A realistic theoretical model of steady Langmuir circulations is constructed. Vorticity in the wind direction is generated by the Stokes drift of the gravity-wave field acting upon spanwise vorticity deriving from the wind-driven current. We believe that the steady Langmuir circulations represent a balance between this generating mechanism and turbulent dissipation.

Nonlinear equations governing the motion are derived under fairly general conditions. Analytical and numerical solutions are sought for the case of a directional wave spectrum consisting of a single pair of gravity waves propagating at equal and opposite angles to the wind direction. Also, a statistical analysis, based on linearized equations, is developed for more general directional wave spectra. This yields an estimate of the average spacing of windrows associated with Langmuir circulations. The latter analysis is applied to a particular example with simple properties, and produces an expected windrow spacing of rather more than twice the length of the dominant gravity waves.

The relevance of our model is assessed with reference to known observational features, and the evidence supporting its applicability is promising.
“There is general agreement among those who have worked on the subject that, as Langmuir first concluded in his pioneering paper of 1938, the circulations are driven by the wind.”

but later in the introduction:

“..... the circulations are a motion forced by the interaction of vorticity in the wind-induced current and the wind-generated gravity waves .... the wind enters only as the source of this parallel current and of the short crested gravity waves.”

My opinion is that wind features only indirectly in the CL theory. CL recognize this, and apologize to the reader in the introduction.
Nevertheless, there are some basic characteristics that seem always to be reported. We shall attempt here to summarize a minimum of features that a theory must be able to explain.

**LC (1)** A parallel system of vortices aligned with the wind must be predicted.

**LC (2)** A means must be given by which these vortices are driven by the wind.

**LC (3)** The resulting cells must have the possibility of an asymmetric structure with downwelling speeds larger than upwelling speeds.

**LC (4)** Downwelling zones must be under lines where the wind-directed surface current is greatest.

**LC (5)** The Langmuir circulations must have maximum downwelling speeds comparable to the mean wind-directed surface drift.
A rational model for Langmuir circulations

\[ u_t + u \cdot \nabla u + \nabla p = \nu_e \Delta u , \]
\[ \nabla \cdot u = 0 . \]

\[ @z = \bar{z} : \quad \bar{z}_t + u\bar{z}_x + \nu \bar{z}_y = w , \quad p = g\bar{z} . \]

(1) We should use stress BCs.
(2) Gravity appears only in the BC.

\[ \omega = \nabla \times u , \quad \text{and} \quad u \cdot \nabla u = \underbrace{\omega \times u} + \nabla \frac{1}{2} |u|^2 , \]

"vortex force"

\[ \therefore \quad u_t + \omega \times u + \nabla \left( p + \frac{1}{2} |u|^2 \right) = \nu_e \Delta u , \]
\[ \nabla \times \Rightarrow \quad \omega_t + \nabla \times (\omega \times u) = \nu_e \Delta \omega . \]

CL76 launches with the vorticity equation.
Stokes (1847) et sequentia uses a simple solution of the vorticity equation:

\[ \omega(x, t) = 0 \]

CL76 is perhaps the first theoretical paper to seriously consider the interaction of irrotational surface gravity waves with non-zero vorticity.

\[ \omega_t + \nabla \times (\omega \times u) = \nu_e \triangle \omega \]

\[ \omega = \nabla \times u \]

Note the short time scale of irrotational surface gravity waves

\[ \sigma = \sqrt{gk}, \quad k = \frac{2\pi}{100 \text{m}} \quad \Rightarrow \quad \sigma = \frac{2\pi}{8 \text{s}} \]

Wave-averaged equations are the goal.
3. Governing equations for weak currents in the presence of surface waves

Note the short time scale of surface gravity waves

\[ \sigma = \sqrt{gk}, \quad k = \frac{2\pi}{100m} \Rightarrow \sigma = \frac{2\pi}{8s} \]

Use space and time scales like these to scale the vorticity equation.

“The small parameter \( \epsilon \) may be regarded as characteristic of the wave slope; that is to say, it is \( O(\alpha m) \) when \( a \) and \( m \) are the amplitude and wavenumber of a typical wave component.”

\[
\left( \epsilon^2 \partial_{\tilde{t}} + \partial_{\tilde{t}} \right) \omega + \nabla \times (\omega \times u) = \epsilon^2 f_2 u_z + \epsilon^2 \nu_2 \Delta \omega
\]

I've included rotation (Huang 1979) and a slow time scale.

\[ \tilde{t} = \sigma t, \quad \bar{t} = \epsilon^2 \sigma t \]

Expand in wave slope:

\[ u = \epsilon u_1 + \epsilon^2 u_2 + \cdots \]
First order in wave slope

$$\partial_{\tilde{t}} \omega_1 = 0 \implies \omega_1 = 0 \text{ and } u_1 = \xi_{\tilde{t}}$$

The wavy particle displacement satisfies

$$\ddot{\xi} = 0, \quad \nabla \cdot \xi = 0, \quad \nabla \times \xi = 0$$

This leading-order solution is any linear, irrotational wavy solution e.g., an infinite plane surface wave, or a sum of waves, or a standing wave etc.

Second order in wave slope

$$\partial_{\tilde{t}} \omega_2 + \nabla \times (\omega_1 \times u_1) = 0$$

But this time CL take a different solution

$$\omega_2 = \Omega(x, \tilde{t})$$

The goal is a slow-time evolution equation for this mean vorticity.

Uncurl:

$$u_2 = U(x, \tilde{t}) + \nabla \tilde{\phi}_2$$

Recall:

$$\left(\epsilon^2 \partial_{\tilde{t}} + \partial_{\tilde{t}}\right) \omega + \nabla \times (\omega \times u) = \epsilon^2 f_2 u_z + \epsilon^2 \nu_2 \Delta \omega$$
Third order in wave slope

\[
\begin{align*}
\partial_t \boldsymbol{\omega}_1 + \partial_t \boldsymbol{\omega}_3 + \nabla \times (\boldsymbol{\omega}_1 \times \boldsymbol{u}_2) + \nabla \times (\boldsymbol{\omega}_2 \times \boldsymbol{u}_1) &= f_2 \boldsymbol{u}_{1z} + \nu_2 \triangle \boldsymbol{\omega}_1 \\
=0 & \quad \Omega = \xi_t = \xi_{iz} = 0
\end{align*}
\]

This integrates to:

\[
\boldsymbol{\omega}_3 = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{\Omega}) + f_2 \boldsymbol{\xi}_z + \tilde{\omega}_3
\]

(Note the convenience of working with wavy displacements.)

Fourth order in wave slope

\[
\begin{align*}
\partial_t \boldsymbol{\Omega} + \partial_t \boldsymbol{\omega}_4 + \nabla \times \boldsymbol{\omega}_1 \times \boldsymbol{u}_3 + \nabla \times \boldsymbol{\omega}_2 \times \boldsymbol{u}_2 + \nabla \times \boldsymbol{\omega}_3 \times \boldsymbol{u}_1 &= f_2 \boldsymbol{u}_{2z} + \nu_2 \triangle \boldsymbol{\omega}_2 \\
=0 & \quad \Omega = \xi_t = \nu_2 = \xi_{iz} = 0
\end{align*}
\]

The time average is:

\[
\partial_t \boldsymbol{\Omega} + \nabla \times (\boldsymbol{\Omega} \times \boldsymbol{U}) + \nabla \times (\boldsymbol{\omega}_3 \times \boldsymbol{u}_1) = f_2 \boldsymbol{U}_z + \nu_2 \triangle \boldsymbol{\Omega}
\]

Recall:

\[
\left(\epsilon^2 \partial_t + \partial_t\right) \boldsymbol{\omega} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) = \epsilon^2 f_2 \boldsymbol{u}_z + \epsilon^2 \nu_2 \triangle \boldsymbol{\omega}
\]
Why is the CL model “difficult”?

After this straightforward perturbation expansion, the main technical problem is showing that:

\[
\nabla \times (\omega_3 \times u_1) = \nabla \times (\Omega \times u^S) - f_2 u^S_z,
\]

where the Stokes drift is:

\[ u^S = \xi \cdot \nabla \tilde{\xi} \]

The main point is that the CL wave-averaged vortex force is expressed completely in terms of the Stokes drift.

Recall:

\[ \omega_3 = \nabla \times (\xi \times \Omega) + f_2 \xi_z + \tilde{\omega}_3 \]

and

\[ u_1 = \xi \tilde{t} \]
Using the fact that, by continuity,
\[ \nabla \cdot u_w = \nabla^2 \phi = 0 \]
vector identities yield
\[ \langle \omega_1 \rangle = (\omega_0 \cdot \nabla) \int u_w \, dt - \left( \int u_w \, dt \cdot \nabla \right) \omega_0, \]
\[ \text{curl} \left( u_w \times \langle \omega_1 \rangle \right) = \langle \omega_1 \rangle \cdot \nabla u_w - \left( u_w \cdot \nabla \right) \langle \omega_1 \rangle. \]
On changing to Cartesian tensor notation, with a comma denoting partial differentiation, the last two results may be combined to yield
\[ [\text{curl} \left( u_w \times \langle \omega_1 \rangle \right)]_t = \bar{\omega}_k^0 \omega^w_{i,j} \int^t u^w_{j,k} \, dt - \bar{\omega}_k^0 \omega^w_{i,k} \int^t u^w_i \, dt \]
\[ + u^w_k \left[ \omega^w_{i,j} \int^t u^w_{j,k} \, dt \right]_{,k} - u^w_k \left[ \omega^w_{i,k} \int^t u^w_i \, dt \right]_{,k}, \tag{11} \]
where \( \omega^w_k \) and \( u^w_i \) denote the \( i \)th components of \( \omega_0 \) and \( u_w \), respectively.

Now, the mean Lagrangian or Stokes drift \( U_s \) due to the irrotational motion (Phillips 1966, p. 31) has \( i \)th component
\[ U^i_s = \epsilon^w u^w_k \int^t u^w_i \, dt \equiv \epsilon^w u^k_i, \tag{12} \]
and (11) may be rearranged as
\[ [\text{curl} \left( u_w \times \langle \omega_1 \rangle \right)]_t = \bar{\omega}_k^0 \omega^w_{i,j} - \bar{\omega}_k^0 \frac{\partial}{\partial t} \left( \int^t u^w_i \, dt \int^t \omega^w_{i,k} \, dt \right) \]
\[ - \bar{\omega}_k^0 \frac{\partial}{\partial t} \left( \int^t u^w_i \, dt \int^t \omega^w_{i,k} \, dt \right) + u^w_k \left[ \omega^w_{i,j} \int^t u^w_{j,k} \, dt \right]_{,k}. \tag{13a} \]
The final term on the right-hand side may be shown to equal
\[ - \bar{\omega}_k^0 u^w_k + \frac{1}{2} \left[ \omega^w_i \frac{\partial}{\partial t} \left( \int^t u^w_i \, dt \int^t u^w_j \, dt \right) \right]_{,jk} - \frac{1}{2} \left[ \omega^w_i \frac{\partial}{\partial t} \left( \int^t u^w_j \, dt \int^t u^w_k \, dt \right) \right]_{,jk}, \]
and since the mean of all time derivatives is zero, it follows that
\[ \text{curl} \left( u_w \times \langle \omega_1 \rangle \right) = (\omega_0 \cdot \nabla) u_s - (u_s \cdot \nabla) \omega_0, \tag{13b} \]
where \( \epsilon^w u_s = U_s. \)
Equation (10) for \( \omega_0 \) therefore becomes
\[ - \alpha \nabla^2 \omega_0 = (\bar{\omega}_0 \cdot \nabla) (\bar{v}_0 + u_s) - (\bar{v}_0 + u_s) \cdot \nabla \omega_0. \tag{14} \]
The wave-averaged CL equations

\[ \partial_t \Omega + \nabla \times \left[ \Omega \times (U + u^S) \right] = f_2 (U + u^S)_z + \nu_2 \Delta \Omega \]

\[ \Omega = \nabla \times U \]

The derivation of this equation is the main reason that CL76 is an important paper. CL76 is 25 pages long, and this result is on the sixth page.

Perhaps CL should have stopped here?
Discussion Question

What has happened to the pre-CL76 theory of mean-flow generation by surface gravity waves?

For example, the Longuet Higgins & Stewart radiation stress. Note that this wave-averaged effect is irrotational.

CL work in terms of vorticity, and obtain a wave-averaged vortex force. But we know there also non-zero irrotational mean flows associated with gravity waves. Is there a unified framework?
The CL one problem

No dependence on the downwind coordinate, $X$

And one must specify the Stokes drift.

The most natural choice is certainly:

$$u^S = (ak)^2 \frac{\sigma}{k} e^{2kz}$$

Instead, CL76 uses the “crossed-wave” model:

$$u^S = \frac{1}{2} (ak)^2 \frac{\sigma}{k} \left[ 1 + \cos^2 \theta \cos(2k \sin \theta \gamma) \right]$$
The CL one problem (continued)

The equations of motion are:

\[
\begin{align*}
\left(\psi_z \nabla \psi_y - \psi_y \nabla \psi_z\right) - \left(u_z^s u_y - u_y^s u_z\right) &= \nu_2 \Delta^2 \psi, \\
\psi_z u_y - \psi_y u_z &= \nu_2 \Delta u.
\end{align*}
\]

The surface BCs are:

\[
@z = 0 : \quad \nu_2 u_z = \tau, \quad \psi = \psi_{zz} = 0
\]

Specify the Stokes drift, and require no motion at great depth. But.....
Why the “crossed-wave” Stokes drift?

\[
(\psi_y \Delta \psi_z) - (u_z^s u_y - u_y^s u_z) = \nu_2 \Delta^2 \psi ,
\]
\[
\psi_z u_y - \psi_y u_z = \nu_2 \Delta u .
\]

From the abstract:

Vorticity in the wind driven current can be modeled by the Stokes drift of the gravity-wave field acting upon spanwise vorticity, \( u_z \), associated with a directly wind driven current. But if \( \partial_y u^s = 0 \), that wind drift does not act on the spanwise vorticity.

The strongest component of vorticity is probably the spanwise vorticity, \( u_z \), associated with a directly wind driven current. But if \( \partial_y u^s = 0 \), that wind drift does not act on the spanwise vorticity.

\[
\text{downwind vorticity} = \boldsymbol{\omega} \cdot \hat{x} = w_y - v_z
\]
\[
\text{spanwise vorticity} = \boldsymbol{\omega} \cdot \hat{y} = u_z
\]
\[
\text{vertical vorticity} = \boldsymbol{\omega} \cdot \hat{z} = -u_y
\]
But CL one has no steady solution

which employs a constant eddy viscosity. (It is important to note at this point that, in a fluid of infinite depth, the existence of an applied stress at the surface leads to a steadily increasing (in time) total mean x momentum, and is therefore not compatible with a steady solution. Our procedure thus constructs a quasisteady state for a limited period of time.) ‘Reasonable’ forms for \( U_h \)

Nonetheless, CL proceed with a steady solution...........

It is natural to attempt solving the \( \psi \) by Fourier series in \( y \):

\[
\psi = \sum_{n=1}^{\infty} \phi_n(y)
\]

\[
u = U_h(z) + \sum_{n=1}^{\infty} u_n(z),
\]

and it is also convenient to expand the x vorticity \( \xi \) in Fourier series as

\[
\xi = -\sum_{n=1}^{\infty} \xi_n(z) \sin 2\ln y.
\]

The types of expansion are dictated by the fact that \( u \) is even in \( y \) while \( \psi \) is odd. The Fourier component in (33) that is independent of \( y \) has been denoted by \( U_h(z) \) and is the horizontally averaged current.
Nevertheless, there are some basic characteristics that seem always to be reported. We shall attempt here to summarize a minimum of features that a theory must be able to explain.

**LC (1)** A parallel system of vortices aligned with the wind must be predicted.

**LC (2)** A means must be given by which these vortices are driven by the wind.

**LC (3)** The resulting cells must have the possibility of an asymmetric structure with downwelling speeds larger than upwelling speeds.

**LC (4)** Downwelling zones must be under lines where the wind-directed surface current is greatest.

**LC (5)** The Langmuir circulations must have maximum downwelling speeds comparable to the mean wind-directed surface drift.

**LC (1)** and **LC (2)** are built into the model by prescribing the crossed-wave Stokes drift, which specifies the roll spacing. The vortices aren't “driven by the wind” - they're “driven” by the crossed-wave Stokes drift.

The numerical solution does not clearly exhibit **LC(3)**. But perhaps the model could produce cells with the correct asymmetry in a more nonlinear parameter range.

The model can produce either **LC (4)**, or the reverse, depending on the wind speed. The prediction is that weak winds reverse **LC (4)**.

**LC (5)** was not exhibited by the solutions that converged.
My discussion questions

1. What has happened to the pre-CL76 theory of mean-flow generation by surface gravity-wave stress?

2. Surface waves balance some (or most of) the wind stress e.g., by a combination of viscous tangential stress and form stress. CL76 applies all the wind stress to the downwind current, leaving none for the waves. Should some of the wind stress be balanced by the Stokes drift?

3. Viscosity generates wave vorticity via the BC. CL ignore this process. Viscosity also damps the waves. So the wind must supply energy to keep the waves steady. How do we extend CL76 to model the effect of wind stress and viscosity on the waves?
My not entirely successful attempt to avoid the longish Craik-Leibovich calculation follows on the next three slides.

Using index notation, Basile Gallet found some errors in the first attempt: I assumed that identity (72) is self-evident. It’s not.
Avoiding index notation.....

A Vector obnoxiousness

Some operator identities involving directional derivatives

Suppose $p$ and $q$ are incompressible vector fields i.e., $\nabla \cdot p = 0$ and $\nabla \cdot q = 0$. Then using a standard vector identity

\[ p \cdot \nabla q \cdot \nabla - q \cdot \nabla p \cdot \nabla = \nabla \times (q \times p) \cdot \nabla \quad (71) \]

On the left of (71) all four $\nabla$’s are hungry and operate on everything to their right. I’ll use the special notation $|$ to limit the action of differential operators: on the right of (71) the operator $\nabla \times$ is stopped by $|$ so that $\nabla \times$ acts only on $p \times q$.

Using this notation $|$, there is a second identity

\[ p \cdot \nabla q | \cdot \nabla - q \cdot \nabla p | \cdot \nabla = p \cdot \nabla q \cdot \nabla - q \cdot \nabla p \cdot \nabla . \quad (72) \]

On the left of (72) the action of the first pair of $\nabla$’s is limited by $|$ i.e., on the left of (72) only the second pair of $\nabla$’s is hungry.

\[ \nabla \times (p \times q) = pm \cdot q \nabla - q \cdot p \nabla + q \cdot \nabla p - p \cdot \nabla q \]
In (13) we have the vortex force

\[ \mathbf{V} \overset{\text{def}}{=} \boldsymbol{\omega}_3 \times \xi_t, \]  

\[ = [\nabla \times (\xi \times \Omega)] \times \xi_t + f_2 \xi_z \times \xi_t. \quad (73) \]  

\[ \nabla \times \vec{V} = \nabla \times (\Omega \times u^S) - f_2 u_z^S, \]  

where the Stokes velocity is

\[ u^S = \bar{\xi} \cdot \nabla \xi_t = \frac{1}{2} [\xi \cdot \nabla \xi_t - \xi_t \cdot \nabla \xi] = -\frac{1}{2} \nabla \times (\xi \times \xi_t). \quad (76) \]

To get to the average equation in (14) we must show that

\[ \nabla \times \bar{V} = \nabla \times (\Omega \times u^S) \]

The Craik-Leibovich term

Expanding the \( \nabla \times \mathbf{V}_1 \) in (74) using a standard vector identity we have

\[ \nabla \times \mathbf{V}_1 = \xi_t \cdot \nabla [\nabla \times (\xi \times \Omega)] - [\nabla \times (\xi \times \Omega)] \cdot \nabla \xi_t, \]  

and again

\[ \nabla \times \mathbf{V}_1 = \xi_t \cdot \nabla \Omega \cdot \nabla \xi - \xi_t \cdot \nabla \xi \cdot \nabla \Omega - \nabla \Omega \cdot \nabla \xi_t + \xi \cdot \nabla \Omega \cdot \nabla \xi_t. \]  

Now use the identity (72) to open the directional derivatives in the final two terms:

\[ \nabla \times \mathbf{V}_1 = \xi_t \cdot \nabla \Omega \cdot \nabla \xi - \xi_t \cdot \nabla \xi \cdot \nabla \Omega - \Omega \cdot \nabla \xi \cdot \nabla \xi_t + \xi \cdot \nabla \Omega \cdot \nabla \xi_t. \]  

Averaging (79), we see that the first and last terms on the right cancel and the middle two terms result in

\[ \nabla \times \bar{V}_1 = u^S \cdot \nabla \Omega - \Omega \cdot \nabla u^S = \nabla \times (\Omega \times u^S). \]  

(80)
The Huang term $V_2$

Start with

$$V_2 = f_2 \partial_z (\xi \times \xi_t) - f_2 \xi \times \xi_{zt},$$  
(81)

$$= f_2 \partial_z (\xi \times \xi_t) - f_2 \partial_t (\xi \times \xi_z) + f_2 \xi_t \times \xi_z$$  
(82)

and hence

$$V_2 = \frac{1}{2} f_2 \partial_z (\xi \times \xi_t) - \frac{1}{2} f_2 \partial_t (\xi \times \xi_z).$$  
(83)

The curl is

$$\nabla \times V_2 = \frac{1}{2} f_2 \partial_z [\xi_t \cdot \nabla \xi - \xi \cdot \nabla \xi_t] - \frac{1}{2} f_2 \partial_t \nabla \times (\xi \times \xi_z),$$  
(84)

$$= -f_2 \partial_z (\xi \cdot \nabla \xi_t) + \frac{1}{2} f_2 \partial_t \left[ \partial_z (\xi \cdot \nabla \xi) - \nabla \times (\xi \times \xi_z) \right],$$  
(85)

$$= -f_2 \partial_z (\xi \cdot \nabla \xi_t) + f_2 \partial_t \xi \cdot \nabla \xi_z.$$  
(86)

The time average of the above is $\nabla \times \bar{V}_2 = -f_2 u_z^S$. 

BC conditions on the surface

We use the notation

$$\phi_s \text{def} = \phi(x, y, z, t)$$  
(87)

i.e., $\phi_s$ is $\phi$ evaluated on the surface. Thus boundary conditions are

$$\left( \partial_t + u_s \partial_x + v_s \partial_y \right) D_s z = w_s,$$  
(88)

and

$$p_s = g_z.$$  
(88)

The operator $D_s$ is the material derivative following the surface flow. Note that derivatives such as $\partial_x$ and $\partial_t$ do not commute with evaluation at the surface:

$$\partial_x \phi_s = \left( \phi_x \right)_s + z_x (\phi_z)_s$$  
(89)

However one can verify that

$$\left( D \phi \right)_s D_t = D_s \phi_s, $$  
(90)

and

$$\nabla \phi_s = \partial_x \phi_s \hat{e}_1 + \partial_y \phi_s \hat{e}_2 + (\phi_z)_s \hat{e}_3 - \nabla z.$$  
(91)

The velocity vector evaluated on the surface is

$$u_s = u_s \hat{e}_1 + v_s \hat{e}_2 + z_t \hat{e}_3$$  
(92)

Basis vectors

We use the non-orthogonal basis vectors

$$\hat{e}_1 = \nabla x,$$  
(93)

$$\hat{e}_2 = \nabla y,$$  
(94)

$$\hat{e}_3 = \nabla (z - z) = \hat{z} - z_x \hat{x} - z_y \hat{y},$$  
(95)