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ON THE THEORETICAL FORM OF OCEAN SWELL ON A ROTATING EARTH

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Summary

In the discussion of the form of ocean swell the waves studied by Gerstner (1802) and Stokes (1847) have received particular attention. In Gerstner waves each fluid particle describes a circle about a fixed point. Their mathematical properties are very simple, but they cannot be generated from rest by conservative forces. On the other hand, Stokes waves can be generated from rest by conservative forces. There is a surface drift associated with Stokes waves; but the mathematical difficulties are considerable. It has been customary to consider swell as a train of Stokes waves.

One of the fundamental assumptions underlying these theories is that the waves are moving on a non-rotating Earth. It is now shown that if the curvature of the Earth can be neglected, the effect of the Earth's rotation makes swell waves in their stationary state differ very little from Gerstner waves. The work suggests that in the general case each particle moves approximately in a horizontal circle of inertia as well as in the nearly vertical Gerstner motion; the diameter of the inertia circles is greatest at the surface, where it may be several hundred metres, and the period of revolution is half a pendulum day (equal to 12 cosec ϕ hours, where ϕ is the latitude).

Except for this circular movement there is no mass transport, and thus there is no ground for supposing that ocean drift currents are due to fluid transport with the waves. On the other hand, waves appear to provide a reasonable mechanism for the generation of inertia currents which have been frequently observed.

Introduction.—It is a well-known fact of observation that, at a distance from a storm, waves separate out into component wave-trains. If the storm area is sufficiently far distant from the region of observation, the wave profile consists at any time of a series of long-crested waves, following each other at constant intervals and keeping a constant period over a time which is long compared with the wave period. Waves of this type, which are generated by a distant agency, are known as swell. In favourable circumstances they have been known to travel several thousand miles, from which it may be deduced that the effect of viscosity is small. Detailed experimental observations of swell have not yet been undertaken, so that little is known about surface drift current due to waves and distribution of vorticity.

Much theoretical work has been done with a view to discovering the properties of swell. In this work it has been convenient to make the following assumptions:—

(*a*) The steepness of the waves is small, so that they offer little resistance to the surrounding air. The pressure on the free surface can therefore be taken as constant and the wave profile is symmetrical about crest and trough.

(*b*) The evident stability of the motion shows that for the purpose of calculation it is reasonable to regard the long-crested waves as part of an infinitely long periodic wave-train in which the crests are at right angles to the direction of propagation. The magnitude and direction of gravity is taken to be constant at all points.

(*c*) The crests are assumed to be so long that the effect of the finite crest-length can be neglected. Waves have therefore in theoretical work been considered as having crests of infinite length.

(*d*) The wave action is negligible at a distance of a few wave-lengths below the surface, so that the water-velocity at this depth is very small. It is also assumed that the sea-bed is beyond the range of wave action, so that the water can be taken to be infinitely deep.

(*e*) The effect of the Earth's rotation is neglected. It is to be supposed that a steady state exists under these conditions, and that swell waves have attained the steady state.

A considerable amount of mathematical research has been carried out on the determination of these steady states. One solution was discovered as early as 1802 by Gerstner (1). His solution is recommended by its mathematical simplicity. Each fluid particle describes a circular orbit in a vertical plane, with a fixed point as centre. Gerstner showed that the surface profile is trochoidal, and that there is no surface drift. The motion is rotational, the vorticity being about a direction parallel to the crests, but in a sense opposed to the orbital motion.

It was remarked by Stokes (2) that such a motion cannot be set up from rest in an inviscid incompressible fluid by the action of conservative forces. For under these conditions motions set up from rest are free from vorticity, by a theorem due to Lagrange, which can also be deduced from theorem 4 (p. 5). It follows that the final steady state must be a state of irrotational motion. Stokes found that in this state the particles move in nearly circular orbits; at the end of a period the particle has not returned to its original position but is slightly displaced in the direction of propagation. This means that there is a small drift velocity in this direction which is greatest near the surface and decreases rapidly as the depth increases. The drift velocity of Stokes waves is approximately

$$c\left(\frac{2\pi a}{\lambda}\right)^2 \exp\left(-4\pi h/\lambda\right),$$

where c is the velocity of propagation, a is the surface amplitude, λ is the wave-length and h the depth of the centre of the orbit below the surface. The pheno-

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menon of surface drift is often referred to as mass transport. The mathematical calculation of the Stokes waves is complicated and was put on a rigorous basis only in 1925 by Levi-Civita (3).

In addition to the wave motions of Gerstner and Stokes there are other wave motions with intermediate vorticity distributions and mass transport velocities. These waves were first treated by Dubreil-Jacotin (4) and are discussed by Miche (5). They have neither the mathematical simplicity of Gerstner waves nor the immediate physical application of Stokes waves.

No observations are available to decide between these wave types. To decide the point it will be necessary to make observations on the mass transport, since the profiles and wave velocities of the various types are much the same. In the absence of observations it has usually been considered (e.g. by Sverdrup and Munk (8) in their theory of wave generation) that swell waves should not differ much from Stokes waves, for the theoretical reasons given. It has also recently been suggested that the surface drift associated with Stokes waves may provide an explanation for the generation of certain ocean currents. A theory on these lines would avoid some of the difficulties associated with Ekman's theory (7, Art. 334 a), particularly the large coefficients of eddy viscosity which are needed to explain the generation of ocean currents in a reasonable time. But when natural phenomena on a large scale are considered, it is necessary to include the effect of the rotation of the Earth. It will be seen that when this effect is taken into account, the steady state is no longer that predicted by Stokes, but approximates very closely to the state discovered by Gerstner. There is no surface drift, so that the suggested theory of ocean currents is untenable.

Waves on a rotating Earth.—It has been mentioned that on a non-rotating Earth there is a variety of wave types, whose mass transport velocity may be positive, zero or negative in the direction of propagation. Of these the Stokes wave is the only type which can be generated by conservative forces. It will be shown that if there is a mass transport in any direction, no steady state can exist if the non-conservative Coriolis force is taken into account. This conclusion is derived without making any assumption about the wave profile. It is also shown that the normal to the mean surface is parallel to the direction of gravity. It follows that, since there is no mass transport, waves on a rotating Earth approximate to Gerstner waves. The normal of the orbital plane has a slight inclination to the direction of the wave crests, of the order of 1 minute of arc, which is negligible.

We shall consider the properties of an infinite long-crested wave-train travelling with constant wave velocity $\mathbf{c}$ in the x-direction. It is assumed that the free surface is cylindrical, with generators in the y-direction defined by a unit vector $\boldsymbol{l}$ perpendicular to $\mathbf{c}$; and also that the motion is two-dimensional in the sense that all sections of the motion by planes perpendicular to $\boldsymbol{l}$ are congruent in their velocity and pressure distributions. The motion is assumed to be periodic in space in the direction of $\mathbf{c}$, the period being the wave-length λ.

We define the mean normal of a periodic wave surface as the normal to the plane touching the wave surface along all the crests. The mean normal $\boldsymbol{n}$ of the free surface is the unit vector parallel to $\boldsymbol{l} \times \mathbf{c}$. This mean normal is not at once identified with the direction of gravity.

The motion can be rendered steady by referring it to axes travelling with velocity $\mathbf{c}$. The free surface is now fixed in space, its mean normal being $\boldsymbol{n}$. Since the motion is two-dimensional, the surfaces of equal pressure, one of which is the free surface, are cylindrical wave surfaces of mean normal $\boldsymbol{n}$, periodic with wave-length λ. It can be shown that the same remark applies to the stream surfaces. This is a consequence of the periodicity of the motion and the equation of continuity, provided that the component of velocity in the x-direction is always less than $|\boldsymbol{c}|$ in absolute magnitude. (The stream surfaces in the steady state are then infinite in the x-direction.)

The notation is explained below :

- $\boldsymbol{r}$ position vector,
- $\boldsymbol{u}$ velocity vector at $\boldsymbol{r}$,
- p pressure,
- ρ density (assumed constant),
- $\boldsymbol{\omega}$ Earth's angular velocity,
- $\boldsymbol{c}$ wave velocity,
- $\boldsymbol{l}$ unit vector along the crest (at right angles to $\boldsymbol{c}$),
- $\boldsymbol{n}$ unit vector at right angles to $\boldsymbol{l}$ and $\boldsymbol{c}$,
- Ψ geopotential including the centrifugal potential.

The Eulerian equation of motion is

$$\frac{\partial}{\partial t}\boldsymbol{u}+(\boldsymbol{u}\,.\,\mathrm{grad})\boldsymbol{u}+2\boldsymbol{\omega}\times\boldsymbol{u}=-\mathrm{grad}\left(\frac{p}{\rho}+\Psi\right). \tag{1}$$

It is assumed that

$$\boldsymbol{g}=\mathrm{grad}\,\Psi \text{ is constant everywhere.} \tag{2}$$

The effect of the Earth's curvature is thus neglected.

The equation of continuity is

$$\mathrm{div}\,\boldsymbol{u}=0 \text{ everywhere.} \tag{3}$$

The boundary conditions are

$$p \text{ is constant on the free surface,} \tag{4}$$

$$\boldsymbol{u}\to 0 \text{ at a great distance from the free surface.} \tag{5}$$

The x-component of $\boldsymbol{u}$ is numerically less than $|\boldsymbol{c}|$ inside the fluid.

The equation (1) is non-linear in the three components of $\boldsymbol{u}$, so a complete solution is not feasible. We shall endeavour to find properties of possible wave motions, paying special attention to the magnitude and direction of mass transport.

We begin by considering the inclination of the mean free surface to the horizontal, or, what is the same thing, the angle between the mean normal $\boldsymbol{n}$ and the gravitational acceleration $\boldsymbol{g}$.

Theorem 1. *The mean free surface is horizontal.*—It has already been remarked that the isobaric surfaces (i.e. the surfaces of equal pressure) are cylindrical wave surfaces of mean normal $\boldsymbol{n}$ but, from equation (1), the normal to the isobaric surface is given by

$$\mathrm{grad}\,\frac{p}{\rho}=-\boldsymbol{g}+O(|\boldsymbol{u}|). \tag{6}$$

It follows from this and equation (5) that the isobaric surfaces at great depth are planes normal to $\boldsymbol{g}$. But the plane through the crests of each isobaric surface is normal to $\boldsymbol{n}$. Therefore $\boldsymbol{n}$ and $\boldsymbol{g}$ are parallel vectors.

The z-axis is taken vertical, along $\boldsymbol{n}$.

We next obtain an analogue of Bernoulli's theorem, which will allow us to conclude immediately that there is no mass transport along the crests. This theorem connects the velocity in the steady motion, $\boldsymbol{u}-\mathbf{c}$, with pressure, gravity and another term representing the effect of the Coriolis force. It will be seen that this last term corresponds to a uniform gravitational field perpendicular to $\mathbf{c}$.

Theorem 2. *Extension of Bernoulli's theorem.*—The first set of axes was chosen so that

$$\boldsymbol{u}=\boldsymbol{u}(x-|\mathbf{c}|t,\ y,\ z), \tag{7}$$

whence

$$\frac{\partial}{\partial t}\boldsymbol{u}=-|\mathbf{c}|\frac{\partial}{\partial x}\boldsymbol{u}=-(\mathbf{c}\,.\,\mathrm{grad})\boldsymbol{u}.$$

Equation (1) can therefore be written

$$[(\boldsymbol{u}-\mathbf{c})\,.\,\mathrm{grad}](\boldsymbol{u}-\mathbf{c})+2\boldsymbol{\omega}\times(\boldsymbol{u}-\mathbf{c})+2\boldsymbol{\omega}\times\mathbf{c}=-\mathrm{grad}\left(\frac{p}{\rho}+\Psi\right),$$

or

$$\tfrac{1}{2}\mathrm{grad}(\boldsymbol{u}-\mathbf{c})^2+[\mathrm{curl}(\boldsymbol{u}-\mathbf{c})]\times(\boldsymbol{u}-\mathbf{c})+2\boldsymbol{\omega}\times(\boldsymbol{u}-\mathbf{c})=-\mathrm{grad}\left(\frac{p}{\rho}+\Psi+2[\boldsymbol{\omega},\mathbf{c},\boldsymbol{r}]\right),$$

whence

$$(\boldsymbol{u}-\mathbf{c})\,.\,\mathrm{grad}\,\{\tfrac{1}{2}(\boldsymbol{u}-\mathbf{c})^2+\frac{p}{\rho}+\Psi+2[\boldsymbol{\omega},\ \mathbf{c},\ \boldsymbol{r}]\}=0.$$

But

$$\frac{D}{Dt}\equiv(\boldsymbol{u}-\mathbf{c})\,.\,\mathrm{grad}$$

denotes differentiation following a particle, whence

$$\tfrac{1}{2}(\boldsymbol{u}-\mathbf{c})^2+\frac{p}{\rho}+\Psi+2[\boldsymbol{\omega},\ \mathbf{c},\ \boldsymbol{r}]$$

is constant for every particle.

Theorem 3. *There is no transport along the crests.*—Consider a particle which at time t_0 is on the crest of a stream surface in the steady motion. There will be an infinite sequence of times t, at which the particle is again on a crest. We have already seen that at time t all the variables except possibly x and y return to their initial values, whence $[\boldsymbol{\omega},\ \mathbf{c},\ \Delta\boldsymbol{r}]=0$, where $\Delta\boldsymbol{r}$ is the change in $\boldsymbol{r}$. This can be written

$$\alpha\Delta y+\beta\Delta z=0 \qquad (\alpha\neq 0 \text{ except on the equator})$$

but Δz is zero. Hence Δy is zero, i.e. there is no transport along the crests.

Theorem 4. *The circulation theorem.*—We now need an extension to rotating axes of a theorem due to Helmholtz, namely that the circulation along any closed curve moving with the fluid is constant, provided that the external forces are conservative and that the density is a function of the pressure only, i.e.

$$\frac{D}{Dt}\oint \boldsymbol{v}\,.\,d\boldsymbol{r}=0.$$

Here $\boldsymbol{v}$ is the velocity referred to fixed axes. The proof of this theorem given by Kelvin depends on the identity (which holds for fixed or moving axes):

$$\frac{D}{Dt}\oint \boldsymbol{u}\,.\,d\boldsymbol{r}=\oint\frac{D\boldsymbol{u}}{Dt}\,.\,d\boldsymbol{r}$$

for any closed curve (*cf.* Lamb, *Hydrodynamics*, Art. 33, 1932). Using this identity, we have from equation (1)

$$\frac{D}{Dt}\oint \boldsymbol{u}\,.\,d\boldsymbol{r} = \oint \frac{D\boldsymbol{u}}{Dt}\,.\,d\boldsymbol{r} = -\oint \operatorname{grad}\left(\frac{p}{\rho}+\Psi\right).\,d\boldsymbol{r} - 2\oint (\boldsymbol{\omega}\times\boldsymbol{u})\,.\,d\boldsymbol{r}$$

$$= -2\oint \boldsymbol{\omega}\,.\,(\boldsymbol{u}\times d\boldsymbol{r})$$

$$= -2\boldsymbol{\omega}\,.\,\frac{D}{Dt}\oint \tfrac{1}{2}\boldsymbol{r}\times d\boldsymbol{r}$$

by integration by parts.

But $\frac{1}{2}\oint \boldsymbol{r}\times d\boldsymbol{r}$ represents the directed area of the circuit, which is the vector whose component in any direction is the area of the projection of the circuit on a plane normal to that direction. Hence

$$\frac{D}{Dt}(C+2|\boldsymbol{\omega}|A)=0,$$

where C is the circulation, A the area of the projection on a plane normal to $\boldsymbol{\omega}$, so that

$$C+2|\boldsymbol{\omega}|A \text{ is constant for all time.} \tag{8}$$

This result is a special case of the circulation theorem due to V. Bjerknes. It has also been derived independently by G. I. Taylor (**9**) in his discussion of the motion of solids in a rotating fluid.

We have already shown that there is no transport along the crests. The circulation theorem will now be used to show that there is no transport in the direction of propagation.

Theorem 5. *There is no transport in the direction of propagation.*—Consider a closed rectangular circuit moving with the fluid and consisting at time t_0 of two lines of particles of length L parallel to $\boldsymbol{l}$, vertically above each other, one at great depth, one near the surface; and of two vertical lines joining their ends. The deep side of this rectangle remains fixed throughout all time, since at a great depth the velocity is zero. On the other hand, if there is a transport of fluid in the direction of propagation, the upper horizontal side of the rectangle is displaced more and more from its initial position. The two vertical sides remain on the whole in a vertical plane, both having the same velocity distribution along their length, since the motion is two-dimensional.

Consider the circulation along this circuit at any time. The circulation along the deep side is zero, the sum of the circulations along the sides which are vertical at t_0 is also zero at all times. The circulation along the circuit is therefore equal to the flow along the upper horizontal side. Let V be the maximum absolute value of the velocity component along $\boldsymbol{l}$. Then

$$|C|<VL. \tag{9}$$

But if there is a mass transport, the projected area of the circuit increases at a rate ultimately proportional to the mass transport. Therefore

$$A\to\infty \tag{10}$$

except on the equator. But

$$C+2|\boldsymbol{\omega}|A \text{ is constant.} \tag{11}$$

Equations (9), (10) and (11) are incompatible. It follows that there is no transport in the direction of progagation.

We might have used a similar argument to prove Theorem 3.

It has now been shown that the drift velocity is zero at all depths, except when the wave-train is near the equator. This conclusion is independent of the condition of constant pressure at the surface, which is not used except in the proof of Theorem 1, where it might have been replaced by a more general condition. It will be assumed without proof that the equations of motion actually have a solution of permanent type of any period and sufficiently small amplitude.

Let us consider the steady state of given period and amplitude as a function of the vector $\boldsymbol{\omega}$, which is now supposed capable of taking a continuous range of values regardless of its physical meaning. As $\boldsymbol{\omega}$ takes a series of values tending to zero, it is assumed that the corresponding steady states will tend to the solution of

$$\frac{\partial}{\partial t}\boldsymbol{u}+(\boldsymbol{u}\,.\,\mathrm{grad})\boldsymbol{u}=-\mathrm{grad}\left(\frac{p}{\rho}+\Psi\right),$$

whose mass transport in all directions is zero, i.e. to Gerstner's solution. If $\boldsymbol{\omega}$ is sufficiently small, the corresponding steady state will differ by very little from the Gerstner wave of the same period and amplitude. An estimate of the deviation can be obtained by considering the motion of a particle along the crest. In Lagrangian coordinates, the corresponding equation of motion is

$$\frac{\partial^2 y}{\partial t^2}+2\omega_z\frac{\partial x}{\partial t}-2\omega_x\frac{\partial z}{\partial t}=0.$$

x and z are given to a close approximation by Gerstner's solution. Substituting, we obtain the result that the normal to the orbital plane is inclined to the direction of $\boldsymbol{l}$ at an angle of order $|\boldsymbol{\omega}|\,T/\pi$, where T is the period of the waves. For 10-second waves the angle is of the order of 1 minute of arc. It appears justified, therefore, to conclude that the steady state of swell waves on a rotating Earth is given, to a high approximation, by Gerstner's solution.

The unsteady state.—The foregoing arguments may be used to obtain information about the unsteady state. It will be shown that in latitude 45° swell waves set up from rest differ significantly from Stokes waves after about $4\frac{1}{2}$ hours.

Before swell arrives in a given area, the water is at rest. Swell waves invading this area will therefore initially be free from vorticity, i.e. they will take the form of Stokes waves; we have seen that this is not a steady state. Consider the circuit of Theorem 5. As the wave-height increases, a surface drift in the direction of propagation develops, leading to an increase in circulation, which means that the flow along the upper horizontal side is increased. This implies a resultant velocity along the crests, so that there is a tendency for the drift current to change its direction. It is easily seen that this tendency, depending on the projected area of the circuit, is effectively the same as for a shearing current, whose velocity at any depth is equal to the corresponding mass transport velocity. Also with the assumption that the velocity at a great depth is zero the change in mean velocity is completely determined. Considering then a laminar shearing current, we find with little difficulty that each layer has a constant mean absolute velocity whose direction turns through four right angles in half a pendulum day (equal to 12 cosec ϕ hours, where ϕ is the latitude, e.g. 17 hours in latitude 45°). Each particle moves in a circle, the circle of inertia, of radius $V/2|\boldsymbol{\omega}|\sin\phi$, where V is the absolute magnitude of the velocity of the particle. The drift is of the order of 5 cm./sec. in the surface layer; the corresponding radius is of the order of 500 metres in latitude 45°. This represents the greatest possible drift of a particle from its mean position; experimental verification should be possible.

In terms of vorticity, the Coriolis force generates a vorticity in just the direction assumed by Gerstner's theory. The Gerstner state of vorticity is reached in 3 cosec ϕ hours, about $4\frac{1}{2}$ hours in latitude 45°. (The mass transport is then along the crest.) This time is short compared with the life of a wave-train, in moderate and high latitudes, where the Earth's rotation must therefore be taken into account. The vorticity continues to increase, oscillating about the Gerstner value, probably with decreasing amplitude.

The radius of the horizontal motion is proportional to the mass transport velocity corresponding to the final amplitude if the waves build up quickly, i.e. in a time short compared with a quarter pendulum day. If the waves build up slowly, each increase in amplitude will probably make an independent contribution to the surface drift, each contribution initially moving in the direction of propagation and rotating with period 12 cosec ϕ hours. The magnitude of the resultant velocity in each layer may therefore be much reduced, with a consequent reduction in the radius of the inertia circle. The interference of circular drift motions established at different times and different places will also have a mutually destructive effect, so that in practice the radius of the horizontal motion should be much reduced. Drifts of the inertia type often persist for several days.

Conclusion.—It has usually been assumed that circulatory inertia currents are generated by viscous or turbulent drag on the surface (see **6**, p. 44). To reconcile observation and theory, a very large coefficient of eddy viscosity is required, of which no direct measurements are so far available. The wave theory of inertia currents outlined in this paper avoids this difficulty; it is suggested that some observed currents may be more easily explained by this theory. Further observations are needed before more definite conclusions can be drawn.

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