta'') + (\zeta^2 + \zeta'^2)\right]$$

$$= \frac{g^2\,\tau^3}{4\,(\tau^2 - 1)^2}\left[\tau^2\,(\zeta^2)'' + \tau\,(3\zeta^2 + \zeta'^2)' + 2\,(\zeta^2 + \zeta'^2)\right]. \tag{43}$$

On integrating over $-\infty < y < \infty$ we find

$$\int_{-\infty}^{\infty} U_s\,\mathrm{d}y = \frac{g^2\,\tau^3}{2\,(\tau^2 - 1)^2}\int_{-\infty}^{\infty}(\zeta^2 + \zeta'^2)\,\mathrm{d}y \tag{44}$$

so that *the Stokes velocity at the surface, integrated across the wave region, is in the same direction as the phase velocity.*

For large wave periods τ the expression on the right of (43) is dominated by the terms containing the highest power of τ. Hence we have approximately

$$U_s \doteqdot \tfrac{1}{4}\,\tau\,g^2\,(\zeta^2)''. \tag{45}$$

Now the double Kelvin wave of lowest order, which is also the mode most likely to occur (LONGUET-HIGGINS, 1968b) has the form of an exponential function in the regions far from the steep slope ($|y| \gg 1$), joined by a rounded curve in the sloping region (see Fig. 5). The curve of ζ^2 is of course similar. Since $U_s \propto (\zeta^2)''$ very nearly it follows that in this mode U_s is positive in the two outer regions, far from the steep slope, but has a rather strong negative jet in the neighbourhood of the transition region as in Fig. 3. The integrated value, as was seen earlier, is positive but relatively small in magnitude since it is proportional to a lower power of τ.

Higher modes also exist, in which the surface profile $\zeta(y)$ has one or more zeros in $-\infty < y < \infty$. In such waves the profile of U_s will reverse more often than in Fig. 3. However, it has been pointed out (1968b) that since the higher modes are more sensitive to the form of the depth profile they are more easily scattered and less likely to be observed in practice.

To take the exact calculation of U_s a step further we may substitute for ζ'' in equation (42) using the differential equation (32). This gives

$$U_s = \frac{g^2\,\tau^3}{2(\tau^2-1)}\left[h\,(\tau\zeta' + \zeta)^2 + (h - \tau h')\,(\zeta' + \tau h)^2 + \frac{\tau^2-1}{g\tau}\,\zeta\,(\zeta' + \tau\zeta)\right]$$
$$= \tfrac{1}{2}\tau\,[u^2 + (1 - \tau h'/h)\,(iv)^2 - (\zeta/h)\,(iv)] \qquad (46)$$

where u and v are given by equations (39). In dimensional form equation (46) may be written

$$(U_s/c) = \tfrac{1}{2}\left[(u/c)^2 + \left(1 - \frac{f\,h'}{\sigma kh}\right)(iv/c)^2 - \frac{f\zeta}{\sigma h}\,(iv/c)\right]. \qquad (47)$$

6. A PARTICULAR DEPTH PROFILE

The above analysis is well illustrated by a particular form of the depth profile:

$$h = \bar{h}\,(1 + \beta \tanh y/W) \qquad (48)$$

in which $\bar{h}$ represents the mean depth and β the proportional change in depth at large distances. Thus, as $y \to \mp\,\infty$, so $h \to h_1$, h_2, where

$$h_1 = \bar{h}\,(1 - \beta), \qquad h_2 = \bar{h}\,(1 + \beta), \qquad \beta = \frac{h_2 - h_1}{h_2 + h_1}. \qquad (49)$$

A method of solving equation (32) numerically was given in a previous paper (LONGUET-HIGGINS, 1968b), which contains also a general discussion of the eigenfunctions and the corresponding periods τ. For any given values of the depth change β, the wave-number k and the nondimensional parameter

$$\varpi \equiv \frac{W^2 f^2}{\beta g \bar{h}} \geqslant 0, \qquad (50)$$

it appears that there are an infinite number of possible modes, with corresponding periods τ, such that the nth mode has exactly n zeros in the interval $(-\infty < y < \infty)$. Of these modes the lowest ($n = 0$) is the most insensitive to changes in the bottom profile. When $\varpi > 0$, the frequency $\sigma\,(= 1/\tau)$ tends to zero both as $kW \to 0$ and as $kW \to \infty$, for all the modes, so that at some intermediate value of kW the group-velocity $\mathrm{d}\sigma/\mathrm{d}k$ vanishes (see Fig. 10 of the paper just quoted). This may have some interesting dynamical consequences. However in most parts of the ocean we have $\varpi \ll 1$. It is found that when $\varpi \to 0$ the value of kW corresponding to zero group-velocity tends to zero for the lowest mode (but not the higher modes); and as $kW \to 0$ the frequency σ and period τ tend to finite, positive values. In what follows we shall consider only the lowest mode, in this limiting, nondivergent case.

The period τ for the lowest mode is shown as a function of kW in Fig. 4, for various values of β. Clearly τ is an increasing function of kW, so that the group-velocity $\mathrm{d}\sigma/\mathrm{d}k$ is always negative. Shown by broken lines are the asymptotes

$$\left.\begin{aligned} \tau &\sim A + B\,(kW) \quad \text{as} \quad kW \to 0 \\ \tau &\sim C\,(kW) + D \quad \text{as} \quad kW \to \infty \end{aligned}\right\} \qquad (51)$$

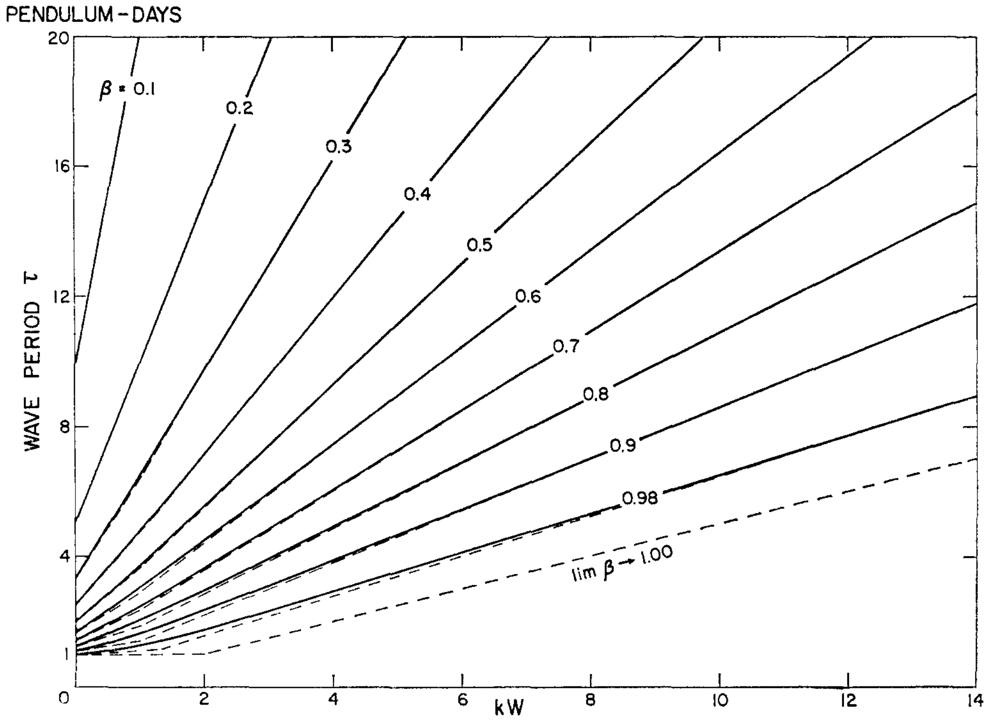


Fig. 4. The wave period τ as a function of kW for the depth profile $h = \bar{h}\,(1 + \beta \tanh y/W)$, when the horizontal divergence is negligible. The broken lines denote the asymptotes as $kW \to 0$ and $kW \to \infty$ (see equations 51).

where A, B, C and D are constants given by

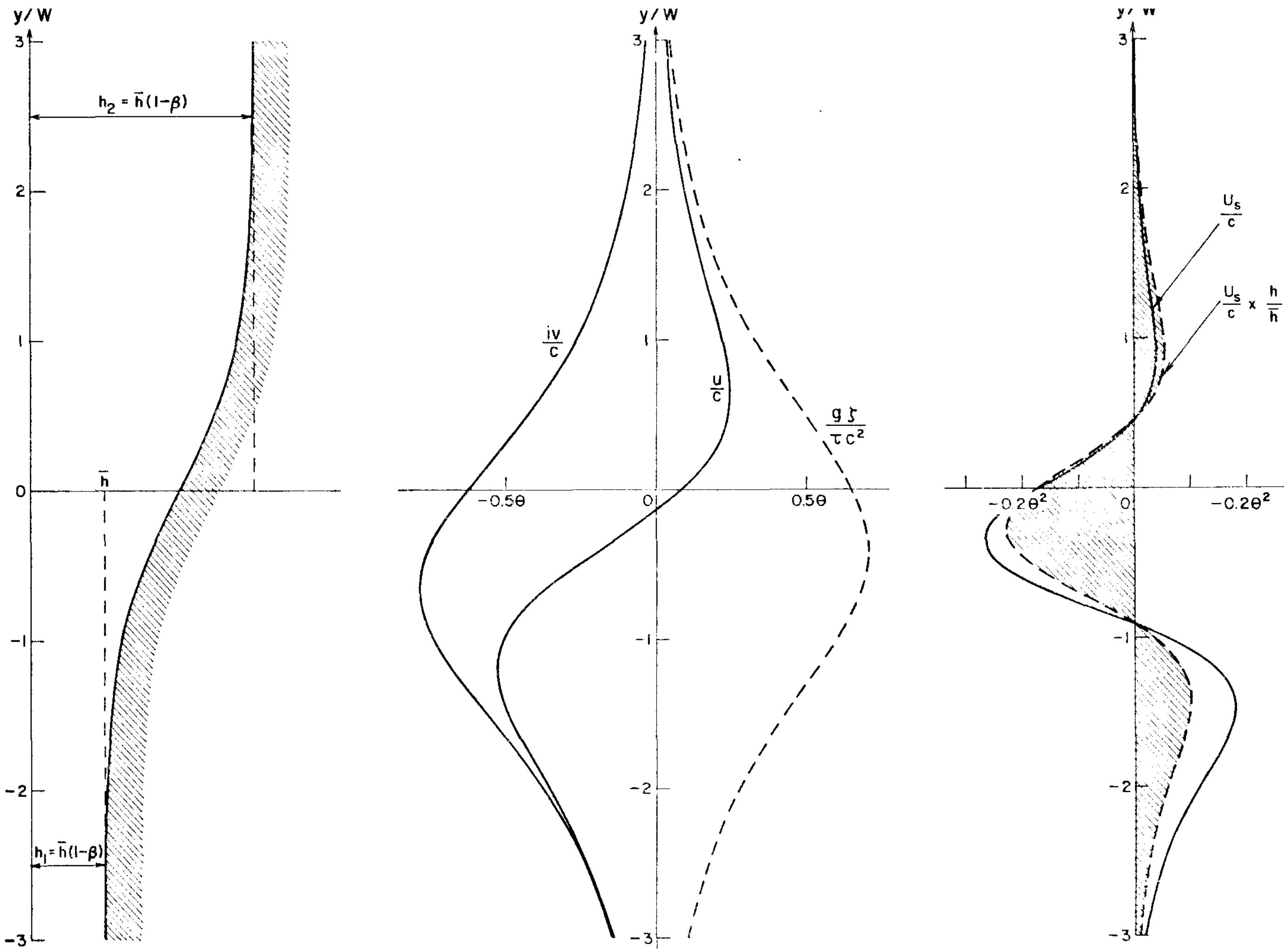
$$\left.\begin{aligned} A &= \frac{1}{\beta}, & B &= \frac{1-\beta^2}{2\beta^2}\log\left(\frac{1+\beta}{1-\beta}\right), \\ C &= \frac{1+\sqrt{(1-\beta^2)}}{2\beta}, & D &= \sqrt{\left[\frac{\sqrt{(1-\beta^2)}}{2\,(1-\sqrt{(1-\beta^2)})}\right]}, \end{aligned}\right\} \tag{52}$$

which expressions may be derived from Section 9 of the paper just referred to. For small values of β *both* the asymptotes (52) have the same equation.

$$\tau \sim \frac{1}{\beta}(1 + kW) \tag{53}$$

and indeed it can be seen from Fig. 4 that for $\beta < 0{\cdot}5$ the calculated wave period τ can hardly be distinguished from either asymptote. When β lies between 0·5 and 1·0 the difference is more palpable. When $\beta = 1{\cdot}0$ no solution with finite energy exists, and the limit $\beta \to 1{\cdot}0$ is therefore singular.

The solution in a typical case when $\beta = 0{\cdot}5$ and $kW = 1{\cdot}0$ is shown in Fig. 5. Figure 5a shows the depth profile; the depth varies from $0{\cdot}5\,\bar{h}$ at $y = -\infty$ to $1{\cdot}5\,\bar{h}$ at $y = +\infty$. In Fig. 5b, the surface elevation $\bar{\zeta}$ (or equivalently the pressure variation $\rho g\bar{\zeta}$) is indicated by a broken line and the two components u and (iv) of the velocity by solid lines. v is of course in quadrature with u and ζ. It can be seen that whereas the transverse component of velocity v has always the same sign independently of y,



(a) PROFILE OF THE MEAN DEPTH (b) ORBITAL VELOCITY AND SURFACE ELEVATION (c) THE STOKES VELOCITY

Fig. 5. (a) The depth profile $h = \bar{h}(1 + \beta \tanh y/W)$ in the typical case $\beta = 0{\cdot}5$ and $kW = 1{\cdot}0$. (b) The components of orbital velocity u, v and the surface elevation ζ for the lowest mode. (c) The Stokes velocity U_s and the depth-integrated transport hU_s.

the longitudinal component u has one zero in $(-\infty, \infty)$. The surface elevation, like (*iv*), also has always the same sign. Since τ is somewhat greater than 1, the longitudinal component u behaves more nearly like ζ' than ζ [see equation (31)]. In other words, the currents are roughly geostrophic. It will be noted that the maximum values of $|v|$ and $|u|$, and hence the maximum energy density, is displaced from the position of mean depth ($y = 0$) somewhat towards shallow water. Essentially this is because the horizontal restoring forces depend on the relative change in depth, that is (h'/h) more nearly than on h', and (h'/h) is greater on the shallower side.

The Stokes velocity U_s is shown in Fig. 5c. The quantity θ in Figs. 5b and 5c denotes a small parameter, proportional to the amplitude of the motion. Thus in Fig. 5b the abscissa is proportional to θ, and in 5c it is proportional to θ^2.

The full curve in Fig. 5c indicates the Stokes velocity as calculated from equation (47). The positive velocity on the two flanks of the profile stand out clearly, and also the negative jet in the sloping zone. The broken curve represents the *vertically integrated* Stokes transport, which is proportional to hU_s. On the shallow side this curve lies inside the full curve; on the deep side it lies outside. The total transport M integrated with respect to both width and depth is proportional to the area under the broken curve. Generally, as we saw in Section 4, this total transport is negative. But in the present case when the divergence is zero ($\varpi = 0$) the total transport is identically zero also.

Figure 6 shows the effect of narrowing the shelf relative to the wavelength. The negative jet is concentrated into an ever narrower zone and increases in intensity relative to the positive motion on the flanks of the escarpment.

Figure 7 shows the effect of increasing the contrast in depth between h_1 and h_2. The motion tends to become more asymmetrical, and the energy of the wave motion, hence also the negative jet, is concentrated more towards the upper part of the slope.

For the higher modes ($n = 1, 2, \ldots$) which were mentioned earlier, the results are qualitatively similar, except that the Stokes velocity changes sign $(n + 2)$ times in $(-\infty < y < \infty)$ instead of only twice.

7. CONCLUSIONS

The general conclusions of this paper in relation to mean velocities may be stated concisely in the form

$$\text{Lagrange} = \text{Euler} + \text{Stokes}. \tag{54}$$

Moreover, Stokes may be of comparable magnitude to Lagrange, eclipsing Euler. It is quite important, therefore, in recording a mean velocity to state the method of measurement and the type of mean referred to. In determining the origin of water masses it is the Lagrangian mean which is most relevant. On the other hand for some dynamical studies Eulerian means are more useful.

The analysis of this paper has been carried through for motions that are assumed periodic in the first place. However the analysis is equally valid for quasi-periodic motions having a more or less broad spectrum, provided that meaningful time averages can be taken while a particle moves through a distance $|\Delta\mathbf{x}|$ which is small compared to a typical length scale of the field of motion. In those formulae which are frequency-

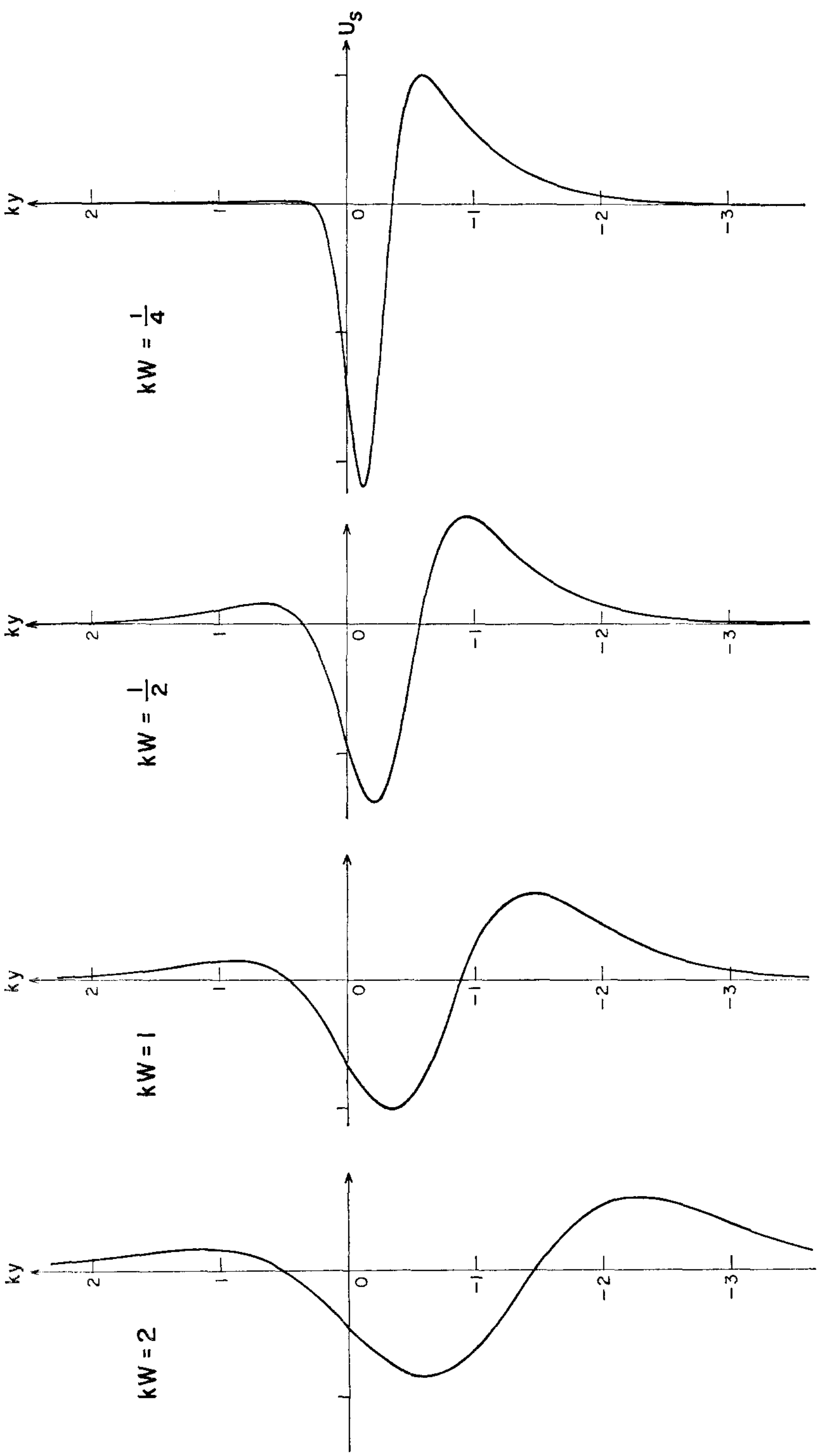


Fig. 6. The profile of the Stokes velocity U_s when $k = 1{\cdot}0$ and for different widths W of the sloping zone. U_s is in arbitrary units.

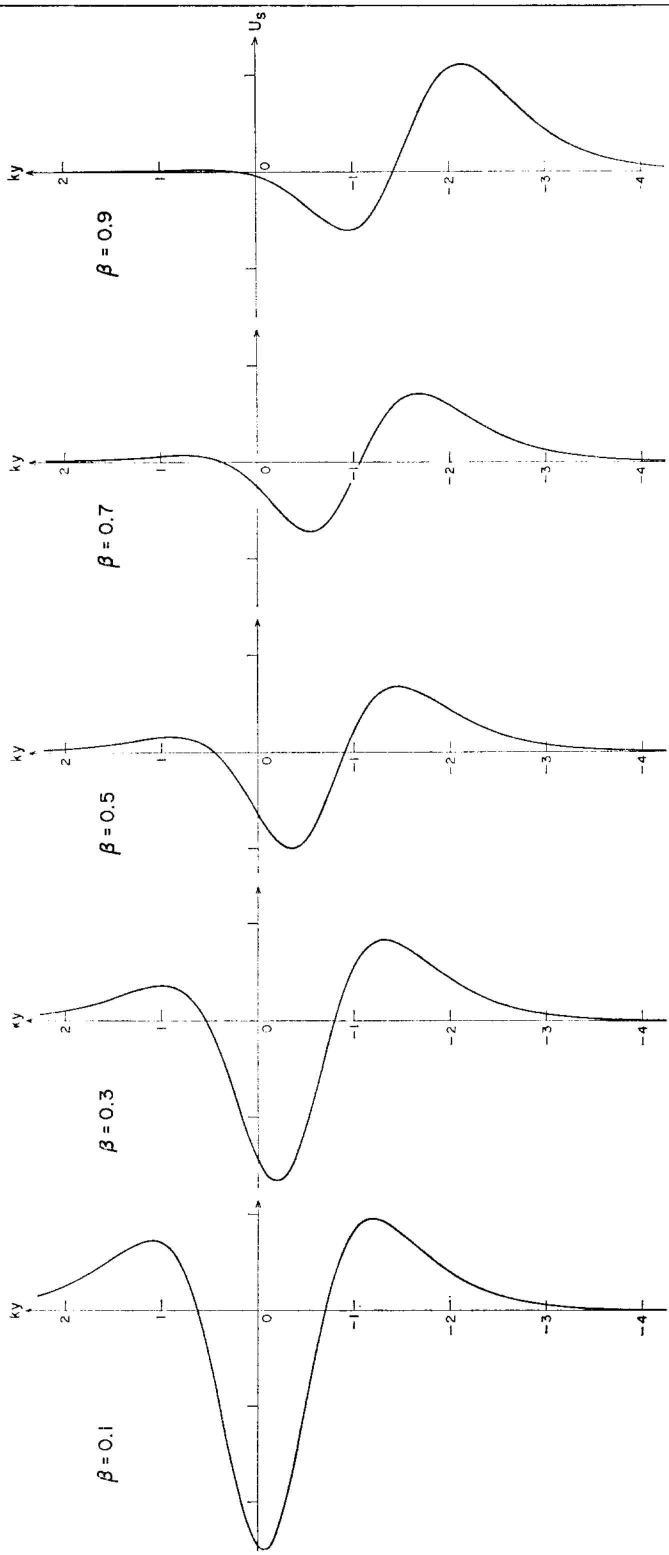


Fig. 7. The profile of the Stokes velocity U_s when $kW = 1\cdot0$ for different contrasts between deep water and shallow water: $\beta = 0\cdot1, 0\cdot3, 0\cdot5, 0\cdot7$ and $0\cdot9$.

dependent, mean values may be expressed in terms of the autospectra and cospectra of u, v, ζ and ζ'.

The correlations involved in estimating the Stokes velocity may of course be small. The greatest and the least values of the correlation will be obtained when the motion is coherent, as in the presence of progressive waves.

The Stokes velocity has been worked out in detail for a particular type of wave motion—namely double Kelvin waves—but it should be emphasized that there is no restriction to this type of motion. At many deep sea stations tidal motions, for example, predominate. The great wavelength of the barotropic tides will in most cases result in only a small Stokes velocity (though in shallow seas it may be appreciable). Nevertheless if an appreciable fraction of the energy is related to baroclinic tides, as is apparently true near the continental slopes and elsewhere, the relatively short wavelength of the baroclinic tides may result in a quite strong Stokes velocity.

In order to estimate the Stokes velocity without assumption as to the type of motion present it appears desirable to establish tripartite stations of recording current-meters. From these not only the velocity field but its spacial gradient can be estimated. Equivalently one can often determine from a tripartite station the direction and speed of propagation of any predominant wave motion.

REFERENCES

LAMB H. (1932) *Hydrodynamics*. Cambridge University Press, 6th ed., 738 pp.

LONGUET-HIGGINS M. S. (1968a) On the trapping of long waves along a discontinuity of depth in a rotating ocean. *J. fluid. Mech.*, **31**, 417–434.

LONGUET-HIGGINS M. S. (1968b) Double Kelvin waves with continuous depth profiles. *J. fluid Mech.* **34** (1), 49–80.

STARR V. P. (1959) Hydrodynamical analogy to $E = mc^2$. *Tellus*, **11**, 135–138.

STOKES G. G. (1847) On the theory of oscillatory waves. *Trans. Cambridge Phil. Soc.*, **8**, 441–455.

Erratum

On page 440, in the first sentence between equations (41) and (42):

ζ should read ζ'.