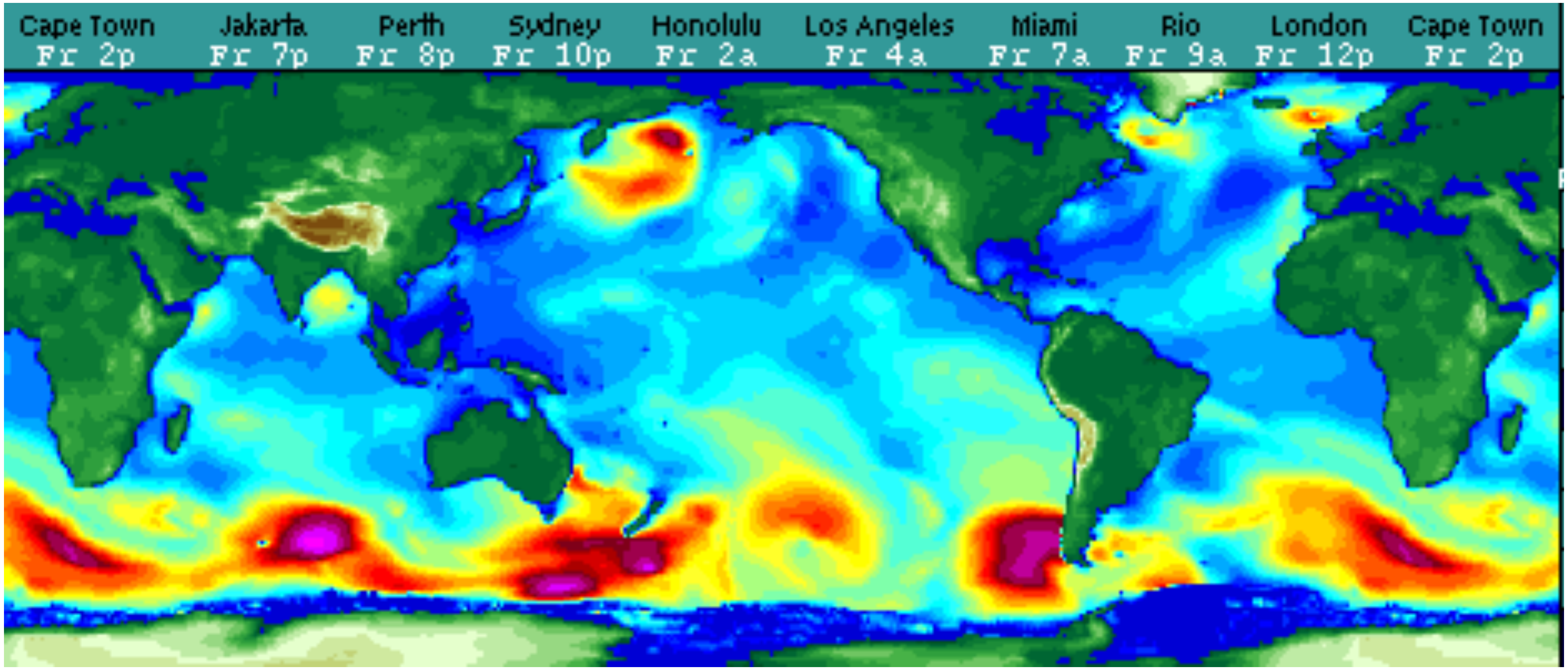
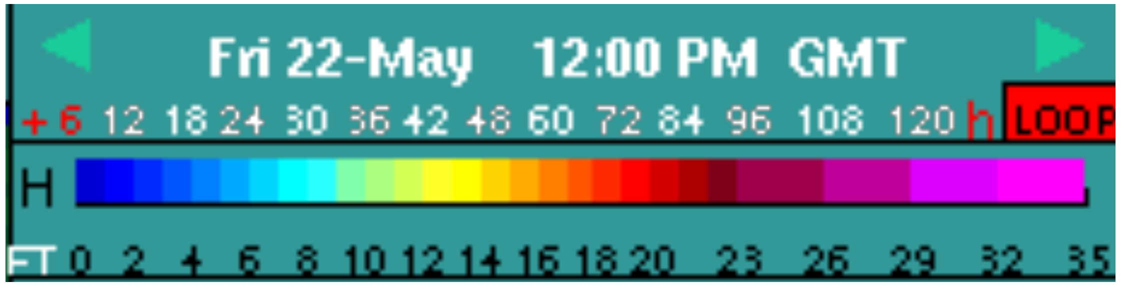
Oceanic wave-balanced surface fronts and filaments

by Jim McWilliams and Baylor Fox-Kemper

(told by Greg)



(predicted significant wave height by Surfline's LOLA)



Craik-Leibovich-Huang theory

predicts wave-averaged modifications to the Boussinesq equations

$$\partial_t \boldsymbol{u} + (f\hat{\boldsymbol{z}} + \boldsymbol{\omega}) \times (\boldsymbol{u} + \boldsymbol{u}_s) = -\nabla(\pi^\dagger + \tfrac{1}{2}\boldsymbol{u}^2) + b^\dagger \hat{\boldsymbol{z}},$$

$$\partial_t b^\dagger + (\boldsymbol{u} + \boldsymbol{u}_s) \cdot \nabla b^\dagger = 0, \qquad \nabla \cdot \boldsymbol{u} = 0$$

“state 0”

an exact steady solution:
ordinary waveless balance

$$f v_0 = p_{0x}$$

$$-f u_0 = p_{0y}$$

geostrophic

$$p_{0z} = b_0$$

hydrostatic

“state 1”

wave-modified balance?

what happens when we add waves?

Wave-averaged Boussinesq equations

$$\partial_t \boldsymbol{u} + (f\hat{\boldsymbol{z}} + \boldsymbol{\omega}) \times (\boldsymbol{u} + \boldsymbol{u}_s) = -\nabla(\pi^\dagger + \tfrac{1}{2}\boldsymbol{u}^2) + b^\dagger \hat{\boldsymbol{z}},$$

$$\partial_t b^\dagger + (\boldsymbol{u} + \boldsymbol{u}_s) \cdot \nabla b^\dagger = 0, \qquad \nabla \cdot \boldsymbol{u} = 0$$

define

$$b \overset{\text{def}}{=} b^\dagger - \bar{b}\,,$$

$$p \overset{\text{def}}{=} \pi^\dagger - \bar{\pi} - \boldsymbol{u}^{\mathrm{S}} \cdot \boldsymbol{u}\,,$$

$$\boldsymbol{u}^{\mathrm{L}} \overset{\text{def}}{=} \boldsymbol{u} + \boldsymbol{u}^{\mathrm{S}}\,.$$

$$\frac{\mathrm{D}^{\mathrm{L}}\boldsymbol{u}^{\mathrm{L}}}{\mathrm{D}t} + f\,\hat{\boldsymbol{z}} \times \boldsymbol{u}^{\mathrm{L}} + \boldsymbol{\nabla} p - b\,\hat{\boldsymbol{z}} = \left(\boldsymbol{\nabla} \times \boldsymbol{u}^{\mathrm{S}}\right) \times \boldsymbol{u}^{\mathrm{L}} + \boldsymbol{u}^{\mathrm{S}}_t\,.$$

$$b_t + \boldsymbol{u}^{\mathrm{L}} \cdot \boldsymbol{\nabla} b + w^{\mathrm{L}} N^2 = 0$$

Traditional balance (state 0)

take $\boldsymbol{u}^{\mathrm{S}} = 0$ *and* $\boldsymbol{u}^{\mathrm{L}} = v_0(x,z)\,\hat{\boldsymbol{y}}$

$$\frac{\mathrm{D}^{\mathrm{L}}\boldsymbol{u}^{\mathrm{L}}}{\mathrm{D}t} + f\,\hat{\boldsymbol{z}} \times \boldsymbol{u}^{\mathrm{L}} + \boldsymbol{\nabla} p - b\,\hat{\boldsymbol{z}} = \left(\boldsymbol{\nabla} \times \boldsymbol{u}^{\mathrm{S}}\right) \times \boldsymbol{u}^{\mathrm{L}} + \boldsymbol{u}^{\mathrm{S}}_t \,.$$

$$b_t + \boldsymbol{u}^{\mathrm{L}} \boldsymbol{\cdot} \boldsymbol{\nabla} b + w^{\mathrm{L}} N^2 = 0$$

Traditional balance (state 0)

take $\boldsymbol{u}^{\mathrm{S}} = 0$ *and* $\boldsymbol{u}^{\mathrm{L}} = v_0(x, z)\,\hat{\boldsymbol{y}}$

a steady solution is $\boxed{f v_0 = p_{0x}}$ and $\boxed{p_{0z} = b_0}$

1. geostrophic balance
2. hydrostatic balance
3. *balance*

note that $\dfrac{\mathrm{D}^{\mathrm{L}}\boldsymbol{u}^{\mathrm{L}}}{\mathrm{D}t} = 0$ for parallel flow

$$\frac{\mathrm{D}^{\mathrm{L}}\boldsymbol{u}^{\mathrm{L}}}{\mathrm{D}t} + f\,\hat{\boldsymbol{z}} \times \boldsymbol{u}^{\mathrm{L}} + \boldsymbol{\nabla} p - b\,\hat{\boldsymbol{z}} = \left(\boldsymbol{\nabla} \times \boldsymbol{u}^{\mathrm{S}}\right) \times \boldsymbol{u}^{\mathrm{L}} + \boldsymbol{u}^{\mathrm{S}}_t \,.$$

$$b_t + \boldsymbol{u}^{\mathrm{L}} \cdot \boldsymbol{\nabla} b + w^{\mathrm{L}} N^2 = 0$$

Wave-modified balance (state 1)

take $\boldsymbol{u}^{\mathrm{S}} = u^{\mathrm{S}}(z)\,\hat{\boldsymbol{x}} + v^{\mathrm{S}}(z)\,\hat{\boldsymbol{y}}$ *and* $\boldsymbol{u}^{\mathrm{L}} = v_1(x,z)\,\hat{\boldsymbol{y}}\,.$

$$\frac{\mathrm{D}^{\mathrm{L}}\boldsymbol{u}^{\mathrm{L}}}{\mathrm{D}t} + f\,\hat{\boldsymbol{z}} \times \boldsymbol{u}^{\mathrm{L}} + \boldsymbol{\nabla} p - b\,\hat{\boldsymbol{z}} = \left(\boldsymbol{\nabla} \times \boldsymbol{u}^{\mathrm{S}}\right) \times \boldsymbol{u}^{\mathrm{L}} + \boldsymbol{u}^{\mathrm{S}}_t\,.$$

$$b_t + \boldsymbol{u}^{\mathrm{L}} \boldsymbol{\cdot} \boldsymbol{\nabla} b + w^{\mathrm{L}} N^2 = 0$$

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$$\boxed{\left(\boldsymbol{\nabla}\times\boldsymbol{u}^{\mathrm{S}}\right)\times\boldsymbol{u}^{\mathrm{L}} = -v^{\mathrm{S}}_z v_1\,\hat{\boldsymbol{z}}\,.}$$

note that $\dfrac{\mathrm{D}^{\mathrm{L}}\boldsymbol{u}^{\mathrm{L}}}{\mathrm{D}t} = 0$ still

a new term.

$$\frac{\mathrm{D}^{\mathrm{L}}\boldsymbol{u}^{\mathrm{L}}}{\mathrm{D}t} + f\,\hat{\boldsymbol{z}}\times\boldsymbol{u}^{\mathrm{L}} + \boldsymbol{\nabla}p - b\,\hat{\boldsymbol{z}} = \left(\boldsymbol{\nabla}\times\boldsymbol{u}^{\mathrm{S}}\right)\times\boldsymbol{u}^{\mathrm{L}} + \boldsymbol{u}^{\mathrm{S}}_t\,.$$

$$b_t + \boldsymbol{u}^{\mathrm{L}}\boldsymbol{\cdot}\boldsymbol{\nabla}b + w^{\mathrm{L}}N^2 = 0$$

Wave-modified balance (state 1)

take $\boldsymbol{u}^{\mathrm{S}} = u^{\mathrm{S}}(z)\,\hat{\boldsymbol{x}} + v^{\mathrm{S}}(z)\,\hat{\boldsymbol{y}}$ *and* $\boldsymbol{u}^{\mathrm{L}} = v_1(x, z)\,\hat{\boldsymbol{y}}\,.$

the steady solution $\boxed{f v_1 = p_{1x}}$ and $\boxed{p_{1z} = b_1 - v^{\mathrm{S}}_z v_1}$!!!!

1. geostrophic balance
2. ***not*** hydrostatic balance

note that $\dfrac{\mathrm{D}^{\mathrm{L}}\boldsymbol{u}^{\mathrm{L}}}{\mathrm{D}t} = 0$ still

$$(\boldsymbol{\nabla} \times \boldsymbol{u}^{\mathrm{S}}) \times \boldsymbol{u}^{\mathrm{L}} = -v^{\mathrm{S}}_z v_1\,\hat{\boldsymbol{z}}\,.$$

$$\frac{\mathrm{D}^{\mathrm{L}}\boldsymbol{u}^{\mathrm{L}}}{\mathrm{D}t} + f\,\hat{\boldsymbol{z}} \times \boldsymbol{u}^{\mathrm{L}} + \boldsymbol{\nabla} p - b\,\hat{\boldsymbol{z}} = -v^{\mathrm{S}}_z v_1\,\hat{\boldsymbol{z}}\,. \qquad + \boldsymbol{u}^{\mathrm{S}}_t\,.$$

$$b_t + \boldsymbol{u}^{\mathrm{L}} \cdot \boldsymbol{\nabla} b + w^{\mathrm{L}} N^2 = 0$$

Wave-modified balance (state 1)

$$p \overset{\text{def}}{=} \pi^{\dagger} - \bar{\pi} - \boldsymbol{u}^{\mathrm{S}} \cdot \boldsymbol{u}\,,$$

compare with (2.6) using

$$-f(v + \underbrace{v_s}_{\text{SC}}) = -\partial_x \pi + \underbrace{v_s \partial_x v}_{\text{SV}},$$

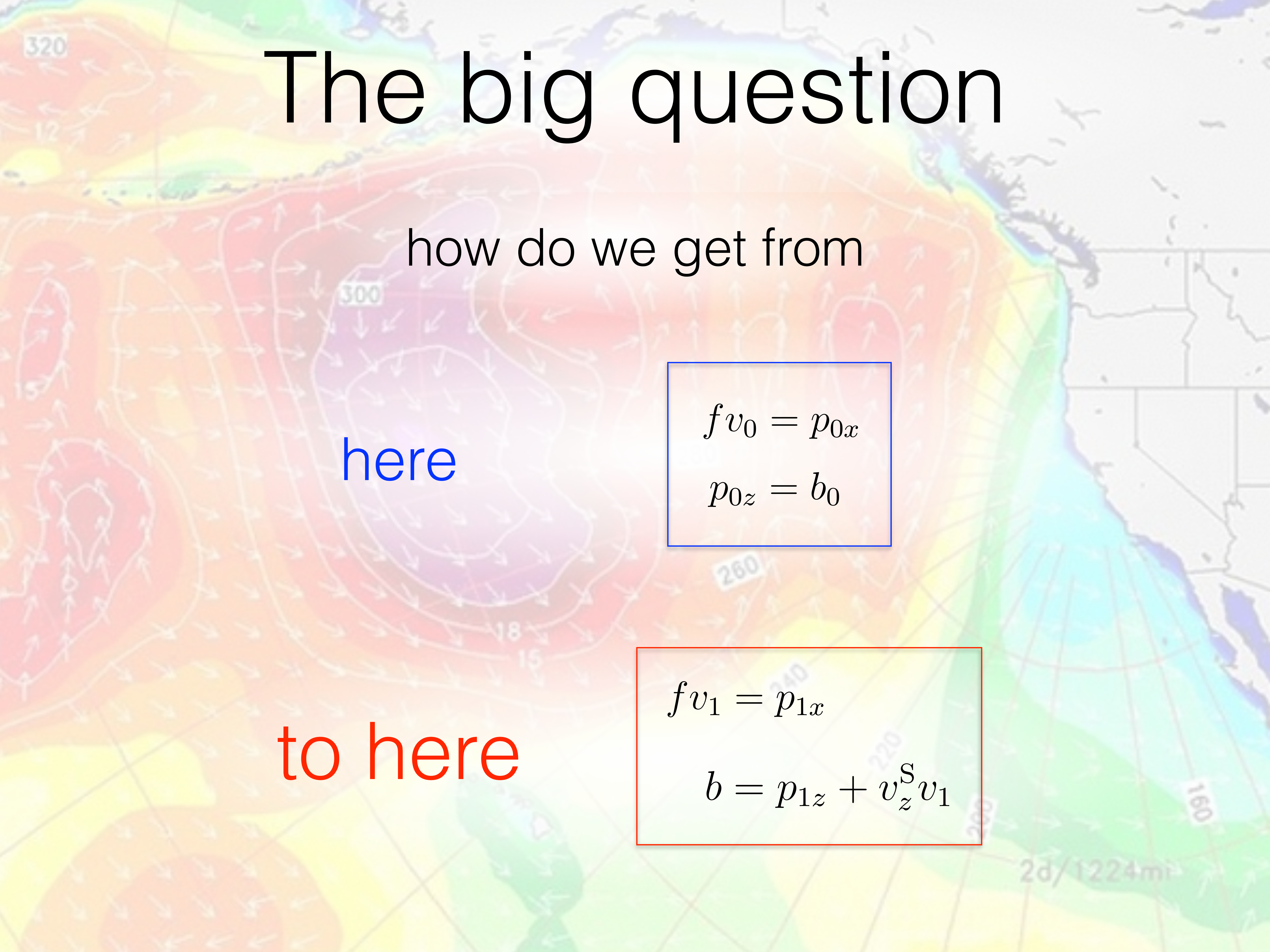
$$\partial_z \pi = b + \underbrace{v_s \partial_z v}_{\text{SV}}$$

$$\boxed{f v_1 = p_{1x}} \quad \text{and} \quad \boxed{p_{1z} = b_1 - v_z^{\mathrm{S}} v_1}$$

$$\boxed{(\boldsymbol{\nabla} \times \boldsymbol{u}^{\mathrm{S}}) \times \boldsymbol{u}^{\mathrm{L}} = -v_z^{\mathrm{S}} v_1\, \hat{\boldsymbol{z}}\,.}$$

$$\frac{\mathrm{D}^{\mathrm{L}} \boldsymbol{u}^{\mathrm{L}}}{\mathrm{D}t} + f\, \hat{\boldsymbol{z}} \times \boldsymbol{u}^{\mathrm{L}} + \boldsymbol{\nabla} p - b\, \hat{\boldsymbol{z}} = -v_z^{\mathrm{S}} v_1\, \hat{\boldsymbol{z}}\,. \qquad + \boldsymbol{u}_t^{\mathrm{S}}\,.$$

$$b_t + \boldsymbol{u}^{\mathrm{L}} \cdot \boldsymbol{\nabla} b + w^{\mathrm{L}} N^2 = 0$$



The big question

how do we get from

here

$$fv_0 = p_{0x}$$

$$p_{0z} = b_0$$

to here

$$fv_1 = p_{1x}$$

$$b = p_{1z} + v^{\mathrm{S}}_z v_1$$

Conservative adjustment

consider a transient adjustment for which

$$\frac{\mathrm{D^L} q}{\mathrm{D}t} = 0\,, \qquad \text{and} \qquad \frac{\mathrm{D^L} b^\dagger}{\mathrm{D}t} = 0$$

Let's solve the adjustment problem for small $\mathscr{R} \overset{\text{def}}{=} \dfrac{V}{fL}$

with one little twist…

A special scaling

for oceanic wave-balanced flows

PV: $$\boxed{q = fN^2 + N^2 v_x + f b_z + \frac{\partial(v,b)}{\partial(x,z)}}$$

Ordinary QG $$\left(\frac{v_x}{f}, \frac{b_z}{N^2}\right) \sim \mathscr{R}\,, \qquad \frac{v_z b_x}{fN^2} \sim \mathscr{R}^2$$

$$\boxed{\mathscr{R} \overset{\text{def}}{=} \frac{V}{fL}}$$

A special scaling

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Ordinary QG

$$\left(\frac{v_x}{f}, \frac{b_z}{N^2}\right) \sim \mathscr{R}\,, \qquad \frac{v_z b_x}{fN^2} \sim \mathscr{R}^2$$

---

here, assume

$$\frac{v_z^{\mathrm{S}} b_x}{fN^2} \sim \epsilon\,\mathscr{R}$$

$$\epsilon = \underbrace{\frac{V_s}{V}}_{\overset{\mathrm{def}}{=}\mu} \underbrace{\frac{h}{h_s}}_{\overset{\mathrm{def}}{=}\lambda} \underbrace{\frac{V}{fL}}_{\overset{\mathrm{def}}{=}\mathscr{R}},$$

$$\boxed{\mathscr{R} \overset{\mathrm{def}}{=} \frac{V}{fL}}$$

$$\boxed{\epsilon \gg \mathscr{R}}$$

strong and *shallow*
drift fields

Potential vorticity

equation (2.5)

$$\boxed{q = fN^2 + N^2 v_x + f b_z + \frac{\partial(v,b)}{\partial(x,z)}}$$

use $v = v^{\mathrm{L}} - v^{\mathrm{S}}$

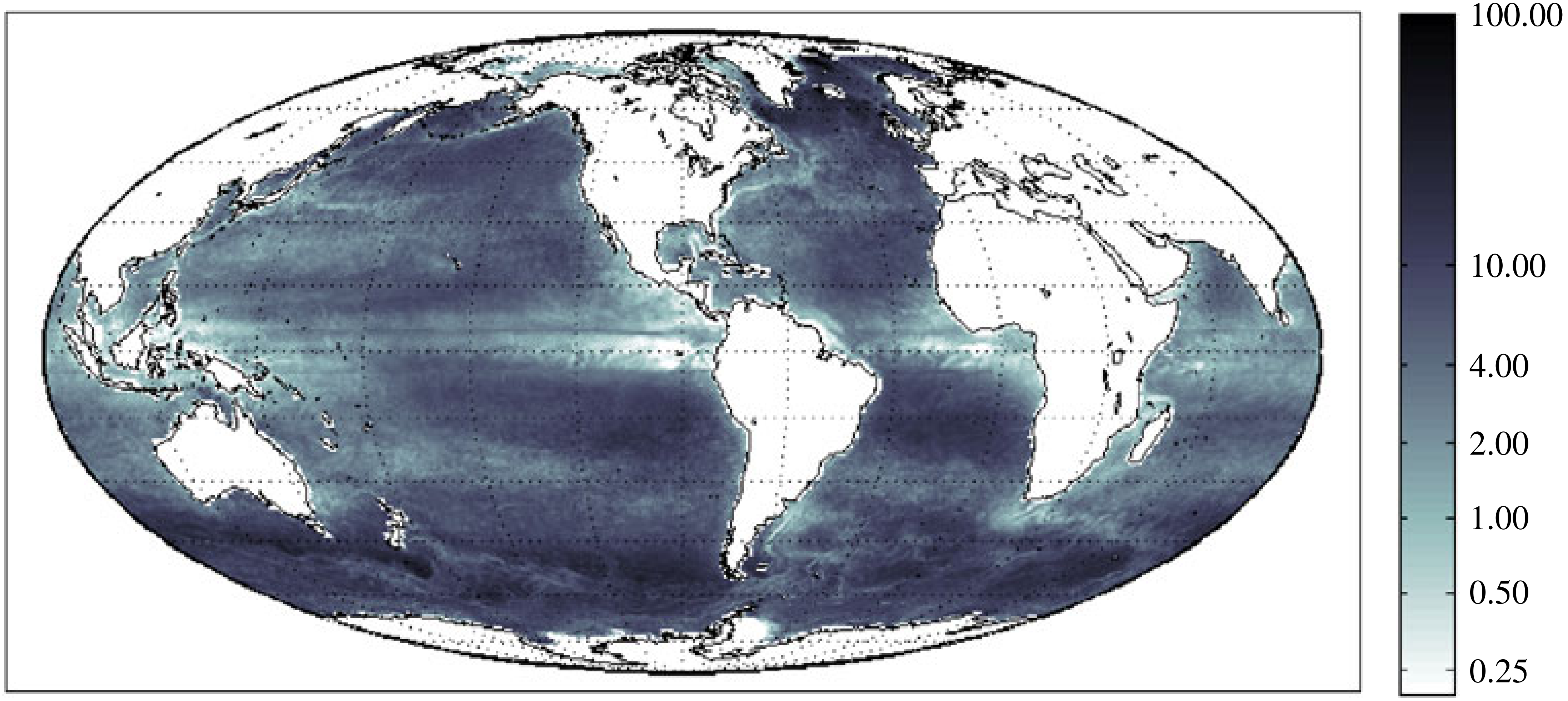
$$q = fN^2 \quad + \quad N^2 v^{\mathrm{L}}_x + f b_z \quad + \quad v^{\mathrm{S}}_z b_x \quad + \quad \frac{\partial(v^{\mathrm{L}},b)}{\partial(x,z)}$$

$$O(1) \qquad O(\mathscr{R}) \qquad O(\epsilon\mathscr{R}) \qquad O(\mathscr{R}^2)$$

↑ note

$$\boxed{\mathscr{R} \overset{\text{def}}{=} \frac{V}{fL}}$$

if $\epsilon \gg \mathscr{R}$, *we should consider this term!*



$\epsilon/\mathscr{R}$

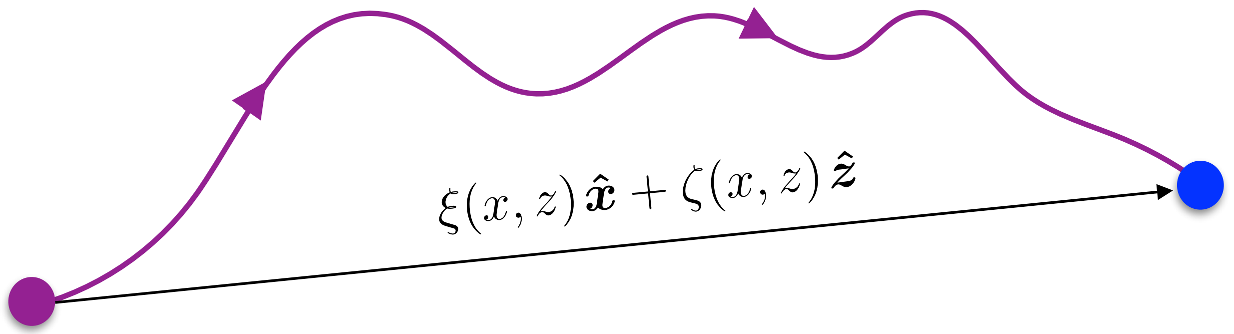
FIGURE 1. (Colour online) Estimated ratio $\epsilon/\mathscr{R} \approx (|\boldsymbol{u}_s \cdot \boldsymbol{u}|h)/(|\boldsymbol{u}|^2 h_s)$ governing the relative importance of Stokes effects versus nonlinearity. Eulerian velocity ($\boldsymbol{u}$) is taken as the AVISO weekly satellite geostrophic velocity or $-\boldsymbol{u}_s$ (for anti-Stokes flow) if $|\boldsymbol{u}_s| > |\boldsymbol{u}|$. The

$$\mathscr{R} \overset{\text{def}}{=} \frac{V}{fL}$$

if $\epsilon \gg \mathscr{R}$, *we should consider this term!*

Conservative adjustment with particle displacements

$\mathscr{R} \ll 1$ implies small displacements



$q_0(x_0, z_0)$

$b_0^\dagger(x_0, z_0)$

waveless

$q(x, z) = q_0(x_0, z_0)$

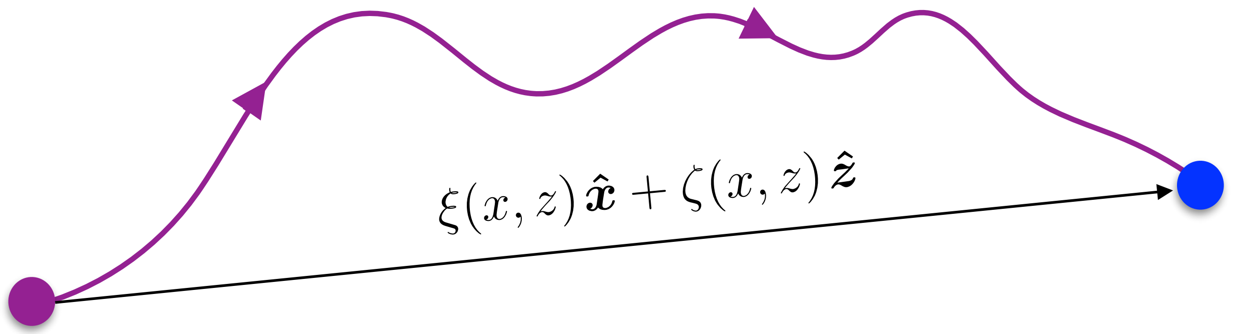
$b^\dagger(x, z) = b_0^\dagger(x_0, z_0)$

wave-balanced

waveless

Conservative adjustment with particle displacements

$\mathscr{R} \ll 1$ implies small displacements



$q_0(x_0, z_0)$

$b_0^\dagger(x_0, z_0)$

waveless

$\xi(x,z)\,\hat{\boldsymbol{x}} + \zeta(x,z)\,\hat{\boldsymbol{z}}$

$q(x,z) = q_0(x_0, z_0)$

$b^\dagger(x,z) = b_0^\dagger(x_0, z_0)$

wave-balanced

conservation of buoyancy

$$
\begin{aligned}
b_0^\dagger(x_0, z_0) &= b^\dagger(x,z) + \xi b_x^\dagger + \zeta b_z^\dagger(x,z) + \cdots \;, \\
&= \bar{b}(x,z) + b_0 + \zeta N^2 + O(\mathscr{R}^2)
\end{aligned}
$$

$$
\longrightarrow \qquad b = b_0 + \zeta N^2 + O(\mathscr{R}^2) \qquad \longrightarrow \qquad \boxed{\zeta = \frac{1}{N^2}(b - b_0)} \qquad (1)
$$

(turn the page)

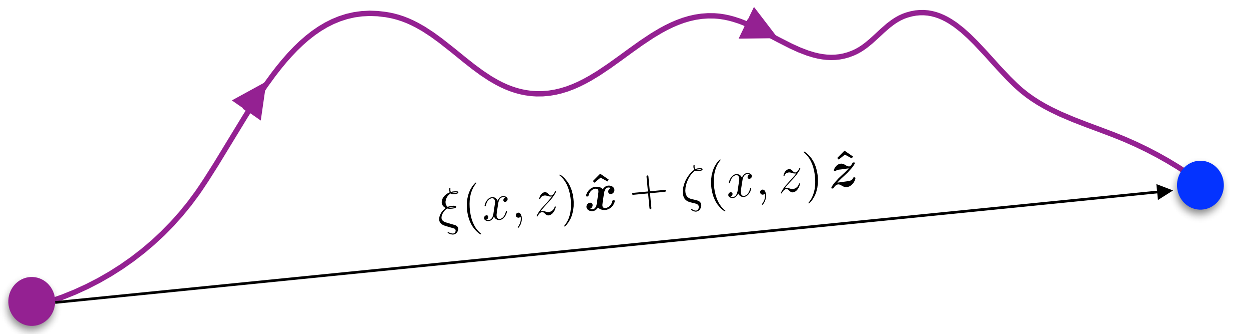
Conservative adjustment with particle displacements

$\mathscr{R} \ll 1$ implies small displacements

$q_0(x_0, z_0)$

$b_0^\dagger(x_0, z_0)$

waveless



$q(x, z) = q_0(x_0, z_0)$

$b^\dagger(x, z) = b_0^\dagger(x_0, z_0)$

wave-balanced

$$\boxed{\zeta = \frac{1}{N^2}(b - b_0)} \qquad (1)$$

potential vorticity

$$q_0(x_0, z_0) = q_0(x, z) + \xi q_{0x} + \zeta q_{0z} + \cdots$$

$$= fN^2 + v_{0x} + fb_{0z} + \zeta\left(fN^2\right)_z + O(\mathscr{R}^2)$$

$$\boxed{q_0(x_0, z_0) = fN^2 + v_{0x} + N^2\left(\frac{fb_0}{N^2}\right)_z + fb\frac{(N^2)_z}{N^2} + O(\mathscr{R}^2)} \qquad (2)$$

(turn the page)

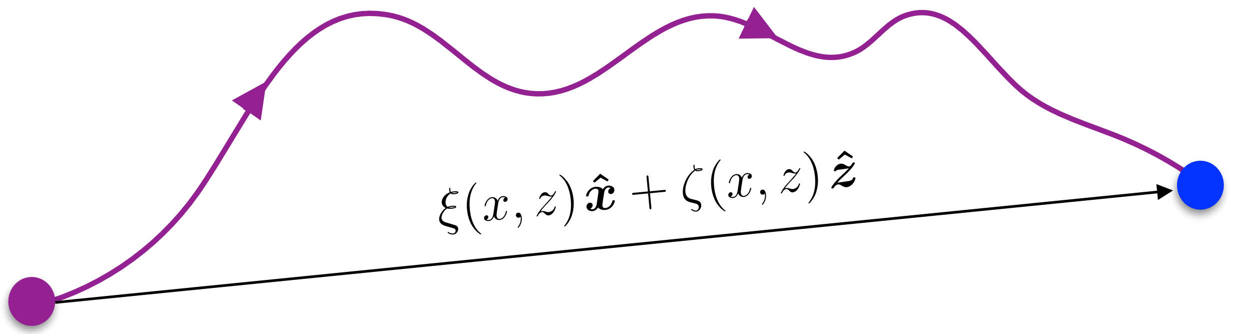
Conservative adjustment with particle displacements

$\mathscr{R} \ll 1$ implies small displacements

$q_0(x_0, z_0)$

$b_0^\dagger(x_0, z_0)$

waveless



$q(x, z) = q_0(x_0, z_0)$

$b^\dagger(x, z) = b_0^\dagger(x_0, z_0)$

wave-balanced

$$\boxed{\zeta = \frac{1}{N^2}(b - b_0)} \qquad (1)$$

$$\boxed{q_0(x_0, z_0) = fN^2 + v_{0x} + N^2\left(\frac{fb_0}{N^2}\right)_z + fb\frac{(N^2)_z}{N^2} + O(\mathscr{R}^2)} \qquad (2)$$

next, observe that

$$q = fN^2 + v_x + fb_z + v_z^{\mathrm{S}} b_x + O(\mathscr{R}^2) \quad \longrightarrow$$

$$\boxed{\underbrace{v_x + \left(\frac{fb}{N^2}\right)_z + \frac{v_z^{\mathrm{S}} b_x}{N^2}}_{\overset{\mathrm{def}}{=} Q} = \underbrace{v_{0x} + \left(\frac{fb_0}{N^2}\right)_z}_{\overset{\mathrm{def}}{=} Q_0}}$$

Section 4: constant N and small ϵ

$$\boxed{v_x + \left(\frac{fb}{N^2}\right)_z + \frac{v_z^{\mathrm{S}} b_x}{N^2} = v_{0x} + \left(\frac{fb_0}{N^2}\right)_z}$$

propose

$$v = v_0 + \epsilon v'$$

$$b = b_0 + \epsilon b'$$

$$\longrightarrow \quad v'_x + \left(\frac{fb'}{N^2}\right)_z = -\frac{v_z^{\mathrm{S}} b_{0x}}{N^2} + O(\epsilon)$$

next, use “imbalance” conditions to obtain elliptic equation

$$v = p_x/f$$

$$b = p_z + v_z^{\mathrm{S}} v_0 + O(\epsilon)$$

simplified

$$\longrightarrow \quad \boxed{b'_{xx} + \left(\frac{fb'}{N^2}\right)_{zz} = \underbrace{f v_z^{\mathrm{S}} v_{0xx} - \left(\frac{f v_z^{\mathrm{S}} v_{0z}}{N^2}\right)_z}_{\text{source term}}}$$

A way that was easier for me.

Exact PV

$$q = \underbrace{fN^2}_{O(1)} + \underbrace{N^2 v_x^{\mathrm{L}} + f b_z + v_z^{\mathrm{S}} b_x}_{O(\mathscr{R})} + \underbrace{\frac{\partial(v^{\mathrm{L}}, b)}{\partial(x, z)}}_{O(\mathscr{R}^2)}$$

PV equation

$$q_t + \boldsymbol{u}^{\mathrm{L}} \boldsymbol{\cdot} \boldsymbol{\nabla} q = 0 \,,$$

buoyancy equation

$$b_t + \boldsymbol{u}^{\mathrm{L}} \boldsymbol{\cdot} \boldsymbol{\nabla} b + w^{\mathrm{L}} N^2 = 0$$

PV and buoyancy combine in the same way as ordinary QGPV, except there is an extra term in the PV. We get:

low Rossby PV

$$Q \overset{\text{def}}{=} v_x^{\mathrm{L}} + \left(\frac{fb}{N^2}\right)_z + \frac{v_z^{\mathrm{S}} b_x}{N^2} \,.$$

imbalance conditions

$$f v^{\mathrm{L}} = p_x$$

$$b = p_z + v_z^{\mathrm{S}} v^{\mathrm{L}}$$

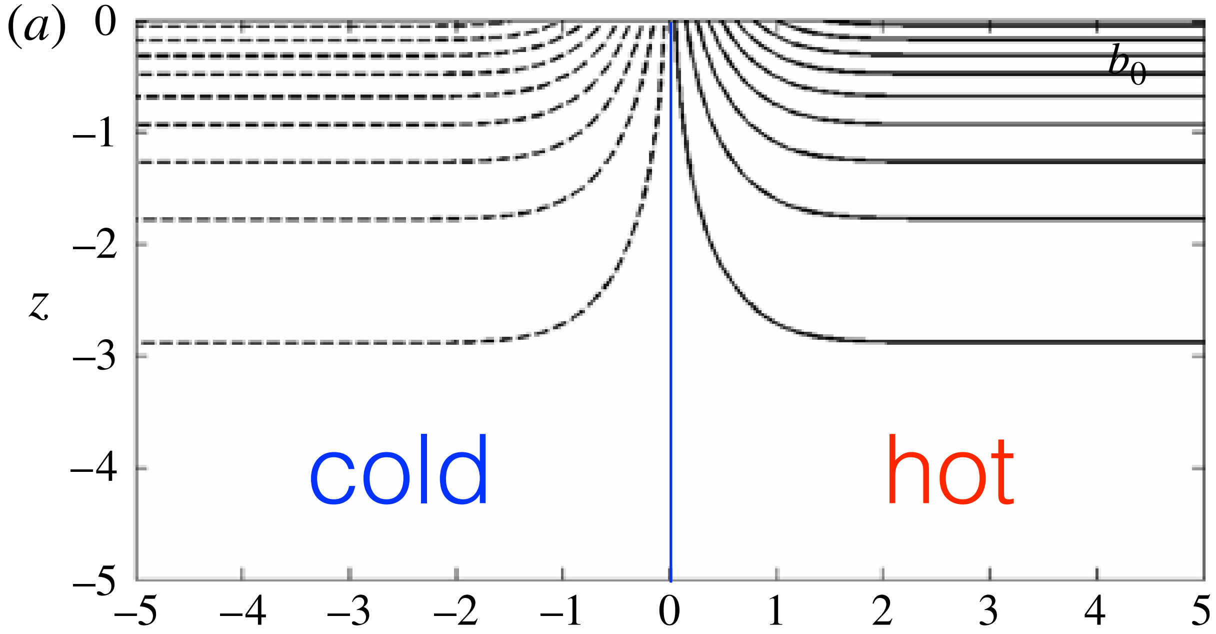
adjustment problem is posed by equating initial and final PV:

$$Q_0 = Q = \underbrace{v_x^{\mathrm{L}} + \left(\frac{fb}{N^2}\right)_z + \frac{v_z^{\mathrm{S}} b_x}{N^2}}_{\text{elliptic problem for } b \text{ or } p} \,.$$

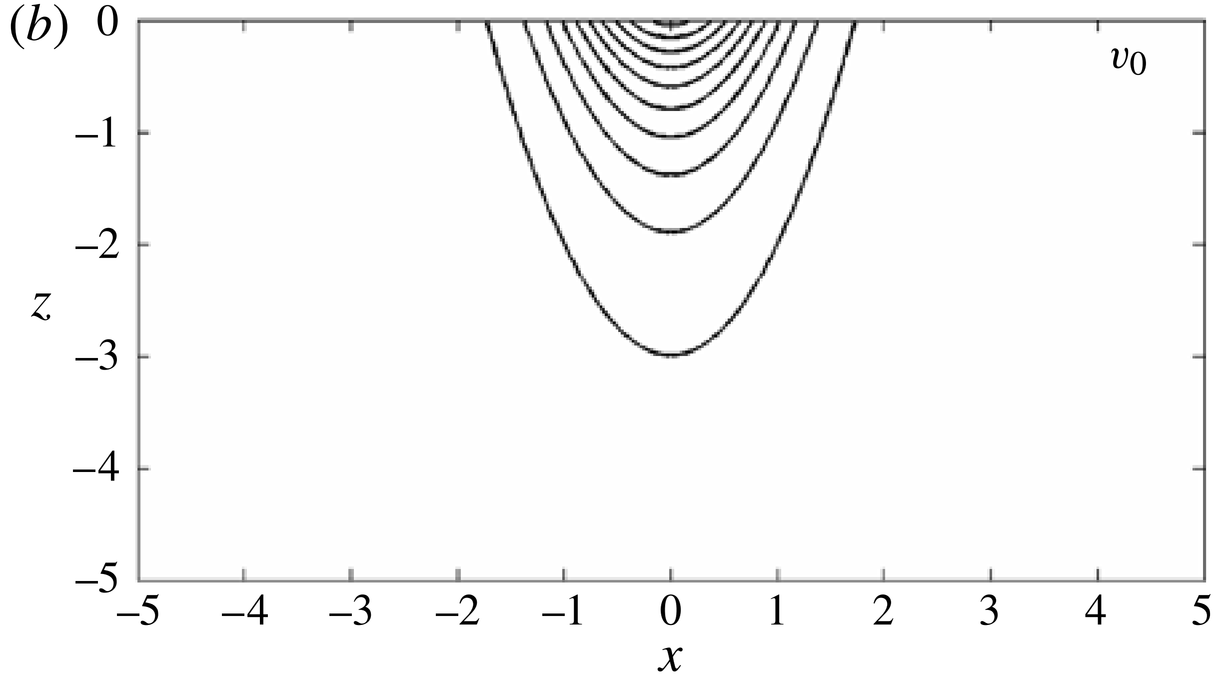
↗ source

Oceanic wave-balanced fronts

initial buoyancy



initial velocity



$$\boxed{b'_{xx} + \left(\frac{f b'}{N^2}\right)_{zz} = f v_z^{\mathrm{S}} v_{0xx} - \left(\frac{f v_z^{\mathrm{S}} v_{0z}}{N^2}\right)_z}$$

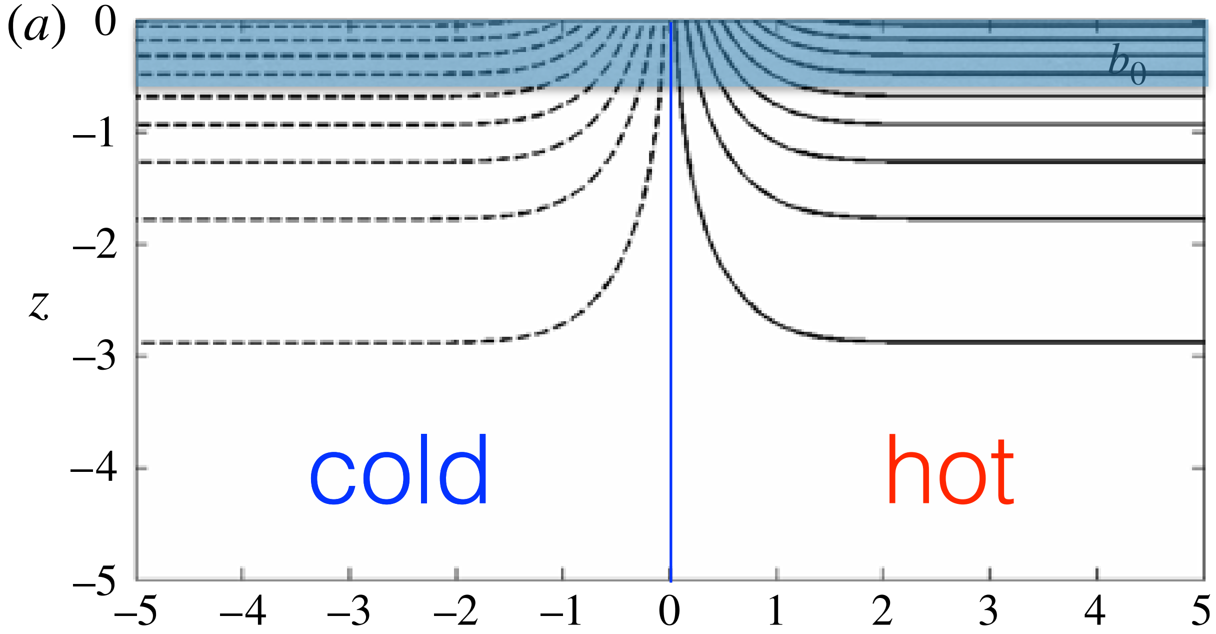
$$v_z^{\mathrm{S}} = \mathrm{e}^{\lambda z} \, .$$

$$v_0(x, z) = \mathrm{e}^{-x^2+z},$$

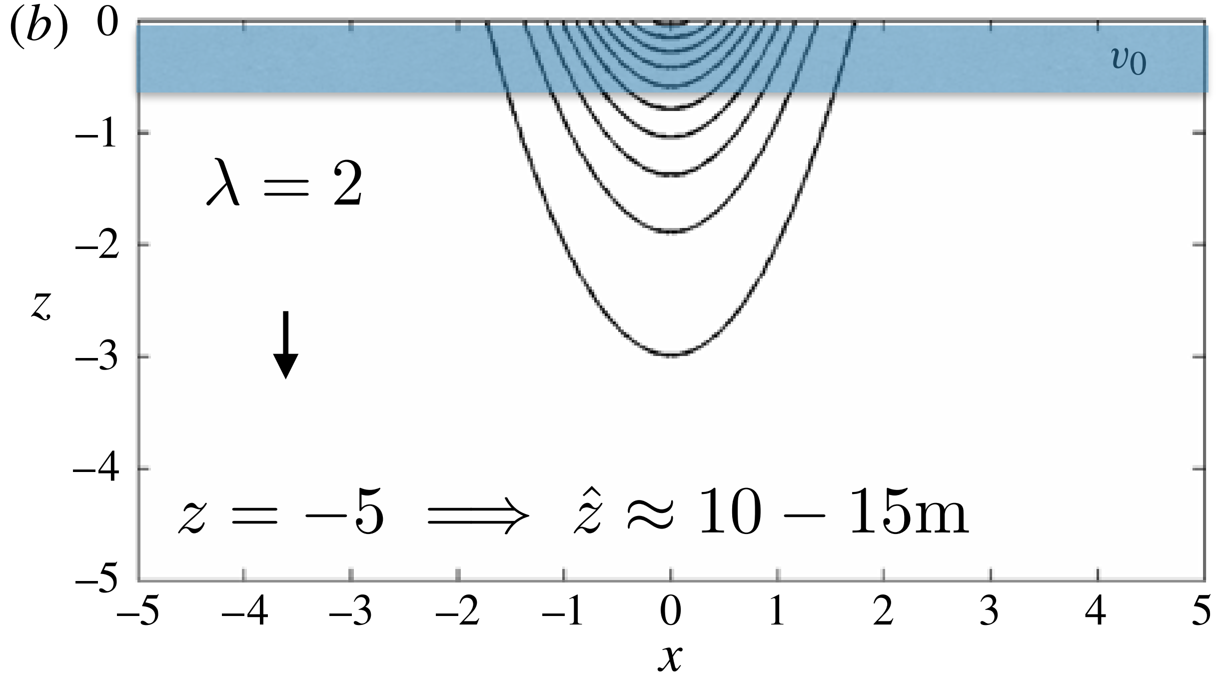
$$b_0(x, z) = \frac{\sqrt{\pi}}{2}\mathrm{erf}[x]\mathrm{e}^{z}.$$

Oceanic wave-balanced fronts

initial buoyancy



initial velocity



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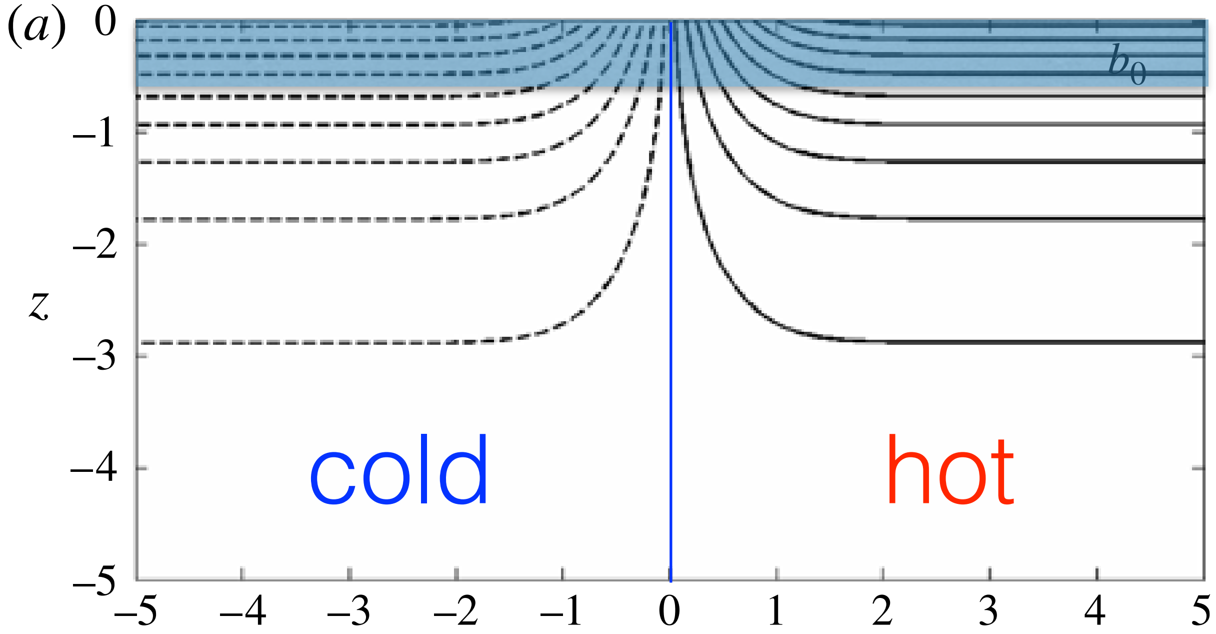
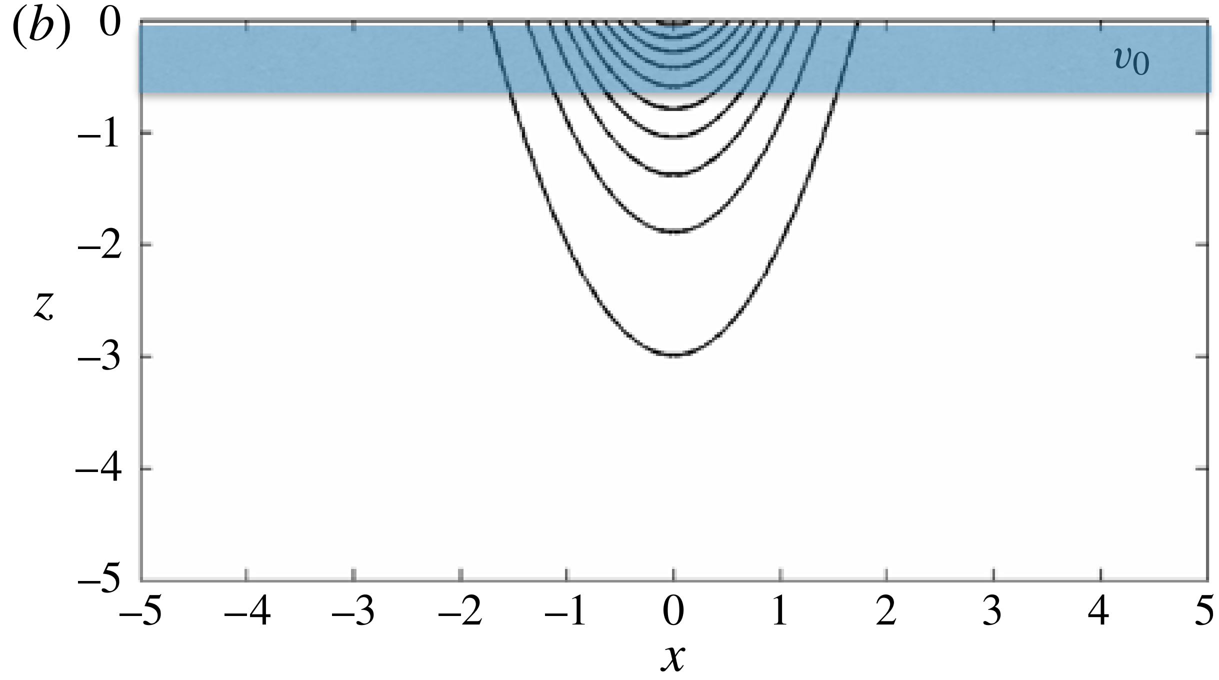
Oceanic wave-balanced fronts

initial buoyancy



initial velocity



$$\boxed{b'_{xx} + \left(\frac{fb'}{N^2}\right)_{zz} = f v_z^{\mathrm{S}} v_{0xx} - \left(\frac{f v_z^{\mathrm{S}} v_{0z}}{N^2}\right)_z}$$

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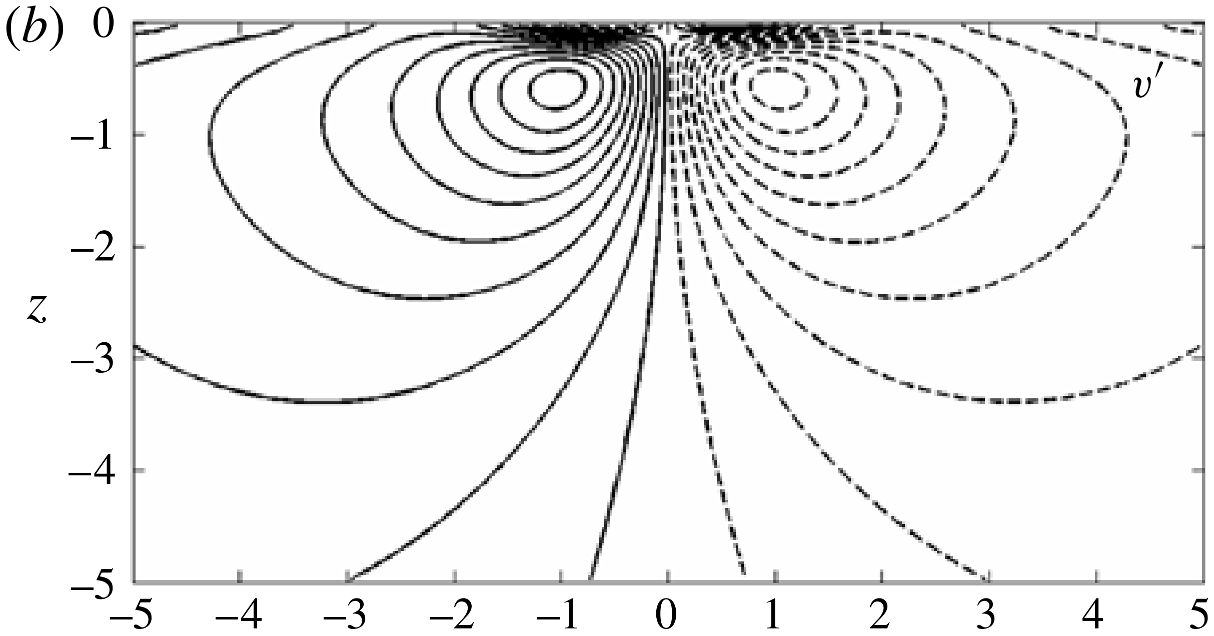
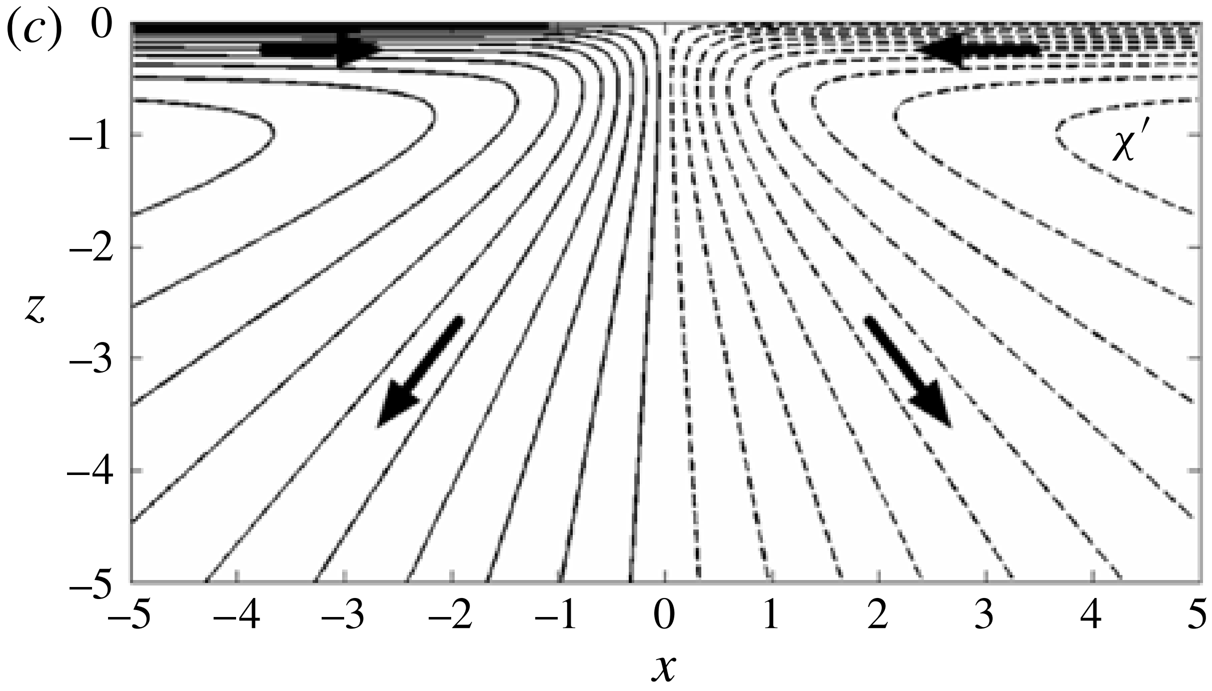
Oceanic wave-balanced fronts

velocity perturbation



particle displacement



$$\boxed{b'_{xx} + \left(\frac{f b'}{N^2}\right)_{zz} = f v_z^{\mathrm{S}} v_{0xx} - \left(\frac{f v_z^{\mathrm{S}} v_{0z}}{N^2}\right)_z}$$

$$v_z^{\mathrm{S}} = \mathrm{e}^{\lambda z}\,.$$

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Oceanic wave-balanced fronts

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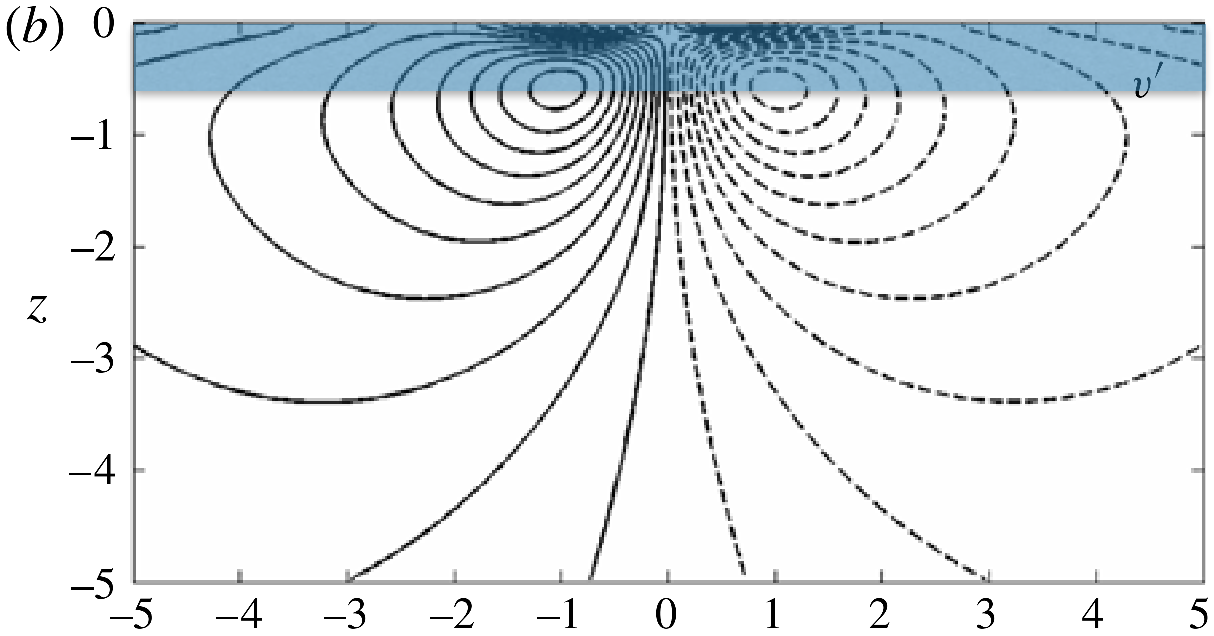
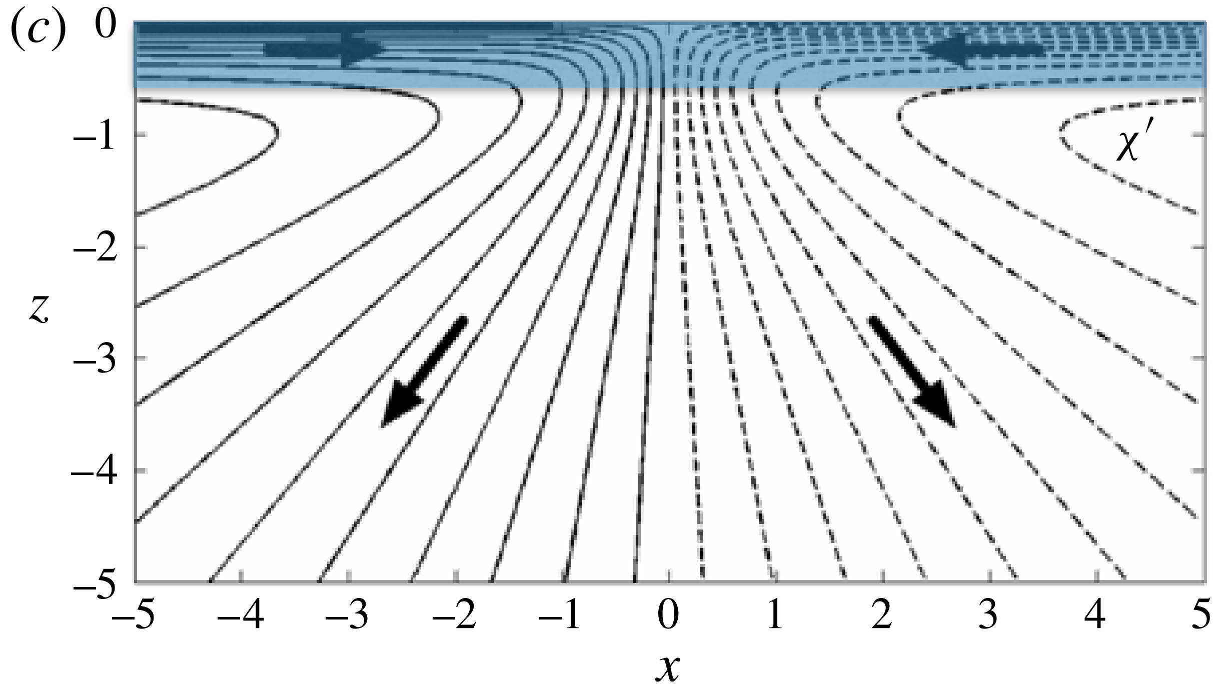


particle displacement



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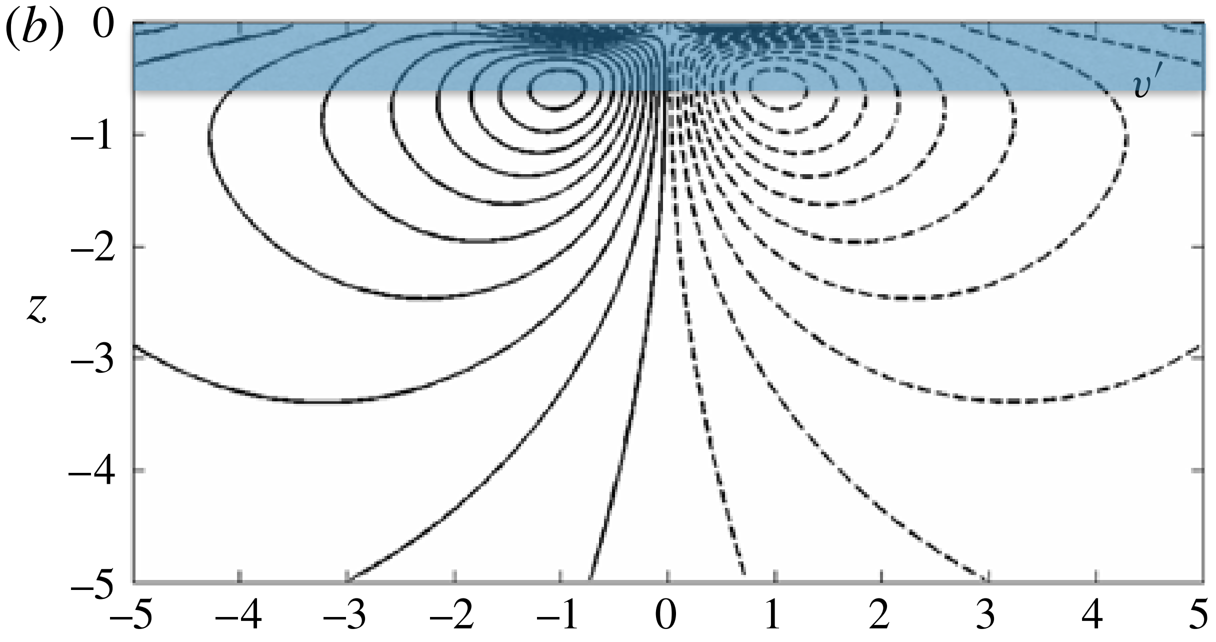
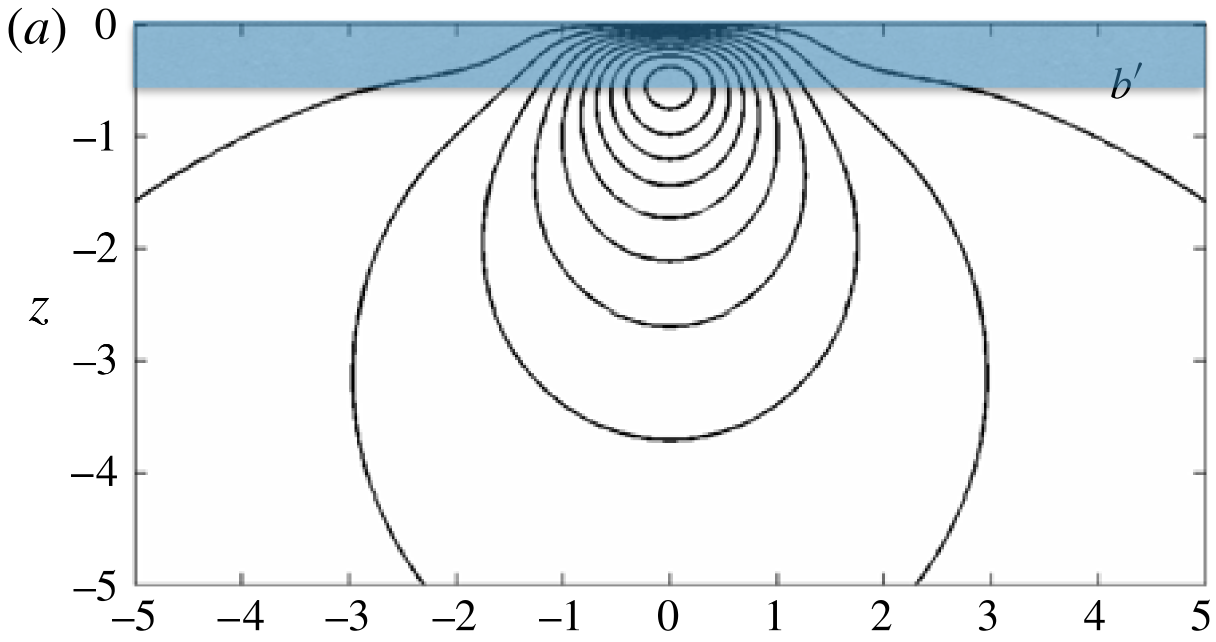
Oceanic wave-balanced fronts

velocity perturbation



buoyancy perturbation



far field

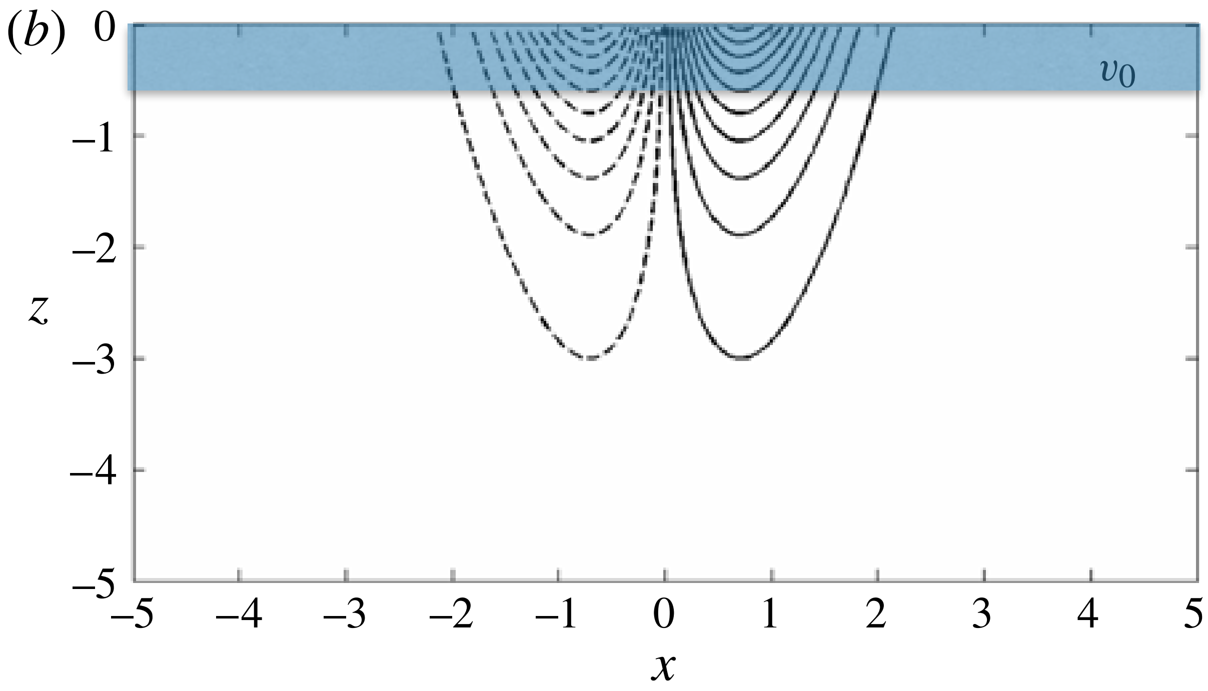
$$b' \sim \frac{\sin[\theta]}{r} = \frac{\mathscr{B}^{1/2} z}{r^2}, \quad v' \sim \frac{\cos[\theta]}{r} = \frac{x}{r^2}.$$

wave-averaging effectively introduces an inhomogeneous boundary condition into the elliptic problem.

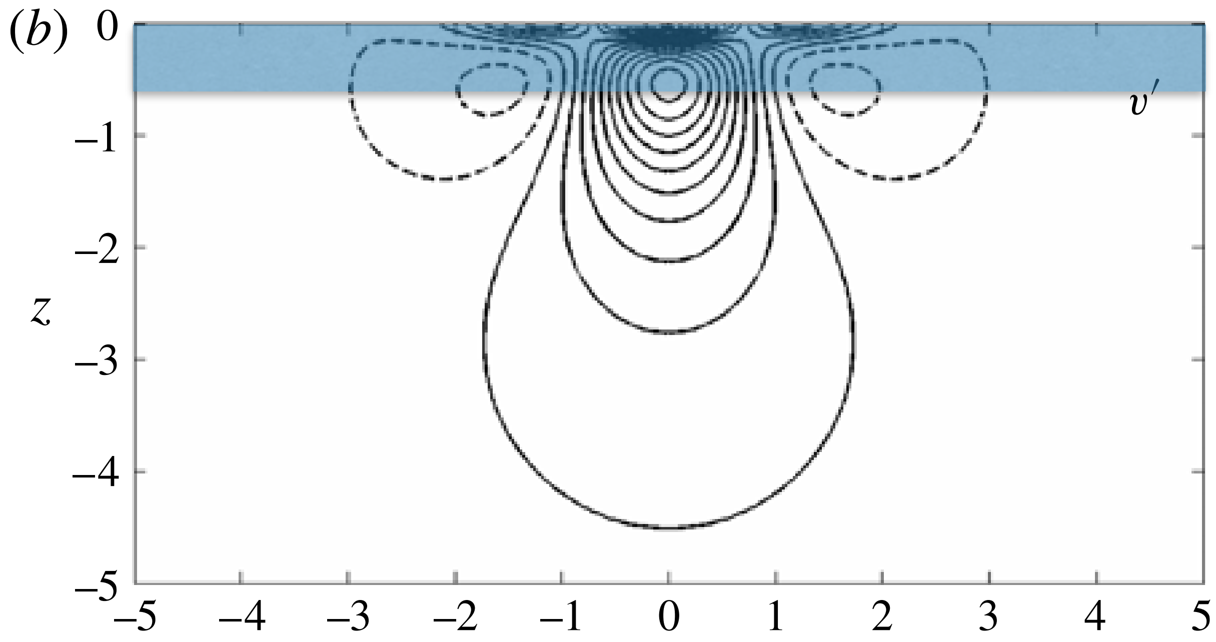
$$\boxed{b'_{xx} + \left(\frac{f b'}{N^2}\right)_{zz} = f v_z^{\mathrm{S}} v_{0xx} - \left(\frac{f v_z^{\mathrm{S}} v_{0z}}{N^2}\right)_z}$$

Oceanic wave-balanced filaments

initial velocity



velocity perturbation



far field $$b' \sim \frac{\sin[2\theta]}{r^2} = \frac{2\mathscr{B}^{1/2}xz}{r^4}, \qquad v' \sim \frac{\cos[2\theta]}{\boxed{r}} = \frac{x^2 - \mathscr{B}z^2}{r^4},$$

typo

wave-averaging effectively introduces an inhomogeneous boundary condition into the elliptic problem.

$$\boxed{b'_{xx} + \left(\frac{fb'}{N^2}\right)_{zz} = f v_z^{\mathrm{S}} v_{0xx} - \left(\frac{f v_z^{\mathrm{S}} v_{0z}}{N^2}\right)_z}$$

Generalization

$$\partial_x^2 b' + \partial_z^2 \left(\frac{b'}{N^2}\right) = \mathscr{F}' - \epsilon \left(\frac{S_s^2}{N^2} \partial_x \partial_z v_0 + 2 S_s \partial_z \left(\frac{\partial_x b'}{N^2}\right) + (\partial_z S_s) \left(\frac{\partial_x b'}{N^2}\right) + \epsilon S_s^2 \left(\frac{\partial_x^2 b'}{N^2}\right)\right)$$

drift layer

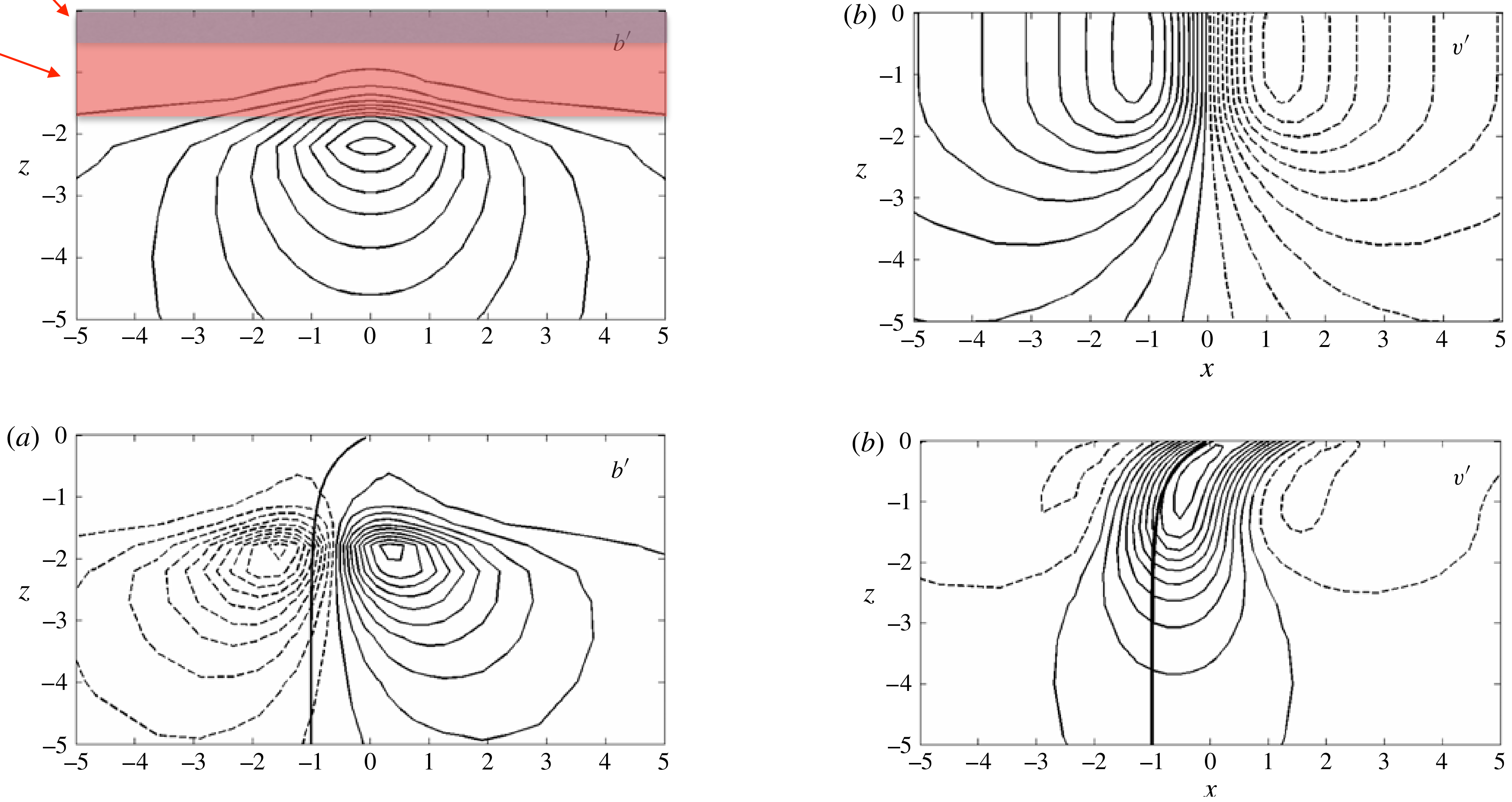
“mixed layer”

buoyancy

velocity

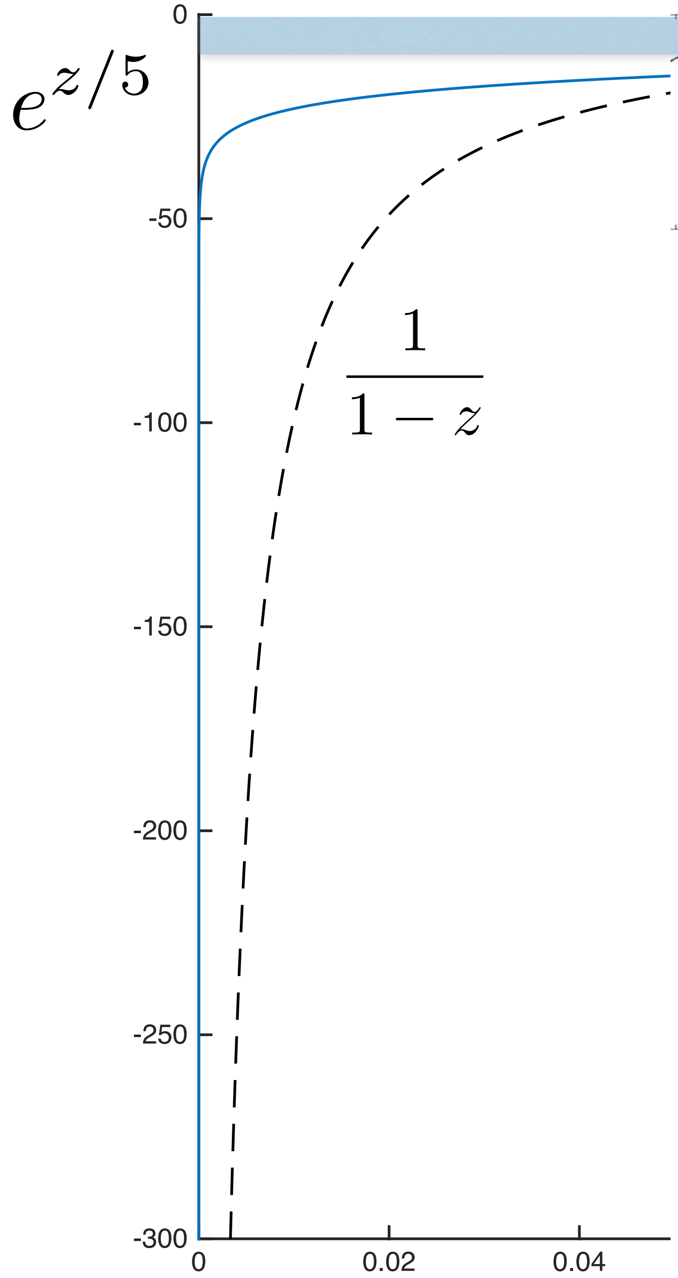
$\epsilon \ll 1$

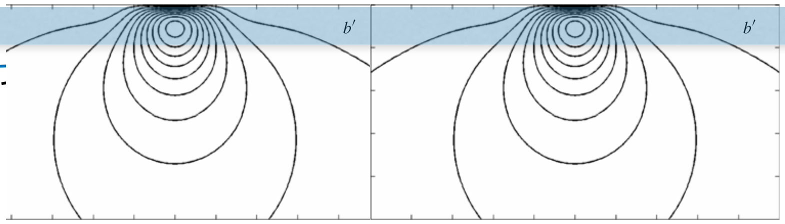
$\epsilon = 2$



The importance of sanity

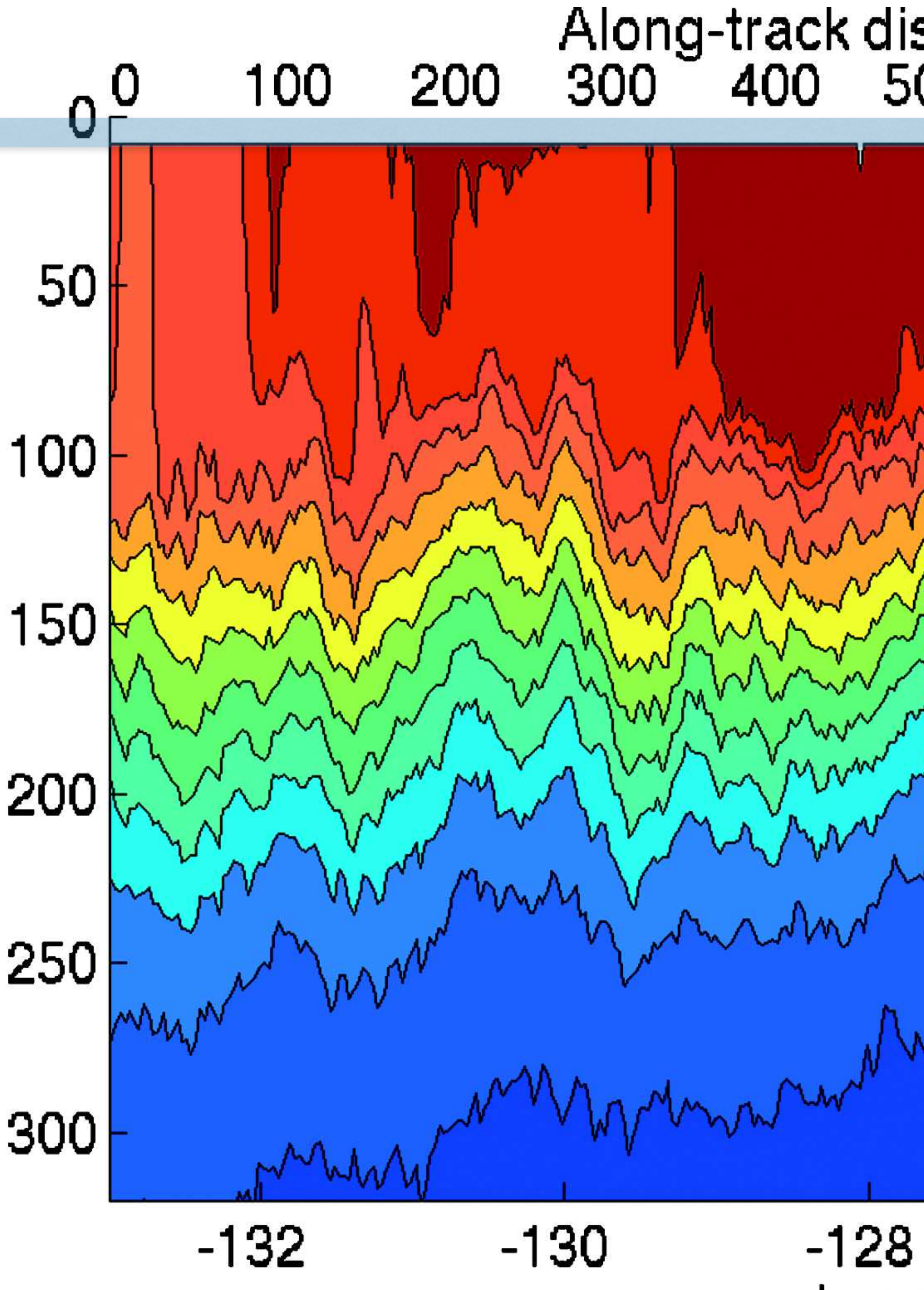
scaled for 5m Stokes drift layer





$$b' \sim \frac{\sin[\theta]}{r} = \frac{\mathscr{B}^{1/2} z}{r^2},$$

$$b' \sim \frac{\sin[2\theta]}{r^2} = \frac{2\mathscr{B}^{1/2} xz}{r^4},$$



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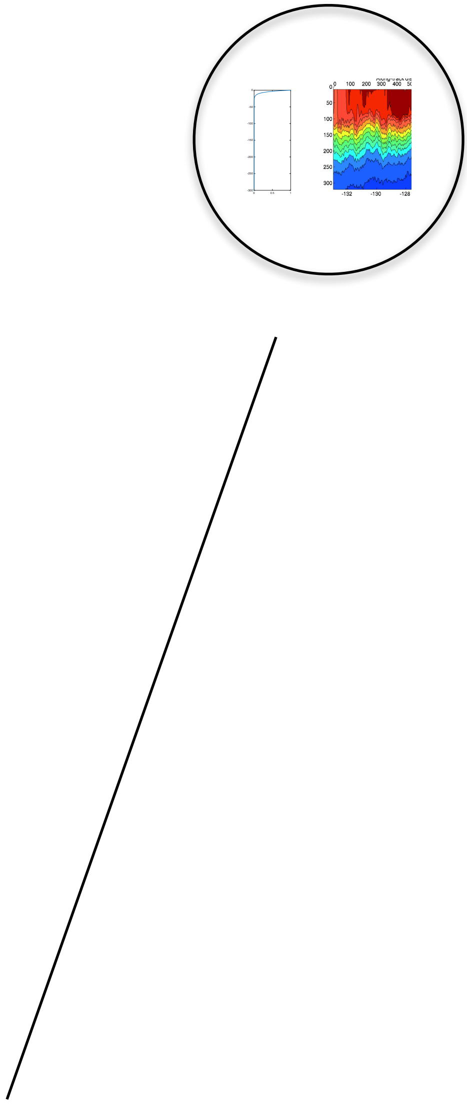
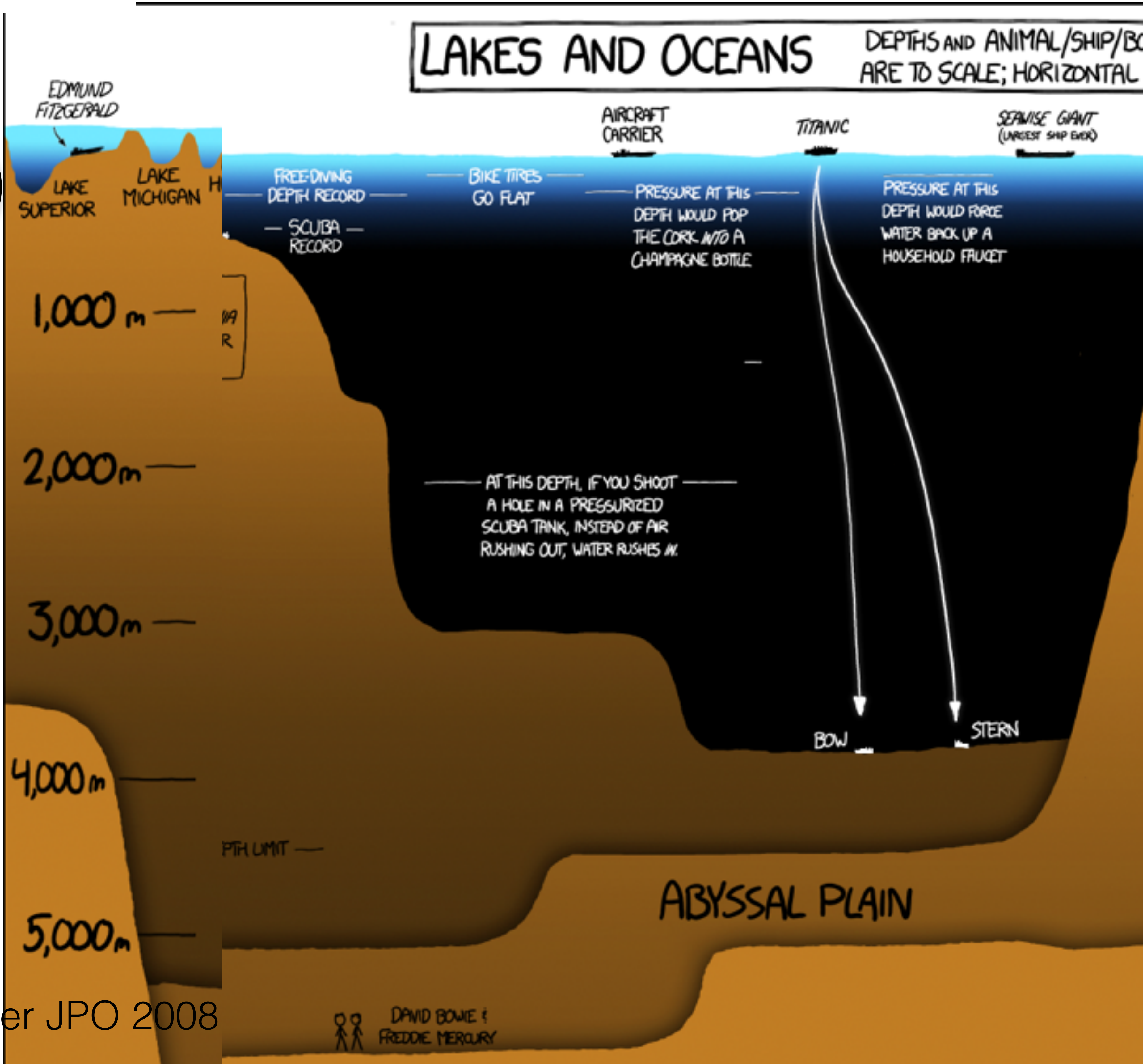
Sanity?!





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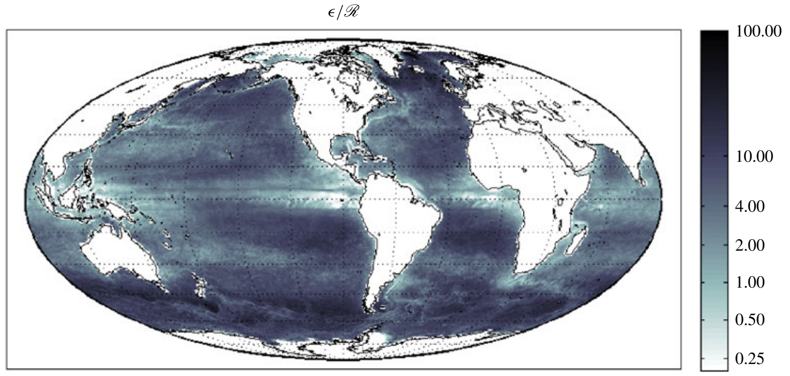
Sanity?!

McWilliams and Kemper’s conclusions

- waves disturb hydrostatic balance



- wave-averaged terms contribute to potential vorticity balance
    - important for strong, shallow Stokes drift fields

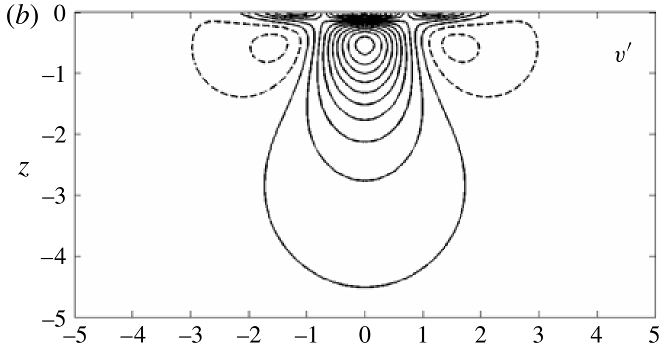
$$f v_1^{\mathrm{L}} = p_{1x}\ ,$$

$$b_1 = p_{1z} + f^{-1} v_z^{\mathrm{S}} p_{1x}$$

- effect of the waves resembles *inhomogeneous boundary condition* at surface

$$Q \overset{\mathrm{def}}{=} v_x^{\mathrm{L}} + \left(\frac{fb}{N^2}\right)_z + \frac{v_z^{\mathrm{S}} b_x}{N^2}\ .$$

- weaker stratification implies deeper response
- with a mixed layer, response is trapped



Greg's questions

- wave field forced by strongly intermittent storms.

  Is spatially uniform wave field realistic?

- why isn't there a second paper

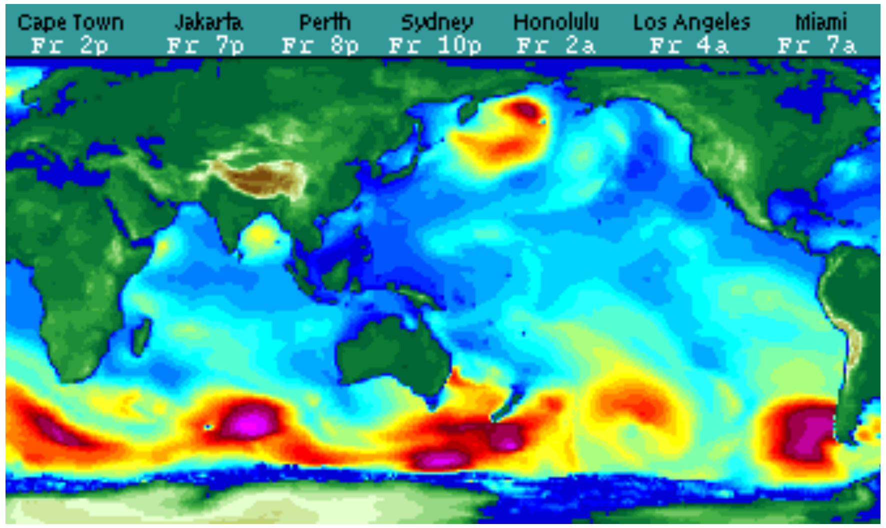
  "Oceanic frontal- and filamental modulation of surface waves"?

- Can we generalize to 3D?

  "Wave-imbalanced" quasigeostrophic flow?

- McWilliams and Kemper mention several times that adjustment radiates internal waves.

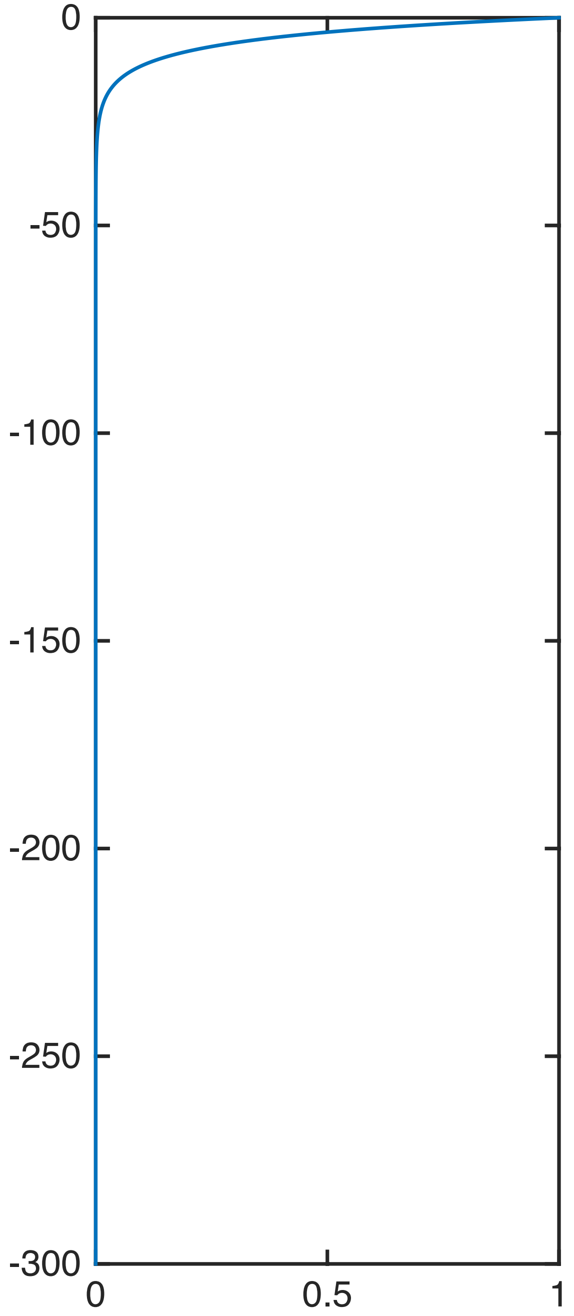
  Does radiated energy come from waves or balanced flow?

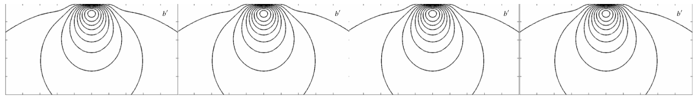


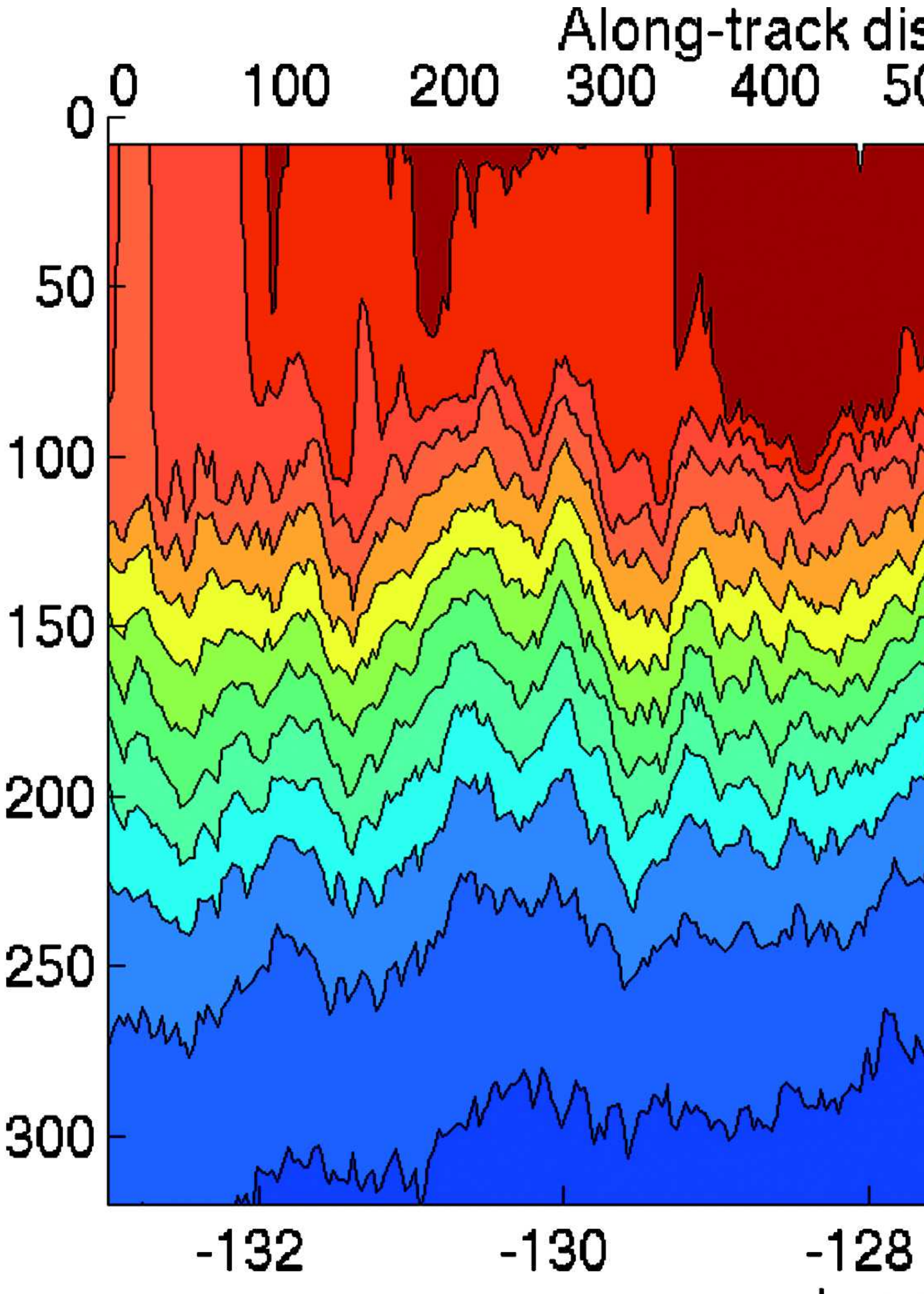
$$Q = v_x^{\mathrm{L}} + \left(\frac{fb}{N^2}\right)_z - \overset{\neq 0}{\overset{\big\downarrow}{v_x^{\mathrm{S}}}} + \frac{v_z^{\mathrm{S}} b_x}{N^2}$$

Sanity

5m Stokes drift layer







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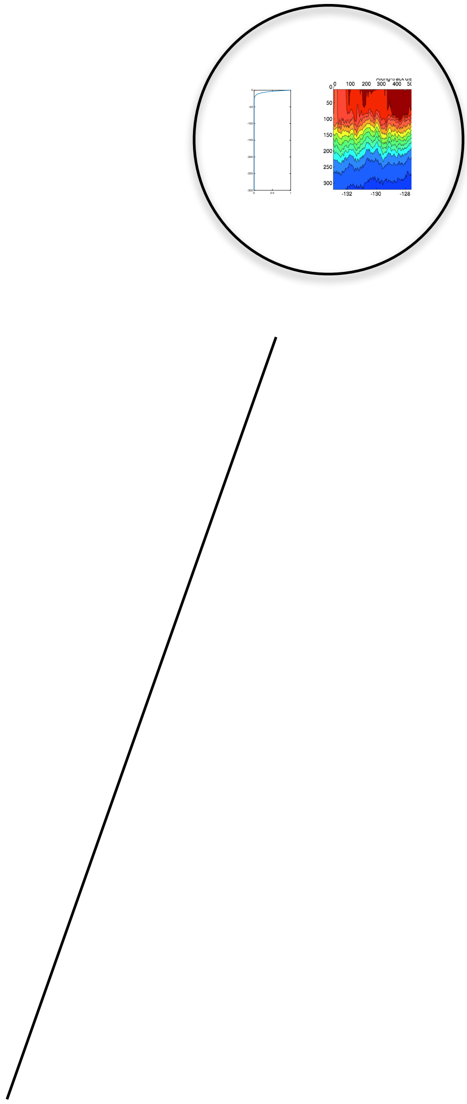
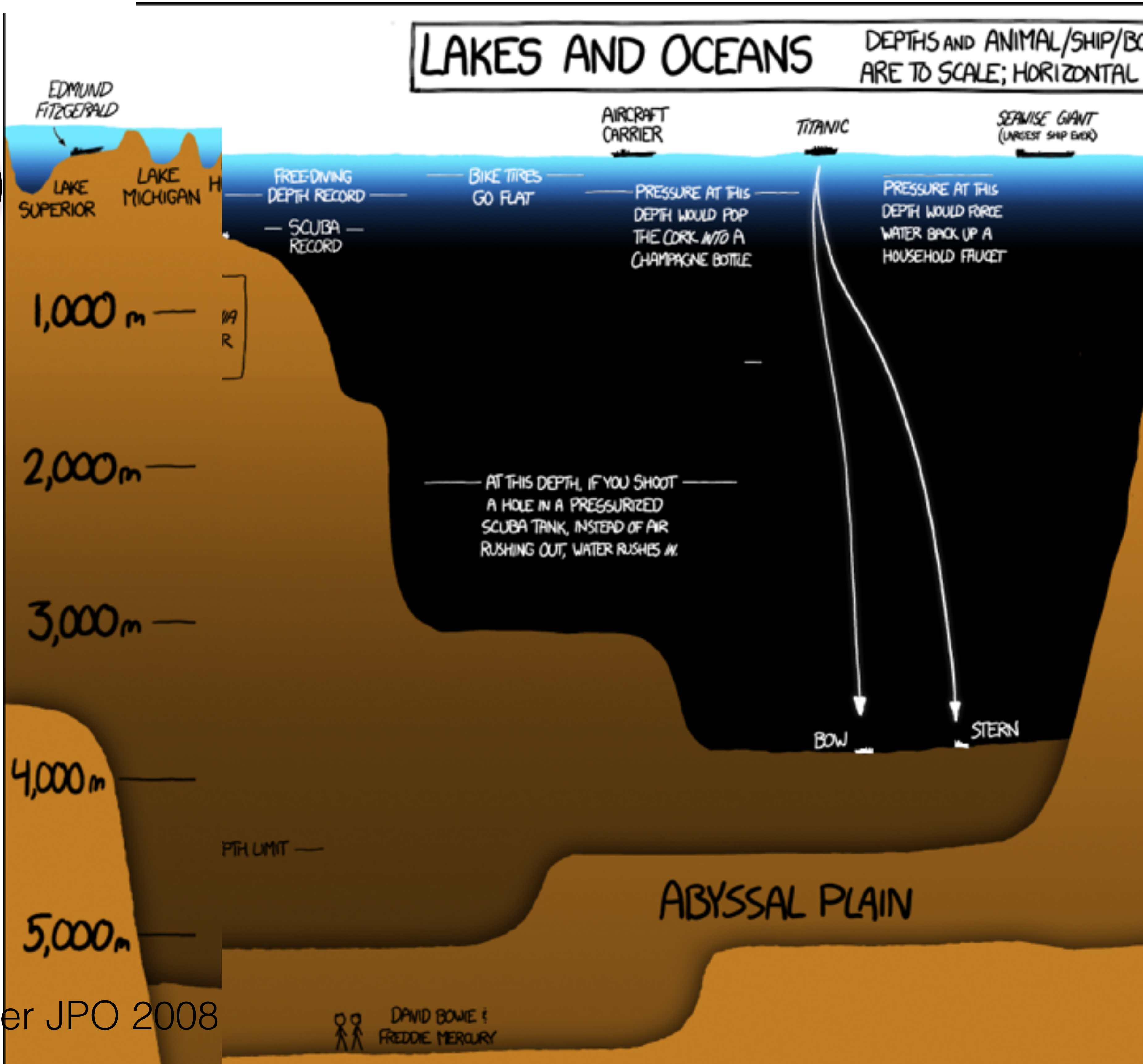
Sanity?!





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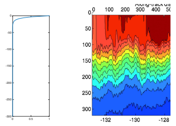
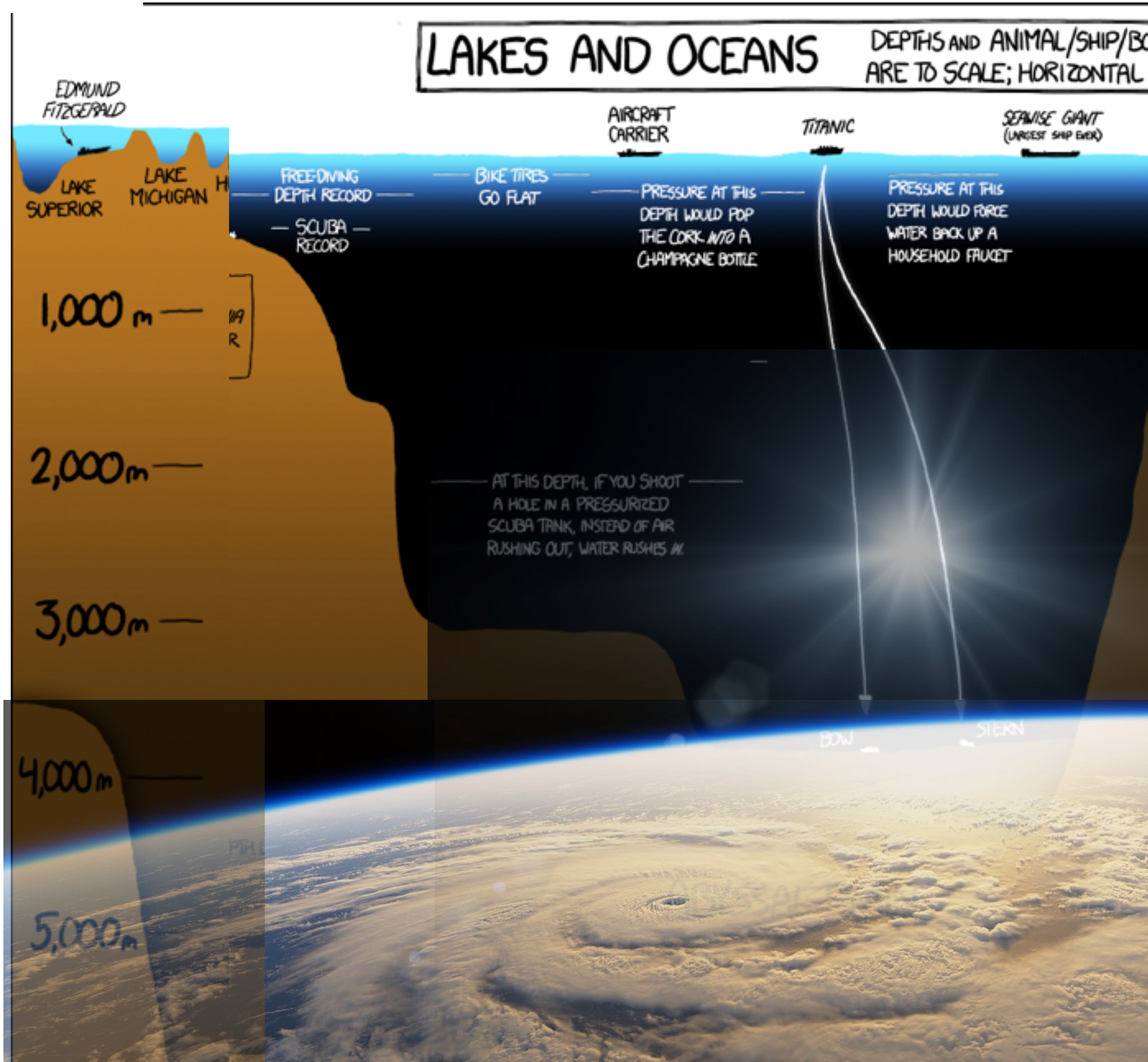


Taylor column



Thanks

Generalization?

effect of the waves resembles inhomogeneous boundary condition at surface

is it possible to derive this effective boundary condition?

in 3D, the "strong, infinite wave" CL-QG theory generalizes to

$$Q_t + \mathsf{J}(\psi, Q) = 0\,.$$

where

definition of potential vorticity

$$\boxed{Q \overset{\text{def}}{=} v_x^{\mathrm{L}} - u_y^{\mathrm{L}} + \left(\frac{fb}{N^2}\right)_z + \frac{v_z^{\mathrm{S}} b_x}{N^2} - \frac{u_z^{\mathrm{S}} b_y}{N^2}\,.}$$

imbalance conditions

$$\boxed{\begin{aligned} v^{\mathrm{L}} &= \psi_x\,, \\ -u^{\mathrm{L}} &= \psi_y\,, \\ b &= f\psi_z + v_z^{\mathrm{S}}\psi_x - u_z^{\mathrm{S}}\psi_y\,. \end{aligned}}$$

Generalization?

effect of the waves resembles inhomogeneous boundary condition at surface

is it possible to derive this effective boundary condition?

in 3D, the “strong, infinite wave” CL-QG theory generalizes to

$$Q_t + \mathsf{J}(\psi, Q) = 0\,.$$

where

$$\begin{aligned} Q &\overset{\mathrm{def}}{=} \triangle\psi + \mathsf{L}\psi + \psi_x \mathsf{L} v^{\mathrm{S}} - \psi_y \mathsf{L} u^{\mathrm{S}} - \omega^{\mathrm{w}} \\ &\quad + \frac{v_z^{\mathrm{S}}}{N^2}\Big(2f\psi_{xz} + (v_z^{\mathrm{S}}\psi_x)_x - (u_z^{\mathrm{S}}\psi_y)_x\Big) + \frac{u_z^{\mathrm{S}}}{N^2}\Big((v_z^{\mathrm{S}}\psi_x)_y - (u_z^{\mathrm{S}}\psi_y)_y\Big) \end{aligned}$$

Two branches in the solution

$$\tilde{b}(z) = \frac{1+\lambda+2\mathscr{B}}{(1+\lambda)^2 - 2\mathscr{B}} \left(\mathrm{e}^{\sqrt{2\mathscr{B}}z} - \mathrm{e}^{(1+\lambda)z} \right),$$

$$\tilde{v}(z) = \frac{1+\lambda+2\mathscr{B}}{\sqrt{2\mathscr{B}}((1+\lambda)^2 - 2\mathscr{B})} \mathrm{e}^{\sqrt{2\mathscr{B}}z} - \frac{2+\lambda}{(1+\lambda)^2 - 2\mathscr{B}} \mathrm{e}^{(1+\lambda)z}.$$

B Deriving the PV equation

When the Stokes drift field is $\boldsymbol{u}^{\mathrm{S}} = u^{\mathrm{S}}(z)\,\hat{\boldsymbol{x}} + v^{\mathrm{S}}(z)\,\hat{\boldsymbol{y}}$, the exact PV and buoyancy equations are

$$\big(\partial_t + \boldsymbol{u}^{\mathrm{L}}\boldsymbol{\cdot\nabla}\big)\, q = 0\,, \qquad \big(\partial_t + \boldsymbol{u}^{\mathrm{L}}\boldsymbol{\cdot\nabla}\big)\, b + w^{\mathrm{L}} N^2 = 0\,. \tag{96}$$

where

$$q \overset{\text{def}}{=} \underbrace{fN^2}_{O(1)} + \underbrace{N^2\hat{\omega} + fb_z + v_z^{\mathrm{S}} b_x - u_z^{\mathrm{S}} b_y}_{O(\mathscr{R})} + \underbrace{\hat{\boldsymbol{\omega}}\boldsymbol{\cdot\nabla} b}_{O(\mathscr{R}^2)}\ . \tag{97}$$

To leading-order in Rossby number, the momentum equations yield the "imbalance" conditions,

$$fv^{\mathrm{L}} = p_x\,, \tag{98}$$

$$-fu^{\mathrm{L}} = p_y\,, \tag{99}$$

$$b = p_z + v_z^{\mathrm{S}} v^{\mathrm{L}} - u_z^{\mathrm{S}} u^{\mathrm{L}}\,. \tag{100}$$

The balance conditions imply the leading-order velocity has no divergence. Note also that we can define a geostrophic streamfunction in terms of which the entire problem can be formulated. At $O(1)$ the buoyancy equation implies that $w_g^{\mathrm{L}} = 0$ ("g" for geostrophic). At $O(\mathscr{R})$ we get

$$w_a^{\mathrm{L}} = \big(\partial_t + \boldsymbol{u}^{\mathrm{L}}\boldsymbol{\cdot\nabla}\big)\,\frac{b}{N^2}\,, \tag{101}$$

where we can move the N^2 around because $w_g^{\mathrm{L}} = 0$. The leading-order PV equation is therefore

$$\big(\partial_t + \boldsymbol{u}^{\mathrm{L}}\boldsymbol{\cdot\nabla}\big)\,\big(N^2\hat{\omega} + fb_z + v_z^{\mathrm{S}} b_x - u_z^{\mathrm{S}} b_y\big) + w_a^{\mathrm{L}}\,\big(fN^2\big)_z = 0\,. \tag{102}$$

We can then use the buoyancy equation to eliminate w_a^{L}. After dividing by N^2, we have a nice result:

$$Q_t + \mathsf{J}(\psi, Q) = 0\,, \tag{103}$$

with

$$Q \overset{\text{def}}{=} v_x^{\mathrm{L}} - u_y^{\mathrm{L}} + \left(\frac{fb}{N^2}\right)_z + \frac{v_z^{\mathrm{S}} b_x}{N^2} - \frac{u_z^{\mathrm{S}} b_y}{N^2}\,, \tag{104}$$

$$Q_t + \mathsf{J}(\psi, Q) = 0\,, \tag{103}$$

with

$$Q \overset{\text{def}}{=} v_x^{\mathrm{L}} - u_y^{\mathrm{L}} + \left(\frac{fb}{N^2}\right)_z + \frac{v_z^{\mathrm{S}} b_x}{N^2} - \frac{u_z^{\mathrm{S}} b_y}{N^2}\,, \tag{104}$$

We insert these balance conditions into the expression for q to yield an expression solely in terms of ψ. We get

$$\begin{aligned}
Q &\overset{\text{def}}{=} \triangle\psi + \mathsf{L}\psi + \left(\frac{f v_z^{\mathrm{S}} \psi_x}{N^2}\right)_z + \left(\frac{f u_z^{\mathrm{S}} \psi_y}{N^2}\right)_z + \frac{v_z^{\mathrm{S}} b_x}{N^2} - \frac{u_z^{\mathrm{S}} b_y}{N^2} - \omega^{\mathrm{w}}\,, && (28)\\
&= \triangle\psi + \mathsf{L}\psi + \left(\frac{f v_z^{\mathrm{S}} \psi_x}{N^2}\right)_z - \left(\frac{f u_z^{\mathrm{S}} \psi_y}{N^2}\right)_z - \omega^{\mathrm{w}}\\
&\quad + \frac{v_z^{\mathrm{S}}}{N^2}\Big(f\psi_{xz} + (v_z^{\mathrm{S}}\psi_x)_x - (u_z^{\mathrm{S}}\psi_y)_x\Big) && (29)\\
&\quad + \frac{u_z^{\mathrm{S}}}{N^2}\Big(f\psi_{yz} + (v_z^{\mathrm{S}}\psi_x)_y - (u_z^{\mathrm{S}}\psi_y)_y\Big)\\
&= \triangle\psi + \mathsf{L}\psi + \psi_x \mathsf{L} v^{\mathrm{S}} - \psi_y \mathsf{L} u^{\mathrm{S}} - \omega^{\mathrm{w}}\\
&\quad + \frac{v_z^{\mathrm{S}}}{N^2}\Big(2f\psi_{xz} + (v_z^{\mathrm{S}}\psi_x)_x - (u_z^{\mathrm{S}}\psi_y)_x\Big) + \frac{u_z^{\mathrm{S}}}{N^2}\Big((v_z^{\mathrm{S}}\psi_x)_y - (u_z^{\mathrm{S}}\psi_y)_y\Big) && (30)
\end{aligned}$$

Wave-averaged quasigeostrophy

$$(\partial_t + \boldsymbol{u}^{\mathrm{L}} \cdot \boldsymbol{\nabla})\, Q = 0\,, \qquad \text{where} \qquad Q \overset{\mathrm{def}}{=} v^{\mathrm{L}}_x + \left(\frac{fb}{N^2}\right)_z + \frac{v^{\mathrm{S}}_z b_x}{N^2}\,.$$

Initial waveless state

$$Q_0 = v_{0x} + \left(\frac{fb_0}{N^2}\right)_z$$

prescribed, with

$$f v_0 = p_{0x}$$

$$b_0 = p_{0z}$$

Final wavy state

$$Q_1 = v^{\mathrm{L}}_{1x} + \left(\frac{fb_1}{N^2}\right)_z + \frac{v^{\mathrm{S}}_z b_{1x}}{N^2}\,.$$

unknown!

$$f v^{\mathrm{L}}_1 = p_{1x}\,,$$

$$b_1 = p_{1z} + f^{-1} v^{\mathrm{S}}_z p_{1x}\,.$$

$$\boxed{Q_0 = Q_1}$$

Wave-averaged quasigeostrophy

$$\left(\partial_t + \boldsymbol{u}^{\mathrm{L}} \cdot \boldsymbol{\nabla}\right) Q = 0\,, \qquad \text{where} \qquad Q \stackrel{\text{def}}{=} v_x^{\mathrm{L}} + \left(\frac{fb}{N^2}\right)_z + \frac{v_z^{\mathrm{S}} b_x}{N^2}\,.$$

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unknown!

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$$b_1 = p_{1z} + f^{-1} v_z^{\mathrm{S}} p_{1x}\,.$$