A Talk on Stokes Drift

1. Longuet-Higgins, Michael S. 1969. "On the Transport of Mass by Time-Varying Ocean Currents" 16 (5). Elsevier: 431-47.
2. Ursell, F., and GER Deacon. 1950. "On the Theoretical Form of Ocean Swell on a Rotating Earth." Geophysical Journal International 6: 1-8.

Yue Wu

April 24, 2015

Outline

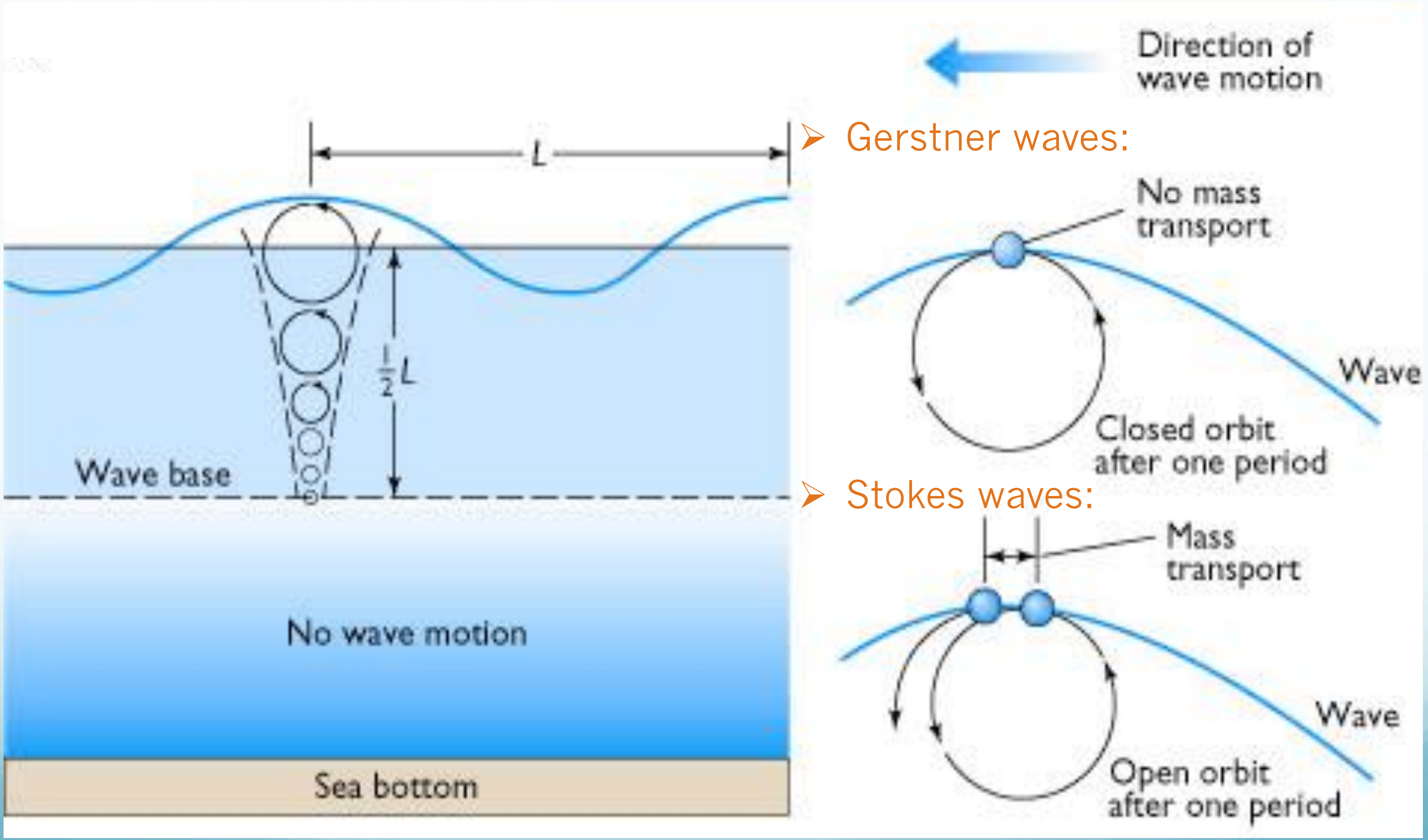
Particle Motions

Stokes drift

Eulerian Mean Velocity

With Rotation

Particle Motions

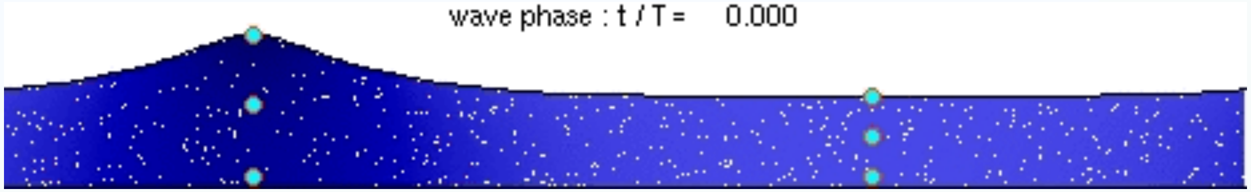


- Gerstner waves:
- Stokes waves:

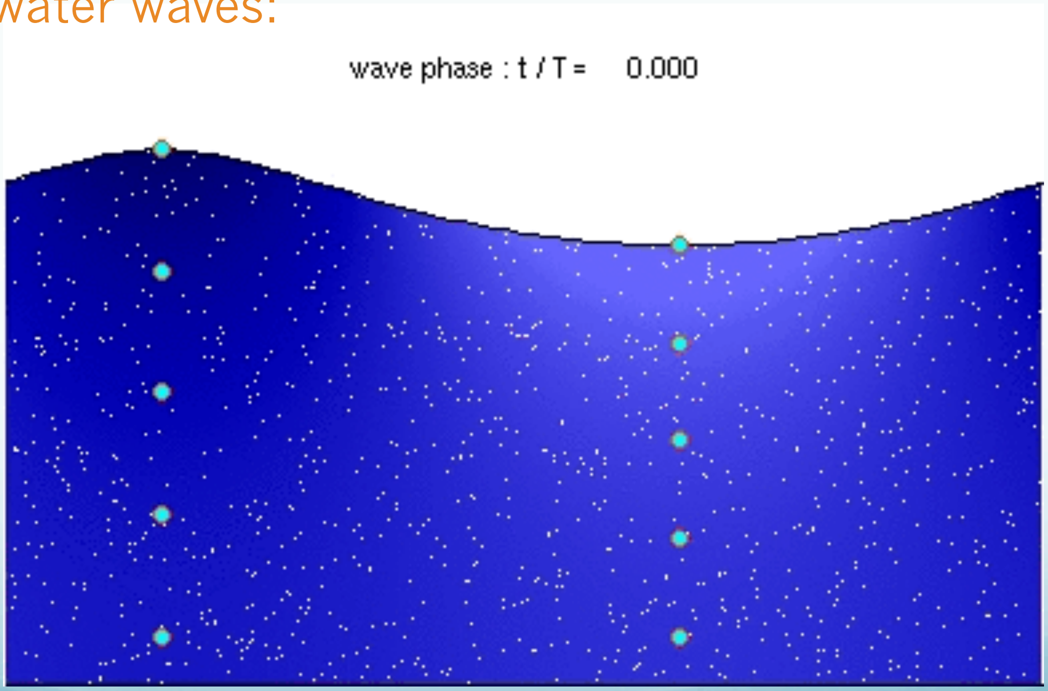
Source: Invitation to oceanography, by Paul R. Pinet.

Particle Motions

- For shallow-water waves:



- For deep-water waves:



Source: http://en.wikipedia.org/wiki/Stokes_drift.

Particle Motions

- Particle displacement in the wave field:

(periodic, infinite plane wave)

$$\dot{x} = \varepsilon c \cos(kx - \omega t)\mathrm{e}^{kz} + \varepsilon^2 u_2(x,z,t) + \cdots,$$
$$\dot{z} = \varepsilon c \sin(kx - \omega t)\mathrm{e}^{kz} + \varepsilon^2 w_2(x,z,t) + \cdots.$$

$$c \equiv \frac{\omega}{k}, \quad \varepsilon \equiv ak.$$

- The Lagrangian mean velocity is:

$$u_L \equiv \overline{\dot{x}},$$
$$= \varepsilon^2 \overline{u_2} + \varepsilon^2 c\,\mathrm{e}^{2kz}.$$

Lagrange = Euler + Stokes

Stokes Drift

(1) Single wave (Stokes, 1847)

$$\mathbf{U}_{\mathrm{S}} = a^2 \omega k \exp(2kz)$$

(2) Wave field (Jenkins, 1989)

$$\mathbf{U}_{\mathrm{S}}(\mathbf{x},t) = 4p \iint fke^{2kc} E(f,q)\, df\, dq$$

(3) Parameterization by wind speed (Wu, 1983)

$$\mathbf{U}_{\mathrm{S}} = 0.0186(gLW_{10}{}^{-2})^{0.03} \mathbf{W}_{10}$$

Laboratory Observations

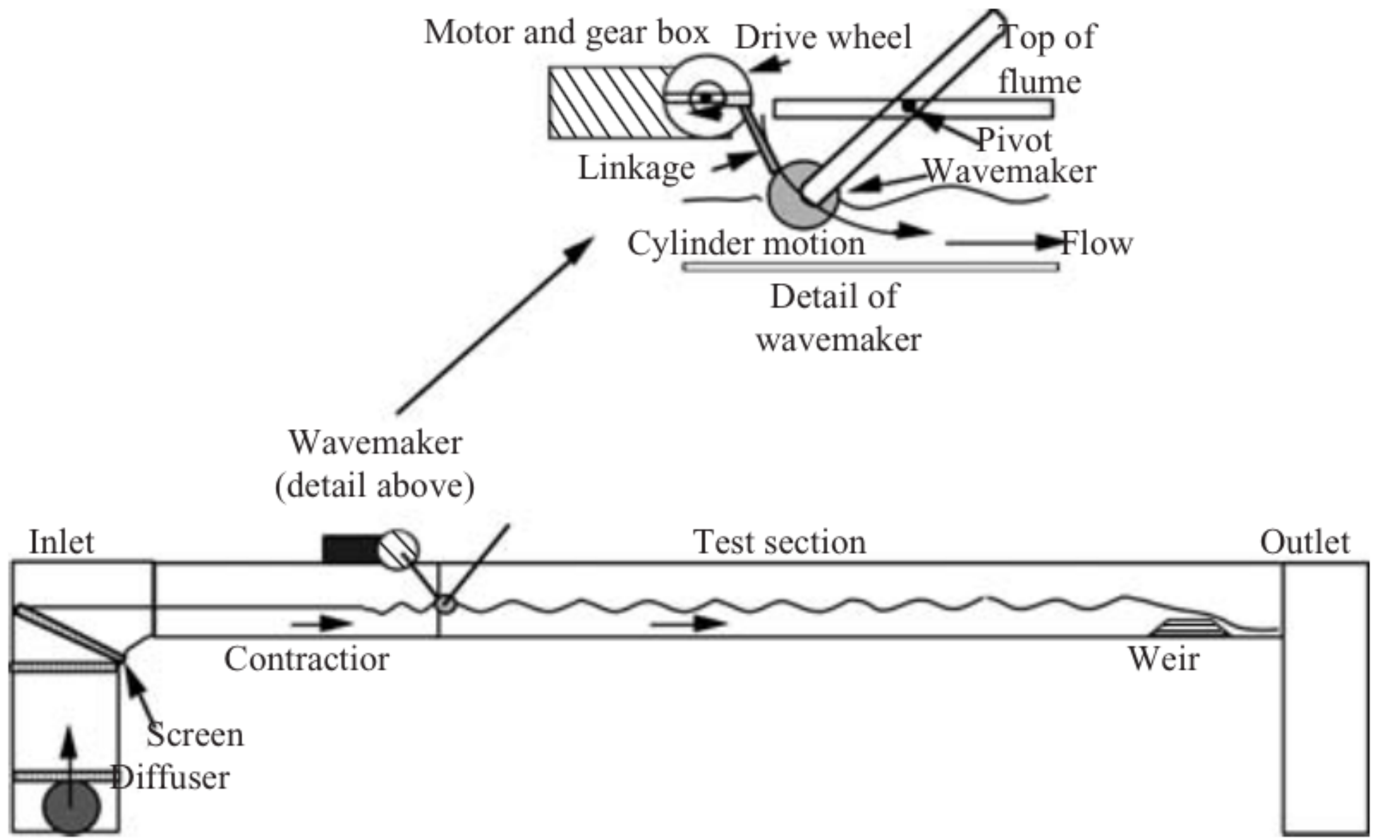


FIGURE 1. Sketch of the wave–current flume used by Nepf *et al.* (1995).

Source: Monismith et al. (2007).

Longuet-Higgins, 1969

(periodic, infinite plane wave)

- For a particle oscillating in the neighborhood of its original position,

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}(\mathbf{x}_0,t) + \Delta\mathbf{x}\cdot\nabla\mathbf{u}(\mathbf{x}_0,t) \ ,$$

$$\Delta\mathbf{x} = \int_{t_0}^{t} \mathbf{u}(\mathbf{x}_0,t)dt \ ,$$

- Taking the mean value over x and t,

$$\overline{\mathbf{u}(\mathbf{x},t)} = \overline{\mathbf{u}_0(\mathbf{x},t)} + \overline{\int \mathbf{u}_0(\mathbf{x},t)\,dt\cdot\nabla\mathbf{u}(\mathbf{x}_0,t)} \ ,$$

$$\mathrm{U} = \bar{\mathbf{u}} + \overline{\int \mathbf{u}\,dt\cdot\nabla\mathbf{u}} \ ,$$

Mass transport velocity

Stokes velocity

Lagrange = Euler + Stokes

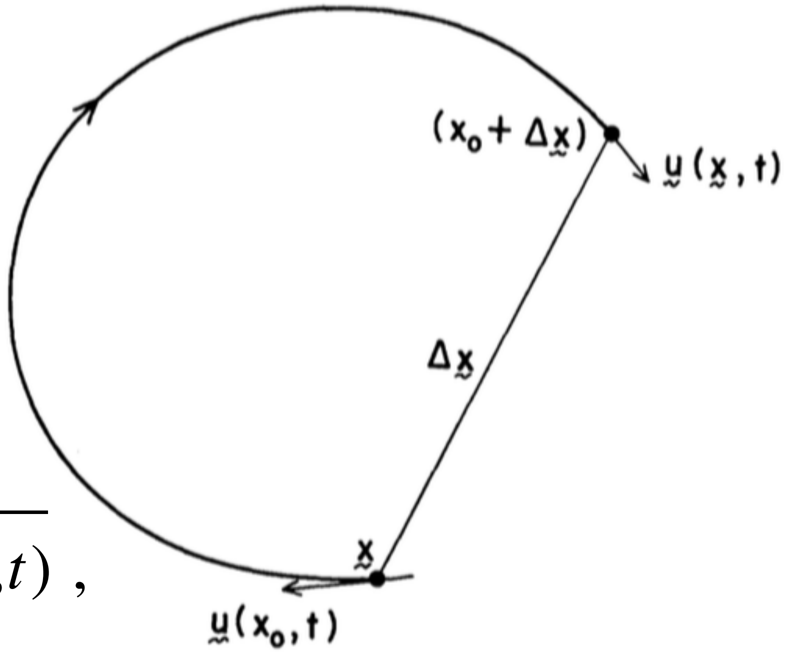


Fig. 1. The trajectory of a marked particle with initial position x_0.

- The Stokes velocity is defined as

$$\mathbf{U}_s = \overline{\int \mathbf{u}\,dt\cdot\nabla\mathbf{u}} \ .$$

Longuet-Higgins, 1969

- *"The mass transport past any fixed point does not depend solely on the mean velocity measured at that point."*
- *"In determining the origin of water masses, it is the Lagrangian mean which is most relevant."*

FLOW OVER A 2D LINEAR BOTTOM TOPOGRAPHY:

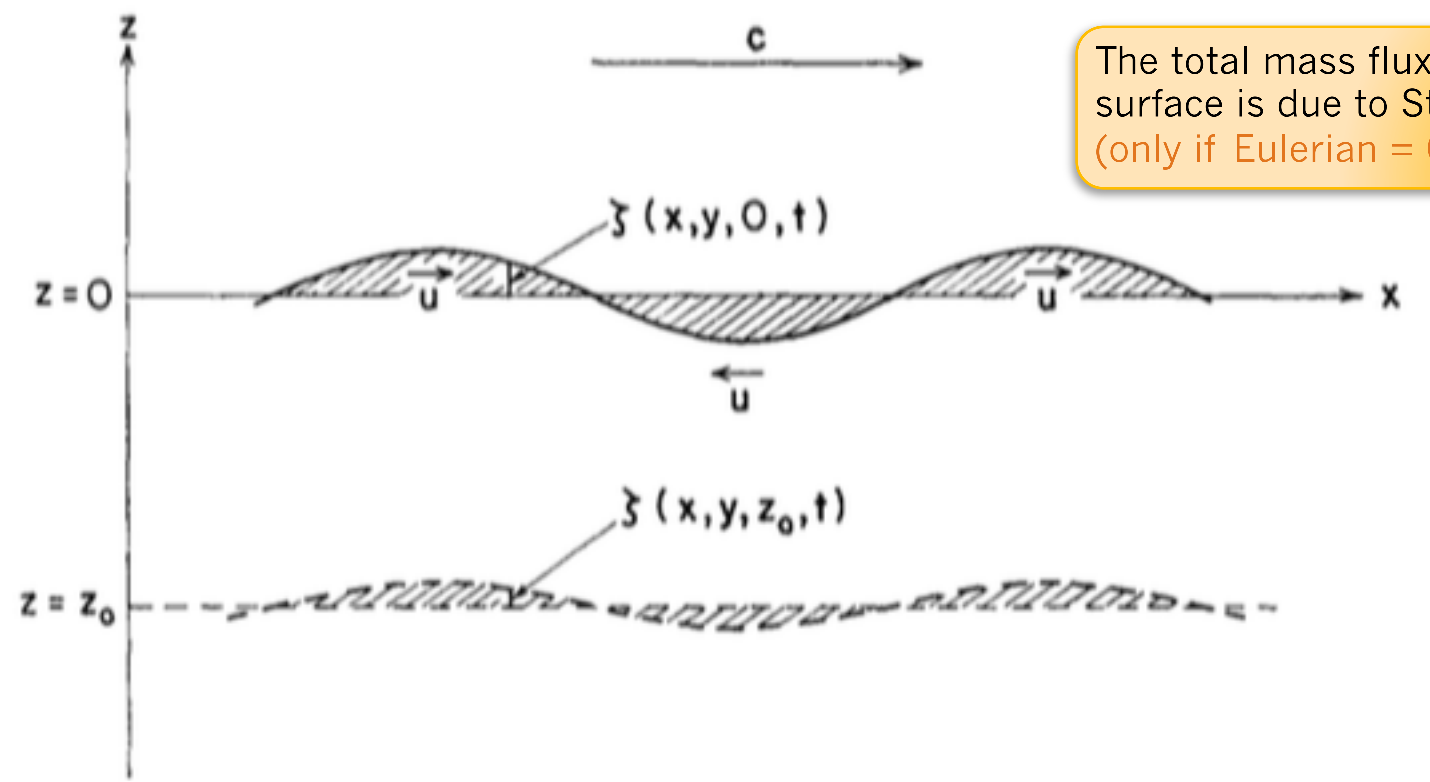
- Particle velocities are calculated from the governing equations,

$$\frac{D\mathbf{u}}{Dt} + \mathbf{f} \times \mathbf{u} = -g\nabla\zeta, \quad \nabla \cdot (h\mathbf{u}) = -\frac{\partial\zeta}{\partial t}.$$

- Total Stokes transport are calculated from definitions.

$$\mathbf{U}_s = \overline{\int \mathbf{u}\, dt \cdot \nabla\mathbf{u}}, \quad M(0) = \overline{u\zeta} = \overline{u\int w\, dt} = \overline{\int_{-h}^{0} \mathbf{U}s dz}.$$

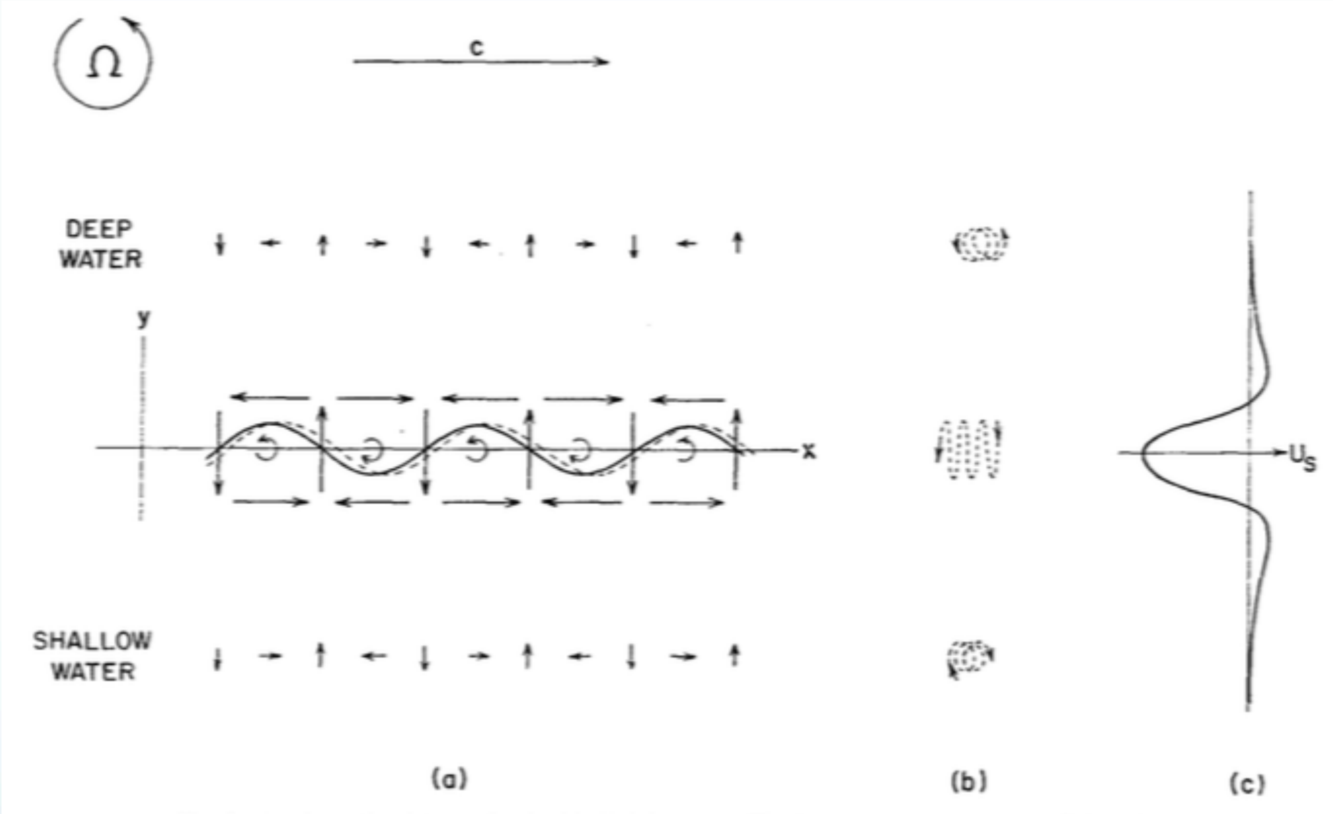
$$M(z_0) = \overline{(u\zeta)_{z_0}} = \overline{\int_{-h}^{z_0} \mathbf{U} s dz}, \quad \mathbf{U}_s = \frac{dM}{dz_0} = \frac{\partial}{\partial z}\overline{(u\zeta)_{z_0}} \; .$$



The total mass flux below the surface is due to Stokes drift: (only if Eulerian = 0)

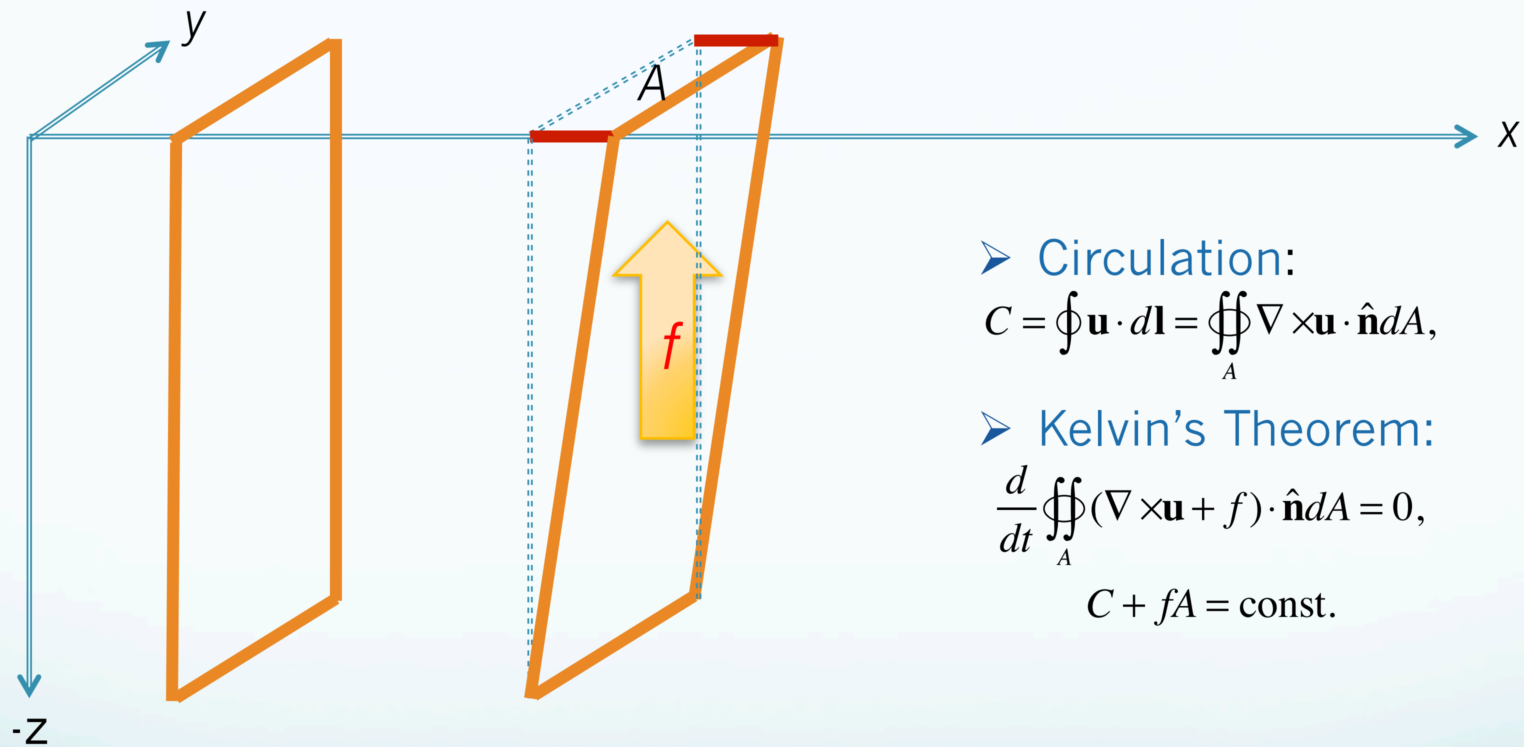
Fig. 2. Derivation of the *total* mass flux below a given level as being due to the positive mass carried by the wave crests minus the mass defect carried by the troughs; and a similar argument applied at a lower level within the fluid.

- He found that: $M = -\dfrac{g\sigma k}{f^2 - \sigma^2}\int_{-\infty}^{\infty} \zeta^2 dy < 0.$ $\qquad (\sigma^2 < f^2)$



- The total Stokes flux could be opposite to the direction of wave propagation.
- *"They (the Lagrangian and the Eulerian mean velocity) may easily be in opposite direction, perhaps leading to false conclusion as to the origins of water masses."*

Ursell and Deacon, 1950



- Circulation:

$$C = \oint \mathbf{u} \cdot d\mathbf{l} = \oiint_A \nabla \times \mathbf{u} \cdot \hat{\mathbf{n}} dA,$$

- Kelvin's Theorem:

$$\frac{d}{dt} \oiint_A (\nabla \times \mathbf{u} + f) \cdot \hat{\mathbf{n}} dA = 0,$$

$$C + fA = \text{const.}$$

- There is no transport along the crests.
- There is NO transport in the direction of propagation.

Eulerian + Stokes = 0

Another proof that

$$f \neq 0 \quad \Rightarrow \quad \bar{u} + \mathbf{U}_s = 0.$$

We assume an infinite plane wave e.g. the Stokes wave.

The Anti-Stokes Flow

- Governing equations (no *y* gradient, incompressible, steady):

$$u_t + uu_x + wu_z - fv = -p_x,$$
$$v_t + uv_x + wv_z + fu = 0,$$
$$w_t + uw_x + ww_z = -p_z,$$
$$u_x + w_z = 0.$$

Taking the mean value over *x*:

$$\overline{u_t} + \overline{(wu)_z} - f\bar{v} = 0,$$
$$\overline{v_t} + \overline{(wv)_z} + f\bar{u} = 0.$$

(1) if there is no rotation,

$$f = 0,$$
$$\overline{wu} = 0, \quad v = 0$$
$$\therefore \bar{u} = 0.$$

Lagrange = Stokes.

(2) if there is rotation,

$$\bar{u} = -\frac{\overline{(wv)_z}}{f}.$$

The Eulerian mean velocity is NOT zero.

Details of the Algebra

- Linearized governing equations (with rotation):

$$\begin{aligned} v_t + fu &= 0, \\ \xi_t &= u, \\ \xi_x + \zeta_z &= 0. \end{aligned} \quad \Rightarrow \quad v = -f\xi$$

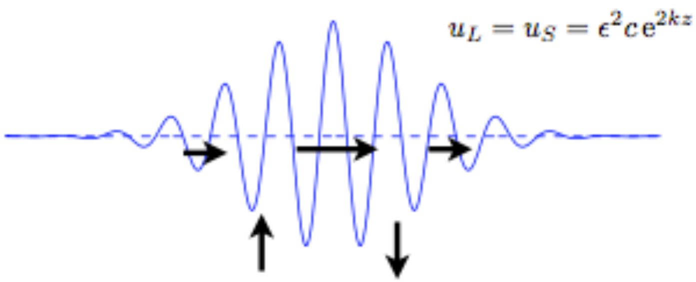
$$\begin{aligned} \bar{u} &= -\frac{1}{f}\overline{(wv)_z} = \overline{(w\xi)_z}, \\ &= \overline{(\zeta_t \xi)_z} = -\overline{(\zeta \xi_t)_z} \\ &= -\overline{(\zeta u)_z} \\ &= -\mathbf{U}s. \end{aligned}$$

Eulerian + Stokes = 0

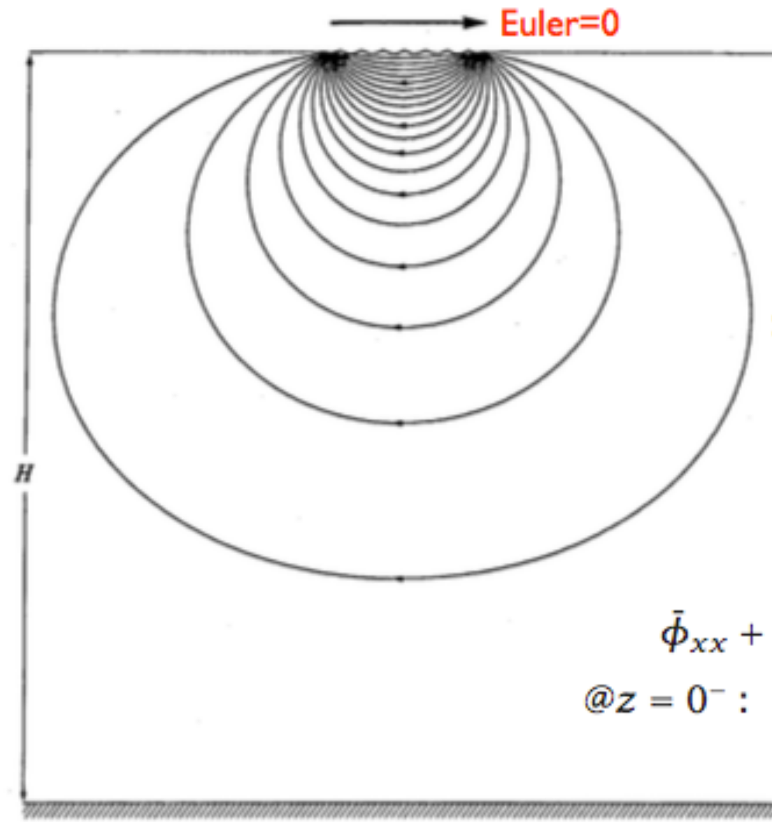
Example 1: A packet of surface waves (but no rotation).

McIntyre, JFM, 1981

On the 'wave momentum' myth 339



A deep, irrotational Eulerian mean flow is driven by Stokes pumping.



$$\bar{\phi}_{xx} + \bar{\phi}_{zz} = 0$$

$$@z = 0^- : \qquad \underbrace{\bar{w}}_{=\bar{\phi}_z} = -S_x$$

FIGURE 2. The irrotational, $O(a^2)$ return flow underneath a packet of surface gravity waves propagating to the right. (The streamlines, plotted at equal intervals, are quantitatively correct for a two-dimensional wave packet whose amplitude is constant except near its ends.)

Lagrange = Euler + Stokes

Source: slides from W. Young's talk.

See also Longuet-Higgins & Stewart (1964)

The wave-averaged CL equations

$$\partial_{\bar{t}}\boldsymbol{\Omega} + \nabla\times\left[\boldsymbol{\Omega}\times\left(\boldsymbol{U}+\boldsymbol{u}^S\right)\right] = f_2\left(\boldsymbol{U}+\boldsymbol{u}^S\right)_z + \nu_2\triangle\boldsymbol{\Omega}$$

$$\boldsymbol{\Omega} = \nabla\times\boldsymbol{U}$$

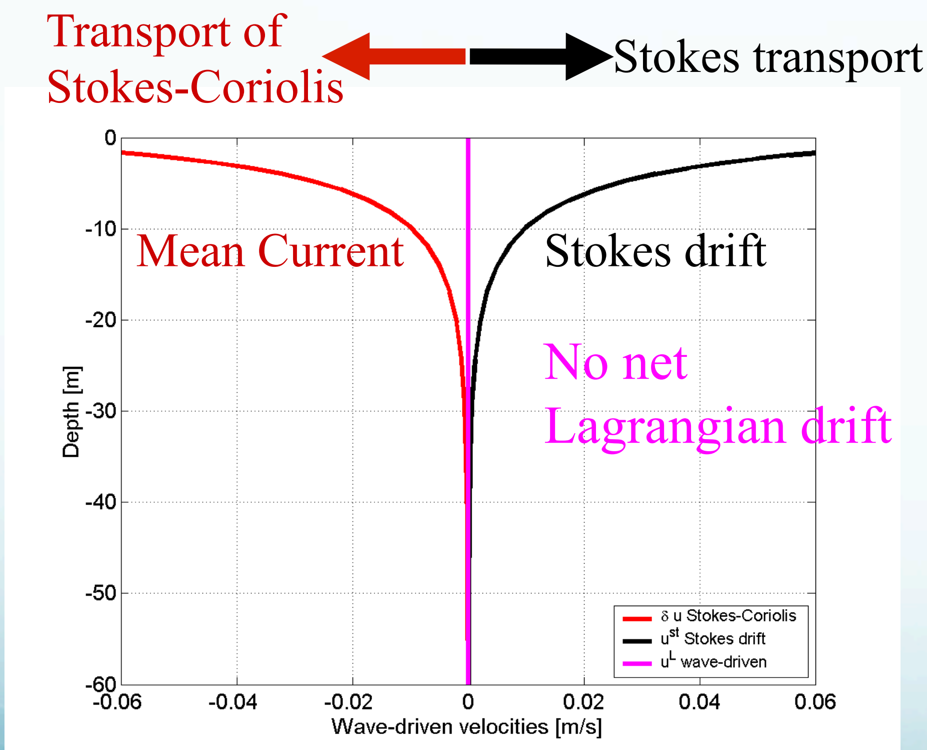
The derivation of this equation is the main reason that CL76 is an important paper. CL76 is 25 pages long, and this result is on the sixth page.

Perhaps CL should have stopped here?

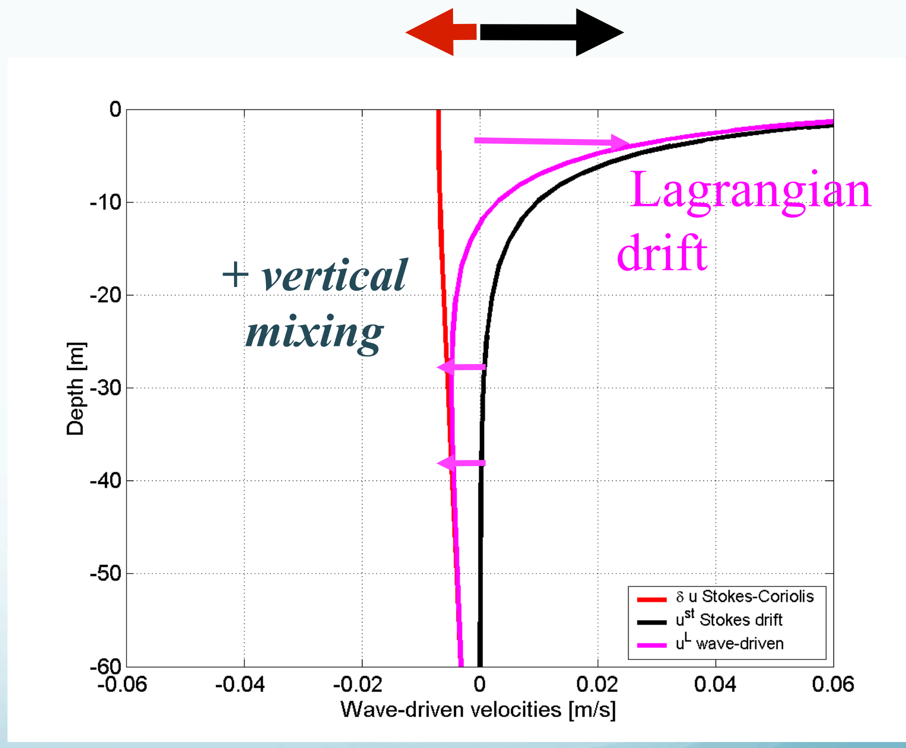
Source: slides from W. Young's talk.

Stokes-Coriolis force (Hasselmann, 1970)

- Vertically integrated, no net transport induced by waves.
- But there still might be a net Lagrangian drift :



Case of a long swell



Case of a wind sea

Which one dominates the surface drift ?

1. Surface drift due to the wind: 2 to 3% of U_{10} (Huang, 1979).
2. Wave-induced Stokes drift at the surface: 1 to 3% of U_{10} (Kenyon, 1969).
3. Ekman currents at the surface strongly depend on the vertical mixing: 0.5 to 4% of $U_{10.}$
4. Affects the upper few tens meters.

THANK YOU FOR YOUR ATTENTION!