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ON THE THEORY OF OSCILLATORY WAVES.

[Read March 1, 1847.]

IN the Report of the Fourteenth Meeting of the British Association for the Advancement of Science it is stated by Mr Russell, as a result of his experiments, that the velocity of propagation of a series of oscillatory waves does not depend on the height of the waves*. A series of oscillatory waves, such as that observed by Mr Russell, does not exactly agree with what it is most convenient, as regards theory, to take as the type of oscillatory waves. The extreme waves of such a series partake in some measure of the character of solitary waves, and their height decreases as they proceed. In fact it will presently appear that it is only an indefinite series of waves which possesses the property of being propagated with a uniform velocity, and without change of form: at least this is the case when the waves are such as can be propagated along the surface of a fluid which was previously at rest. The middle waves, however, of a series such as that observed by Mr Russell agree very nearly with oscillatory waves of the standard form. Consequently, the velocity of propagation determined by the observation of a number of waves, according to Mr Russell's method, must be very nearly the same as the velocity of propagation of a series of oscillatory waves of the standard form, and whose length is equal to the mean length of the waves observed, which are supposed to differ from each other but slightly in length.

* Page 369 (note), and page 370.

On this account I was induced to investigate the motion of oscillatory waves of the above form to a second approximation, that is, supposing the height of the waves finite, though small. I find that the expression for the velocity of propagation is independent of the height of the waves to a second approximation. With respect to the form of the waves, the elevations are no longer similar to the depressions, as is the case to a first approximation, but the elevations are narrower than the hollows, and the height of the former exceeds the depth of the latter. This is in accordance with Mr Russell's remarks at page 448 of his first Report*. I have proceeded to a third approximation in the particular case in which the depth of the fluid is very great, so as to find in this case the most important term, depending on the height of the waves, in the expression for the velocity of propagation. This term gives an increase in the velocity of propagation depending on the square of the ratio of the height of the waves to their length.

There is one result of a second approximation which may possibly be of practical importance. It appears that the forward motion of the particles is not altogether compensated by their backward motion; so that, in addition to their motion of oscillation, the particles have a progressive motion in the direction of propagation of the waves. In the case in which the depth of the fluid is very great, this progressive motion decreases rapidly as the depth of the particle considered increases. Now when a ship at sea is overtaken by a storm, and the sky remains overcast, so as to prevent astronomical observations, there is nothing to trust to for finding the ship's place but the dead reckoning. But the estimated velocity and direction of motion of the ship are her velocity and direction of motion relatively to the water. If then the whole of the water near the surface be moving in the direction of the waves, it is evident that the ship's estimated place will be erroneous. If, however, the velocity of the water can be expressed in terms of the length and height of the waves, both which can be observed approximately from the ship, the motion of the water can be allowed for in the dead reckoning.

As connected with this subject, I have also considered the motion of oscillatory waves propagated along the common surface of two liquids, of which one rests on the other, or along the upper

* *Reports of the British Association*, Vol. VI.

surface of the upper liquid. In this investigation there is no object in going beyond a first approximation. When the specific gravities of the two fluids are nearly equal, the waves at their common surface are propagated so slowly that there is time to observe the motions of the individual particles. The second case affords a means of comparing with theory the velocity of propagation of oscillatory waves in extremely shallow water. For by pouring a little water on the top of the mercury in a trough we can easily procure a sheet of water of a small, and strictly uniform depth, a depth, too, which can be measured with great accuracy by means of the area of the surface and the quantity of water poured in. Of course, the common formula for the velocity of propagation will not apply to this case, since the motion of the mercury must be taken into account.

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1. In the investigations which immediately follow, the fluid is supposed to be homogeneous and incompressible, and its depth uniform. The inertia of the air, and the pressure due to a column of air whose height is comparable with that of the waves are also neglected, so that the pressure at the upper surface of the fluid may be supposed to be zero, provided we afterwards add the atmospheric pressure to the pressure so determined. The waves which it is proposed to investigate are those for which the motion is in two dimensions, and which are propagated with a constant velocity, and without change of form. It will also be supposed that the waves are such as admit of being excited, independently of friction, in a fluid which was previously at rest. It is by these characters of the waves that the problem will be rendered determinate, and not by the initial disturbance of the fluid, supposed to be given. The common theory of fluid motion, in which the pressure is supposed equal in all directions, will also be employed.

Let the fluid be referred to the rectangular axes of x, y, z, the plane xz being horizontal, and coinciding with the surface of the fluid when in equilibrium, the axis of y being directed downwards, and that of x taken in the direction of propagation of the waves, so that the expressions for the pressure, &c. do not contain z. Let p be the pressure, ρ the density, t the time, u, v the resolved parts of the velocity in the directions of the axes of x, y; g the force of gravity, h the depth of the fluid when in equilibrium. From the character of the waves which was mentioned last, it follows by a known theorem that $udx + vdy$ is an exact differential $d\phi$. The equations by which the motion is to be determined are well known. They are

$$p = g\rho y - \rho\frac{d\phi}{dt} - \frac{\rho}{2}\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right\} \text{ ....(1);}$$

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0 \text{......................(2);}$$

$$\frac{d\phi}{dy} = 0, \text{ when } y = h \text{..................(3);}$$

$$\frac{dp}{dt} + \frac{d\phi}{dx}\frac{dp}{dx} + \frac{d\phi}{dy}\frac{dp}{dy} = 0, \text{ when } p = 0 \text{ ....(4);}$$

where (3) expresses the condition that the particles in contact with the rigid plane on which the fluid rests remain in contact with it, and (4) expresses the condition that the same surface of particles continues to be the free surface throughout the motion, or, in other words, that there is no generation or destruction of fluid at the free surface.

If c be the velocity of propagation, u, v and p will be by hypothesis functions of $x - ct$ and y. It follows then from the equations $u = d\phi/dx$, $v = d\phi/dy$ and (1), that the differential coefficients of ϕ with respect to x, y and t will be functions of $x - ct$ and y; and therefore ϕ itself must be of the form

$$f(x - ct,\ y) + Ct.$$

The last term will introduce a constant into (1); and if this constant be expressed, we may suppose ϕ to be a function of $x - ct$ and y. Denoting $x - ct$ by x', we have

defn of x'

$$\frac{dp}{dx} = \frac{dp}{dx'}, \quad \frac{dp}{dt} = -c\frac{dp}{dx'},$$

and similar equations hold good for ϕ. On making these substitutions in (1) and (4), omitting the accent of x, and writing $-gk$ for C, we have

$$p = g\rho\,(y + k) + c\rho\frac{d\phi}{dx} - \frac{\rho}{2}\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right\} \text{ ......(5),}$$

$$\left(\frac{d\phi}{dx} - c\right)\frac{dp}{dx} + \frac{d\phi}{dy}\frac{dp}{dy} = 0, \text{ when } p = 0 \text{.........(6).}$$

Substituting in (6) the value of p given by (5), we have

$$g\frac{d\phi}{dy} - c^2\frac{d^2\phi}{dx^2} + 2c\left(\frac{d\phi}{dx}\frac{d^2\phi}{dx^2} + \frac{d\phi}{dy}\frac{d^2\phi}{dxdy}\right)$$

$$- \left(\frac{d\phi}{dx}\right)^2\frac{d^2\phi}{dx^2} - 2\frac{d\phi}{dx}\frac{d\phi}{dy}\frac{d^2\phi}{dxdy} - \left(\frac{d\phi}{dy}\right)^2\frac{d^2\phi}{dy^2} = 0 \ldots (7),$$

when $$g(y+k) + c\frac{d\phi}{dx} - \tfrac{1}{2}\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right\} = 0 \ldots\ldots\ldots (8).$$

The equations (7) and (8) are exact; but if we suppose the motion small, and proceed to the second order only of approximation, we may neglect the last three terms in (7), and we may easily eliminate y between (7) and (8). For putting ϕ', $\phi_{,}$, &c. for the values of $d\phi/dx$, $d\phi/dy$, &c. when $y = 0$, the number of accents above marking the order of the differential coefficient with respect to x, and the number below its order with respect to y, and observing that k is a small quantity of the first order at least, we have from (8)

Def of accents for derivatives

$$g(y+k) + c(\phi' + \phi_{,}'y) - \tfrac{1}{2}(\phi'^2 + \phi_{,}^2) = 0,$$

whence $$y = -k - \frac{c}{g}\phi' + \frac{c}{g}\phi_{,}'\left(k + \frac{c}{g}\phi'\right) + \frac{1}{2g}(\phi'^2 + \phi_{,}^2)^* \ldots (9).$$

Substituting the first approximate value of y in the first two terms of (7), putting $y = 0$ in the next two, and reducing, we have

$$g\phi_{,} - c^2\phi'' - (g\phi_{,,} - c^2\phi_{,}'')\left(k + \frac{c}{g}\phi'\right) + 2c(\phi'\phi'' + \phi_{,}\phi_{,}') = 0 \ldots (10).$$

ϕ will now have to be determined from the general equation (2) with the particular conditions (3) and (10). When ϕ is known, y, the ordinate of the surface, will be got from (9), and k will then be determined by the condition that the mean value of y shall be zero. The value of p, if required, may then be obtained from (5).

2. In proceeding to a first approximation we have the equations (2), (3) and the equation obtained by omitting the small terms in (10), namely,

$$g\frac{d\phi}{dy} - c^2\frac{d^2\phi}{dx^2} = 0, \text{ when } y = 0 \ldots\ldots\ldots\ldots (11).$$

* The reader will observe that the y in this equation is the ordinate of the surface, whereas the y in (1) and (2) is the ordinate of any point in the fluid. The context will always shew in which sense y is employed.

The general integral of (2) is

$$\phi = \Sigma A \epsilon^{mx+ny},$$

the sign Σ extending to all values of A, m and n, real or imaginary, for which $m^2 + n^2 = 0$: the particular values of ϕ, $Cx + C'$, $Dy + D'$, corresponding respectively to $n = 0$, $m = 0$, must also be included, but the constants C', D' may be omitted. In the present case, the expression for ϕ must not contain real exponentials in x, since a term containing such an exponential would become infinite either for $x = -\infty$, or for $x = +\infty$, as well as its differential coefficients which would appear in the expressions for u and v; so that m must be wholly imaginary. Replacing then the exponentials in x by circular functions, we shall have for the part of ϕ corresponding to any one value of m,

$$(A\epsilon^{my} + A'\epsilon^{-my}) \sin mx + (B\epsilon^{my} + B'\epsilon^{-my}) \cos mx,$$

and the complete value of ϕ will be found by taking the sum of all possible particular values of the above form and of the particular value $Cx + Dy$. When the value so formed is substituted in (3), which has to hold good for all values of x, the coefficients of the several sines and cosines, and the constant term must be separately equated to zero. We have therefore

$$D = 0,\ A' = \epsilon^{2mh} A,\ B' = \epsilon^{2mh} B;$$

so that if we change the constants we shall have

$$\phi = Cx + \Sigma\, (\epsilon^{m(h-y)} + \epsilon^{-m(h-y)}) (A \sin mx + B \cos mx) \ldots (12),$$

the sign Σ extending to all real values of m, A and B, of which m may be supposed positive.

3. To the term Cx in (12) corresponds a uniform velocity parallel to x, which may be supposed to be impressed on the fluid in addition to its other motions. If the velocity of propagation be defined merely as the velocity with which the wave form is propagated, it is evident that the velocity of propagation is perfectly arbitrary. For, for a given state of relative motion of the parts of the fluid, the velocity of propagation, as so defined, can be altered by altering the value of C. And in proceeding to the higher orders of approximation it becomes a question what we shall define the velocity of propagation to be. Thus, we might define it to be the velocity with which the wave form is propa-

gated when the mean horizontal velocity of a particle in the upper surface is zero, or the velocity of propagation of the wave form when the mean horizontal velocity of a particle at the bottom is zero, or in various other ways. The following two definitions appear chiefly to deserve attention.

First, we may define the velocity of propagation to be the velocity with which the wave form is propagated in space, when the mean horizontal velocity *at each point of space occupied by the fluid* is zero. The term "mean" here refers to the variation of the time. This is the definition which it will be most convenient to employ in the investigation. I shall accordingly suppose $C=0$ in (12), and c will represent the velocity of propagation according to the above definition.

Secondly, we may define the velocity of propagation to be the velocity of propagation of the wave form in space, when the mean horizontal velocity of the mass of fluid comprised between two very distant planes perpendicular to the axis of x is zero. The mean horizontal velocity of the mass means here the same thing as the horizontal velocity of its centre of gravity. This appears to be the most natural definition of the velocity of propagation, since in the case considered there is no current in the mass of fluid, taken as a whole. I shall denote the velocity of propagation according to this definition by c'. In the most important case to consider, namely, that in which the depth is infinite, it is easy to see that $c'=c$, whatever be the order of approximation. For when the depth becomes infinite, the velocity of the centre of gravity of the mass comprised between any two planes parallel to the plane yz vanishes, provided the expression for u contain no constant term.

4. We must now substitute in (11) the value of ϕ.

$$\phi = \Sigma \left(\epsilon^{m(h-y)} + \epsilon^{-m(h-y)}\right) (A \sin mx + B \cos mx) \ldots (13);$$

but since (11) has to hold good for all values of x, the coefficients of the several sines and cosines must be separately equal to zero: at least this must be true, provided the series contained in (11) are convergent. The coefficients will vanish for any one value of m, provided

$$c^2 = \frac{g}{m} \frac{\epsilon^{mh} - \epsilon^{-mh}}{\epsilon^{mh} + \epsilon^{-mh}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (14).$$

Putting for shortness $2mh = \mu$, we have

$$\frac{d \log c^2}{d\mu} = -\frac{1}{\mu} + \frac{2}{\epsilon^{\mu} - \epsilon^{-\mu}},$$

which is positive or negative, μ being supposed positive, according as

$$2\mu >< \epsilon^{\mu} - \epsilon^{-\mu} >< 2\left(\mu + \frac{\mu^3}{1.2.3} + \ldots\ldots\right),$$

and is therefore necessarily negative. Hence the value of c given by (14) decreases as μ or m increases, and therefore (11) cannot be satisfied, for a given value of c, by more than one positive value of m. Hence the expression for ϕ must contain only one value of m. Either of the terms $A \cos mx$, $B \sin mx$ may be got rid of by altering the origin of x. We may therefore take, for the most general value of ϕ,

$$\phi = A\left(\epsilon^{m(h-y)} + \epsilon^{-m(h-y)}\right) \sin mx \ldots\ldots\ldots\ldots (15).$$

Substituting in (8), we have for the ordinate of the surface

$$y = -\frac{mAc}{g}\left(\epsilon^{mh} + \epsilon^{-mh}\right) \cos mx \ldots\ldots\ldots\ldots (16),$$

k being $=0$, since the mean value of y must be zero. Thus everything is known in the result except A and m, which are arbitrary.

5. It appears from the above, that of all waves for which the motion is in two dimensions, which are propagated in a fluid of uniform depth, and which are such as could be propagated into fluid previously at rest, so that $udx + vdy$ is an exact differential, there is only one particular kind, namely, that just considered, which possesses the property of being propagated with a constant velocity, and without change of form; so that a solitary wave cannot be propagated in this manner. Thus the degradation in the height of such waves, which Mr Russell observed, is not to be attributed wholly, (nor I believe chiefly,) to the imperfect fluidity of the fluid, and its adhesion to the sides and bottom of the canal, but is an essential characteristic of a solitary wave. It is true that this conclusion depends on an investigation which applies strictly to indefinitely small motions only: but if it were true in general that a solitary wave could be propagated uniformly, without degradation, it would be true in

the limiting case of indefinitely small motions; and to disprove a general proposition it is sufficient to disprove a particular case.

6. In proceeding to a second approximation we must substitute the first approximate value of ϕ, given by (15), in the small terms of (10). Observing that $k=0$ to a first approximation, and eliminating g from the small terms by means of (14), we find

$$g\phi_{,} - c^2\phi'' - 6A^2m^3c \sin 2mx = 0 \text{ ....(17).}$$

The general value of ϕ given by (13), which is derived from (2) and (3), must now be restricted to satisfy (17). It is evident that no new terms in ϕ involving $\sin mx$ or $\cos mx$ need be introduced, since such terms may be included in the first approximate value, and the only other term which can enter is one of the form

$$B\left(\epsilon^{2m(h-y)} + \epsilon^{-2m(h-y)}\right)\sin 2mx.$$

Substituting this term in (17), and simplifying by means of (14), we find

$$B = \frac{3mA^2}{c\left(\epsilon^{mh} - \epsilon^{-mh}\right)^2}.$$

Moreover since the term in ϕ containing $\sin mx$ must disappear from (17), the equation (14) will give c to a second approximation.

If we denote the coefficient of $\cos mx$ in the first approximate value of y, the ordinate of the surface, by a, we shall have

$$A = -\frac{ga}{mc\left(\epsilon^{mh} + \epsilon^{-mh}\right)} = -\frac{ca}{\left(\epsilon^{mh} - \epsilon^{-mh}\right)};$$

and substituting this value of A in that of ϕ, we have

$$\phi = -ac\,\frac{\epsilon^{m(h-y)} + \epsilon^{-m(h-y)}}{\epsilon^{mh} - \epsilon^{-mh}}\sin mx + 3ma^2c\,\frac{\epsilon^{2m(h-y)} + \epsilon^{-2m(h-y)}}{\left(\epsilon^{mh} - \epsilon^{-mh}\right)^4}\sin 2mx \text{ ...... (18).}$$

The ordinate of the surface is given to a second approximation by (9). It will be found that

$$y = a\cos mx - ma^2\,\frac{\left(\epsilon^{mh} + \epsilon^{-mh}\right)\left(\epsilon^{2mh} + \epsilon^{-2mh} + 4\right)}{2\left(\epsilon^{mh} - \epsilon^{-mh}\right)^3}\cos 2mx \text{ ...(19),}$$

$$k = \frac{ma^2}{\epsilon^{2mh} - \epsilon^{-2mh}}.$$

7. The equation to the surface is of the form

$$y = a\cos mx - Ka^2\cos 2mx \text{ ....(20),}$$

where K is necessarily positive, and a may be supposed to be positive, since the case in which it is negative may be reduced to that in which it is positive by altering the origin of x by the quantity π/m or $\lambda/2$, λ being the length of the waves. On referring to (20) we see that the waves are symmetrical with respect to vertical planes drawn through their ridges, and also with respect to vertical planes drawn through their lowest lines. The greatest depression of the fluid occurs when $x=0$ or $=\pm\lambda$, &c., and is equal to $a-a^2K$: the greatest elevation occurs when $x=\pm\lambda/2$ or $=\pm 3\lambda/2$, &c., and is equal to $a+a^2K$. Thus the greatest elevation exceeds the greatest depression by $2a^2K$. When the surface cuts the plane of mean level, $\cos mx - aK\cos 2mx = 0$. Putting in the small term in this equation the approximate value $mx=\pi/2$, we have $\cos mx = -aK = \cos(\pi/2 + aK)$, whence

$$x = \pm(\lambda/4 + aK\lambda/2\pi),\ = \pm(5\lambda/4 + aK\lambda/2\pi),\ \&c.$$

We see then that the breadth of each hollow, measured at the height of the plane of mean level, is $\lambda/2 + aK\lambda/\pi$, while the breadth of each elevated portion of the fluid is $\lambda/2 - aK\lambda/\pi$.

It is easy to prove from the expression for K, which is given in (19), that for a given value of λ or of m, K increases as h decreases. Hence the difference in form of the elevated and depressed portions of the fluid is more conspicuous in the case in which the fluid is moderately shallow than in the case in which its depth is very great compared with the length of the waves.

8. When the depth of the fluid is very great compared with the length of a wave, we may without sensible error suppose h to be infinite. This supposition greatly simplifies the expressions already obtained. We have in this case

$$\phi = -ac\epsilon^{-my}\sin mx \quad\text{.......................}(21),$$

$$y = a\cos mx - \tfrac{1}{2}ma^2\cos 2mx \quad\text{.......................}(22),$$

$$k=0,\quad K=\frac{m}{2}=\frac{\pi}{\lambda},\quad c^2=\frac{g\lambda}{2\pi},$$

the y in (22) being the ordinate of the surface.

It is hardly necessary to remark that the state of the fluid at any time will be expressed by merely writing $x-ct$ in place of x in all the preceding expressions.

9. To find the nature of the motion of the individual particles, let $x+\xi$ be written for x, $y+\eta$ for y, and suppose x and y to be independent of t, so that they alter only in passing from one particle to another, while ξ and η are small quantities depending on the motion. Then taking the case in which the depth is infinite, we have

$$\frac{d\xi}{dt} = u = -mac\epsilon^{-m(y+\eta)}\cos m(x+\xi-ct) = -mac\epsilon^{-my}\cos m(x-ct)$$

$$+ m^2ac\epsilon^{-my}\sin m(x-ct)\,.\,\xi + m^2ac\epsilon^{-my}\cos m(x-ct)\,.\,\eta, \text{ nearly,}$$

$$\frac{d\eta}{dt} = v = mac\epsilon^{-m(y+\eta)}\sin m(x+\xi-ct) = mac\epsilon^{-my}\sin m(x-ct)$$

$$+ m^2ac\epsilon^{-my}\cos m(x-ct)\,.\,\xi - m^2ac\epsilon^{-my}\sin m(x-ct)\,.\,\eta, \text{ nearly.}$$

To a first approximation

$$\xi = a\epsilon^{-my}\sin m(x-ct), \qquad \eta = a\epsilon^{-my}\cos m(x-ct),$$

the arbitrary constants being omitted. Substituting these values in the small terms of the preceding equations, and integrating again, we have

$$\xi = a\epsilon^{-my}\sin m(x-ct) + m^2a^2ct\epsilon^{-2my},$$

$$\eta = a\epsilon^{-my}\cos m(x-ct).$$

Hence the motion of the particles is the same as to a first approximation, with one important difference, which is that in addition to the motion of oscillation the particles are transferred forwards, that is, in the direction of propagation, with a constant velocity depending on the depth, and decreasing rapidly as the depth increases. If U be this velocity for a particle whose depth below the surface in equilibrium is y, we have

$$U = m^2a^2c\epsilon^{-2my} = a^2\left(\frac{2\pi}{\lambda}\right)^{\frac{3}{2}} g^{\frac{1}{2}}\epsilon^{-\frac{4\pi y}{\lambda}} \text{.....................} (23).$$

The motion of the individual particles may be determined in a similar manner when the depth is finite from (18). In this case the values of ξ and η contain terms of the second order, involving respectively $\sin 2m(x-ct)$ and $\cos 2m(x-ct)$, besides the term in ξ which is multiplied by t. The most important thing to consider is the value of U, which is

$$U = m^2a^2c\,\frac{\epsilon^{2m(y-h)} + \epsilon^{-2m(y-h)}}{(\epsilon^{mh} - \epsilon^{-mh})^2} \text{.......................} (24).$$

Since U is a small quantity of the order a^2, and in proceeding to a second approximation the velocity of propagation is given to the order a only, it is immaterial which of the definitions of velocity of propagation mentioned in Art. 3 we please to adopt.

10. The waves produced by the action of the wind on the surface of the sea do not probably differ very widely from those which have just been considered, and which may be regarded as the typical form of oscillatory waves. On this supposition the particles, in addition to their motion of oscillation, will have a progressive motion in the direction of propagation of the waves, and consequently in the direction of the wind, supposing it not to have recently shifted, and this progressive motion will decrease rapidly as the depth of the particle considered increases. If the pressure of the air on the posterior parts of the waves is greater than on the anterior parts, in consequence of the wind, as unquestionably it must be, it is easy to see that some such progressive motion must be produced. If then the waves are not breaking, it is probable that equation (23), which is applicable to deep water, may give approximately the mean horizontal velocity of the particles; but it is difficult to say how far the result may be modified by friction. If then we regard the ship as a mere particle, in the first instance, for the sake of simplicity, and put U_0 for the value of U when $y=0$, it is easy to see that after sailing for a time t, the ship must be a distance $U_0 t$ to the lee of her estimated place. It will not however be sufficient to regard the ship as a mere particle, on account of the variation of the factor ϵ^{-2my}, as y varies from 0 to the greatest depth of the ship below the surface of the water. Let δ be this depth, or rather a depth something less, in order to allow for the narrowing of the ship towards the keel, and suppose the effect of the progressive motion of the water on the motion of the ship to be the same as if the water were moving with a velocity the same as all depths, and equal to the mean value of the velocity U from $y=0$ to $y=\delta$. If U_1 be this mean velocity,

$$U_1 = \frac{1}{\delta}\int_0^\delta U dy = \frac{ma^2c}{2\delta}\left(1-\epsilon^{-2m\delta}\right).$$

On this supposition, if a ship be steered so as to sail in a direction making an angle θ with the direction of the wind, supposing the water to have no current, and if V be the velocity with which the ship moves through the water, her actual velocity will be the resultant of a velocity V in the direction just mentioned, which, for shortness, I shall call the direction of steering, and of a velocity U_1 in the direction of the wind. But the ship's velocity as estimated by the log-line is her velocity relatively to the water at the surface, and is therefore the resultant of a velocity V in the direction of steering, and a velocity $U_0 - U_1$ in a direction opposite to that in which the wind is blowing. If then E be the estimated velocity, and if we neglect U^2,

$$E = V - (U_0 - U_1) \cos \theta.$$

But the ship's velocity is really the resultant of a velocity $V + U_1 \cos \theta$ in the direction of steering, and a velocity $U_1 \sin \theta$ in the perpendicular direction, while her estimated velocity is E in the direction of steering. Hence, after a time t, the ship will be a distance $U_0 t \cos \theta$ ahead of her estimated place, and a distance $U_1 t \sin \theta$ aside of it, the latter distance being measured in a direction perpendicular to the direction of steering, and on the side towards which the wind is blowing.

I do not suppose that the preceding formula can be employed in practice; but I think it may not be altogether useless to call attention to the importance of having regard to the magnitude and direction of propagation of the waves, as well as to the wind, in making the allowance for lee-way.

11. The formulæ of Art. 6 are perfectly general as regards the ratio of the length of the waves to the depth of the fluid, the only restriction being that the height of the waves must be sufficiently small to allow the series to be rapidly convergent. Consequently, they must apply to the limiting case, in which the waves are supposed to be extremely long. Hence long waves, of the kind considered, are propagated without change of form, and the velocity of propagation is independent of the height of the waves to a second approximation. These conclusions might seem, at first sight, at variance with the results obtained by Mr Airy for the case of long waves*. On proceeding to a second approximation, Mr Airy finds that the form of long waves alters as they proceed, and that the expression for the velocity of propagation contains a

* *Encyclopædia Metropolitana, Tides and Waves*, Articles 198, &c.

term depending on the height of the waves. But a little attention will remove this apparent discrepancy. If we suppose mh very small in (19), and expand, retaining only the most important terms, we shall find for the equation to the surface

$$y = a \cos mx - \frac{3a^2}{4m^2h^3} \cos 2mx.$$

Now, in order that the method of approximation adopted may be legitimate, it is necessary that the coefficient of $\cos 2mx$ in this equation be small compared with a. Hence a/m^2h^3, and therefore $\lambda^2 a/h^3$, must be small, and therefore a/h must be small compared with $(h/\lambda)^2$. But the investigation of Mr Airy is applicable to the case in which λ/h is very large; so that in that investigation a/h is large compared with $(h/\lambda)^2$. Thus the difference in the results obtained corresponds to a difference in the physical circumstances of the motion.

12. There is no difficulty in proceeding to the higher orders of approximation, except what arises from the length of the formulæ. In the particular case in which the depth is considered infinite, the formulæ are very much simpler than in the general case. I shall proceed to the third order in the case of an infinite depth, so as to find in that case the most important term, depending on the height of the waves, in the expression for the velocity of propagation. Third order

For this purpose it will be necessary to retain the terms of the third order in the expansion of (7). Expanding this equation according to powers of y, and neglecting terms of the fourth, &c. orders, we have

$$g\phi_{,} - c^2\phi'' + (g\phi_{,,} - c^2\phi_{,}'')\,y + (g\phi_{,,,} - c^2\phi_{,,}'')\,\frac{y^2}{2} + 2c(\phi'\phi'' + \phi_{,}\phi_{,}')$$

$$+ 2c(\phi_{,}'\phi'' + \phi'\phi_{,}'' + \phi_{,,}\phi_{,}' + \phi_{,}\phi_{,,}')\,y - \phi'^2\phi'' - 2\phi'\phi_{,}\phi_{,}' - \phi_{,}^2\phi_{,,} = 0 \qquad \text{............(25).}$$

In the small terms of this equation we must put for ϕ and y their values given by (21) and (22) respectively. Now since the value of ϕ to a second approximation is the same as its value to a first approximation, the equation $g\phi_{,} - c^2\phi'' = 0$ is satisfied to terms of the second order. But the coefficients of y and $y^2/2$, in the first line of (25), are derived from the left-hand member of the preceding equation by inserting the factor ϵ^{-my}, differentiating either once or twice with respect to y, and then putting $y=0$. Consequently these coefficients contain no terms of the second order, and therefore the terms involving y in the first line of (25) are to be neglected. The next two terms are together equal to $cd(\phi'^2+\phi_{,}^2)/dx$. But

$$\phi'^2+\phi_{,}^2=m^2a^2c^2,$$

which does not contain x, so that these two terms disappear. The coefficient of y in the second line of (25) may be derived from the two terms last considered in the manner already indicated, and therefore the terms containing y will disappear from (25). The only small terms remaining are the last three, and it will easily be found that their sum is equal to $m^4a^3c^3\sin mx$, so that (25) becomes

$$g\phi_{,}-c^2\phi''+m^4a^3c^3\sin mx=0\ldots\ldots\ldots\ldots(26).$$

The value of ϕ will evidently be of the form $A\epsilon^{-my}\sin mx$. Substituting this value in (26), we have

$$(m^2c^2-mg)A+m^4a^3c^3=0.$$

Dividing by mA, and putting for A and c^2 their approximate values $-ac$, g/m respectively in the small term, we have

$$mc^2=g+m^2a^2g,$$

whence $$c=\left(\frac{g}{m}\right)^{\frac{1}{2}}(1+\tfrac{1}{2}m^2a^2)=\left(\frac{g\lambda}{2\pi}\right)^{\frac{1}{2}}\left(1+\frac{2\pi^2a^2}{\lambda^2}\right).$$

The equation to the surface may be found without difficulty. It is

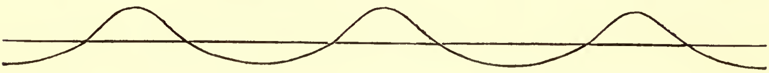
$$y=a\cos mx-\tfrac{1}{2}ma^2\cos 2mx+\tfrac{3}{8}m^2a^3\cos 3mx^{*}\ldots\ldots(27):$$

we have also

$$k=0,\ \phi=-ac(1-\tfrac{5}{8}m^2a^2)\,\epsilon^{-my}\sin mx.$$

* It is remarkable that this equation coincides with that of the prolate cycloid, if the latter equation be expanded according to ascending powers of the distance of the tracing point from the centre of the rolling circle, and the terms of the fourth order be omitted. The prolate cycloid is the form assigned by Mr Russell to waves of the kind here considered. *Reports of the British Association*, Vol. VI. p. 448. When the depth of the fluid is not great compared with the length of a wave, the form of the surface does not agree with the prolate cycloid even to a second approximation.

The following figure represents a vertical section of the waves propagated along the surface of deep water. The figure is drawn



for the case in which $a = \frac{7\lambda}{80}$. The term of the third order in (27) is retained, but it is almost insensible. The straight line represents a section of the plane of mean level.

13. If we consider the manner in which the terms introduced by each successive approximation enter into equations (7) and (8), we shall see that, whatever be the order of approximation, the series expressing the ordinate of the surface will contain only cosines of mx and its multiples, while the expression for ϕ will contain only sines. The manner in which y enters into the coefficient of $\cos rmx$ in the expression for ϕ is determined in the case of a finite depth by equations (2) and (3). Moreover, the principal part of the coefficient of $\cos rmx$ or $\sin rmx$ will be of the order a^r at least. We may therefore assume

$$\phi = \Sigma_1^{\infty} a^r A_r (\epsilon^{rm(h-y)} + \epsilon^{-rm(h-y)}) \sin rmx,$$
$$y = a \cos mx + \Sigma_2^{\infty} a^r B_r \cos rmx,$$

and determine the arbitrary coefficients by means of equations (7) and (8), having previously expanded these equations according to ascending powers of y. The value of c^2 will be determined by equating to zero the coefficient of $\sin mx$ in (7).

Since changing the sign of a comes to the same thing as altering the origin of x by $\frac{1}{2}\lambda$, it is plain that the expressions for A_r, B_r and c^2 will contain only even powers of a. Thus the values of each of these quantities will be of the form

$$C_0 + C_1 a^2 + C_2 a^4 + \ldots$$

It appears also that, whatever be the order of approximation, the waves will be symmetrical with respect to vertical planes passing through their ridges, as also with respect to vertical planes passing through their lowest lines.

14. Let us consider now the case of waves propagated at the common surface of two liquids, of which one rests on the

other. Suppose as before that the motion is in two dimensions, that the fluids extend indefinitely in all horizontal directions, or else that they are bounded by two vertical planes parallel to the direction of propagation of the waves, that the waves are propagated with a constant velocity, and without change of form, and that they are such as can be propagated into, or excited in, the fluids supposed to have been previously at rest. Suppose first that the fluids are bounded by two horizontal rigid planes. Then taking the common surface of the fluids when at rest for the plane xz, and employing the same notation as before, we have for the under fluid

$$\frac{d^2\phi}{d^2x}+\frac{d^2\phi}{dy^2}=0 \text{.....................} (28),$$

$$\frac{d\phi}{dy}=0 \text{ when } y=h \text{.................} (29),$$

$$p=C+g\rho y+c\rho\frac{d\phi}{dx},$$

neglecting the squares of small quantities. Let $h_{,}$ be the depth of the upper fluid when in equilibrium, and let $p_{,}$, $\rho_{,}$, $\phi_{,}$, $C_{,}$ be the quantities referring to the upper fluid which correspond to p, ρ, ϕ, C referring to the under: then we have for the upper fluid

$$\frac{d^2\phi_{,}}{dx^2}+\frac{d^2\phi_{,}}{dy^2}=0 \text{........................} (30),$$

$$\frac{d\phi_{,}}{dy}=0 \text{ when } y=-h_{,} \text{................} (31),$$

$$p_{,}=C_{,}+g\rho_{,}y+c\rho_{,}\frac{d\phi_{,}}{dx}.$$

We have also, for the condition that the two fluids shall not penetrate into, nor separate from each other,

$$\frac{d\phi}{dy}=\frac{d\phi_{,}}{dy}, \text{ when } y=0 \text{..............} (32).$$

Lastly, the condition answering to (11) is

$$g\left(\rho\frac{d\phi}{dy}-\rho_{,}\frac{d\phi_{,}}{dy}\right)-c^2\left(\rho\frac{d^2\phi}{dx^2}-\rho_{,}\frac{d^2\phi_{,}}{dx^2}\right)=0 \text{........} (33),$$

when $$C-C_{,}+g(\rho-\rho_{,})y+c\left(\rho\frac{d\phi}{dx}-\rho_{,}\frac{d\phi_{,}}{dx}\right)=0 \text{......} (34).$$

Since $C - C'$ is evidently a small quantity of the first order at least, the condition is that (33) shall be satisfied when $y = 0$. Equation (34) will then give the ordinate of the common surface of the two liquids when y is put $= 0$ in the last two terms.

The general value of ϕ suitable to the present case, which is derived from (28) subject to the condition (29), is given by (13) if we suppose that the fluid is free from a uniform horizontal motion compounded with the oscillatory motion expressed by (18). Since the equations of the present investigation are linear, in consequence of the omission of the squares of small quantities, it will be sufficient to consider one of the terms in (13). Let then

$$\phi = A\left(\epsilon^{m(h-y)} + \epsilon^{-m(h-y)}\right)\sin mx \dots\dots\dots (35).$$

The general value of $\phi_{,}$ will be derived from (13) by merely writing $-h_{,}$ for h. But in order that (32) may be satisfied, the value of $\phi_{,}$ must reduce itself to a single term of the same form as the second side of (35). We may take then for the value of $\phi_{,}$

$$\phi_{,} = A_{,}\left(\epsilon^{m(h_{,}+y)} + \epsilon^{-m(h_{,}+y)}\right)\sin mx \dots\dots\dots\dots (36).$$

Putting for shortness

$$\epsilon^{mh} + \epsilon^{-mh} = S, \quad \epsilon^{mh} - \epsilon^{-mh} = D,$$

and taking $S_{,}$, $D_{,}$ to denote the quantities derived from S, D by writing $h_{,}$ for h, we have from (32)

$$DA + D_{,}A_{,} = 0 \dots\dots\dots\dots\dots\dots\dots (37),$$

and from (33)

$$\rho\,(gD - mc^2S)\,A + \rho_{,}\,(gD_{,} + mc^2S_{,})\,A_{,} = 0 \dots\dots\dots (38).$$

Eliminating A and $A_{,}$ from (37) and (38), we have

$$c^2 = \frac{g}{m}\,\frac{(\rho - \rho_{,})\,DD_{,}}{\rho SD_{,} + \rho_{,}S_{,}D} \dots\dots\dots\dots\dots\dots (39).$$

The equation to the common surface of the liquids will be obtained from (34). Since the mean value of y is zero, we have in the first place

$$C_{,} = C \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (40).$$

We have then, for the value of y,

$$y = a\cos mx \dots\dots\dots\dots\dots\dots\dots\dots\dots (41),$$

where

$$a = \frac{mc}{g}\,\frac{\rho_, A_, S_, - \rho AS}{\rho - \rho_,} = \frac{DD_,}{c}\,\frac{\rho_, A_, S_, - \rho AS}{\rho SD_, + \rho_, S_, D} \quad\text{.........(42).}$$

Substituting in (35) and (36) the values of A and $A_,$ derived from (37) and (42), we have

$$\phi = -\frac{ac}{D}\left(\epsilon^{m(h-y)} + \epsilon^{-m(h-y)}\right)\sin mx \quad\text{...........(43),}$$

$$\phi_, = \frac{ac}{D_,}\left(\epsilon^{m(h_,+y)} + \epsilon^{-m(h_,+y)}\right)\sin mx \quad\text{............(44).}$$

Equations (39), (40), (41), (43) and (44) contain the solution of the problem. It is evident that C remains arbitrary. The values of p and $p_,$ may be easily found if required.

If we differentiate the logarithm of c^2 with respect to m, and multiply the result by the product of the denominators, which are necessarily positive, we shall find a quantity of the form $P\rho + P_,\rho_,$, where P and $P_,$ do not contain ρ or $\rho_,$. It may be proved in nearly the same manner as in Art. 4, that each of the quantities P, $P_,$ is necessarily negative. Consequently c will decrease as m increases, or will increase with λ. It follows from this that the value of ϕ cannot contain more than two terms, one of the form (35), and the other derived from (35) by replacing $\sin mx$ by $\cos mx$, and changing the constant A: but the latter term may be got rid of by altering the origin of x.

The simplest case to consider is that in which both h and h' are regarded as infinite compared with λ. In this case we have

$$\phi = -ac\epsilon^{-my}\sin mx, \quad \phi_, = ac\epsilon^{my}\sin mx,$$

$$c^2 = \frac{\rho - \rho_,}{\rho + \rho_,}\,\frac{g}{m}, \quad y = a\cos mx,$$

the latter being the equation to the surface.

15. The preceding investigation applies to two incompressible fluids, but the results are applicable to the case of the waves propagated along the surface of a liquid exposed to the air, provided that in considering the effect of the air we neglect terms which, in comparison with those retained, are of the order of the ratio of the length of the waves considered to the length of

a wave of sound of the same period in air. Taking then ρ for the density of the liquid, $\rho_{,}$ for that of the air at the time, and supposing $h_{,}=\infty$, we have

$$c^2 = \frac{g}{m}\frac{(\rho-\rho_{,})\,D}{\rho S+\rho_{,}D} = \frac{gD}{mS}\left\{1-\left(1+\frac{D}{S}\right)\frac{\rho_{,}}{\rho}\right\}, \text{ nearly.}$$

If we had considered the buoyancy only of the air, we should have had to replace g in the formula (14) by $\frac{\rho-\rho_{,}}{\rho}g$. We should have obtained in this manner

$$c^2 = \frac{g}{m}\frac{(\rho-\rho_{,})\,D}{\rho S} = \frac{gD}{mS}\left(1-\frac{\rho_{,}}{\rho}\right).$$

Hence, in order to allow for the inertia of the air, the correction for buoyancy must be increased in the ratio of 1 to $1+D/S$. The whole correction therefore increases as the ratio of the length of a wave to the depth of the fluid decreases. For very long waves the correction is that due to buoyancy alone, while in the case of very short waves the correction for buoyancy is doubled. Even in this case the velocity of propagation is altered by only the fractional part $\rho_{,}/\rho$ of the whole; and as this quantity is much less than the unavoidable errors of observation, the effect of the air in altering the velocity of propagation may be neglected.

16. There is a discontinuity in the density of the fluid mass considered in Art. 14, in passing from one fluid into the other; and it is easy to shew that there is a corresponding discontinuity in the velocity. If we consider two fluid particles in contact with each other, and situated on opposite sides of the surface of junction of the two fluids, we see that the velocities of these particles resolved in a direction normal to that surface are the same; but their velocities resolved in a direction tangential to the surface are different. These velocities are, to the order of approximation employed in the investigation, the values of $d\phi/dx$ and $d\phi_{,}/dx$ when $y=0$. We have then from (43) and (44), for the velocity with which the upper fluid slides along the under,

$$mac\left(\frac{S_{,}}{D_{,}}+\frac{S}{D}\right)\cos mx.$$

17. When the upper surface of the upper fluid is free, the equations by which the problem is to be solved are the same as those of Art. 14, except that the condition (31) is replaced by

$$g\frac{d\phi_{,}}{dy} - c^2\frac{d^2\phi_{,}}{dx^2} = 0, \text{ when } y = -h_{,} \ldots\ldots\ldots (45);$$

and to determine the ordinate of the upper surface, we have

$$C_{,} + g\rho_{,}y + c\rho_{,}\frac{d\phi_{,}}{dx} = 0,$$

where y is to be replaced by $-h_{,}$ in the last term. Let us consider the motion corresponding to the value of ϕ given by (35). We must evidently have

$$\phi_{,} = (A_{,}\epsilon^{my} + B_{,}\epsilon^{-my})\sin mx,$$

where $A_{,}$ and $B_{,}$ have to be determined. The conditions (32), (33) and (45) give

$$DA + A_{,} - B_{,} = 0,$$

$$\rho\,(gD - mc^2S)\,A + \rho_{,}\,(g + mc^2)\,A_{,} - \rho_{,}\,(g - mc^2)\,B_{,} = 0,$$

$$(g + mc^2)\,\epsilon^{-mh_{,}}A_{,} - (g - mc^2)\,\epsilon^{mh_{,}}B_{,} = 0.$$

Eliminating A, $A_{,}$ and $B_{,}$ from these equations, and putting

$$c^2 = \frac{g\zeta}{m},$$

we find

$$(\rho SS_{,} + \rho_{,}DD_{,})\,\zeta^2 - \rho\,(SD_{,} + S_{,}D)\,\zeta + (\rho - \rho_{,})\,DD_{,} = 0 \ldots (46).$$

The equilibrium of the fluid being supposed to be stable, we must have $\rho_{,} < \rho$. This being the case, it is easy to prove that the two roots of (46) are real and positive. These two roots correspond to two systems of waves of the same length, which are propagated with the same velocity.

In the limiting case in which $\rho/\rho_{,} = \infty$, (46) becomes

$$SS_{,}\zeta^2 - (SD_{,} + S_{,}D)\,\zeta + DD_{,} = 0,$$

the roots of which are D/S and $D_{,}/S_{,}$, as they evidently ought to be, since in this case the motion of the under fluid will not be affected by that of the upper, and the upper fluid can be in motion by itself.

When $\rho_{,} = \rho$ one root of (46) vanishes, and the other becomes $\dfrac{SD_{,} + S_{,}D}{SS_{,} + DD_{,}}$ or $\dfrac{\epsilon^{m(h+h_{,})} - \epsilon^{-m(h+h_{,})}}{\epsilon^{m(h+h_{,})} + \epsilon^{-m(h+h_{,})}}$. The former of these roots corresponds to the waves propagated at the common surface of the fluids, while the latter gives the velocity of propagation belonging to a single fluid having a depth equal to the sum of the depths of the two considered.

When the depth of the upper fluid is considered infinite, we must put $D_{\prime\prime}/S_{\prime}=1$ in (46). The two roots of the equation so transformed are 1 and $\frac{(\rho-\rho')D}{\rho S+\rho_{\prime}D}$, the former corresponding to waves propagated at the upper surface of the upper fluid, and the latter agreeing with Art. 15.

When the depth of the under fluid is considered infinite, and that of the upper finite, we must put $D/S=1$ in (46). The two roots will then become 1 and $\frac{(\rho-\rho_{\prime})D_{\prime}}{\rho S_{\prime}+\rho_{\prime}D_{\prime}}$. The value of the former root shews that whatever be the depth of the upper fluid, one of the two systems of waves will always be propagated with the same velocity as waves of the same length at the surface of a single fluid of infinite depth. This result is true even when the motion is in three dimensions, and the form of the waves changes with the time, the waves being still supposed to be such as could be excited in the fluids, supposed to have been previously at rest, by means of forces applied at the upper surface. For the most general small motion of the fluids in this case may be regarded as the resultant of an infinite number of systems of waves of the kind considered in this paper. It is remarkable that when the depth of the upper fluid is very great, the root $\zeta=1$ is that which corresponds to the waves for which the upper fluid is disturbed, while the under is sensibly at rest; whereas, when the depth of the upper fluid is very small, it is the other root which corresponds to those waves which are analogous to the waves which would be propagated in the upper fluid if it rested on a rigid plane.

When the depth of the upper fluid is very small compared with the length of a wave, one of the roots of (46) will be very small; and if we neglect squares and products of $mh_{\prime}$ and ζ, the equation becomes $2\rho D\zeta-2(\rho-\rho_{\prime})mh_{\prime}D=0$, whence

$$\zeta=\frac{\rho-\rho_{\prime}}{\rho}mh_{\prime},\quad c^2=\frac{\rho-\rho_{\prime}}{\rho}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(47).$$

These formulæ will not hold good if mh be very small as well as $mh_{\prime}$, and comparable with it, since in that case all the terms of

(46) will be small quantities of the second order, $mh_{,}$ being regarded as a small quantity of the first order. In this case, if we neglect small quantities of the third order in (46), it becomes

$$4\rho\zeta^2 - 4m\rho\,(h + h_{,})\,\zeta + 4(\rho - \rho_{,})\,m^2 hh_{,} = 0,$$

whence

$$c^2 = \frac{g}{2}\left\{h + h_{,} \pm \sqrt{(h - h_{,})^2 + \frac{4\rho_{,}}{\rho}\,hh_{,}}\right\} \dots\dots\dots\dots\dots (48).$$

Of these values of c^2, that in which the radical has the negative sign belongs to that system of waves to which the formulæ (47) apply when $h_{,}$ is very small compared with h.

If the two fluids are water and mercury, $\rho/\rho_{,}$ is equal to about 13·57. If the depth of the water be very small compared both with the length of the waves and with the depth of the mercury, it appears from (47) that the velocity of propagation will be less than it would have been, if the water had rested on a rigid plane, in the ratio of ·9624 to 1, or 26 to 27 nearly.

APPENDIX.

[A. *On the relation of the preceding investigation to a case of wave motion of the oscillatory kind in which the disturbance can be expressed in finite terms.*

In the *Philosophical Transactions* for 1863, p. 127, is a paper by the late Professor Rankine in which he has shewn that it is possible to express in finite terms, without any approximation, the motion of a particular class of waves of the oscillatory kind. It is remarkable that the results for waves of this kind were given as long ago as in 1802, by Gerstner*, whose investigation however seems to have been but little noticed for a long time. This case of motion has latterly attracted a good deal of attention, partly no doubt from the facility of dealing with it, but partly, it would seem, from misconceptions as to its intrinsic importance.

* See Weber's *Wellenlehre auf Experimente gegründet*, p. 338.

The investigation may be presented in very short compass in the following manner.

Let us confine our attention to the case of a mass of liquid, regarded as a perfect fluid of a depth practically infinite, in which an indefinite series of regular periodic waves is propagated along the surface, the motion being in two dimensions, and vanishing at an infinite depth. Taking the plane of motion for the plane of xy, y being measured vertically downwards, let us seek to express the actual co-ordinates x, y of any particle in terms of two parameters h, k particularising that particle, and of the time t. Let us assume for trial

$$x = h + K \sin m(h - ct), \qquad y = k + K \cos m(h - ct) \quad \text{......(49)},$$

where m, c are two constants, and K a function of k only. It will be easily seen that these equations, regarded merely as expressing the geometrical motion of points, and apart from the physical possibility of the motion, represent a wave disturbance of periodic character travelling in the direction of OX with a velocity of propagation c.

As the disturbance is in two dimensions, we may speak of areas as representing volumes. Let us consider first the condition of constancy of the mass. The four loci corresponding to constant values h, $h + dh$, k, $k + dk$, of the two parameters respectively enclose a quadrangular figure which is ultimately a parallelogram, the area of which must be independent of the time. Now the area is $S\,dh\,dk$ where

$$S = \frac{dx}{dh}\frac{dy}{dk} - \frac{dx}{dk}\frac{dy}{dh} \quad \text{........................(50)}.$$

On performing the differentiations we find

$$S = 1 + (mk + K')\cos m(h - ct) + mKK' \quad \text{.........(51)},$$

where K' stands for dK/dk. In order that this may be independent of the time it is necessary and sufficient that

$$mK + K' = 0 \quad \text{........................(52)},$$

whence

$$K = a\epsilon^{-mk} \quad \text{..........................(52')},$$

and

$$S = 1 - m^2K^2 = 1 - m^2a^2\epsilon^{-2mk} \quad \text{............(53)}.$$

The dynamical equations give

$$\begin{aligned}\frac{dp}{\rho} &= g\,dy - \left(\frac{d^2x}{dt^2}\,dx + \frac{d^2y}{dt^2}\,dy\right)\\ &= g\,dy + m^2c^2K\{\sin m(h-ct)\,dx + \cos m(h-ct)\,dy\}\\ &= g\,dy + m^2c^2\{(x-h)\,d(x-h) + (y-k)\,d(y-k)\}\\ &\qquad + m^2c^2\{(x-h)\,dh + (y-k)\,dk\}.\end{aligned}$$

The last line becomes by (49) and (52),

$$mc^2\{mK\sin m(h-ct)\,dh - K'\cos m(h-ct)\,dk\},$$

or

$$-mc^2d\,.\,K\cos m(h-ct).$$

The dynamical equations are therefore satisfied, the expression for dp being a perfect differential, and we have

$$\begin{aligned}\frac{p}{\rho} &= gy + \tfrac{1}{2}m^2c^2\{(x-h)^2 + (y-k)^2\} - mc^2K\cos m(h-ct) + C\\ &= gk + \tfrac{1}{2}m^2c^2K^2 + (g - mc^2)K\cos m(h-ct) + C.\end{aligned}$$

It remains to consider the equations of condition at the boundaries of the fluid. The expression for K satisfies the condition of giving a disturbance which decreases indefinitely as the depth increases, and we have only to see if it be possible to satisfy the condition at the free surface. Now the particles at the free surface differ only by the value of the parameter h, as follows from the fundamental conception of wave motion, and therefore for some one value of k we must have $p = 0$ independently of the time. This requires that

$$c^2 = \frac{g}{m} = \frac{g\lambda}{2\pi},$$

and if we please to take $k = 0$ at the surface, and determine C accordingly, we have

$$\frac{p}{\rho} = gk - \tfrac{1}{2}ga^2m(1 - \epsilon^{-2mk}) \quad \text{.........(54).}$$

Since p is independent of the time, not merely for $k = 0$, but for *any* constant value of k, it follows that when the wave motion is converted into steady motion by superposing a velocity equal and opposite to that of propagation, it is not merely the line of motion or stream-line which forms the surface but *all* the stream-lines that are lines of constant pressure. This is undoubtedly no necessary property of wave-motion converted into steady motion, which only requires that the particular stream-line at the surface shall be one for which the pressure is constant, though Gerstner has expressed himself as if he supposed it necessarily true; it is merely a character of the special case investigated by Gerstner and Rankine. Nevertheless in the case of *deep* water it must be very approximately true. For in the first place it is strictly true at the surface, and in the second place, it must be sensibly true at a very moderate depth and for all greater depths, since the disturbance very rapidly diminishes on passing from the surface downwards; so that unless the amount of disturbance be excessive the supposition that all the stream-lines are lines of constant pressure will not be much in error.

In the case investigated by the mathematicians just mentioned, each particle returns periodically to the position it had at a given instant; there is no progressive motion combined with a periodic disturbance, such as was found in the case investigated in the present paper: and for deep water the absence of progressive motion is doubtless peculiar to the former case, as will presently more clearly appear.

If we suppose a regular periodic wave motion to be going on, and then suppose small suitable pressures applied to the surface in such a manner as to check the motion, we may evidently produce a secular subsidence of the wave disturbance while still leaving it at any moment regular and periodic, save as to secular change, provided the opposing pressures are suitably chosen. The wave-length will be left unchanged, but not so, in general, the periodic time. If the amount of disturbance in one wave period be insensible, the particles which at one time have a common mean depth must at any future time have a common mean depth, and must ultimately lie in a horizontal plane when the wave motion has wholly subsided. In this condition therefore there can be no motion except a horizontal flow with a velocity which is some function of the depth. By a converse process we may imagine a regular periodic wave motion of given wave-length excited in a fluid in which there previously was none; and according to the nature of the arbitrary flow with which we start, we shall obtain as the result a wave motion of such or such a kind*.

In any given case of wave motion, the flow which remains

* To prevent possible misconception I may observe that I am not here contemplating the actual mode of excitement of waves by wind, which in some respects is essentially different.

when the waves have been caused to subside in the manner above explained is easily determined, since we know that in the motion of a liquid in two dimensions the angular velocity is not affected by forces applied to the surface. If ω be the angular velocity

$$2\omega = \frac{dv}{dx} - \frac{du}{dy} = \frac{1}{S}\left\{\frac{dy}{dk}\frac{dv}{dh} - \frac{dy}{dh}\frac{dv}{dk} + \frac{dx}{dk}\frac{du}{dh} - \frac{dx}{dh}\frac{du}{dk}\right\}$$

S being defined by (50). In Gerstner and Rankine's solution

$$u = -mac\epsilon^{-mk}\cos m(h - ct), \quad v = mac\epsilon^{-mk}\sin m(h - ct),$$

and on effecting the differentiations and substituting for S from (53) we find

$$\omega = -\frac{m^3a^2c\epsilon^{-2mk}}{1 - m^2a^2\epsilon^{-2mk}} \quad \text{.................(55).}$$

Let y' be the depth and u' the horizontal velocity, after the wave-motion has been destroyed as above explained, of the line of particles which had k for a parameter; then we must have

$$\omega = -\tfrac{1}{2}\frac{du'}{dy'} \quad \text{.............................(56).}$$

Since in a horizontal length which may be deemed infinite compared with λ the area between the ordinates y', $y' + dy'$ must be the same as between the lines of particles which have k, $k + dk$ for their k-parameter

$$dy' = Sdk,$$

S being defined by (50). Putting for S its value given by (53) we have

$$dy' = (1 - m^2a^2\epsilon^{-2mk})\,dk \quad \text{....................(57),}$$

$$y' = k - \tfrac{1}{2}ma^2(1 - \epsilon^{-2mk}) \quad \text{..................(58).}$$

We have then from (56) by (55) and (57),

$$u' = 2m^3a^2c\int\epsilon^{-2mk}dk = -m^2a^2c\epsilon^{-2mk} \quad \text{...........(59),}$$

since u' vanishes when $k = \infty$.

It appears then that in order that it should be possible to excite these waves in deep water previously free from wave disturbance, by means of pressures applied to the surface, a preparation must be laid in the shape of a horizontal velocity decreasing from the surface downwards according to the value of ϵ^{-2mk}, where k is a function of the depth y' determined by the transcendental equation in k (58), and moreover a velocity decreasing downwards according to this law will serve for waves of the present kind of only one particular height depending on the coefficient of the exponential in the expression for the flow. Under these conditions the horizontal velocity depending (when we adopt approximations) on the square and higher powers of the elevation, which belongs to the wave-motion, is exactly neutralized by the pre-existing horizontal velocity in a contrary direction, pre-existing, that is, when we think of the waves as having been excited in a fluid previously destitute of wave-motion, not as having gone on as they are from a time indefinitely remote. The absence of any forward horizontal motion of the individual particles in waves of this kind, though attractive at first sight, is not of any real physical import, because we are not concerned with the *biographies* so to speak of the individual particles.

The oscillatory waves which most naturally present themselves to our attention are those which are excited in the ocean or on a lake by the action of the wind, or those which having been so excited are propagated into (practically, though not in a rigorous mathematical sense) still water. Of the latter kind are the surf which breaks upon our western coasts as a result of storms out in the Atlantic, or the grand rollers which are occasionally observed at St Helena and Ascension Island. The motion in these cases having been produced from rest, by forces applied to the surface, there is no molecular rotation, and therefore the investigation of the present paper strictly applies. Moreover, if we conceive the waves gradually produced by suitable forces applied to the surface, in the manner explained at p. 222, the investigation applies to the waves (secular change apart) at any period of their growth, and not merely when they have attained one particular height.

There can be no question, it seems to me, that this is the class of oscillatory waves which on merely physical grounds we should naturally select for investigation. The interest of the solution first given by Gerstner, and it *is* of great interest, arises not from any physical pre-eminence of the class of waves to which it relates, but from the imperfection of our analysis, which renders it important to discuss a case in which all the circumstances of the motion can be simply expressed in mathematical terms without any approximation. And though this motion is not exactly that which on purely physical grounds we should prefer to investigate, namely, that in which the molecular rotation is *nil*, yet unless the height of the waves be extravagant, it agrees so nearly with it that for many purposes the simpler expressions of Rankine may be used without material error, even when we are investigating wave motion of the irrotational kind.

B. *Considerations relative to the greatest height of oscillatory irrotational waves which can be propagated without change of form.*

In a paper published in the *Philosophical Magazine*, Vol. XXIX. (1865), p. 25, Rankine gave an investigation which led him to the conclusion that in the steepest possible oscillatory waves of the irrotational kind, the crests become at the vertex infinitely curved in such a manner that a section of the crest by the plane of motion presents two branches of a curve which meet at a right angle*.

In this investigation it is assumed in the first place that the steepness may be pushed to the limit of an infinite curvature at a particular point, and in the second place that the variations

* It is not quite clear whether Rankine supposed his proposition, that "all waves in which molecular rotation is null, begin to break when the two slopes of the crest meet at right angles," to apply only to free waves, or to forced waves as well. One would have supposed the former, were it not that a figure is referred to representing forced waves of one particular kind. It is readily shewn that the contour of a forced wave is arbitrary, even though the motion be restricted to be irrotational. Let $U=C$ (p. 4) be the general equation of the stream lines when the wave motion is converted into steady motion. Then in the general case of a finite depth, which includes as a limiting and therefore particular case that of an infinite depth, the parameter C has one constant value at the upper surface, and another at the bottom, and it satisfies the partial differential equation (5) of p. 4. Hence the problem of finding U is the same as that of determining the permanent temperature, varying in two dimensions only, of a homogeneous isotropic solid the section of which is bounded below by a horizontal line at a finite or infinite depth, and above by a given arbitrary contour, the bounding surfaces being at two given constant temperatures. The latter problem is evidently determinate, and therefore also the former, so that *forced* waves may present in their contour sharp angles, not merely of 90^0, but of any value we please to take.

of the components of the velocity, in passing from the crest to a point infinitely close to it, may be obtained by differentiation, or in other words from the second terms of the expansion by Taylor's Theorem applied to infinitely small increments of the variables.

The first assumption might perhaps be called in question, but it would appear likely to give at any rate a superior limit to the steepest form possible, if not the steepest form itself. But as regards the second it would seem *à priori* very likely that the crest might just be one of those singular points where Taylor's Theorem fails; and that such must actually be the case may be shewn by simple considerations.

Let us suppose that a fluid of either finite or infinite depth is disturbed by a wave motion which is propagated uniformly without change, the motion of the fluid being either rotational or not, and let us suppose further that the crests are perfectly sharp, so that a crest is formed by two branches of a curve which either meet at a finite angle (their prolongations belonging to the region of space where the fluid is not), or else touch, forming a cusp.

Reduce the wave motion to steady motion by superposing a velocity equal and opposite to that of propagation. Then a particle at the surface may be thought of as gliding along a fixed smooth curve: this follows directly from physical considerations, or from the ordinary equation of steady motion. On arriving at a crest the particle must be momentarily at rest, and on passing it must be ultimately in the condition of a particle starting from rest down an inclined or vertical plane. Hence the velocity must vary ultimately as the square root of the distance from the crest.

Hitherto the motion has been rotational or not, let us now confine ourselves to the case of irrotational motion. Place the origin at the crest, refer the function ϕ to polar co-ordinates r, θ; θ being measured from the vertical, and consider the value of ϕ very near the origin, where ϕ may be supposed to vanish, as the arbitrary constant may be omitted. In general ϕ will be of the form $\Sigma A_n r^n \sin n\theta + \Sigma B_n \cos n\theta$. In the present case ϕ must contain sines only on account of the symmetry of the motion, as already shewn (p. 212), so that retaining only the most important term we may take $\phi = Ar^n \sin n\theta$. Now for a point in the section of the profile we must have $d\phi/d\theta = 0$, and $d\phi/dr$ varying ultimately as $r^{\frac{1}{2}}$. This requires that $n = \frac{3}{2}$, and for the profile that $\frac{3}{2}\theta = \frac{1}{2}\pi$, so that the two branches are inclined at angles of $\pm 60^\circ$ to the vertical, and at an angle of 120° to each other, not of 90° as supposed by Rankine.

This however leaves untouched the question whether the disturbance can actually be pushed to the extent of yielding crests with sharp edges, or whether on the other hand there exists a limit, for which the outline is still a smooth curve, beyond which no waves of the oscillatory irrotational kind can be propagated without change of form.

After careful consideration I feel satisfied that there is no such earlier limit, but that we may actually approach as near as we please to the form in which the curvature at the vertex becomes infinite, and the vertex becomes a multiple point where the two branches with which alone we are concerned enclose an angle of 120°. But whether in the limiting form the inclination of the wave to the horizon continually increases from the trough to the summit, and is consequently limited to 30°, or whether on the other hand the points of inflexion which the profile presents in the general case remain at a finite distance from the summit when the limiting form is reached, so that on passing from the trough to the summit the inclination attains a maximum from which it begins to decrease before the summit is reached, is a question which I cannot certainly decide, though I feel little doubt that the former alternative represents the truth.

In Rankine's case of wave motion the limiting form presents crests which are cusped. For the maximum wave $ma = 1$ or $a = \lambda/2\pi$. We see from (55) that in this case the angular velocity becomes infinite at the surface, where k vanishes; and if we suppose such waves excited in the manner already explained in a fluid initially destitute of wave motion, the horizontal velocity u' which must exist in preparation for the waves must be such that du'/dy' becomes infinite at the surface. It appears to be this circumstance which renders it possible for even rotational waves to attain in the limit to an infinite thinness of crest without losing the property of uniform propagation.

When swells are propagated towards a smooth, very gently shelving shore, the height increases when the finiteness of depth begins to take effect. Presently the limiting height for uniformly propagated irrotational waves is passed, and then the form of the wave changes independently of the mere secular change due to diminishing depth. The tendency is now for the high parts to overtake the less high in front of them, and thereby to become higher still, until at last the crest topples over and the wave finally breaks. The breaking is no doubt influenced by friction against the bottom (denoting by "friction" the effect of the eddies produced), but I do not believe that it is wholly or even mainly due to this cause. Before the wave breaks altogether the top gets very thin, but the maximum height for uniform propagation is probably already passed by a good deal, so that we must guard against being misled by this observation as to the character of the limiting form.

In watching many years ago a grand surf which came rolling in on a sandy beach near the Giant's Causeway, without any storm at the place itself, I recollect being struck with the blunt wedge-like form of the waves where they first lost their flowing outline, and began to show a little broken water at the very summit. It is only I imagine on an oceanic coast, and even there on somewhat rare occasions, that the form of waves of this kind, of nearly the maximum height, can be studied to full advantage. The observer must be stationed nearly in a line with the ridges of the waves where they begin to break.

C. *Remark on the method of Art. I.*

There appears to be a slight advantage in employing the function U or ψ $(=\int(udy-vdx))$ instead of ϕ, the wave motion having been reduced to steady motion as is virtually done in Art. 1. The general equation for ψ is the same as for ϕ, (2), and the general expression for ψ answering to that given for ϕ on p. 212 is

$$\psi=-cy+\Sigma_1^\infty C_r\left(\epsilon^{rm(h-y)}-\epsilon^{-rm(h-y)}\right)\cos rmx.$$

The expression for p in terms of ψ is almost identical with that in terms of ϕ. So far there is nothing to choose between the two. But for the two equations which have to be satisfied simultaneously at the surface, instead of $p=0$ and the somewhat complicated equation (7), we have $p=0$ and $\psi=$ const., which constant we may take $=0$ if we leave open the origin of y. The substitution of this equation of simpler form for (7) is a gain in proceeding to higher orders of approximation. I remember however thinking as I was working at the paper that as far as the approximation there went the gain was not such as to render it worth while to make the change.

But while these sheets were going through the press I devised a totally different method of conducting the approximation, which I find possesses very substantial advantages in proceeding to higher orders of approximation. The reader will find this new method after the paper "on the critical values of the sums of periodic series."]

[From the *Report of the British Association* for 1847, Part II. p. 6.]

On the Resistance of a Fluid to Two Oscillating Spheres.

The object of this communication was to shew the application of Professor Thomson's method of images to the solution of certain problems in hydrodynamics. Suppose that there exists in an infinite mass of incompressible fluid a point from which, or to which the fluid is flowing with a velocity alike in all directions. Conceive now two such points, of intensities equal in magnitude and opposite in sign, to coexist in the fluid; and then suppose these points to approach, and ultimately coalesce, their intensities varying inversely as the distance between them. Let the resulting point be called a *singular point of the second order*. The motion of a fluid about a solid, oscillating sphere is the same as if the solid sphere were replaced by fluid, in the centre of which existed such a point. It is easy to shew that the motion of the fluid due to a point of this kind, when the fluid is interrupted by a sphere having its centre in the axis of the singular point, is the same as if the sphere's place were occupied by fluid containing one singular point of the second order. By the application of this principle may be found the resistance experienced by a sphere oscillating in presence of a fixed sphere or plane, or within a spherical envelope, the oscillation taking place in the line joining the centres, or perpendicular to the plane. In a similar manner may be found the resistance to two spheres which touch, or are connected by a rod, or to the solid made up of two spheres which cut, provided the exterior angle of the surfaces be a submultiple of two right angles, the oscillation in these cases also taking place in the line joining the centres. The numerical calculation is very simple, and may be carried to any degree of accuracy.

The investigation mentioned in the preceding paper arose out of the communication to me by Sir William Thomson of his beautiful method of electrical images before he had published it. Having myself paid more attention to the motion of fluids than to electricity, I endeavoured to find if it would in any manner apply to the solution of problems in the motion of fluids. I found that what is called above a singular point of the second order had a perfect image in a sphere when its axis was in the direction of a radius, which led to a complete solution of the problem mentioned in the paper when one sphere lay wholly outside or inside the other. I shewed this to Professor Thomson, who pointed out to me that a solution was also attainable, and that in finite terms, when the spheres intersected, provided the angle of intersection was a submultiple of two right angles. He saw that the property of a singular point of the second order of giving a perfect image in the case mentioned, admitted of an application to the theory of magnetism, which he has published in a short paper in the second volume of the *Cambridge and Dublin Mathematical Journal*, (1847) p. 240.

Although the mathematical result is contained in the paper just mentioned, I subjoin the process by which I found it out.

The expression (see p. 41) for the function ϕ around a sphere which moves in a perfect fluid previously at rest may be thought of as applying to the whole of an infinite mass of fluid, provided we conceive what has here been called a singular point of the second order to exist at the origin. Let us conceive a spherical surface S with its centre at O and having a radius a to exist in the fluid; let P be the singular point, lying either within or without the sphere S, and having its axis in the line OP. Let r', θ' be polar co-ordinates originating at P, θ' being measured from OP produced, and let r, θ be polar co-ordinates originating at O; let m be a constant, and $OP = c$, then ϕ being the function due to the singular point we have

$$\phi = -\frac{m\cos\theta'}{r'^2} = -\frac{m\,.\,r'\cos\theta'}{r'^3} = -m\frac{r\cos\theta - c}{(r^2 - 2cr\cos\theta + c^2)^{\frac{3}{2}}}$$

$$= -m\frac{d}{dc}(r^2 - 2cr\cos\theta + c^2)^{-\frac{1}{2}}.$$

Now if e be less than 1,

$$(1 - 2e\cos\theta + e^2)^{-\frac{1}{2}} = P_0 + eP_1 + e^2P_2 + \dots,$$

where P_0, P_1, P_2... are Laplace's, or in this case more properly Legendre's, coefficients*. Hence by expanding and differentiating with respect to c, we have

$$\phi = -m\left(\frac{1P_1}{r^2} + \frac{2cP_2}{r^3} + \frac{3c^2P_3}{r^4} + \dots\right), \text{ if } r > c \dots\dots\dots(1),$$

$$\phi = \quad m\left(\frac{1P_0}{c^2} + \frac{2rP_1}{c^3} + \frac{3r^2P_2}{c^4} + \dots\right), \text{ if } r < c \dots\dots\dots(2).$$

We are not of course concerned with the constant term in the latter of these two expressions. For the normal velocity (ν) at the surface of the sphere we get by differentiating with respect to r, and then putting $r = a$

$$\nu = m\left(\frac{1\,.\,2P_1}{a^3} + \frac{2\,.\,3cP_2}{a^4} + \frac{3\,.\,4c^2P_3}{a^5} + \dots\right), \text{ if } a > c \dots\dots(3),$$

$$\nu = m\left(\frac{1\,.\,2P_1}{c^3} + \frac{2\,.\,3aP_2}{c^4} + \frac{3\,.\,4a^2P_3}{c^5} + \dots\right), \text{ if } a < c \dots(4).$$

First suppose the point P outside the sphere, let the sphere be thought of as a solid sphere, and consider the motion "reflected" (p. 28) from it. The reflected motion being symmetrical about the axis, we must have for it

$$\phi = \frac{Q_0}{r} + \frac{Q_1}{r^2} + \frac{Q_2}{r^3} + \dots\dots\dots\dots\dots\dots\dots\dots(5),$$

where Q_0, Q_1, Q_2... are Laplace's functions involving θ only. This gives for the normal velocity (ν') in the reflected motion at the surface of the sphere

$$\nu' = -\frac{1Q_0}{a^2} - \frac{2Q_1}{a^3} - \frac{3Q_2}{a^4} - \dots\dots\dots\dots\dots\dots(6);$$

and since we must have $\nu' = -\nu$ we get from (4) and (6)

$$Q_0 = 0,\ Q_1 = m\,\frac{1a^3P_1}{c^3},\quad Q_2 = m\,\frac{2a^5P_2}{c^4},\quad Q_3 = m\,\frac{3a^7P_3}{c^5}\dots$$

which reduces (5) to

$$\phi = m\,\frac{a^3}{c^3}\left(\frac{1P_1}{r^2} + \frac{2a^2P_2}{cr^3} + \frac{3a^4P_3}{c^2r^4} + \dots\right)\dots\dots\dots\dots\dots\dots(7).$$

* The functions which in Art. 9 of the paper "On some Cases of Fluid Motion" (p. 38) I called "Laplace's coefficients," following the nomenclature of Pratt's *Mechanical Philosophy*, are more properly called "Laplace's functions;" the term "Laplace's coefficients" being used to mean the coefficients in the expansion of

$$[1 - 2e\,\{\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\omega - \omega')\} + e^2]^{-\frac{1}{2}},$$

to be understood according to the usual notation and not as in the text.

This is identical with what (1) becomes on writing m', c' for m, c provided that

$$m' = -m\frac{a^3}{c^3}, \qquad c' = \frac{a^2}{c}.$$

Hence the reflected motion is perfectly represented by supposing the sphere's place occupied by fluid within which, at the point P' in the line OP determined by $OP' = c'$, there exists a singular point of the same character as P, but of opposite sign, and of intensity less in the ratio of a^3 to c^3.

The case of a spherical mass of fluid within a rigid enclosure and containing a singular point of the second order with its axis in a radial direction might be treated in a manner precisely similar, by supposing the space exterior to the sphere filled with fluid, taking to represent the reflected motion in this case, instead of (5), the corresponding expression according to ascending powers of r, and comparing the resulting normal velocity at the surface of the sphere with (3) instead of (4). This is however unnecessary, since we see that the relation between the two singular points P, P' is reciprocal, so that either may be regarded as the image of the other.

Suppose now that we have two solid spheres, S, S', exterior to each other, immersed in a fluid. Suppose that S' is at rest, and that S moves in the direction of the line joining the centres, the fluid being at rest except as depends on the motion of S. The motion of the fluid may be determined by the method of successive reflections (p. 28), which in this case becomes greatly simplified in consequence of the existence of a perfect image representing each reflected motion, so that the process is identical with that of Thomson's method of images, except that the decrease of intensity of the successive images takes place according to the cubes of the ratios of the successive quantities such as a, c, instead of the first powers.

If a sphere move inside a spherical envelope, in the direction of the line joining the centres, the space between being filled with fluid which is otherwise at rest, the motion may be determined in a precisely similar manner.

If two spheres outside each other, or just touching, be connected by an infinitely thin rod, and move in a fluid in the direction of the line joining their centres, we have only to find the motion due to the motion of each sphere supposing the other at rest, and to superpose the results.

I should probably not have thought of applying the method to the solid bounded by the outer portions of two intersecting spheres, had not Professor Thomson shewn me that it was not limited to the cases in which each sphere is complete; and that although it fails from non-convergence when the spheres intersect, yet when the exterior angle of intersection is a submultiple of two right angles the places of the successive images recur in a cycle, and a solution of the problem may be obtained in finite terms by placing singular points of the second order at the places of the images in a complete cycle.

The simplest case is that in which the spheres are generated by the revolution round their common axis of two circles which intersect at right angles. In this case if S, S' are the spheres, O, O' their centres, O_1 the middle point of the common chord of the circles, the image of O in S' will be at O_1, and the image of O_1 in S will be at O'.

Let a, b be the radii of the spheres; c the distance $\sqrt{(a^2+b^2)}$ of their centres; e, f the distances a^2/c, b^2/c of O_1 from O, O'; C the velocity of the spheres; r, θ the polar co-ordinates of any point measured from O; r_1, θ_1 the co-ordinates measured from O_1; r', θ' the co-ordinates measured from O'; θ, θ_1, θ' being all measured from the line OO'. If S' were away, we should have for the fluid exterior to S

$$\phi = -Ca^3\frac{\cos\theta}{2r^2}.$$

For the image of this in S' we have a singular point at O_1 for which

$$\phi = \frac{Ca^3b^3}{c^3}\cdot\frac{\cos\theta_1}{2r_1^{\,2}},$$

and for the image of this again in S we have a singular point at O' for which

$$\phi = -Cb^3\frac{\cos\theta'}{2r'^2},$$

which is precisely what is required to give the right normal velocity at the surface of S'. Moreover all the singular points lie inside the space bounded by the exterior portions of the intersecting spheres. Hence the three motions together satisfy all the conditions of the problem, so that for the complete solution we have

$$\phi = -\tfrac{1}{2} C \left\{ \frac{a^3 \cos\theta}{r^2} - \frac{a^3 b^3 \cos\theta_1}{c^3 r_1^{\,2}} + \frac{b^3 \cos\theta'}{r'^2} \right\}.$$

Just as in the case of a sphere, if a force act on the solid in the direction of its axis, causing a change in the velocity C, the only part of the expression for the resistance of the fluid which will have a resultant will be that depending upon dC/dt. This follows at once, as at pp. 50, 51, from the consideration that when there is no change of C the *vis viva* is constant, and therefore the resultant pressure is *nil*. If we denote by $M' dC/dt$ the resultant pressure acting backwards, we get for the part of M' due to the pressure of the fluid on the exposed portion of the surface of S',

$$\pi\rho b^2 \int \left\{ \frac{a^3 \cos\theta}{r^2} - \frac{a^3 b^3 \cos\theta_1}{c^3 r_1^{\,2}} + b \cos\theta' \right\} \cos\theta' \sin\theta' \, d\theta',$$

taken between proper limits. Putting $b \cos\theta' = x$, we have

$$r \cos\theta = c + x, \quad r_1 \cos\theta_1 = f + x,$$

$$r^2 = b^2 + c^2 + 2cx, \quad r_1^{\,2} = b^2 + f^2 + 2fx.$$

Expressing $\cos\theta$, $\cos\theta_1$, $\cos\theta'$ in terms of x and r, x and r_1, x, and changing the independent variable, first to x, and then in the first term to r and in the second to r_1, we have for the indefinite integral with sign changed

$$\frac{\pi\rho a^3}{12c^3} \left\{ r^3 - 6b^2 r + 3(c^4 - b^4)\frac{1}{r} \right\}$$

$$- \frac{\pi\rho a^3 b^3}{12 c^3 f^3} \left\{ r_1^{\,3} - 6b^2 r_1 + 3(f^4 - b^4)\frac{1}{r_1} \right\} + \frac{\pi\rho x^3}{3},$$

which is to be taken between the limits $r = a$ to $r = c + b$, $r_1 = ab/c$ to $f + b$, $x = -f$ to b. The part of M' due to the integral over the exposed part of the surface of S will be got from the above by interchanging; and on adding the two expressions together, and putting $f = b^2/c$, $c = \sqrt{(a^2 + b^2)}$, we get for the final result

$$M' = \frac{\pi\rho}{3c^3} \{4c^3(a^3 + b^3) - 2a^6 - 3a^4 b^2 - 6a^3 b^3 - 3a^2 b^4 - 2b^6\}.$$

When one of the radii, as b, vanishes, we get $M' = \tfrac{2}{3}\pi\rho a^3$ as it ought to be.

[From the *Transactions of the Cambridge Philosophical Society*,
Vol. VIII. p. 533.]

On the Critical Values of the Sums of Periodic Series.

[Read December 6, 1847.]

There are a great many problems in Heat, Electricity, Fluid Motion, &c., the solution of which is effected by developing an arbitrary function, either in a series or in an integral, by means of functions of known form. The first example of the systematic employment of this method is to be found in Fourier's *Theory of Heat*. The theory of such developements has since become an important branch of pure mathematics.

Among the various series by which an arbitrary function $f(x)$ can be expressed within certain limits, as 0 and a, of the variable x, may particularly be mentioned the series which proceeds according to sines of $\pi x/a$ and its multiples, and that which proceeds according to cosines of the same angles. It has been rigorously demonstrated that an arbitrary, but finite function of x, $f(x)$, may be expanded in either of these series. The function is not restricted to be continuous in the interval, that is to say, it may pass abruptly from one finite value to another; nor is either the function or its derivative restricted to vanish at the limits 0 and a. Although however the *possibility* of the expansion of an arbitrary function in a series of sines, for instance, when the function does not vanish at the limits 0 and a, cannot but have been contemplated, the *utility* of this form of expansion has hitherto, so far as I am aware, been considered to depend on the actual evanescence of the function at those limits. In fact, if the conditions of the problem require that $f(0)$ and $f(a)$ be equal to zero, it has been considered that these conditions were satisfied by selecting the form of expansion referred to. The chief object of the following paper is to develope the principles according to which the expansion of an arbitrary function is to be treated when the conditions at the limits which determine the particular form of the expansion are apparently violated; and to shew, by examples, the advantage that frequently results from the employment of the series in such cases.

In Section I. I have begun by proving the possibility of the expansion of an arbitrary function in a series of sines. Two methods have been principally employed, at least in the simpler cases, in demonstrating the possibility of such expansions. One, which is that employed by Poisson, consists in considering the series as the limit of another formed from it by multiplying its terms by the ascending powers of a quantity infinitely little less than 1; the other consists in summing the series to n terms, that is, expressing the sum by a definite integral, and then considering the limit to which the sum tends when n becomes infinite. The latter method certainly appears the more direct, whenever the summation to n terms can be effected, which however is not always the case; but the former has this in its favour, that it is thus that the series present themselves in physical problems. The former is the method which I have followed, as being that which I employed when I first began the following investigations, and accordingly that which best harmonizes with the rest of the paper. I should hardly have ventured to bring a somewhat modified proof of a well-known theorem before the notice of this Society, were it not for the doubts which some mathematicians seem to feel on this subject, and because there are some points which Poisson does not seem to have treated with sufficient detail.

I have next shewn how the existence and nature of the discontinuity of $f(x)$ and its derivatives may be ascertained merely from the series, whether of sines or cosines, in which $f(x)$ is developed, even though the summation of the series cannot be effected. I have also given formulæ for obtaining the developements of the derivatives of $f(x)$ from that of $f(x)$ itself. These developements cannot in general be obtained by the immediate differentiation of the several terms of the developement of $f(x)$, or in other words by differentiating under the sign of summation.