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Systematic forcing of large-scale geophysical flows by eddy-topography interaction

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The interaction of eddies with variations in topography, together with a tendency for large-scale wave propagation, generates a systematic stress which acts upon large-scale mean flows. This stress resists the midlatitude tropospheric westerlies, resists the oceanic Antarctic Circumpolar Current, and may be a dominant mechanism in driving coastal undercurrents. Associated secondary circulation provides a systematic upwelling in coastal oceans, pumping deeper water onto continental shelf areas. The derivation rests in turbulence closure theory and is supported by numerical experiments.

1. Introduction

The role of variable topography is recognized to be important to circulation of Earth's and other planetary atmospheres and of Earth's oceans. Importance is made clear, in part, by the large anomalies of potential vorticity given by variation in total depth of fluid. This article is concerned with subinertial vorticity dynamics of eddy–topography interaction. Gravity lee wave phenomena are not included, though these may certainly be significant in many circumstances. The goal of the present article is to show that the vorticity interaction provides an effective rectification mechanism for large-scale mean flows.

Section 2 provides heuristic motivation, followed in §3 by a specific derivation under the idealization of barotropic, quasi-geostrophic flow. When statistics of topographic roughness or of eddy energy are inhomogeneous, a mean torque is obtained in §4. Mean flow rectification is accompanied by secondary circulation which is particularly important for cases of coastal flow where secondary circulation appears as systematic upwelling of deeper water onto continental shelves, as discussed in §5. Statistical projection onto a low-order subsystem is considered in §6. The derivation is theoretical; results are illustrated in §7 from numerical experiments. A wide range of geophysical flows appear to exhibit phenomena much like those here derived; examples and some implications are discussed in §8.

2. Heuristic motivation

This section is a physically motivated overview with two purposes. First, one identifies important processes with a relatively simple argument which obtains qualitatively the rectification mechanism. This may provide some interpretation and guidance for the mathematical analysis in §3. Secondly, it may be that the physical processes described in this section are more robust than the limiting idealizations which are imposed in §3.

There are three processes involved:

(*a*) Tendency for vorticity–topography correlation. Viewed simply, if eddy motions tend to randomize locations of water columns, then on average a water column which overlies a topographic elevation will be foreshortened and so tend to exhibit negative (CW) vorticity in the northern hemisphere. Tendency toward geostrophic compensation, even if incomplete, will induce positive pressure anomaly on a geopotential surface associated with negative vorticity. The overall result, valid in either hemisphere, is to induce a positive correlation between pressure anomaly on geopotential surface and local anomaly of topographic elevation.

(*b*) Large-scale vorticity wave propagation. If there is a background gradient in potential vorticity, perhaps due to meridional variation of Coriolis parameter or perhaps due to larger-scale changes in fluid depth, then there will occur a tendency for Rossby wave propagation. On a continental margin, such waves are sometimes called shelf waves. Vorticity perturbations will tend to phase propagate in a preferred direction, here labelled 'forward'. Nonlinear interaction, topographic scattering and any explicit dissipation will attenuate the vorticity perturbation in its forward phase propagation. Meanwhile, eddy processes tend to re-establish the vorticity–topography correlation. At statistical stationarity, the result is that negative vorticity (in northern hermisphere) and hence positive pressure anomaly (in either hemisphere) will tend to lie 'forward' of topographic elevations. Negative pressure anomalies will tend to lie 'forward' of topographic depressions.

(*c*) Momentum transfer by pressure–slope correlation. Positive pressure anomalies on 'forward' topographic faces, with negative anomalies on 'backward' faces, transfer 'forward' momentum from solid earth into the overlying fluid. Momentum so transferred to the fluid may appear in different forms. The deep fluid may be driven to the right or left of 'forward' such that Coriolis reaction on the mass translation will accommodate the transferred momentum. Other external forces may be applied which will resist the transferred momentum. Finally, momentum transfer which cannot be accommodated otherwise will appear as a tendency to propel the fluid in the 'forward' direction.

3. Derivation

Processes described in the preceding section, including eddy-wave interaction, topographic scattering, etc., are not readily analytically tractable. For general environments and circumstances, resort to numerical simulation experiments will surely be required. In this section I adopt restrictive idealizations for reasons of tractability and clarity.

Consider barotropic, quasi-geostrophic motion on an unbounded, rough-bottomed β-plane, where β is due to variation of Coriolis parameter and/or mean bottom slope. For sufficiently small amplitude topography and sufficiently gentle mean slope, horizontal velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ is given by a stream function field $\boldsymbol{u} = \hat{z} \times \nabla\psi$ where $\hat{z}$ is the unit vertical and ∇ is the horizontal gradient operator. Relative vorticity $\zeta = \hat{z}\cdot\nabla \times \boldsymbol{u}$ is $\zeta = \nabla^2\psi$ where ∇^2 is the two-dimensional Laplacian in horizontal coordinates $\boldsymbol{x} = (x, y)$. In addition to the variable velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$, we assume a temporally evolving but spatially uniform velocity $\boldsymbol{U}(t)$. Variable velocity $\boldsymbol{u}$ and stream-function ψ are defined by departure from spatially uniform motion.

Following customary β-plane notation, increasing y is 'north' and increasing x is 'east'. If β is due to bottom slope then increasing y is 'up-slope'.

Bottom roughness is given by a potential vorticity $h(\boldsymbol{x})$ such that the total depth

$H(\boldsymbol{x})$ is given by $H = H_0(1-h/f_0)$ where H_0 is an average depth and f_0 is reference value for Coriolis parameter assuming β-plane approximation. Alternatively to β-plane, one may suppose total depth given by $H = H_0\,e^{-\beta y/f_0}(1-h/f_0)$ assuming constant Coriolis parameter f_0.

Under restrictions that $\beta y'/f_0 \ll 1$, $h'/f_0 \ll 1$ and $\zeta^{-1}/f_0 \ll 1$ where y' is the scale length of the flow domain in the y-coordinate, h' is the characteristic height of h, and ζ' is a characteristic scale for relative vorticity, the vorticity tendency equation is

$$\partial_t \zeta + J(\psi - Uy, \zeta + \beta y + h) = q - \mathrm{D}\zeta, \tag{1}$$

J is the Jacobian determinant $J(A,B) = |\partial(A,B)/\partial(x,y)|$, q expresses any externally applied torques and D is an operator which may act to dissipate fluctuations of ζ. Explicit forms for q and D will be given as needed.

Convenient boundary conditions may be posed either

(*a*) as a zonal channel requiring ψ be constant along two latitudes y_1 and y_2, with possible further conditions depending upon specific choice of D, and requiring ζ and h to be periodic in x over periodicity length L_x, or

(*b*) as a doubly periodic domain with ζ and h periodic both in x and in y.

Concern in this paper will be directed toward the latter (*b*) formulation. In either case and if $\beta \neq 0$, velocity $\boldsymbol{U}$ is required to have no y-directed component. Only the x-directed component, denoted U, has been retained in (1).

Evolution of U may be obtained from consideration of the x-directed momentum budget equation

$$\partial_t U = E(\hat{U} - U) - \overline{\psi\,\partial_x h}, \tag{2}$$

where $E\hat{U}$ is an externally imposed zonal momentum source. If we think of an idealized ocean, $E\hat{U}$ may be spatially uniform part of the x-directed surface wind stress. For simplicity, a linear frictional drag $-EU$ is included, where E is an Ekman coefficient of surface drag.

The essential term in (2) is $-\overline{\psi\,\partial_x h}$ where the overbar denotes spatial average over the flow domain. Geostrophic stream function ψ being proportional to pressure, $-\overline{\psi\,\partial_x h}$ is the exchange of x-directed momentum across the bottom boundary by pressure forces. Imbalance on the right-hand side of (2) results in net tendency $\partial_t U$, cf. Hart (1979).

Alternatively, an integration by parts gives $-\overline{\psi\,\partial_x h} = \overline{h\,\partial_x \psi}$ under the condition of periodicity in x. With ψ the velocity streamfunction, $-\overline{h\,\partial_x\psi}$ is y-directed geostrophic mass transport while $E\hat{U}$ and EU are ageostrophic transports associated with external force and a frictional Ekman layer. At stationary $\partial_t U = 0$, a zero on the right-hand side of (2) is the statement of no net mass translation.

With double periodicity in each of h, q and ζ, we Fourier expand on basis functions $\exp \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}$, denoting the complex Fourier coefficients at wavelength $\boldsymbol{k}$ as $h_{\boldsymbol{k}}$, $q_{\boldsymbol{k}}$ and $\zeta_{\boldsymbol{k}}$. Equations (1) and (2) transform to

$$(\partial_t + \mathrm{i}\omega_{\boldsymbol{k}} + \nu_\kappa)\,\xi_{\boldsymbol{k}} + J_{\boldsymbol{k}}(\psi, \zeta + h) + \mathrm{i}k_x U h_{\boldsymbol{k}} = q_{\boldsymbol{k}}, \tag{3}$$

and

$$\partial_t U = E(\hat{U} - U) + \mathrm{Im} \sum_{\boldsymbol{k}} k_x \psi^*_{\boldsymbol{k}} h_{\boldsymbol{k}}, \tag{4}$$

where

$$\omega_{\boldsymbol{k}} = \left(U - \frac{\beta}{k^2}\right) k_x. \tag{5}$$

Dissipation operator D transforms to $\nu_{\boldsymbol{k}}$, a function of wavenumber k. $J_{\boldsymbol{k}}$ is the Fourier coefficient of the transformed Jacobian with arguments as listed. Streamfunction coefficient ψ_k is related to ζ_k as $\zeta_k = -k^2\psi_k$. The asterisk denotes complex conjugation.

Equations (3) and (4) express a range of physical phenomena including wave propagation as ω_k, turbulent self-advection as $J(\psi, \zeta)$, and eddy–topography interaction as $J(\psi, h)$. Explicit solutions by numerical integration will be given in §7 below. Details of each solution depend upon the particular $h(\boldsymbol{x})$ and upon the particular realization of $q(\boldsymbol{x}, t)$. For these reasons, explicit solutions may not be of general interest beyond serving as illustrations. Instead our goal is to seek the statistical solution for the evolving statistics of (3), (4).

Given the variance spectrum $H_k = |h_k|^2$ of topographic fluctuation, given a spectrum of external torques, given an explicit dissipation function ν_k, and given explicit parameters β, E and U, we seek the statistical distribution of vorticity variance $Z_k = \langle|\zeta_k|^2\rangle$, the vorticity–topography cross-correlation $C_k = R_k + \mathrm{i}I_k = \langle\zeta_k^* h_k\rangle$ and the average uniform translation $\langle U\rangle$, hereinafter denoted U. Angle brackets denote ensemble averages over realizations of h_k and of q_k. An implicit assumption is that fluctuations may be close, in some sense, to multivariate Gaussian. Departure from random phase in the vorticity field will be taken into account; however, topographic fluctuations with far-from-Gaussian statistics should be regarded with caution.

Approaches to such problems have fallen into two categories. If one strictly omits all external forcing and all dissipation while retaining a finite, though possibly quite large, number of Fourier modes, then methods of classical equilibrium statistical mechanics may be applied in the context of large-scale geophysical flows (Salmon, Holloway & Hendershott 1976). When forcing and dissipation are present, the problem is one of disequilibrium statistical mechanics which is, to date, unsolved in the context of geophysical flows. While equilibrium statistical mechanics provides valuable insights (Carnevale & Frederiksen 1987), omission of forcing and dissipation yields results which are systematically unrealistic. This paper addresses the disequilibrium problem, especially seeking to estimate the average topographic force, which would vanish under equilibrium statistical mechanics.

I follow a line of research begun in turbulence theory by Kraichnan (1959) and abbreviated in plausible though *ad hoc* fashion by Orszag (1970). A number of geophysical applications were developed over recent years and are reviewed by Salmon (1982) or Holloway (1986). In particular I here draw upon the theories of Rossby wave turbulence (Holloway & Hendershott 1977) and of topographic turbulence (Herring 1977; Holloway 1978).

From (3)

$$(\partial_t + 2\nu_k) Z_k = \sum_\Delta 2A_{kp} \operatorname{Re}\langle\zeta_k \zeta_p \zeta_q + \zeta_k \zeta_p h_q\rangle + 2k_x U I_k + Q_k, \tag{6}$$

$$(\partial_t - \mathrm{i}\omega_k + \nu_k) C_k = \sum_\Delta A_{kp} \langle h_k \zeta_p \zeta_q + h_k \zeta_p h_q\rangle + \mathrm{i}k_x U H_k. \tag{7}$$

Symbol Σ_Δ indicates summation over wavevectors $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfying $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q} = \boldsymbol{0}$. $A_{kp} = p^{-2}\boldsymbol{k}\times\boldsymbol{p}\cdot\hat{\boldsymbol{z}}$ results from J_k. Q_k is the effective source of vorticity variance due to external torques q_k. We assume that q_k are uncorrelated with h_k.

Equations (6) and (7), together with (4) when $\operatorname{Im}\langle\psi_k^* h_k\rangle = -k^{-2}I_k$ cannot be solved on account of unknown triple correlations of types $\langle\zeta\zeta\zeta\rangle$, $\langle\zeta\zeta h\rangle$ and $\langle\zeta hh\rangle$. Further equations for the evolution of the triple correlations may be obtained from (3) but these equations will involve a variety of quadruple correlations. Algebraic burden increases rapidly. Continuation of this process leads to an unclosed hierarchy in which equations for evolution of correlations of any order involve unknown correlations of higher order.

Solution requires the introduction of a 'closure hypothesis'. Extensive literature deals with this problem (e.g. Leslie 1973; Orszag 1977) which is not resolved satisfactorily. For present purposes I follow Holloway (1978). Suppressing algebraic detail, the procedure is as follows:

Evolution equations for triple correlations may be obtained from (3) with explicit inclusion of quadruple correlations of types $\langle\zeta\zeta\zeta\zeta\rangle$, $\langle\zeta\zeta\zeta h\rangle$, $\langle\zeta\zeta hh\rangle$ and $\langle\zeta hhh\rangle$. In part such quadruple correlations may be expressed as binary products from Z, C and H; a part which cannot be so expressed defines 'fourth cumulant' statistics. The closure hypothesis involves two steps:

(*a*) it is assumed that the net effect of fourth cumulants is to induce a tendency for triple correlations to relax toward vanishing, and

(*b*) it is assumed that a characteristic relaxation time for triple correlations is short compared with evolution timescales of Z or C. The latter assumption is only the restriction that we consider systems near statistical stationarity.

For simplicity and tractability, assumption (*a*) is modelled by a linear triple correlation process such that each triple correlation evaluated at wavevectors $\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q}$ will realize a relaxation timescale θ_{kpq}. Explicit specification of θ_{kpq}, following Holloway & Hendershott (1977) is

$$\theta_{kpq} = \frac{\mu_{kpq}}{\omega^2_{kpq} + \mu^2_{kpq}}, \tag{8}$$

where

$$\mu_{kpq} = \mu_k + \mu_p + \mu_q,$$

$$\omega_{kpq} = \omega_k + \omega_p + \omega_q,$$

$$\mu_k(\mu_k - \nu_k) = \lambda \sum_p \left(\frac{k^2}{k^2+p^2} Z_p + \frac{p^2}{k^2+p^2} H_p\right), \tag{9}$$

and ω_k is given by (5). Because of the condition $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q} = \boldsymbol{0}$ in (3), U which appears in (5) does not affect (8). A turbulent decorrelation rate μ_k is adopted here as a compromise between a more complicated formulation after Kraichnan (1971) and a relatively simpler heuristic form after Pouquet *et al.* (1975). A contribution from H_p results from incoherent topographic wave propagation after Holloway (1978). λ is a single, adjustable scalar coefficient.

Under assumption (*b*), triple correlations achieve their stationary response to forcing by binary products from Z, C and H. Substitution of stationary values of triple correlations into (6) and (7) yields

$$\tfrac{1}{2}\partial_t Z_k + (\eta_k + \nu_k) Z_k - k_x U I_k + \gamma_k R_k = Q_k + X_k, \tag{10}$$

$$(\partial_t - \mathrm{i}\omega_k + \eta_k + \nu_k) C_k - \mathrm{i}k_x U H_k + \gamma_k H_k = 0, \tag{11}$$

where

$$\eta_k = \sum_\Delta \theta_{kpq} |\boldsymbol{k}\times\boldsymbol{p}|^2 \left[\left(\frac{1}{p^2}-\frac{1}{q^2}\right)\left(\frac{1}{p^2}-\frac{1}{k^2}\right) Z_p + \frac{1}{k^2 q^2} H_p\right], \tag{12}$$

$$\gamma_k = \sum_\Delta \theta_{kpq} |\boldsymbol{k}\times\boldsymbol{p}|^2 \left(\frac{1}{p^2}-\frac{1}{q^2}\right)\frac{1}{p^2} Z_p, \tag{13}$$

$$X_k = \sum_\Delta \theta_{kpq} |\boldsymbol{k}\times\boldsymbol{p}|^2 \left[\left(\frac{1}{p^2}-\frac{1}{q^2}\right)^2 Z_p Z_q + \frac{2}{p^4} Z_p H_q\right]. \tag{14}$$

In (12)–(14) contributions from C_k have been omitted, based in part upon C_k^2 small compared with $Z_k H_k$ and also upon the earlier evaluation of Holloway (1978, see especially figure 10) in which contributions due to R_k were retained under circumstances $\beta = U = 0$. Of central concern in this article is the x-directed force which the topography exerts upon the mean flow. From (4) this force is

$$F = \operatorname{Im} \sum_k k_x \langle \psi_k^* h_k \rangle = -\sum_k k_x k^{-2} I_k. \tag{15}$$

From (11), stationary C_k is

$$C_k = \frac{\mathrm{i}k_x U H_k - \gamma_k H_k}{-\mathrm{i}\omega_k + \eta_k + \nu_k}. \tag{16}$$

If one writes F as a sum of contributions from each $\boldsymbol{k}$, i.e. $F = \Sigma_k F_k$, then

$$F_k = -\frac{\eta_k + \nu_k}{\omega_k^2 + (\eta_k + \nu_k)^2} \frac{k_x^2}{k^2} U H_k + \frac{\omega_k k_x}{\omega_k^2 + (\eta_k + \nu_k)^2} \frac{\gamma_k}{k^2} H_k. \tag{17}$$

Consider the two contributions on the right-hand side of (17). Expression (12) for η_k is not necessarily positive at all k, depending upon the distributions of Z_p and H_p. However calculations of Holloway (1978) show η_k to be positive and an increasing function with increasing k for a wide range of circumstances. Then the first term on the right-hand side of (17) takes a sign opposing U. This is a form drag which is unsymmetrical with respect to U on account of the role of U in ω_k^2. For $U > 0$, ω_k^2 may be small whereas $U < 0$ yield larger ω_k^2 for given magnitude of U. The drag strongly resists $U > 0$, especially near $U = \beta/k^2$, but weakly resists $U < 0$.

The second term on the right-hand side of (17) appears to be novel. This force takes the sign of $\omega_k k_x$ and so is negative for all $U < \beta/k^2$. Importantly, in the absence of mean U, the second term tends to accelerate mean U to negative values. It is the role of this bias term which separates the present theory from that of Brink (1986) who argues that unsymmetric drag in the presence of time-periodic forcing results in non-zero mean flow. The sense of the mean flow anticipated by Brink (1986) is realized in numerical experiments by Haidvogel & Brink (1986); at issue however is whether topographic stress acts as 'drag', i.e. opposed in sense to U.

A question arises concerning the relative magnitude of the two terms in (17). Especially, if $U < 0$ then the first term is positive while the second term is negative. Perhaps the net is insignificant? A careful answer requires detailed evaluation of η_k and γ_k for assigned H_k and for U and Z_k, all self-consistently determined from (4), (10) and (11). That is beyond the scope of the present effort. However, an interesting 'approximation' can be made based upon previous evaluations by Holloway (1978) which indicated that 'roughly' $\eta_k \approx \gamma_k$ over a wide range of $\boldsymbol{k}$ and for some range of conditions. The actual skill of this 'approximation' is not known but the suggestion is that the net in (17) may be given 'roughly' by

$$F_k \approx \frac{\eta_k \sigma_k k_x k^{-2}}{\omega_k^2 + \eta_k^2} H_k, \tag{18}$$

where $\sigma_k = \omega_k - U k_x$ is the intrinsic wave frequency and explicit drag ν_k is here omitted. F_k takes the sign of $\sigma_k k_x < 0$ and so tends to accelerate U toward more negative values independently of the sign of U. This strange result is only approximate; if U is more negative than the equilibrium value from the inviscid equipartition solution of Carnevale & Frederiksen (1987), then the net force will turn to positive after the H-theorem from Carnevale, Frisch & Salmon (1981). Also, the approximate form (18) fails when $\beta \to 0$ since cancellation of the two terms in (17) does not occur precisely.

An important comment on this section is that it is inappropriate to speak of F as a 'form drag', since F may not oppose U at all. To the extent that (18) is a valid approximation, F as a 'form stress' is only weakly a function of U.

4. Statistical inhomogeneity and 'form torque'

A limitation of the foregoing derivation is the assumption of statistical homogeneity in fluctuations of h and of ζ. To some extent this limitation can be relaxed in the sense of a two-scale analysis. One supposes that fluctuations on 'short' lengthscales x, y are slowly modulated in their statistics on 'long' lengthscales X, Y. Fourier transforming on the short scales, statistics Z_k or C_k depend parametrically upon X, Y while β or U may be functions of X, Y. To make such substitutions into the full equations (1) and (2) is an imposing challenge for the theory of sheared, inhomogeneous turbulence. However, we may make some simple observations based upon a statistically inhomogeneous form stress (17) or approximately from (18).

A mean torque acting on the large-scale flow will be given by $\Gamma = -\partial_Y F \equiv \Sigma_k \Gamma_k$ where Fourier contributions $\Gamma_k = -\partial_Y F_k$. 'Roughly' from (18),

$$\Gamma_k = \Big[\sigma_k(\eta_k^2-\omega_k^2)\,\partial_Y\,\eta_k + \frac{k_x\,\eta_k}{k^2}(\eta_k^2-\omega_k^2+2\omega_k\,Uk_x)\,\partial_Y\,\beta + 2\eta_k\,\sigma_k\,\omega_k\,k_x\,\partial_Y\,U\Big](\omega_k^2+\eta_k^2)^{-2}\,H_k - \frac{\eta_k\,\sigma_k\,\partial_Y\,H_k}{(\omega_k^2+\eta_k^2)}, \quad (19)$$

where η_k depends parametrically upon Y through the dependence of Z_p and H_p in (12). Terms on the right-hand side of (19) are grouped as those proportional to H_k and a term proportional to $\partial_Y H_k$. Terms multiplied by H_k include effects of statistical inhomogeneity of eddy properties expressed by $\partial_Y \eta_k$, effects of change in mean bottom slope $\partial_Y \beta$ and mean shear effects $\partial_Y U$. With respect to the relationship between form torque and mean shear, one ought to calculate $\partial_Y F_k$ from (17) with full evaluation of η_k and γ_k rather than from (18).

The contribution of $\partial_Y H_k$ in (19) may be interesting. Even in a flow with constant β and homogeneous eddy statistics, gradients of topographic roughness are seen to act as torques on the mean circulation. A physical interpretation of the source of such torques can be given in terms of secondary circulation as described in the next section.

5. Secondary circulation

Following (2) it was noted that the form stress $-\overline{\psi\,\partial_x h}$ may be written $\overline{h\,\partial_x\,\psi}$, given the condition of periodicity in x. The correlation of fluctuations of total fluid depth with fluctuations in y-directed geostrophic velocity supports a y-directed geostrophic mass translation $-\overline{h\,\partial_x\,\psi}$. If one requires no net y-directed mass translation, then ageostrophic transports $E\hat{U}$ and EU must compensate.

Theoretical derivation in this article is given for barotropic flow. However the implied secondary circulations are quite structured in the vertical. Bottom topographic fluctuations occur only in a small fraction, approximately h'/f_0, of the total fluid depth. At depths above the range of topographic fluctuations, the assumed x-periodicity of pressure assures no y-directed geostrophic mass transport. Therefore the entire geostrophic mass transport $-\overline{h\,\partial_x \psi}$ occurs in the deep fluid over a depth range given by the envelope of topographic fluctuations.

Properties that may be more concentrated in the deeper fluid will be preferentially transported. If β is determined by the gradient of Coriolis parameter, then $-\overline{h\,\partial_x \psi}$ has the sense of poleward transport in either hemisphere. If β is given by large-scale variation of fluid depth, as in oceans overlying continental margins, then $-\overline{h\,\partial_x \psi}$ has the sense of upslope or onshore transport of deep waters. The latter case may have considerable consequences since this mechanism has a systematic bias favouring upwelling of deeper, nutrient laden waters in coastal oceans.

Lastly the 'form torque' obtained in §4 is easily understood in terms of secondary circulations. Where there are gradients of topographic fluctuation variance, the mass transport is divergent, exerting net torque on the water column. Likewise changes in large-scale slope result in divergent transport. A particularly interesting case is that of oceanic banks. These shoal regions are predicted from this theory to experience persistent upslope secondary circulation, the convergent flow inducing anticyclonic vorticity.

6. Statistical projection

One implication of present theory is that idealized models of planetary flows based upon highly truncated spectral equations may be systematically defective. A good deal of interest has been stimulated concerning possible multiple equilibrium solutions following original analyses by Charney & DeVore (1979), Hart (1979) or Wiin-Nielsen (1979). These analyses were based upon systems of only three degrees of freedom, raising the question of how well such idealization may reflect the actual atmosphere where some thousands to perhaps a million degrees of freedom are believed to be dynamically significant. While one approach is to increase the number of degrees of freedom from three to tens or hundreds in numerical models, an alternative is to seek a statistical projection of the large degrees of freedom system onto a small subsystem. One fixes attention on the subsystem, imagining that subsystem embedded in the larger system but with the couplings given only in probability. In this sense Egger (1981) sought to extend three component models of multiple equilibria to more realistic context by hypothesizing that couplings to other atmospheric motions would be realized as additive eddy noise. Also, the damping in three component models may be thought to reflect, in part, an eddy viscosity due to other motions. Amplitudes of eddy noise and of eddy damping have been chosen on phenomenological grounds.

Closure theoretical development as §3 provides a systematic basis for assigning eddy noise and eddy damping. More importantly, §3 reveals an effect which has been overlooked in phenomenologically motivated models. Equations (10)–(14) were developed to approximate statistics of (3). We may observe that (10)–(14) are the exact equations for statistics of a stochastic model equation

$$(\partial_t + \mathrm{i}\omega_k + \nu_k + \eta_k)\,\tilde{\zeta}_k + \gamma_k h_k + \mathrm{i}k_x U h_k = q_k + \xi_k, \tag{20}$$

where η_k appears as eddy damping while ξ_k is a random realization of an eddy noise with statistics given by $\langle \xi_k(t)\,\xi_k^*(t')\rangle = X_k\,\delta(t-t')$. One might take η_k and X_k from (12) and (14).

The essentially 'new' term that appears in (20) is $\gamma_k h_k$ for which there was no corresponding term in (3), since γ_k is real whereas $\mathrm{i}k_x U$ is imaginary. $\tilde{\zeta}_k$ in (20) is not ζ_k in (3); rather $\tilde{\zeta}_k$ is a stochastic model variable whose first and second moment statistics are determined from (20) to approximate statistics of ζ_k from (3). In this sense (20) is the statistical projection of (3), with its many degrees of freedom coupled by J_k terms, onto a single-mode (two degree of freedom) system.

Corresponding to (20), the statistical projection for $\partial_t U$ is only (4) with F from (15) for the topographic stress. To construct a three component system in the style of Charney & DeVore (1979), Hart (1979) or Wiin-Nielsen (1979), extended to include statistically the effects of other degrees of freedom after Egger (1981), one could write

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}+\nu+\eta\right)M-\omega N+\gamma h = \xi_{\mathrm{r}}, \tag{21}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}+\nu+\eta\right)N+\omega M+k_* Uh = \xi_{\mathrm{i}}, \tag{22}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}+E\right)U-\frac{k_*}{k^2}hN = E\hat{U}+\tilde{F}, \tag{23}$$

where we have considered only a particular $\boldsymbol{k}$, say $\tilde{\boldsymbol{k}}$, and written $\zeta_{\tilde{k}} = M+\mathrm{i}N$. Subscripts $\tilde{\boldsymbol{k}}$ are implied in (21)–(23). ξ_{r} and ξ_{i} are the real and imaginary parts of $q_{\tilde{k}}+\xi_{\tilde{k}}$. $\tilde{F}$ is F from (15) omitting the contribution from $\tilde{\boldsymbol{k}}$.

The suggestion is that, if one sought a probabilistic treatment of multiple equilibria in terms of three component systems, then (21)–(23) would be more appropriate. A deterministic calculation which might neglect ξ_{r}, ξ_{i} and $\tilde{F}$, while seeking steady solutions, will resemble previous results in certain regards. Depending on parameter values, including η and γ, there may be three solutions, two of which will be stable and one unstable. However, on account of the 'new' term γh in (21), there will occur circumstances when $\hat{U} > 0$ when one of the stable solutions (corresponding to 'blocked' flow) occurs at $U < 0$. The other stable ('streaming') solution occurs at $0 < U < \hat{U}$.

7. Numerical experiments

The statistical hypotheses leading from (3) to (10), (11) are only hypotheses while the result (17) may seem peculiar, especially that the net stress may propel U toward more negative values even when $U < 0$. A straightforward approach to this question is by direct modelling.

Equations (3) and (4) have been integrated in a spectral domain truncated isotropically at radial wavenumber 30. Jacobian terms are evaluated by dealised pseudospectral method (Orszag 1971) and timestepping is filtered leapfrog with exact evaluation of dissipative terms. Dissipation in the vorticity equation is given the form $\nu_k = E+Ak^4$. Topographic fluctuations are assigned randomly such that expected modal variances are given by

$$\langle|h_k|^2\rangle = Bk^{-1}(k_0+k)^{-\frac{5}{2}} \quad \text{with } k_0 = 3.$$

Two sets of experiments were performed. In the first set, steady forcing $\hat{U}$ is applied in (4) while no external torques q_k are applied in (3). The flow is integrated to statistical stationarity and the stationary U is plotted against $\hat{U}$ in figure 1. In a second set of experiments, fields of statistically homogeneous, isotropic torques q_k are applied over a wavenumber band $4 < k < 7$. This raises the eddy energy levels in the flow and results in more negative values of stationary U for given $\hat{U}$, as also shown in figure 1.

Of particular interest are occurrences of negative U with positive $\hat{U}$, i.e. instances

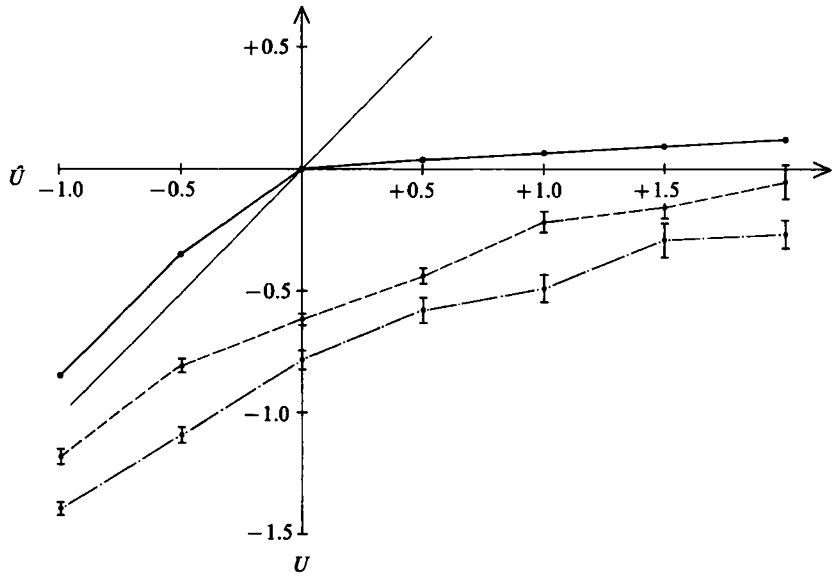

FIGURE 1. Numerical experimental values of mean flow U are plotted against mean driving force $\hat{U}$. ——, force $\hat{U}$ is applied with no other forcing. – – – –, random torques as well as force $\hat{U}$ are applied. —·—, variance of random torques is about twice that shown in the dashed trace. Time averaged U are plotted with bars indicating $\pm$ one standard deviation at statistical stationarity. A light trace indicates the frictional solution $U = \hat{U}$ which would be realized in absence of topography.

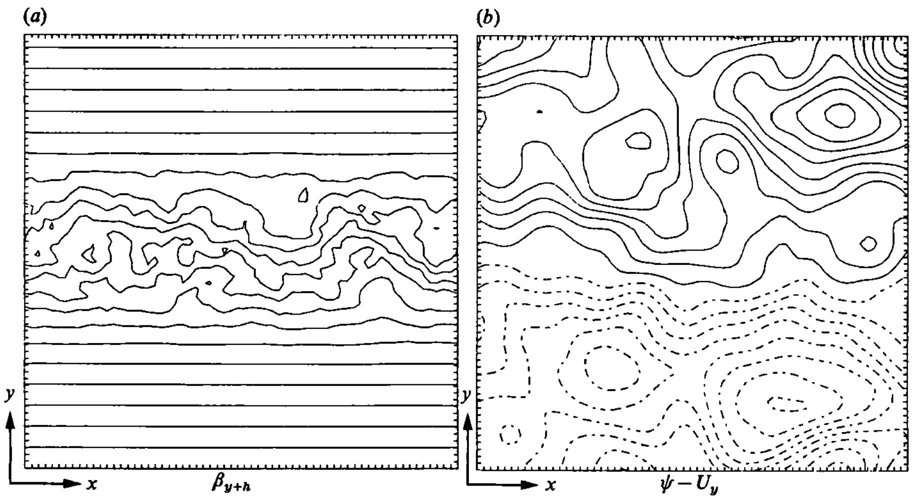

FIGURE 2. (a) A plan view of $\beta y+h$ where h is the same random topography used in figure 1 but here modulated as $\cos^2 y$ for $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$ and zero elsewhere. (b) A plan view of instantaneous total streamfunction $\psi - Uy$ after the flow has achieved statistical stationarity in the presence of random torques and with $\hat{U} = 0$.

where the topographic stress drives persistent mean flow against the mean forcing. The random torques supply no mean momentum but only make up an energy source which must balance both internal dissipation and the energy sink $U\hat{U}$ due to the mean force. Of interest also is the negative sign of topographic stress where U is negative, supporting the argument after (17) that the topographic stress is not just an unsymmetrical drag but rather is biased to take negative values on either sign of U.

Further experiments exhibit the form torque from §4. Random topographic roughness as described in the previous experiments is modulated by $\cos^2 y$ for $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$ and is zero elsewhere. No mean stress is applied, i.e. $\hat{U} = 0$. Statistically homogeneous torques q_k are applied as previously. Positive form torques arise along $y \approx -\frac{1}{4}\pi$, while negative torques arise along $y \approx \frac{1}{4}\pi$ building a negatively x-directed jet along $y \approx 0$. Figure 2 shows a plan view of the modulated topographic roughness and of instantaneous total streamfunction $\psi - U_y$ when the flow achieved statistical stationarity.

Numerical experiments described here do not provide detailed quantitative testing of the foregoing theory. Such effort, requiring full numerical evaluation of equations (8)–(15), lies outside the scope of present investigation. What is given here is a qualitative suggestion of phenomena predicted on closure theoretical grounds.

8. Discussion

The theory developed in this paper offers, I believe, a novel explanation for a wide variety of observed geophysical flows. Especially there are oceanic observations of a propensity for coastal undercurrents overlying the continental margin to flow in the direction of shelf-wave propagation. Such regions are characterized (*a*) by gradients of f_0/H, where H is mean depth averaged in the largescale longshore direction, (*b*) by substantial topographic 'roughness' in the forms of canyons, transverse ridges and other topographic irregularities, and (*c*) substantial eddy activity.

In the northern hemisphere, predicted currents should tend to circulate in counterclockwise sense around ocean basin peripheries. Since the driving force is topographic stress at the bottom of the water column, the phenomenon should be more apparent in the deeper circulation which may sometimes be of opposite sense to surface circulation. An example is the northward flowing California Undercurrent (Kundu & Allen 1976) which is but one example of prevalent occurrences of poleward-flowing eastern margin undercurrents. Along western margins of ocean basins, the present theory suggests a tendency for equatorward flows, as seen for example in the Mid-Atlantic Bight (Beardsley & Winant 1979). In the Arctic Ocean, the predicted tendency is for eastward flow on the basin margins, as might be evidenced by a suggested Beaufort Undercurrent (Aagaard 1984). In the southern hemisphere, the predicted tendency is for westward circulation on the Antarctic margin, (Carmack 1977) opposed to the sense of the Antarctic Circumpolar Current. Evidence for upwelled water associated with a canyon topography on a continental margin is discussed by Freeland & Denman (1982).

Agreement or disagreement in sense only between theory and observation is not a very strong test. However, several of the examples mentioned above flow in a sense opposite to the more apparent driving forces such as wind forcing. For cases of coastal undercurrents, there is often appeal to possible longshore pressure gradients since direct observation of the near longshore tilt of sea surface with respect to geopotential level is difficult. Uncertainty concerning the role of longshore pressure gradients might be ameliorated either in the Arctic Ocean or around Antarctica where one could consider mean flow along a closed path which is smoothly deformed to follow an average isobath.

Another class of oceanic observations is the propensity to find anticyclonic circulation around offshore banks such as the Georges Bank. The form torque obtained in §4 was seen to induce such flows. Again there are other possible explanations, mostly in terms of tidal rectification processes such as Wright & Loder (1985).

A last comment from oceanic observations would concern abyssal circulations where present theory predicts slope currents at the base of continental margins as well as on flanks of mid-ocean ridges following the sense of cyclonic circulation around the peripheries of deep basins (Dickson *et al.* 1985). A number of deep currents are known from their distinctive bottom-water characteristics and it is usually supposed that these are gravity-driven flows, constrained by rotation. If the frictional retardation can be estimated, then a streamwise momentum balance may indicate the role of theorized topographic stress in driving such flows. It is also in the abyssal ocean that one might test for mean westward circulation due to topographic stress associated with planetary β.

In each of the examples mentioned, the outstanding question is the quantitative one: is the expected amplitude of the form stress sufficient to account for the observed flow? At the present stage of development, this important question remains unanswered. However, it is possible to address a more limited question, namely a 'rough' calculation might already show if the stress derived here is inconsequentially small. Let us begin at (18), already leaving aside the question of how well (18) approximates (17). For lengthscales near a dominant eddy scale, η_k takes values 'roughly' as ζ' (the typical vorticity). If Doppler shift $\omega_k - \sigma_k$ is not too great, the expression $\eta\sigma(\omega^2+\eta^2)^{-1}$ in (18) will find its maximum value at the scale for which $|\sigma_k| \approx \eta_k$. If $|\sigma_k| \approx \beta/k$, that scale is $k_\beta = \beta/\zeta'$. In their analysis of flat bottom, β-plane turbulence, Holloway & Hendershott (1977) find k_β to be a transitional scale between wavelike behaviour at larger scales and more turbulent behaviour at shorter scales, with k_β tending to approach the dominant energetic eddy scale. In more weakly turbulent environments, $\eta\sigma/(\omega^2+\eta^2)$ will be dominated nearer the stationary wave resonance $\omega = 0$ at scale $k_0^2 = \beta/U$ where $U > 0$. For present purpose of gross estimation I adopt k_β, taking F_k 'roughly' as $\frac{1}{3}k_\beta^{-1}H_{k\beta}$ where $\frac{1}{3}$ reflects an approximate maximum of $\eta\sigma/(\omega^2+\eta^2)$. Summed over $\boldsymbol{k}$, F may be given roughly as $\frac{1}{3}k_\beta^{-1}H_\beta$ where H_β is the sum of all H_k for which $\frac{1}{2}k_\beta < |\boldsymbol{k}| < 2k_\beta$, say. Since F is acceleration of the depth-averaged flow, stress is $\frac{1}{3}k_\beta^{-1}H_\beta$ multiplied by fluid depth H_0, taking fluid density to be 10^3 kg . m^{-3} for oceanic examples.

Consider the Antarctic Circumpolar Current for which the mean wind stress is eastward at order of 1 dyn/cm^2 or 10^{-4} m^2 s^{-2} at 10^3 kg . m^{-3}. k_β^{-1}, based upon planetary β, is roughly an abyssal eddy Rossby number, say 10^{-2}, times earth radius. Hence $k_\beta \approx 6\times10^4$ m. H_β is the topographic variance as fractional depth of fluid, say 10^{-4} for 50 r.m.s. fluctuations in 5000 m depth, times f_0^{-2} as defined before (1). Hence $H_\beta \approx 10^{-12}$ s^{-2}. With $H_0 \approx 5000$ m, the westward topographic stress is estimated as $\frac{1}{3}k_\beta^{-1}H_\beta H_0 \approx 10^{-4}$ m^2 s^{-2}. There is no pretence to quantitative precision here, only the remark that the theoretically predicted stress is feasibly sufficient to close a momentum budget (Munk & Palmen 1951). A caution is that baroclinic effects will certainly need to be considered in a more complete treatment (Cox 1975).

Consider a coastal circulation, seaward of the shelf break at, say, $H_0 \approx 300$ m. k_β^{-1} is again an eddy Rossby number, perhaps of order 10^{-1}, but multiplied by a lengthscale characteristic of offshore variation of $\ln H_0$, say 10^5 m. Hence $k_\beta^{-1} \approx 10^4$ m. Suppose the fractional topographic variance is 10^{-2}, corresponding to 30 m r.m.s. fluctuations in 300 m. Then $H_\beta \approx 10^{-10}$ s^{-2}. The mean stress acting in the direction of shelf wave propagation is then 10^{-4} m^2 s^{-2} or 1 dyn/cm^2. Such a mean stress would dominate many coastal current regimes. Again I emphasize that calculations in this section are not meant to be precise. Neglect of Doppler shift is certainly not tolerable for many coastal currents, especially those exhibiting $U < 0$ in the deeper water column, for which $\omega^2 \gg \sigma^2$. Then the stress will be significantly reduced in magnitude; however the negative sign of stress is expected to remain despite $U < 0$. Also, I may have chosen uncharacteristically large Rossby number or topographic variance. The present point is only to estimate that the theoretical stress may indeed be large enough to account for the variety of observations mentioned previously.

To close with a more speculative remark, let us recall the discussion of secondary circulation from §5. It was seen that topographic form stress on continental margins provides a systematic upwelling tendency. Implications for biological productivity are apparent. Another major concern in ocean science is the formation of bottom waters at high latitude with the question of mechanisms to raise subsequently the denser fluid, maintaining the mean gravitational potential energy of the ocean mass. such mechanisms are usually considered with respect either to interior mixing, possibly by internal wave breaking, or to turbulent mixing in the benthic boundary layer. Present theory suggests that systematic upwelling on continental margins may be another mechanism. While a barotropic derivation in §3 does not address such a question, an inference is that form stress driven upwelling represents a conversion from horizontal eddy kinetic energy to gravitational mean potential energy.

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REFERENCES

AAGAARD, K. 1984 The Beaufort Undercurrent. In *The Alaskan Beaufort Sea: Ecosystems and Environment*. Academic.

BEARDSLEY, R. C. & WINANT, C. D. 1979 On the mean circulation in the Mid-Atlantic Bight. *J. Phys. Oceanogr.* **9**, 612–619.

BRINK, K. H. 1986 Topographic drag due to barotropic flow over the continental shelf and slope. *J. Phys. Oceanogr.* **16**, 2150–2158.

CARMACK, E. C. 1977 Water characteristics of the Southern Ocean south of the Polar Front. In *A Voyage of Discovery* (ed. M. Angel), 696 pp. Pergamon.

CARNEVALE, G. F. & FREDERIKSEN, J. S. 1987 Nonlinear stability and statistical mechanics of flow over topography. *J. Fluid Mech.* **175**, 157–181.

CARNEVALE, G. F., FRISCH, U. & SALMON, R. 1981 *H*-theorems in statistical fluid dynamics. *J. Phys.* A **14**, 1701–1718.

CHARNEY, J. G. & DE VORE, J. G. 1979 Multiple flow equilibria in the atmosphere and blocking. *J. Atmos. Sci.* **36**, 1205–1216.

COX, M. D. 1975 A baroclinic numerical model of the world ocean: preliminary results. In *Numerical Models of Ocean Circulation*, 364 pp. National Academy of Sciences, Washington, DC.

DICKSON, R. R., GOULD, W. J., MULLER, T. J. & MAILLARD, C. 1985 Estimates of the mean circulation in the deep (> 2000 m) layer of the eastern North Atlantic. *Prog. Oceanogr.* **14**, 103–127.

EGGER, J. 1981 Stochastically driven large-scale circulations with multiple equilibria. *J. Atmos. Sci.* **38**, 2606–2618.

FREELAND, H. J. & DENMAN, K. L. 1982 A topographically controlled upwelling center off southern Vancouver Island. *J. Mar. Res.* **40**, 1069–1093.

HAIDVOGEL, D. B. & BRINK, K. H. 1986 Mean currents driven by topographic drag over the continental shelf and slope. *J. Phys. Oceanogr.* **16**, 2159–2171.

HART, J. E. 1979 Barotropic quasigeostrophic flow over anisotropic mountains. *J. Atmos. Sci.* **36**, 1736–1746.

HERRING, J. R. 1977 Two-dimensional topographic turbulence. *J. Atmos. Sci.* **34**, 1731–1750.

HOLLOWAY, G. 1978 A spectral theory of nonlinear barotropic motion above irregular topography. *J. Phys. Oceanogr.* **8**, 414–427.

HOLLOWAY, G. 1986 Eddies, waves, circulation and mixing: statistical geofluid mechanics. *Ann. Rev. Fluid Mech.* **18**, 91–147.

HOLLOWAY, G. & HENDERSHOTT, M. C. 1977 Stochastic closure for nonlinear Rossby waves. *J. Fluid Mech.* **82**, 747–765.

KRAICHNAN, R. H. 1959 The structure of isotropic turbulence at very high Reynolds Numbers. *J. Fluid Mech.* **5**, 497–543.

KRAICHNAN, R. H. 1971 An almost-Markovian, Galilean-invariant turbulence model. *J. Fluid Mech.* **47**, 513–524.

KUNDU, P. J. & ALLEN, J. S. 1976 Some three-dimensional characteristics of low frequency current fluctuations near the Oregon coast. *J. Phys. Oceanogr.* **6**, 181–199.

LESLIE, D. C. 1973 *Developments in the theory of turbulence.* Clarendon.

MUNK, W. H. & PALMEN, E. 1951 Note on the dynamics of the Antarctic Circumpolar Current. *Tellus* **3**, 53–55.

ORSZAG, S. A. 1970 Analytical theories of turbulence. *J. Fluid Mech.* **41**, 363–386.

ORSZAG, S. A. 1971 Numerical simulation of incompressible flows within simple boundaries. I. Galerkin (spectral) representations. *Stud. Appl. Maths* **50**, 293–327.

ORSZAG, S. A. 1977 Statistical theory of turbulence. In *Fluid Dynamics, 1973, Les Houches Summer School of Theoretical Physics* (ed. R. Balian & J.-L. Peube), 677 pp. Gordon and Breach.

POUQUET, A., LESIEUR, M., ANDRE, J. C. & BASDEVANT, C. 1975 Evolution of high Reynolds number two-dimensional turbulence. *J. Fluid Mech.* **72**, 305–319.

SALMON, R. 1982 Geostrophic turbulence. In *Topics in Ocean Physics, Proc. Intl School Physics Enrico Fermi, Varenna, Italy*, pp. 30–78.

SALMON, R., HOLLOWAY, G. & HENDERSHOTT, M. C. 1976 The equilibrium statistical mechanics of simple quasi-geostrophic models. *J. Fluid Mech.* **75**, 691–703.

WIIN-NIELSEN, A. 1979 Steady states and stability properties of a low order barotropic system with forcing and dissipation. *Tellus* **31**, 375–386.

WRIGHT, D. G. & LODER, J. W. 1985 A depth-dependent study of the topographic rectification of tidal currents. *Geophys. Astrophys. Fluid Dyn.* **31**, 169–220.