Barotropic Quasi-Geostrophic Flow Over Anisotropic Mountains

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ABSTRACT

This paper discusses the nature of quasi-geostrophic β-plane flow over an idealized set of ridges with height $h = F(y/L_y) \cos x/L_x$. When the mountains are highly anisotropic, with scale factor ratio $L_x/L_y \ll 1$, the asymptotically exact forced solution is governed by a simple set of three nonlinear ordinary differential equations similar to those obtained by Charney and DeVore (1978). For fixed forcing, the region of parameter space where multiple, stable steady solutions exist is mapped out. A cusp catastrophe occurs in which a rapid zonal flow over the ridges drops to a very low value as a parameter like the driving Rossby number decreases slightly below a certain critical point; and the zonal flow then remains at this low value for a large range of Rossby number on either side of the bifurcation value. The existence of limit cycle solutions is discussed. Such periodic solutions are shown to exist for the f-plane case, and probably exist for the β-plane as well. However, numerical solutions indicate that the limit cycles are unstable, with the steady solutions being favored. The stationary solutions are also shown to be stable with respect to barotropic isotropic perturbations.

1. Introduction

The influence of large-scale topographic variations on the slow flow of a homogeneous fluid on a rotating planet has long been a problem of great interest to meteorologists and oceanographers. One aspect of this problem which deserves special attention is the question of uniqueness and bifurcation of finite-amplitude steady solutions of the nonlinear hydrodynamic equations for forced motion over topography. Suppose it is possible that more than one stable steady solution exists for a particular set of external parameters. Then it is reasonable to suppose that the actual response will reside in one of these, say, S_1, if the initial conditions were close to it. If now the external parameters change slightly (and quasi-statically) to a point beyond which S_1 exists, the system will either go into a periodic or non-periodic oscillation, or may rapidly make a transition to one of the other stable steady solutions, say S_2. If it does the latter, it may remain in S_2 for a wide range of parameters, particularly if the bifurcation point(s) for S_2 are well removed from that for S_1. Now if S_1 is very different in structure from S_2 we have a mechanism by which the flow may lock onto very different states and persist in these until rather large changes in external parameters are made, or until some large-amplitude perturbation kicks the system out of one of these locked states.

Such ideas as outlined above have their origins in the theory of nonlinear ordinary differential equations often associated with problems in electrical circuit design. Application of such principals in geophysical fluid dynamics has been rare because it has not been possible in very many instances to find even one rigorous steady nonlinear solution for a practical problem involving forcing and dissipation. Pedlosky (1976) found multiple steady-amplitude solutions for nonlinear baroclinic waves in a two-layer fluid. Although the stability of each of these steady solutions was not determined, numerical integrations of his time-dependent equations suggested that only the lowest or first nonlinear steady mode was attained, even if initial conditions were close to the second mode. The suggestion was that the higher modes were unstable and that in a real system only the first mode would be observed. Charney and DeVore (1978, hereafter referred to as CD) discussed barotropic quasi-geostrophic β-plane flow over topography which had a simple form

$$h = \epsilon H_0 \sin \frac{y^*}{L} \cos \frac{x^*}{L} ,$$

where x^* is the zonal variable, y^* the meridional variable and L the horizontal length scale. They represented the streamfunction for the flow in a channel between $y = 0$ and $y = \pi L$ as

$$\psi = A(t) \cos \frac{y^*}{L} + B(t) \sin \frac{y^*}{L} \sin \frac{x^*}{L} + C(t) \sin \frac{y^*}{L} \cos \frac{x^*}{L} .$$

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This is thus a severely truncated Fourier series representation of the flow structure. For their problem it is easily seen that nonlinear wave interactions will generate a whole series of higher harmonics in y and x. Nonetheless, once the spatial structure is specified the hydrodynamic equations are reduced to a set of three nonlinear differential equations for the amplitude coefficients A, B and C. As opposed to Pedlosky's problem the authors demonstrated that two stable steady solutions could exist for the same set of external parameters. These two states were very different in that one had large zonal velocity and one had quite small zonal velocity. The suggestion was that forced flow over topography could either be unblocked or blocked and the actual state attained would be related to the finite-amplitude stability of these states. Realizing that the severe truncation may influence these conclusions, CD conducted numerical integrations of the full shallow-water equations to verify their results from studying the reduced model. There was agreement for several but not all of the cases studied.

We consider in this paper a model similar to that of CD for flow over mountains, but with anisotropic topography in the form of long north-south ridges. This problem is not entirely without geophysical relevance because one might note, for example, that although the east-west scale of the Rocky Mountains is of order 1000 km, the north-south ridge line extends a distance of order 7000 km in the Northern Hemisphere. One can also envision this situation as a prototype, subject to laboratory or numerical experimentation. In any case the beauty of this model is that it is described by a theoretical set of equations almost identical to the spectrally truncated ones proposed by CD but which is derived here without any *ad hoc* truncation. Stable multi-equilibria are shown to exist. By constructing plots of these in parameter space it is suggested that bifurcation may be just as important as the transitions between metastable equilibria proposed by CD in generating blocking or intermittency.

2. The model

We consider barotropic quasi-geostrophic flow over shallow topography on a β-plane. That is we let the fluid mean depth be H_0 and assume a topography with height above $z = 0$ such that

$$h = h_m^*(y^*) \cos x^*/L_x,$$

where h_m^*/H_0 is of the same order as the Rossby number based on the driving velocity U_w. If we use scales U_w for velocity, L_x for horizontal distance and L_x/U_w for time, the governing nondimensional vorticity equation is

$$\frac{\partial \nabla^2 \psi}{\partial t} + J(\psi, \nabla^2\psi + \beta y + h) = -r\nabla^2(\psi - \psi_w), \quad (1)$$

where

$$v = \partial\psi/\partial x, \quad u = -\partial\psi/\partial y, \quad \beta = L_x/a\,\mathrm{Ro}, \quad r = \mathrm{E}^{1/2}/\mathrm{Ro}$$

and

$$h = \frac{h_m^*(y^*)}{H_0\,\mathrm{Ro}} \cos(x) \equiv S(y)\cos(x).$$

In the above, the Rossby number

$$\mathrm{Ro} = U_w/2\Omega L_x,$$

and the Ekman number

$$\mathrm{E} = \nu/\Omega H_0^2,$$

with Ω being the basic rotation rate, a the radius of the earth and ν the effective boundary-layer viscosity. We imagine that the flow may be forced by Ekman suction resulting from the curl of an imposed velocity distribution on the upper horizontal surface such that $\nabla^2\psi_w = 1$ (e.g., the laboratory case where $v_w^* = 0.5U_w r^*/L\hat{\theta}$) or that if $\nabla^2\psi_w = 0$, the flow is uniform when $h = 0$ with velocity $u = 1$.

Since we first want to investigate *forced* solutions generated by the interaction of a zonal flow with topography, we suppose that such solutions will have the same y scale as the topography itself. When h_m^* varies with y scale L_y the forced solutions should have this scale too. Let $\epsilon = L_x/L_y$. When $\epsilon \ll 1$, we are then concerned with long north-south ridges, and the y variable should be rescaled such that

$$\bar{y} = \epsilon y. \quad (2)$$

We also write

$$\psi = \epsilon^{-1}\psi_0(\bar{y},t) + \phi_0(x,\bar{y},t) + \psi_1 + \epsilon\phi_1 + \cdots .$$

That is, the zonal flow U is of order 1 (as suggested by the forcing or initial value), the meridional velocity induced by the mountains is order 1 (the mountain wave is finite amplitude), but the mountain-induced zonal velocity $u \ll U$ as a consequence of the anisotropy of h. Formally,

$$(u_0, v_0) = \left(-\frac{\partial\psi_0}{\partial\bar{y}} - \epsilon\frac{\partial\phi_0}{\partial\bar{y}}, \frac{\partial\phi_0}{\partial x}\right).$$

Introducing these expansions into (1) yields a sequence of problems ordered in powers of ϵ. As ϵ becomes small asymptotically the leading order problem should describe the flow very accurately. Within errors of order ϵ then, this problem can be split up into two equations for ψ_0 and ϕ_0. Dropping the subscripts we have, with the overbar denoting an average over one period of the topography in x,

$$\psi_{\bar{y}\bar{y}t} + r(\psi_{\bar{y}\bar{y}} - \psi_{w\bar{y}\bar{y}}) = \overline{\phi_{\bar{y}}h_x - \phi_x h_{\bar{y}}}^x = \overline{-(\phi_x h)_{\bar{y}}}^x, \quad (3)$$

$$\phi_{xxt} - \psi_{\bar{y}}\phi_{xxx} + \beta\phi_x + r\phi_{xx} = \psi_y \operatorname{Re}(ie^{ix})S(\bar{y}). \quad (4)$$

Note that we can integrate (3) once with respect to $\bar{y}$. With either of the two forcing situations described above we arrive at[1]

$$\frac{\partial U}{\partial t} + r(U - 1) = -\overline{\phi_x h}^x, \quad (5)$$

where

$$U = -\frac{\partial\psi}{\partial\bar{y}}.$$

Now Eq. (4) can be solved by writing

$$\phi = \operatorname{Re}[(f_r(t) + if_i(t))e^{ix}]. \quad (6)$$

Because of the anisotropy, the y dependence only enters parametrically so that the functions f_r and f_i will depend on S, which replaces $\bar{y}$ as a variable. These solutions are exact to order ϵ. The final set of equations are then

$$\dot{U} = -r(U - 1) - \frac{S}{2}f_i, \quad (7)$$

$$\dot{f}_i = -rf_i + US - (U - \beta)f_r, \quad (8)$$

$$\dot{f}_r = -rf_r + (U - \beta)f_i. \quad (9)$$

Apart from numerical values of the coefficients, these equations are identical in structure to those proposed by CD. Similar multi-equilibria are then expected to exist. The next section explores this question. Note first that v goes to zero at $y = (0, L_y)$ because we have specified $S = 0$ there. If this were not so, or if $U(S)$ is discontinuous over the range of S, boundary layers in which y scales with L_x would be needed to turn the flow near such latitudes. It is assumed that this is dynamically possible. Clearly, the simplest physical situation to imagine is one in which S is constant over a large range of y, and we assume that whatever happens at the ends where S goes to zero (or if h intersects a wall) does not affect the solutions appreciably near the center of the channel.

[1] The constant 1 in the parenthesis of Eq. (5) reflects our particular forcing; the upper surface is in either uniform rectilinear motion or in uniform solid body rotation in cylindrical coordinates (e.g., the laboratory case). The additional constant of integration obtained in integrating (3) is zero for the following reason. If we integrate the full vorticity equation (1) over area inside a vertical boundary placed at some low latitude, say, using Stokes' theorem and the fact that the boundary must be a streamline ψ_B,

$$\int_{\substack{\text{area}\\ \text{inside } \psi_B}} J(\psi, \nabla^2\psi + \beta y + h)dA = 0,$$

so that

$$\oint_{\psi_B}\left[\frac{\partial\mathbf{v}}{\partial t} - r\mathbf{v} + r\mathbf{v}_w\right]\cdot d\mathbf{l} = 0.$$

Finally, since $\phi_x = 0$ at ψ_B in our geometry, the integration constant in (5) must be set equal to zero.

3. Critical Points

A discussion of the solution topology for this problem rests on a determination of the critical points of Eqs. (7)–(9). Since the critical points are the allowable (e.g., real) steady solutions of the equations, their determination is of considerable physical significance as well. Setting the time derivatives equal to zero, it is found that stationary solutions are given by the real roots of

$$(1 - \bar{U})(r^2 + (\bar{U} - \beta)^2) - \tfrac{1}{2}S^2\bar{U} = 0, \quad (10)$$

where in this section the overbar simply denotes a steady solution. Since this equation is cubic, there are either *three* distinct real roots or *one* real root. It is especially interesting that even as $r \to 0$ the flow is still governed by a cubic. The *steady* zonal flow friction term is proportional to r but so is the wave flux in general (Hart, 1977) so the net effect of the wave on the zonal flow is dependent on the existence of friction but independent of its magnitude in the inviscid limit. Since all the external parameters in (10) are squared it is easily seen that the real solutions for $\bar{U}$ must fall between zero and one. From (9) it can be shown that the phase α of the mountain wave high-pressure ridge with respect to the peaks is given by

$$\tan\alpha = -f_i/f_r = \frac{-r}{(\bar{U} - \beta)}$$

for steady flow. Thus roots with $\bar{U} > \beta$ involve mountain waves with ridges upstream of the peaks, while if $\bar{U} < \beta$, the ridge is in the lee.

It is necessary to examine the stability of the critical point solutions. We set

$$\begin{pmatrix} U \\ f_r \\ f_i \end{pmatrix} = \begin{pmatrix} \bar{U}_k \\ \bar{f}_{rk} \\ \bar{f}_{ik} \end{pmatrix} + e^{(\sigma - r)t}\begin{pmatrix} U' \\ f'_i \\ f'_r \end{pmatrix}$$

and linearize (7)–(9). In the above, k ranges over all the real solutions $\bar{U}_k$ of (10). The eigenvalue σ is determined from the roots of

$$\sigma^3 + a\sigma^2 + b = 0, \quad (13)$$

where

$$a = \tfrac{1}{2}S^2 + (\bar{U}_k - \beta)^2 + (\bar{U}_k - 1)(\bar{U}_k - \beta), \quad (14)$$

$$b = r(\bar{U}_k - 1)(\bar{U}_k - \beta). \quad (15)$$

Clearly, a steady solution will be stable if $(\sigma - r)$ all have negative real parts. As r becomes small, stability depends primarily on the sign of a. If a is negative the stationary flow may be unstable on an order 1 time scale. Since $\bar{U}$ is always less than one, we see that small values will tend to make a negative (for some S) unless $\bar{U} < \beta$. This suggests that stable steady states may include those for which $\bar{U}$ is either close to one or quite small. If $(\sigma - r)$ crosses through zero along a steady curve $\bar{U}(S)$, another steady solution is expected to branch from the original. Otherwise a bifurcation to a periodic solution (which may itself be unstable) may be expected. Extensive numerical solutions of the nonlinear equations (7)–(9) discussed in Section 5 suggest that there are no stable limit cycles for this problem and that loss of stability at a branch point will be accompanied by a transition from one steady solution to another. Indeed it appears that the solution topology is a cusp catastrophe. At the bifurcation point the cubic equation for $\bar{U}$ factors into the product of a linear and quadratic form. Two solutions coalesce and become complex so that sudden transitions from one steady mode to another may be expected as the bifurcation point is crossed.

For example, consider the f-plane case with $\beta = 0$ and small friction parameter r. The steady solutions of Eq. (10) are, approximately,

$$\bar{U}_{1,2} \approx \tfrac{1}{2} \pm \tfrac{1}{2}(1 - 2S^2)^{1/2}, \qquad (11)$$

$$\bar{U}_3 \approx 2r^2/S^2. \qquad (12)$$

Thus for the mountain height parameter $S < 0.707$ [recall $S = h_m^*(y)/H_0R_0$], three steady solutions exist, two with moderate to large values of U and one with a very small value of U (the blocked state). If $S > 0.707$, only the blocked solution is admissible. Note that if we had concentrated only on the inviscid problem for small r we would have missed once exact solution entirely. Davey (1978) has shown that it is possible to calculate such steady inviscid flows since as $r \rightarrow 0$,

$$\nabla^2\psi + h = F(\psi),$$

in general, and he shows that

$$F(\psi) = \oint_{\psi} hds/|\vartheta| \Big/ \oint_{\psi} (ds/|\vartheta|) + 1.$$

This method focuses on the inviscid solutions and misses entirely the viscous one. Eq. (11) suggests there may even be two solutions to the inviscid problem (suggesting even more may exist with more complicated topography), but as we shall see the lower root is unstable.

The wave structure is very different for $\bar{U}_1$ and $\bar{U}_3$. In $\bar{U}_1$ the wave amplitude is order 1 and the wave is shifted only slightly upstream. $\bar{f}_{r1} \approx S$ and $\bar{f}_{i1} \approx rS/U_1$. For the $\bar{U}_3$ solution $\bar{f}_{i3} \approx 2r/S$ and $\bar{f}_{r3} \approx 4r^2/S^2$, so although the flow is blocked, the wave amplitude is much larger than the zonal flow and the wave is 90° out of phase with h. These two situations correspond closely to the two equilibria presented by CD where in the blocked state the wave is out of phase with the topography.

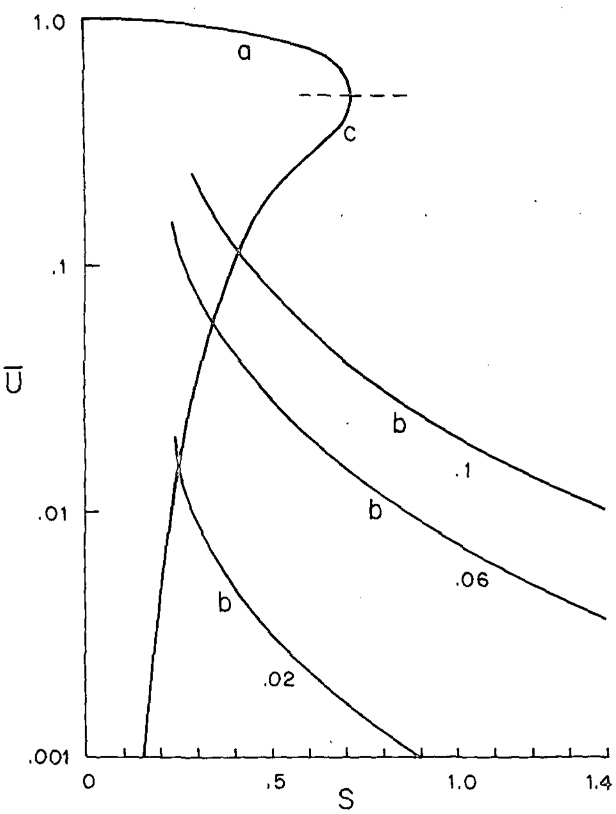

FIG. 1. Approximate zonal flow equilibria for $\beta = 0$, small r (values attached). Curves a and c are the inviscid solutions and b is the viscous one. Curve c is an unstable root.

Fig. 1 shows these approximate solutions plotted against S for various values of r. The approximation breaks down where the viscous roots and the lower inviscid one cross. However, it is clear that a catastrophic transition can occur if $\bar{U}$ is initially large and S increases slowly across the bifurcation point $S = 0.707$ where the two inviscid roots coalesce. The zonal flow should then drop down to the blocked state $\bar{U}_3$. Note that once the flow is in this stable state it will remain there over a fairly wide range of S. There will be a marked persistence. Of course, once the bifurcation value is exceeded there is no assurance the system will indeed drop down to curve (b) even though it is a linearly stable state. It could go into a limit cycle. We show below that some limit cycle solutions are possible, although numerical simulations of the equations show that these are unstable and that direct transitions as postulated here always occur.

Fig. 2 shows the exact roots of Eq. (10) for various values of r with $\beta = 0$. Multiple stable solutions are generally possible provided r is less than about 0.17. The double cusp structure indicates that the per-

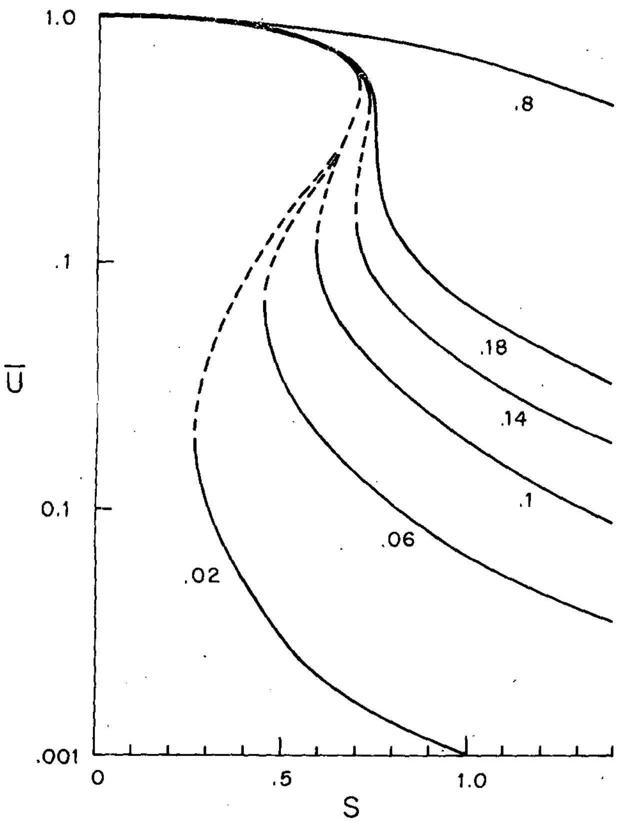

FIG. 2. Solution curves determined from Eq. (10) with $\beta = 0$ and values of r attached. Solid lines give stable equilibrium values of U. The dotted portions are unstable.

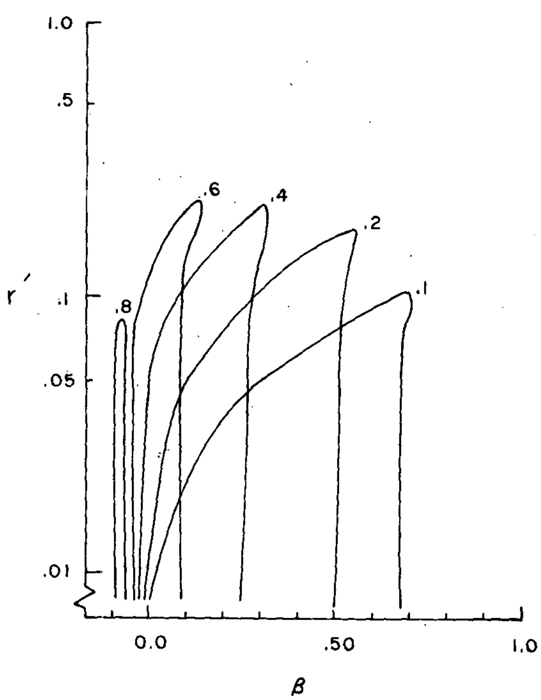

FIG. 3. Boundaries in parameter space where multiple stable steady solutions occur. Inside each loop there are *two* stable equilibria for the value of S attached, outside *one*.

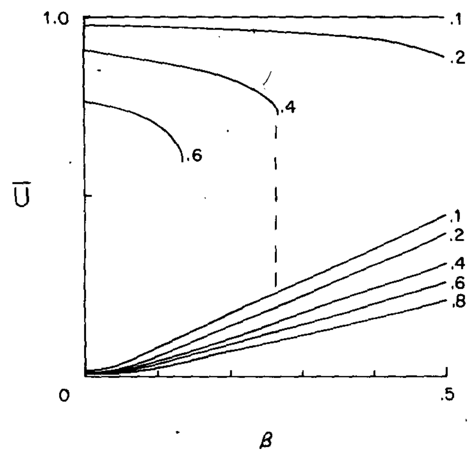

FIG. 4. The two stable roots $\bar{U}_1$ and $\bar{U}_3$ for $r = 0.01$ as functions of the S values attached.

sistence will be more pronounced for smaller values of r. It should be noted that in practice these values of r are substantially less than any reasonable value of β. Thus except for laboratory study where the β-effect can be totally eliminated, these solutions may be of academic interest only.

However, after solving (10) with $\beta \neq 0$, it is quickly found that there are large regions of parameter space where stable multi-equilibria exist. Fig. 3 indicates where they may be found. Generally two stable states exist for moderate to small values of r, S and β. Outside the loops the stable root is always the small one (e.g., $\bar{U}_3$).

Fig. 4 gives the steady zonal flows possible when $\beta \neq 0$ and small r for various values of S. The situation is very similar to that in Fig. 4. If β is fixed equal to that value for which the $S = 0.4$ upper solution terminates, and then S increases from below to just past 0.4, the solution may drop down the dashed line to the lower curve and sit in this lower regime for a wide range of S or β. In a typical laboratory situation β, S and r are all inversely proportional to the driving velocity U_w. If this is varied one moves along slanted trajectories in parameter space. The same catastrophes can be shown to occur. In the atmosphere one can hypothesize (with a little imagination) that slow changes in U_w or ν might cause the required variations in β, S or r. This hypothetical scenario is different from that proposed in CD. There, transitions between steady equilibria were suggested whenever *finite* perturbations, perhaps caused by baroclinic eddies, kicked the solution vector out of the capture range of one of the steady solutions. Although this mechanism may be important, it is suggested here that a system with many equilibria, such as that described in Appendix B, may undergo several catastrophes for relatively small changes in S. This may provide the foundation for an explanation of intermittency and persistence in more complicated physical situations.

4. Limit cycles

The existence and stability of cyclic solutions to Eqs. (7)–(9) is an important but difficult subject of considerable mathematical complexity. Only a few somewhat restricted statements will be made here. The discussion is confined to almost adiabatic (small r) limit cycles for the f-plane. The search for others is done by numerical experimentation.

First note that an "energy" integral in the usual form does not exist because the system is dissipative and driven. However, by multiplying (7)–(9) by $(2U, f_r, f_i)$, respectively, and adding, we see that

$$\frac{d}{dt}(U^2 + \tfrac{1}{2}f_r^2 + \tfrac{1}{2}f_i^2) = r(2U - 2U^2 - f_i^2 - f_r^2). \quad (16)$$

Now for a limit cycle with period T, $T^{-1}\int_0^T (dg/dt) \times dt = 0$, where g is any (cyclic) variable. Thus for limit cycles

$$\overline{2U^2 + f_r^2 + f_i^2} = 2\bar{U}, \quad (17)$$

where the overbar now denotes an average over the period of the limit cycle. This equation implies that $\bar{U} > 0$. A scenario for determining the cycles follows Pedlosky (1972). An inviscid limit cycle involves two free constants of integration A and B because for the inviscid system there are two "energy" integrals.[2] After a fairly long time over which the flow has forgotten its initial values (because $r \neq 0$), inviscid limit cycles can be found by using Eq. (17) and another condition derived from the full equations,

$$\bar{f}_r = -2(\bar{U}^2 - \bar{U})/S, \quad (18)$$

to evaluate A and B. Inviscid solutions to (7)–(9) with $\beta = 0$ are governed by

$$\dot{U}^2 = \tfrac{1}{2}S(A - S)U^2 - \tfrac{1}{4}U^4 + B. \quad (19)$$

The approach is to solve (19) giving $U(t,A,B)$ and then use (8), (9), (17) and (18) to eliminate A and B and determine the values of S that allow cyclic solutions to exist. First note that if B is positive, the solution $U(t)$ is proportional to the elliptic function cn(t). For this possibility $\bar{U} = 0$ and (17) cannot be satisfied. For $B < 0$ the solution is

$$U = 2a[1 - \kappa^2 \,\mathrm{sn}(at,\kappa)]^{1/2} = 2a \,\mathrm{dn}(at,\kappa), \quad (20)$$

where sn(t,k) and dn(t,k) are again elliptic functions,

$$a = \{\tfrac{1}{4}(A - S)S - \tfrac{1}{2}[\tfrac{1}{4}(A - S)^2S^2 + 4B]^{1/2}\}^{1/2}, \quad (21)$$

[2] The inviscid versions of (7)–(9) conserve kinetic energy and enstrophy.

and the modulus k^2 is just $1 + B$. It is apparent from (20) that this solution is made up of a time-averaged part and a periodic oscillation about $\bar{U} = \pi a/K$, where K is the complete elliptic integral of the first kind. The solvability conditions (17) and (18) yield

$$A = \frac{2a\pi}{SK} - \frac{2a^2E}{K}, \quad (22)$$

$$A^2 + 4(\kappa^2 - 1)/S^2 = 2\pi a/\bar{K}, \quad (23)$$

where $E(k)$ is the complete elliptic integral of the second kind. Finally, since $a^2 = (A - S)S/2\kappa^2$ these can be considered as two equations for A, S and κ. Eliminating A would give an equation of form $F(S,\kappa) = 0$, and because κ ranges from zero to 1, the roots of this equation are readily determined. Thus it is possible to find the values of S for which limit cycles of the above type exist, and knowing κ, their properties are given as well.

For small S an asymptotic solution of the transcendental equation is possible. If we suppose that $A = S + q$, where $q \ll 1$, Eqs. (22) and (23) lead to

$$q = \frac{S^3K^2\kappa^2}{2\pi^2}$$

and an eigenvalue relation

$$\ln \frac{4}{(1 - \kappa^2)^{1/2}} = 2\sqrt{2}\pi S^{-6}(1 - \kappa^2)^{1/2}$$

that is easily solved. Since $q \ll 1$, a is small and (22) directly gives the amplitude of the limit cycle oscillation. Hence

$$\bar{U} = \frac{a\pi}{K} = \frac{S^2}{2}$$

and this is just the small S limit for the lower inviscid steady root from (11). Thus one possible cyclic solution just oscillates around the (unstable) steady solution.

For larger S, numerical determination of the roots shows that for $0.17 < S < 0.87$ two limit cycle solutions are allowed, one being just the extension of the asymptotic solution above, and the second being a solution with substantially larger mean U, but smaller excursions about the mean. As S increases toward 0.87 the two limit cycles merge, much in the manner that the two inviscid roots merge in Fig. 2, and for $S > 0.87$ no cyclic solutions exist. In Table 1 some typical properties of the limit cycles are given.

Ideally one would like to compute the stability of these solutions. Analytically, this is an extremely formidable task. One can attempt to answer this question by initiating a numerical solution of (7)–(9) at small r using the values of U, f_r and f_i corresponding to a limit cycle as found above. Such calculations always evolve to one of the steady

TABLE 1. Limit cycle characteristics.

S	a (amplitude/2)	κ^2 (range)	r (period)
0.20	0.02	0.9999	314.
	8.85	0.015	0.35
0.40	0.14	0.998	63.4
	4.33	0.065	0.75
0.60	0.30	0.978	23.13
	2.62	0.169	1.27
0.80	0.55	0.876	9.02
	1.55	0.389	2.27

roots, indicating that although cyclic solutions are possible, they are not stable.

5. Numerical solutions

The ideas of the previous sections can be explored further by solving Eqs. (7)–(9) numerically subject to specified initial conditions for the three variables U, f_r and f_i. The numerical algorithm employs a fourth-order Runge-Kutta scheme that reduces the time step as required by an error norm.

It was first verified that if the initial vector (U, f_r, f_i) is reasonably close to a stable steady solution, the system will converge to this solution. With reference to Fig. 3, it was found that for $\beta = 0$, $r = 0.025$, say, solutions starting near the upper equilibrium spiral into it. That is, the $U(t)$ value, for example, approaches $\bar{U}_1$ in the form of a damped oscillation about $\bar{U}_1$. It is interesting that solutions starting near $\bar{U}_3$ converge to the lower root monotonically. This is probably because this is a "frictional" root [see Eq. (12)]. Monotonic equilibration was found for all cases run at large (>0.25)r. Such behavior is reminiscent of that found for equilibration of nonlinear baroclinic waves at various values of r (Hart, 1973; Pedlosky, 1971).

To test the bifurcation theory for $\beta = 0$ an integration was started using as initial values the *steady* upper solution $(\bar{U}_1, \bar{f}_{r1}, \bar{f}_{i1})$ for the termination point at S_c, say, of the upper $\beta = 0$, $r = 0.08$ curve in Fig. 3, except with $S = 1.05\ S_c$. Fig. 5 (curve a) shows that the solution does make a transition to the lower root in a time on the order of r^{-2}, although most of the switch occurs in a time of order r^{-1}. Once in the lower state a system subject to only modest perturbations cannot recover to $\bar{U}_1$ unless $S \lesssim 0.58$. Fig. 5 (curve b) illustrates a solution starting with the lower steady solution obtained for $\beta = 0$, $r = 0.08$, $S = 0.58$ as the initial condition but with $S = 0.54$. A monotonic transition back to $\bar{U}_1$ is evident.

A similar situation ensues when $\beta \neq 0$. With reference to Fig. 4, an integration was initiated with starting values obtained from the $S = 0.4$ end point at $r = 0.01$, $\beta = 0.27$. With β increased by 7% the solution evolves to the lower root as shown in Fig. 6. At this small value of r there are many more oscillations and overshoots, but the basic transition to lower values of the zonal flow U occurs fairly rapidly.

These numerical calculations seem to indicate that although we have not proven the absence of limit cycles when $\beta \neq 0$, they are not attained by passing a bifurcation point. If they do exist their capture range may be very small, although admittedly this is only a conjecture based on a finite number of numerical experiments. Even attempts to establish the capture ranges of two coexistent steady solutions are difficult because they depend on all three initial values. If initial values f_r and f_i are moderately close to $\bar{U}_3$, say, the system is found to evolve to $\bar{U}_3$ even if $U_1(t = 0) = \bar{U}_1$.

An extensive numerical study was made by conducting many integrations starting at different randomly chosen initial conditions with fixed external parameters. Fig. 7 shows typical phase space trajectories. The arrows show the direction of increasing time. In one case the initial conditions are close to an unstable steady root. In both ex-

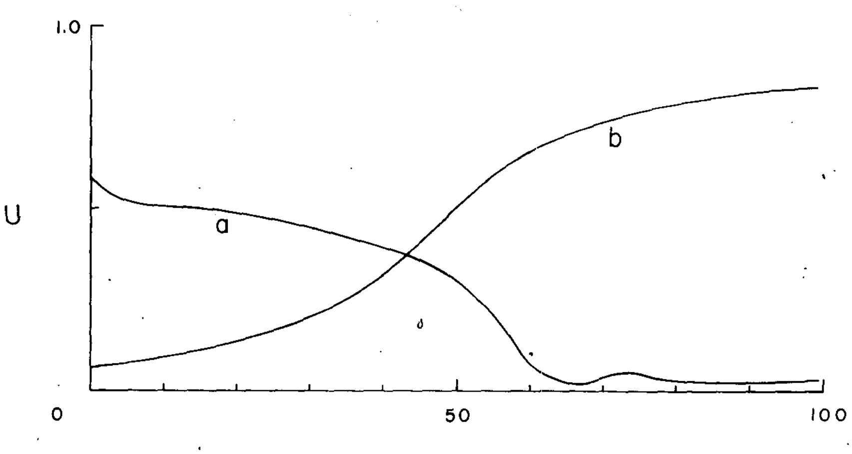

FIG. 5. Numerical integrations for transition between upper and lower states with $\beta = 0$, $r = 0.08$. In (a) $S = 0.74$, in (b) $S = 0.53$.

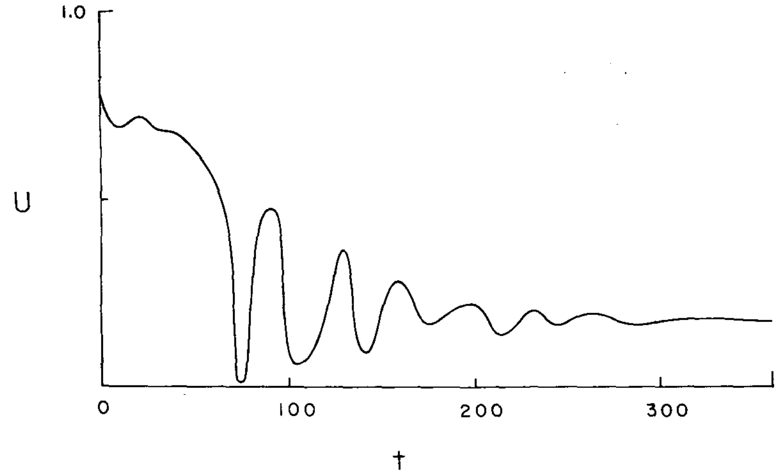

FIG. 6. Numerical integration for $r = 0.01$, $\beta = 0.29$, $S = 0.4$ starting from the steady solution at $\beta = 0.27$.

amples the solution vector spirals in to the lower steady solution. Although one cannot use a finite number of experiments of this type to prove that such spirals always occur, in no initial value problem did a limit cycle or an aperiodic motion typical of a system with a strange attractor appear.

6. Stability to other modes

I have discussed the stability of the forced solution with respect to other highly anisotropic perturbations. That is, the linear stability results of Section 3 are effectively concerned with stability of the mountain induced flow to long-wave perturbations. However, stability of the stationary states with respect to disturbances whose y scales are of the same order as L_x requires additional discussion.

If the vorticity equation (1) is linearized about a basic state $\bar{U}$, $\bar{v}$, noting that $\bar{U}$ is just the zonal velocity (since $\bar{u}$ wave $\ll \bar{U}$), and introducing perturbations of form

$$\phi' = Q(x)e^{il(y-ct)},$$

it is found that

$$\left[il(\bar{v} - c) + \bar{U}\frac{\partial}{\partial x} + r\right]\left(\frac{\partial^2 Q}{\partial x^2} - l^2 Q\right) + \beta\frac{\partial Q}{\partial x} + ilSQ\sin x - il\frac{\partial^2 \bar{v}}{\partial x^2}Q = 0, \quad (21)$$

where $\bar{U}(S)$ is constant and

$$\bar{v} = \frac{\partial \phi_0}{\partial x} = -\bar{f}_r \sin x - \bar{f}_i \cos x.$$

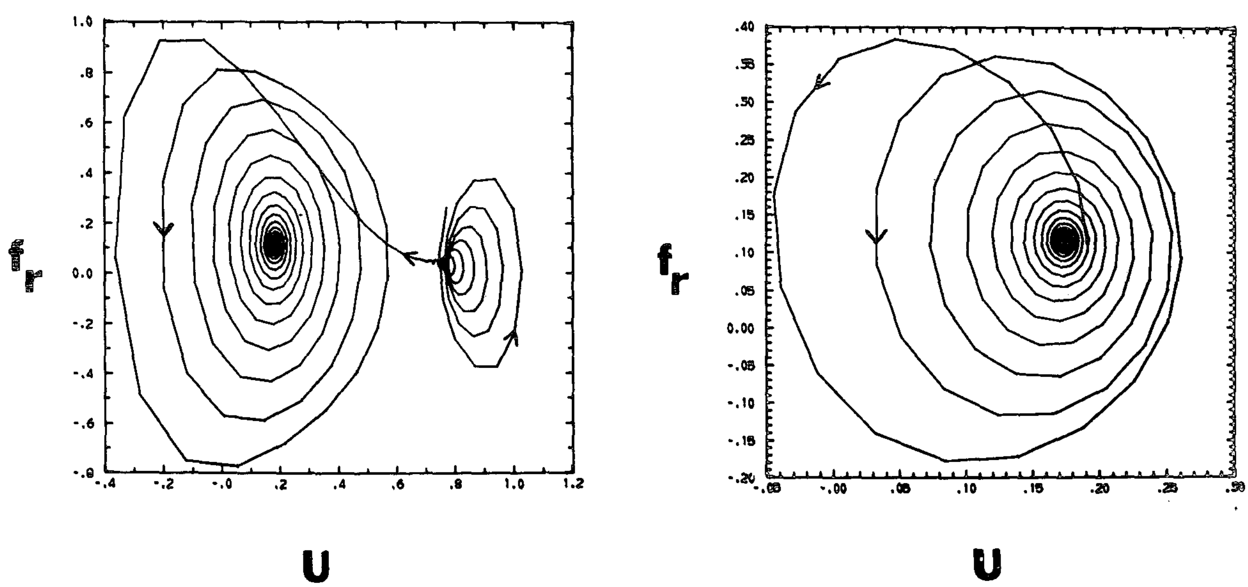

FIG. 7. Typical U–f_r space trajectories for different initial conditions: $\beta = 0.308$, $S = 0.385$, $r = 0.027$.

The above stability problem is in general very difficult because the coefficients are periodic in x. It is somewhat related to the Rossby wave stability problems discussed by Lorenz (1972), Gill (1974) and Coaker (1976). Here, however, friction and advection by the zonal flow are important. The latter is not removable by Galilean transformation. Also, the east-west topography variation plays a crucial role in establishing the basic state potential vorticity gradients.

Since multi-equilibria occur when r is fairly small we only look at this situation. First let $\beta = 0$, and consider the stability of the upper root $\bar{U}_1$. From (11) we have $\bar{U}_1 \sim O(1)$, $\bar{f}_{r1} = S + O(r^2)$ and $\bar{f}_{i1} = O(r)$. Thus the stability equation to leading order in r reduces to

$$\left(\bar{U}_1 \frac{\partial}{\partial x} - ilc\right)\left(\frac{\partial^2 Q}{\partial x^2} - l^2 Q\right) = 0 \quad (22)$$

since the wave basic potential vorticity gradient is only of order r. It is a simple matter to show that if $c_i \neq 0$ the solution of (22) will not be bounded in x so that the actual value of c_i must be of order r or less. Solving (22) we find that

$$Q \propto e^{ilxc/\bar{U}_1}.$$

Thus if c_i is nonzero the perturbation is not bounded in x. This means at worst c_i is of order r, and that then friction will represent an O(1) damping effect on the perturbations. By expanding c and Q in powers of r, and following the procedure used in Appendix A for determining the stability of the frictional root (see below), it is found that $c_i = -r/l + O(r^2)$. The bottom friction dominates these perturbations and renders the f-plane flow stable to linear perturbations at all y wavelengths consistent with the original quasi-geostrophic vorticity equation. Again, this result arises only because the nonlinear mountain wave has essentially zero potential vorticity gradient associated with it.

If $\beta \neq 0$, the $\bar{v}$ field has $\bar{f}_r \sim S\bar{U}/(\bar{U} - \beta)$ and in this case the basic state has an O(1) spatially periodic potential vorticity gradient. Only detailed calculations will divulge the stability characteristics. However, if β is only of O(r) and both are small, the line of reasoning followed above applies and the upper $\bar{U}_1$ flow is stable.

For the lower root ($\bar{f}_{r3}, \bar{U}_3$) are much smaller than $\bar{f}_{i3}$ which itself is of order r when r and β are small. It then is suggested that here too friction will be dominant in determining the stability of the mountain wave. In Appendix A I give the detailed analysis which verifies that the perturbation growth rate $lc_i = -r$ to leading order in r, and hence the flow is stable. Of course, in a real atmosphere other forms of perturbation (e.g., baroclinic ones) may evolve and cause the system to jump out of one of the stationary states in a manner outlined by CD. In an extremely crude way we might lump such effects into r or into random initial condition variation. However a detailed discussion of this aspect of the problem is beyond the scope of this paper.

7. Conclusions

It has been shown that forced, steady, barotropic zonal flow over a set of long ridges can be multivalued. The stable multi-equilibria are located in the space of the three external parameters β, r and $S(y)$ which measure the effects of curvature of the earth, friction and local ridge height relative to the Rossby number characterizing the driving. Since the multi-equilibria regions are finite in this parameter space, and since at least one stable steady solution exists for all values of the parameters, bifurcation points must occur. When the external parameters pass these bifurcation points the flow may make a fairly abrupt transition to an alternate steady state, which if the bifurcation points are fairly far apart, can demonstrate marked persistence. It is suggested that such bifurcation phenomena may be important in atmospheric blocking and in generating short-term climate fluctuations.

It is interesting that for $\beta = 0$ this model predicts that the $\bar{U}_3$ solution effectively dominates at values of S of order 1 when r is small. The forcing, in the form of an Ekman suction velocity in the laboratory case, is balanced by the wave form drag on the topography. This is not Taylor-Proudman blocking since not only is S not large, but the flow is not directed around the topography since $\bar{f}_3$ is only of order r. In a sense this solution represents a Sverdrup-type balance between Ekman suction, and vorticity generation in the zonal mean by wave-induced flow over the topography (which plays the role of β). Under the physical situation described at the beginning of the paper this balance is an alternative exact solution to the one with order 1 zonal flow. When $\beta \neq 0$ and r is small the low velocity solution $\bar{U}_3$ represents a kind of mixed Sverdrup balance with both topographic and β-effects contributing.

A shortcoming of this model is that the topography has form $F(y)$ cos$x$. This assumption was necessary to allow simple exact (as $L_x/L_y \to 0$) solutions of the barotropic vorticity equation. In Appendix B it is argued that more general topography $h \sim \Sigma S_j \cos jx$ does not alter the character of the expansion nor the qualitative nature of the solutions. Although such representations may be adequate for controlled laboratory experimentation, clearly the appropriate meteorological problem involves topography consisting of a few large-scale ridges with small L_x/L_y separated by large distances. The low $\hat{x}$ wavenumbers will thus be excited and contribute to wave interactions scaled out of the present theory. How these affect the qualitative nature of the solution topology needs to be determined.

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APPENDIX A

Stability of $\bar{U}_3$

Let $\beta = r\beta_0$. We have, to leading order in r,

$$\bar{U}_3 = 2r^2/S^2,$$

$$\bar{f}_{r3} = -\frac{2}{S}\beta_0 r,$$

$$\bar{f}_i = 2r/S.$$

We then expand Q and c in powers of r:

$$Q = Q_0 + rQ_1 + \ldots,$$

$$c = c_0 + rc_1 + \ldots.$$

Eq. (21) yields the $O(r^0)$ and $O(r)$ problems:

$$c_0(Q_0'' - l^2Q_0) = S \sin x Q_0, \qquad \text{(A1)[3]}$$

$$c_0(Q_1'' - l^2Q_1) - S \sin x Q_1$$
$$= -(c_1 + irl^{-1} - \bar{f}_{r3}\sin x/r - \bar{f}_{i3}\cos x/r)(Q_0'' - l^2Q_0) + \frac{i\beta_0 Q_0'}{l} - (\bar{f}_{r3}\sin x + \bar{f}_{i3}\cos x)Q/r. \qquad \text{(A2)}$$

First note that c_0 must be real. We multiply (A1) by Q_0^* and (A1)* by Q_0 and subtract, and integrate over a period in x. Thus

$$\text{Im}(c_0)\int_0^{2\pi}(|Q_0'|^2 + l^2|Q_0|^2)dx = 0,$$

so $c_{0i} = 0$. Without loss of generality, then, we can let Q_0 be real. We now multiply (A1) by Q_1, (A2) by Q_0, integrate over a period and subtract. Since

$$\int_0^{2\pi} Q_0 Q_0' dx = 0,$$

the imaginary part of the resulting equation yields $c_{1i} = -rl^{-1}$. Thus the growth rate $\sigma_i \equiv lc_i = -r$, and the basic state $\bar{U}_3$ is stable for $\beta \lesssim O(r)$.

[3] One could in principal solve this Mathieu equation for allowed phase speeds c_0.

The above integrations assume periodicity over one wavelength of the topography. Floquet's theorem suggests that actual solutions may be periodic over some multiple of the basic wavelength, but one can extend the range of integration to encompass a longer interval over which solutions are periodic without changing the result. Physically, one would not expect the growth rates to depend on the sign of $\bar{v}$. Thus, positive contributions to c_i should not enter the expansion until order r^2. Of course, the sign of the friction itself is crucial and thus the friction term does contribute to c_i at order r. There is always the admitted risk that there may be some perturbations that are not describable by a regular perturbation expansion of the proposed form. Short of solving the whole problem numerically this question will remain unanswered, although one has the intuitive feeling that such singular behavior should not occur in a system with fairly strong frictional damping.

APPENDIX B

General Topography

We consider a more general representation of h such that

$$h = \sum_{j=1}^{N} S_j(y)\cos jx.$$

The sum must be finite so that scales of topography that are too small to be considered geostrophic are not included. The governing equations corresponding to (7)–(9) are

$$\dot{U} + r(U - 1) = -\sum_{j=1}^{N} jS_j f_{ij}/2,$$

$$\dot{f}_{ij} + rf_{ij} = US_j/j - (jU - \beta/j)f_{rj},$$

$$\dot{f}_{rj} + rf_{rj} = (jU - \beta/j)f_{ij}.$$

For stationary solutions, equilibria are determined from the roots of

$$U^{-1}(1 - U) = \tfrac{1}{2}\sum_{j=1}^{N} S_j^2/[r^2 + j^2(U - \beta/j^2)^2]. \qquad \text{(A3)}$$

When $\beta = 0$ and r is small the two inviscid roots are given by solutions of

$$U^{-1}(1 - U) = \tfrac{1}{2}\sum_{j=1}^{N} S_j^2/j^2.$$

Thus

$$U_{1,2} = \tfrac{1}{2} \pm \tfrac{1}{2}\Big(1 - 2\sum_{j=1}^{N} S_j^2/j^2\Big)^{1/2}$$

and it is seen, by comparing this result to (11), that S is simply replaced by $\sum_1^N S_j^2/j^2$. The viscous root corresponding to that in Eq. (12) is just

$$U = 2r^2/\sum S_j^2.$$

In this case the more complicated topography has only a minor influence. Some sample calculations of linear stability with $N = 2$, $S_2 = \frac{1}{2}S_1$, indicate that over ranges similar to those in Fig. 2, the upper inviscid root and the viscous one are stable at the same external parameter values.

When β is nonzero the situation is more complicated. The series in (A3) converges very fast for reasonable S_j and one suspects the situation to be well-described by the analysis presented in the body of the paper. However, the order of the characteristic equation (A3) is equal to $2N + 1$, so the possibility of even more equilibria exists. In fact, if r is sufficiently small it is seen that there generally will be a pair of roots near every critical value of $U = \beta/j^2$, i.e., near each zonal wind giving a linear inviscid Rossby wave resonance at nondimensional zonal wavenumber j. It is not known (though easily calculated numerically for specific cases) whether or not any of these additional roots will be stable.

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