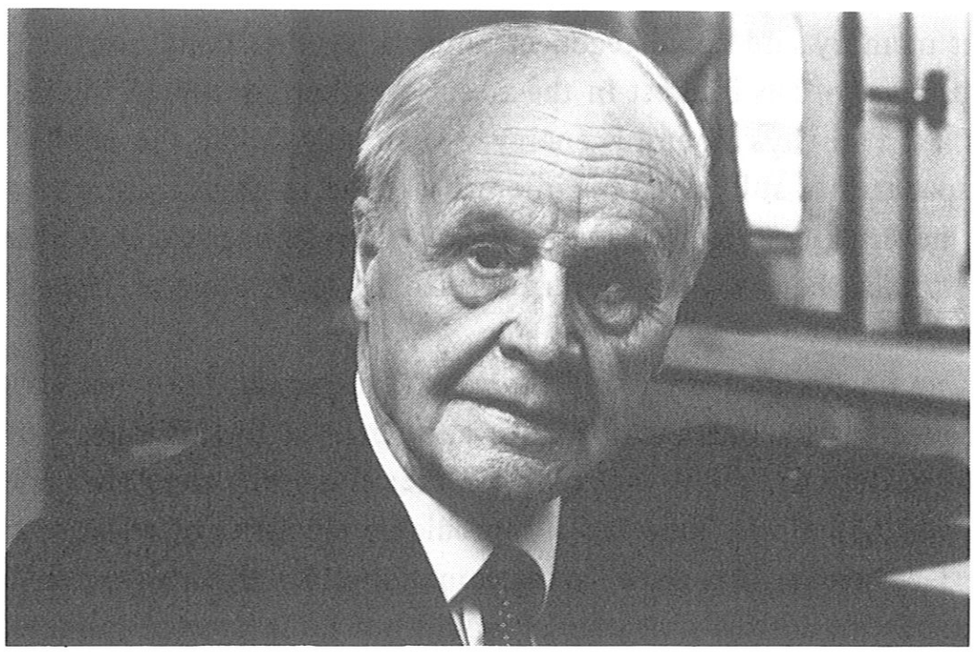
SHORTER CONTRIBUTION

Note on the Dynamics of the Antarctic Circumpolar Current[1]

By W. H. MUNK[2] and E. PALMÉN[3]

Abstract

Unlike all other major ocean currents, the Antarctic Circumpolar Current probably does not have sufficient frictional stress applied at its lateral boundaries to balance the wind stress. The balancing stress is probably applied at the bottom, largely where the major submarine ridges lie in the path of the current. The meridional circulation provides a mechanism for extending the current to a large enough depth to make this possible.



Erik Herbert Palmén

M&P start with a model inspired by Munk's earlier 1950 paper.

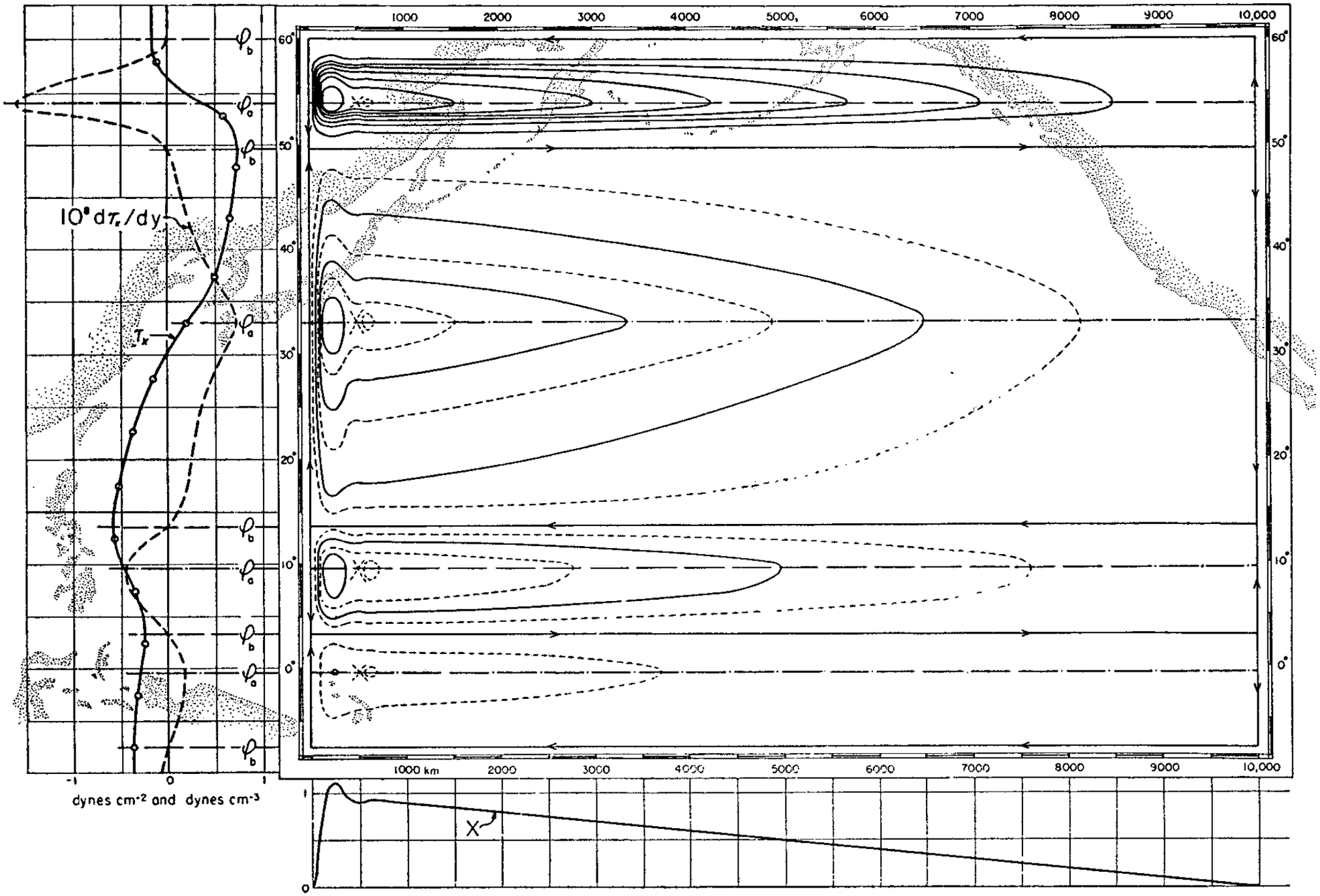
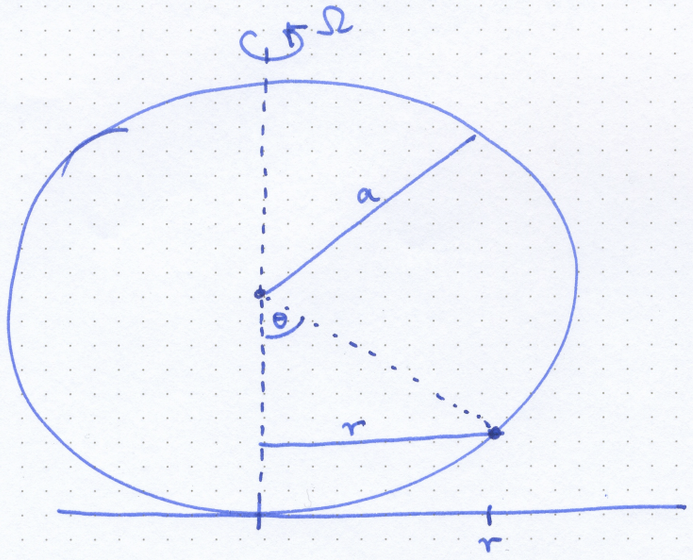


FIG. 2. The mean annual zonal wind stress $\tau_x(y)$ over the Pacific and its curl $d\tau_x/dy$ are plotted on the left, the function $X(x)$ on the lower part. These functions have been combined graphically according to equation (22) to give mass transport streamlines $\psi(x, y)$. The transport between adjacent solid lines is 10×10^{12} g sec^{-1}, or 10 million metric tons sec^{-1}. The total transport between the coast and any point x, y is $\psi(x, y)$. The chart of the Pacific has a uniform distance scale throughout. In the relatively narrow northern portion, the transport is greatly exaggerated.



$r = a \sin\theta$
= cylindrical radius

ϕ = longitude

$$M = \rho \int_{-h}^{0} u_\phi(r,z)\, dz$$

Then

$$A\,(\nabla^2 M - r^{-2} M) = -\tau$$

$\nabla^2 = \frac{1}{r}\partial_r r \partial_r$

$$\rightarrow \quad A\left(M_{rr} + \tfrac{1}{r} M_r - \tfrac{1}{r^2} M\right) = -\tau$$

$$A\,\partial_r\left(M_r + \frac{M}{r}\right) = -\tau$$

Note M = constant × r is solid-body rotation

r is the distance from the axis of rotation

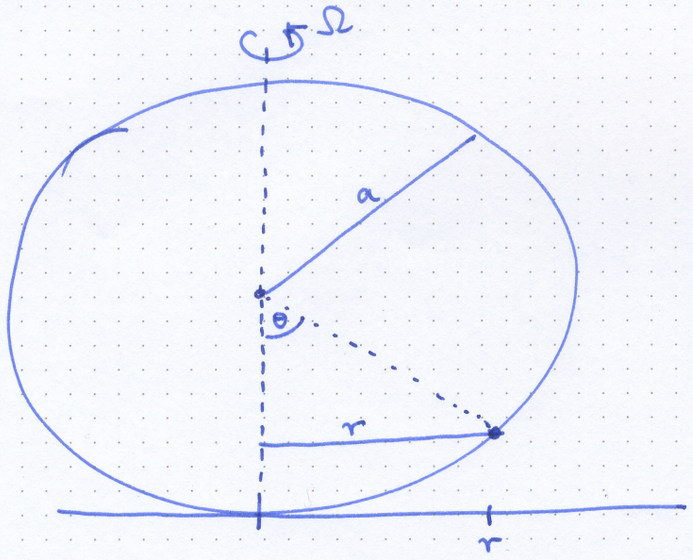
A zonal channel with no-slip at the walls

In its simplest form we may regard the AC current as an eastward flow on a plane tangent to the earth at the South Pole. The flow is induced by constant eastward winds, and depends only on the distance r from the pole on this plane. In this system a balance between the wind stress τ and the lateral friction is expressed by

$$\tau + A \frac{d}{dr}\left(\frac{dM}{dr} + \frac{M}{r}\right) = 0$$

where A is the lateral kinematic eddy viscosity and M the eastward mass transport across a normal vertical plane of unit width extending from surface to bottom.[4] The solution which vanishes at the Antarctic continent ($r = r_0$) and at some other latitude ($r = r_1$) is

$$M = \frac{\tau}{3A}\, r\,(r - r_0)\left(\frac{r + r_0}{r_1 + r_0}\,\frac{r_1^2}{r^2} - 1\right).$$



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of the x-co-ordinate line, but are written out here in view of the frequency of their use. The co-ordinates are

$$\xi_1 = r, \quad \xi_2 = \theta, \quad \text{and} \quad h_1 = 1, \quad h_2 = r,$$

$$\frac{\partial \mathbf{a}}{\partial r} = 0, \quad \frac{\partial \mathbf{a}}{\partial \theta} = \mathbf{b}, \quad \frac{\partial \mathbf{b}}{\partial r} = 0, \quad \frac{\partial \mathbf{b}}{\partial \theta} = -\mathbf{a}.$$

$$\nabla V = \mathbf{a}\frac{\partial V}{\partial r} + \frac{\mathbf{b}}{r}\frac{\partial V}{\partial \theta},$$

$$\mathbf{n}.\nabla\mathbf{F} = \mathbf{a}\left(\mathbf{n}.\nabla F_r - \frac{n_\theta F_\theta}{r}\right) + \mathbf{b}\left(\mathbf{n}.\nabla F_\theta + \frac{n_\theta F_r}{r}\right),$$

$$\nabla.\mathbf{F} = \frac{1}{r}\frac{\partial (rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta},$$

$$\nabla \times \mathbf{F} = \left\{\frac{1}{r}\frac{\partial (rF_\theta)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \theta}\right\}\mathbf{a} \times \mathbf{b},$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2},$$

$$\nabla^2\mathbf{F} = \mathbf{a}\left(\nabla^2 F_r - \frac{F_r}{r^2} - \frac{2}{r^2}\frac{\partial F_\theta}{\partial \theta}\right) + \mathbf{b}\left(\nabla^2 F_\theta + \frac{2}{r^2}\frac{\partial F_r}{\partial \theta} - \frac{F_\theta}{r^2}\right).$$

Rate-of-strain tensor:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad e_{r\theta} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial \theta}.$$

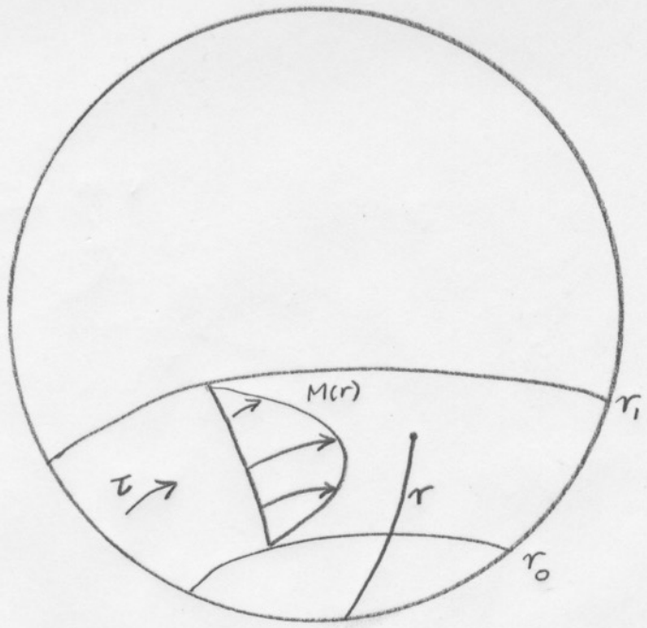
Equation of motion for an incompressible fluid, with no body force:

$$\frac{\partial u_r}{\partial t} + \left(u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}\right)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right),$$

$$\frac{\partial u_\theta}{\partial t} + \left(u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}\right)u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right).$$

$$\nabla^2 F_r - \frac{F_r}{r^2} = \frac{\partial^2 F_r}{\partial r^2} + \frac{1}{r}\frac{\partial F_r}{\partial r} - \frac{F_r}{r^2}$$

The MP zonal channel



$$1\text{dyne}\,\text{cm}^{-2} = 0.1\text{N}\,\text{m}^{-2}$$

$$1\text{Sverdrup} = 10^6\text{m}^3\,\text{s}^{-1}$$
$$\approx 10^9\text{kg}\,\text{s}^{-1}$$
$$= 10^{12}\text{g}\,\text{s}^{-1}$$

M&P use:

$$A = 10^8\text{cgs}$$
$$\tau = 2\text{cgs}$$

The model transport is then:

$$\theta_S = 70°\text{S} \quad \text{and} \quad \theta_N = 45°\text{S}$$
$$\Rightarrow T = 5 \times 10^{16}\text{cgs} = 5 \times 10^4\text{Sv}$$

or

$$\theta_S = 65°\text{S} \quad \text{and} \quad \theta_N = 55°\text{S}$$
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M&P: “The observed transport is 10^{14} grams per second…”
That is100Sv

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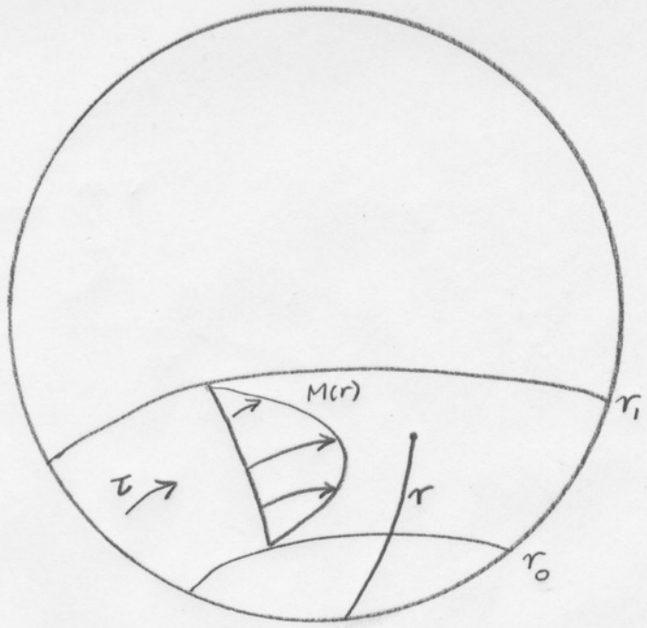

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Discuss: Subsequent estimates of ACC ranged between -15Sv and 250Sv. What is the currently accepted value?

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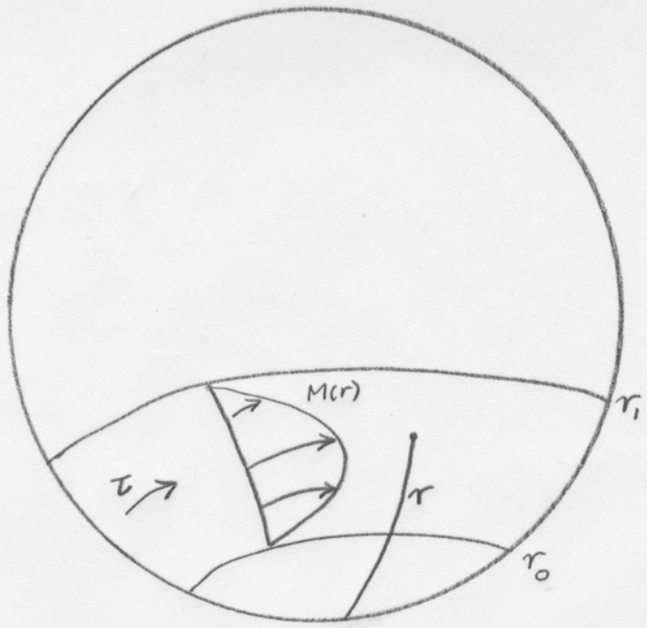

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That is100Sv

Discuss: Subsequent estimates of ACC ranged between -15Sv and 250Sv. What is the currently accepted value?

Discuss: How did M&P estimate A?

The M&P eddy viscosity A

$$\delta_{\mathrm{M}} = \left(\frac{A}{\beta}\right)^{1/3}$$

$$\beta = 10^{-11}\mathrm{m}^{-1}\mathrm{s}^{-1} \quad \Longrightarrow \quad \delta_{\mathrm{M}} = 100\mathrm{km}$$

$$A = 10^{4}\mathrm{m}^{2}\,\mathrm{s}^{-1}$$

This is the number used in Munk (1950)
On the Wind Driven Ocean Circulation.

There is a big problem…

Perhaps A is too small? According to M&P probably not….

Perhaps “skin friction” is important?

$$U_t = \frac{\tau}{\rho H} - \frac{U}{t_{\mathrm{drag}}}$$

$10^7\mathrm{s} = 120\mathrm{days}$

$$U = \frac{t_{\mathrm{drag}}\tau}{\rho H} = \frac{10^7 \times 0.1}{1000 \times 1000} = 1\mathrm{m\ s}^{-1}$$

The transport is:

$$T = LH\frac{t_{\mathrm{drag}}\tau}{\rho H} = \frac{Lt_{\mathrm{drag}}\tau}{\rho}$$
$$= \frac{10^6 \times 10^7 \times 0.1}{10^3} = 10^3\mathrm{Sv}$$

M&P say the “mountain effect” is probably the crucial missing process

The “mountain effect” is also known as as “form stress”, “form drag”, “topographic drag”, “bottom form drag” etc,

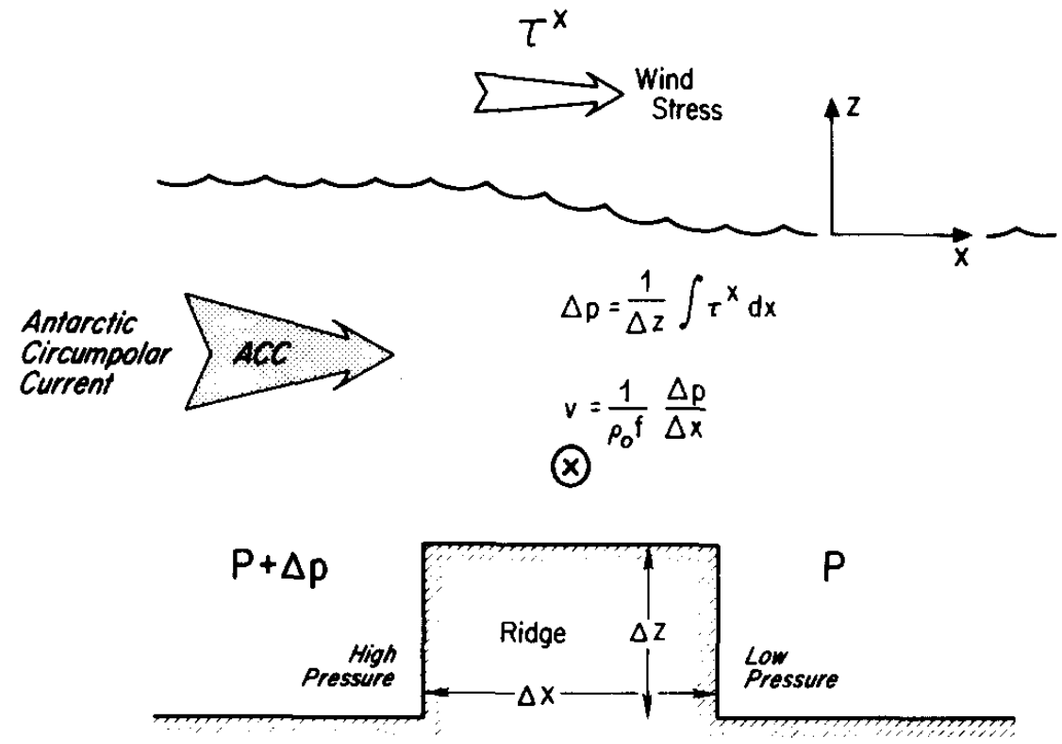

Fig. 3. Schematic presentation of bottom form drag or mountain drag. Wind stress imparted eastward momentum in the water column is removed by the pressure difference across the ridge. Note that geostrophic balance $\rho_0 f v = \partial p/\partial x$ demands an equatorward flow (symbolized by ⊗) along the ridge, evidence of which may be seen in Fig. 1.

from Johnson & Bryden (1989)

The ocean is “leaning” on the ridge.

The column-integrated momentum budget

The x-momentum equation is:

$$u_t + (u^2)_x + (uv)_y + (wu)_z - fv + p_x = \tau_z$$

Key identity:

$$\int_{-h}^{0} p_x \,\mathrm{d}z = \partial_x \int_{-h}^{0} p \,\mathrm{d}z - h_x p(-h)$$

After vertical integration:

$$\partial_t \int_{-h}^{0} u \,\mathrm{d}z + \partial_x \left[\int_{-h}^{0} u^2 + p \,\mathrm{d}z \right] + \partial_y \int_{-h}^{0} uv \,\mathrm{d}z - f \int_{-h}^{0} v \,\mathrm{d}z$$

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Discuss: wind stress is also form stress.

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Discuss: wind stress is also form stress.

Discuss: why don't the advective terms also couple the fluid to the solid Earth? What happened to the vertical integral of (w u)_z?

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Discuss: wind stress is also form stress.

Discuss: no surface is ever perfectly flat. So is skin friction really different from form stress?

Discuss: why don't the advective terms also couple the fluid to the solid Earth? What happened to the vertical integral of (w u)_z?

Recall the column integral:

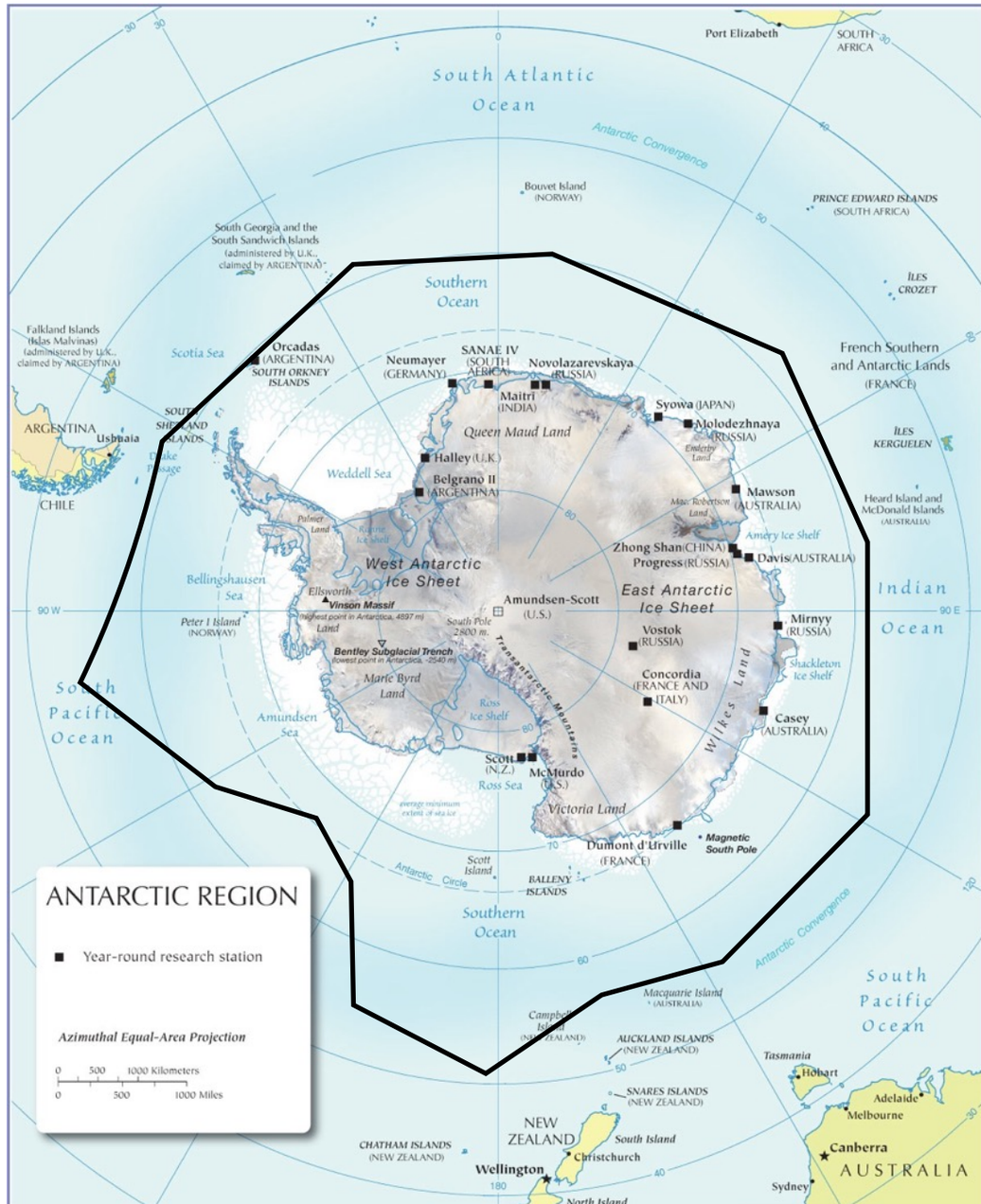
$$\partial_t \int_{-h}^{0} u \,\mathrm{d}z + \partial_x \left[\int_{-h}^{0} u^2 + p \,\mathrm{d}z \right] + \partial_y \int_{-h}^{0} uv \,\mathrm{d}z - f \int_{-h}^{0} v \,\mathrm{d}z$$
$$= \underbrace{\tau(0)}_{\text{wind stress}} - \underbrace{\tau(-h)}_{\text{skin friction}} + \underbrace{h_x p(-h)}_{\text{form stress}}$$

The ACC is special because if one integrates right round the Earth:

$$\oint \left[\int_{-h}^{0} v \,\mathrm{d}z \right] \mathrm{d}x = 0 \quad \text{(almost)}$$

$$\oint \left[\int_{-h}^{0} u^2 + p \,\mathrm{d}z \right] \mathrm{d}x = 0$$

In fact we can integrate round any closed path encircling Antarctica and the integrals above would vanish.



After the contour integral there is not much left

$$\underbrace{\partial_t \langle u \rangle}_{\to 0} + \underbrace{\partial_y \langle uv \rangle}_{\text{negligible}} = \oint \tau(0) - \tau(-h) - h_x p(-h)\,\mathrm{d}x$$

So M&P are saying that the surface wind stress is balanced by "skin friction" and "form stress". And form stress is much greater than skin friction.

What's so special about the ACC?

The ACC is deep….

$$u_t + (u^2)_x + (uv)_y + (wu)_z - fv + p_x = \tau_z$$

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If the wind stress is to be balanced by stresses along the sea bottom, the *AC* current must be deep enough to reach to the bottom. Sverdrup has pointed out that the sharp leftward displacement of the surface current over the submarine ridges would indicate that this is the case (SVERDRUP et al, 1942, pp. 466—469). We may consider two possible explanations:

1. Each layer induces, by turbulent interchange of momentum, motion in the layer beneath, and in this manner the wind stress is transmitted to the sea bottom. Because the fetch of the wind is essentially infinite this mechanism is more effective in the *AC* current than elsewhere.

2. The meridional circulation exports absolute angular momentum in the upper layer, and imports it in a lower layer. Suppose the loss of angular momentum in the upper layer alone associated with a flow of Q g sec^{-1} northward across the 45° S parallel of latitude is balanced entirely by the wind *torque* over the ocean between the Antarctic continent (70° S) and 45° S:

$$Q\,(\omega\, R\cos 45^\circ)\,(R\cos 45^\circ) = \int_{45^\circ\,S}^{70^\circ\,S} \tau\,(2\pi R^2\cos\varphi)\,(R\cos\varphi)\,d\varphi,$$

where ω is the earth's angular velocity and R its

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Discuss: do we really have to rely on vertical eddy viscosity in \tau_z?

$$u_t + (u^2)_x + (uv)_y + (wu)_z - fv + p_x = \tau_z$$

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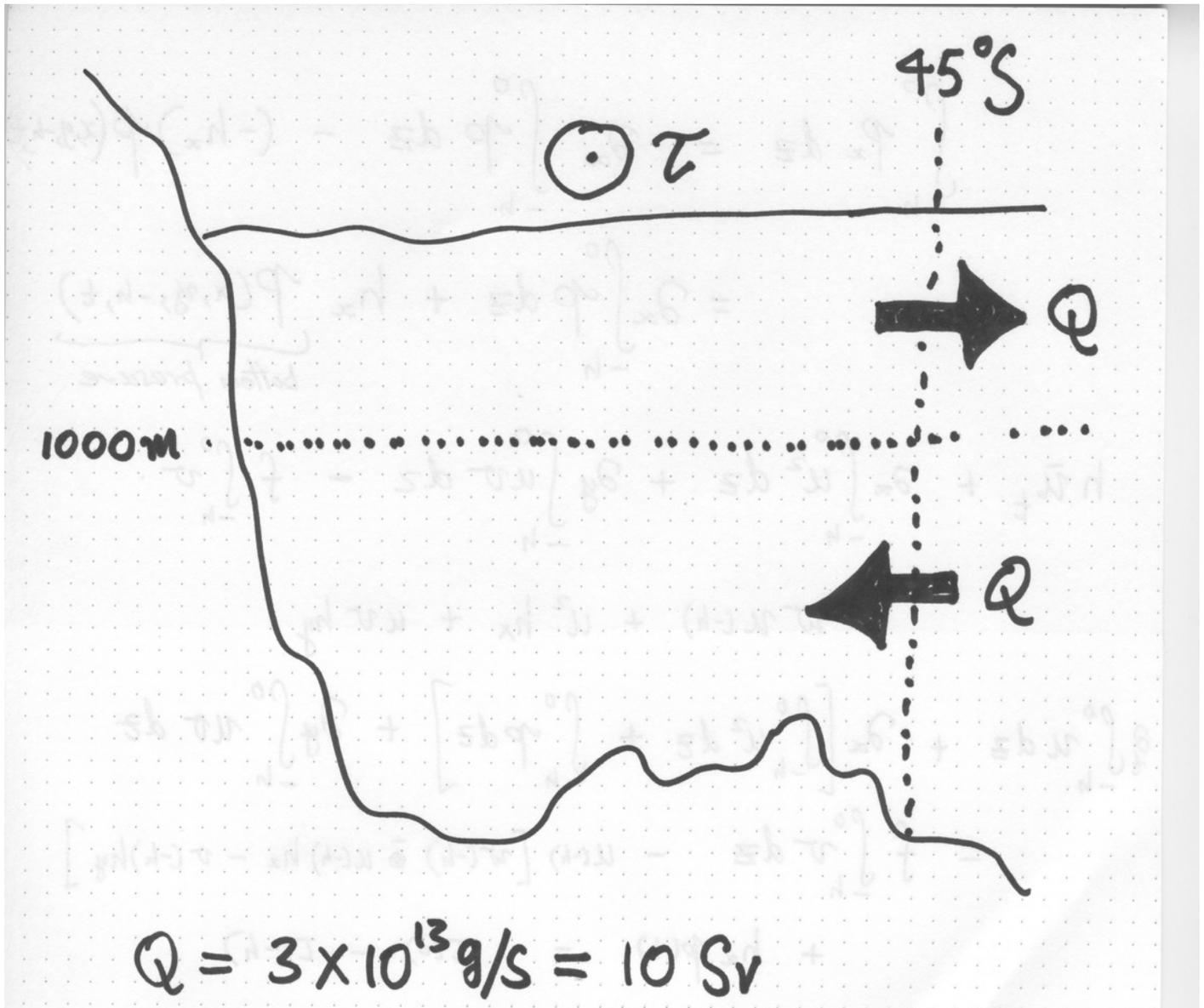

The Coriolis force can balance the wind stress in the upper layer, but not in the full column.

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where ω is the earth's angular velocity and R its radius. Setting $\tau = 2$ dynes cm^{-2} leads to $Q = 3 \times 10^{13}$ g sec^{-1}. This value is in agreement with the observed northward transport in the upper 500—1,000 meters, due mostly to the Benguela and Peru Currents. If we suppose for the moment that the difference between precipitation and evaporation over the whole region south of the latitude 45° S and in addition a possible decrease of the Antarctic ice cap could provide all this water, then the wind stress could be accounted for by the export of absolute angular momentum into the southern oceans. Actually not more than about 0.3–0.5 per cent of the above value of Q could be provided in this way[5], so that virtually all of the water lost in the upper layers must be returned to the Antarctic Sea at a lower level[6], apparently at depths between 1,000 and 4,000 meters. Conservation of angular momentum then requires an increasing cyclonic (eastward) circulation at these great depths with decreasing distance from the pole. Deacon's analysis of the

The ACC is deeper than the Gulf Stream.

Gulf Stream section

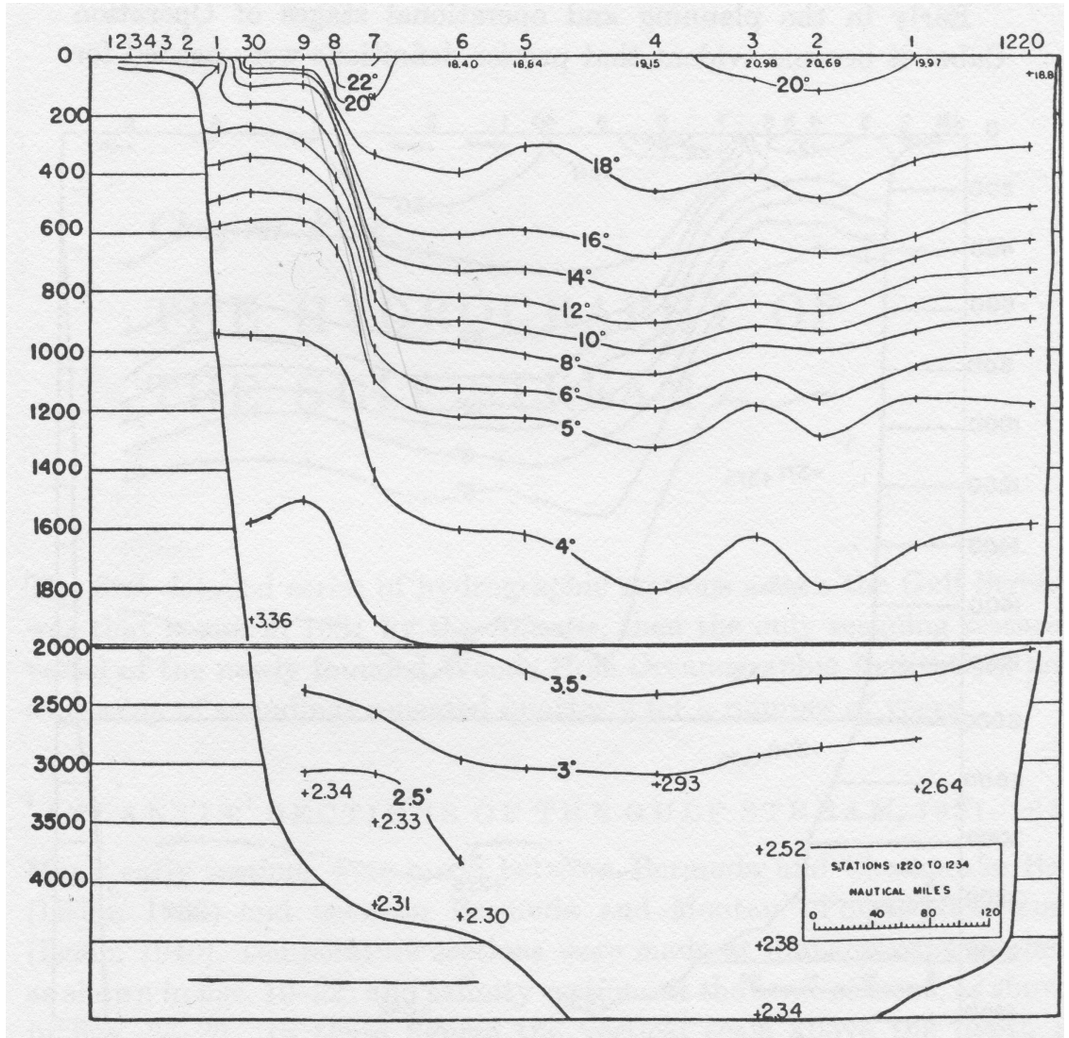


Fig. 20. Temperature section across the Gulf Stream, Chesapeake Bay to Bermuda, April 17–23, 1932, according to Iselin (1936, fig. 5).

from Stommel "The Gulf Stream "

distance from the pole. Deacon's analysis of the *DISCOVERY* observations in fact shows a remarkably uniform horizontal density gradient throughout most of the water column, whereas, for example, Iselin's sections across the Gulf Stream reveal a sharp decrease in horizontal gradients at a thousand meters (compare figs. 159 and 186, SVERDRUP et al. 1942). From this point of view the principal action of the wind is not so much to induce cyclonic circulation throughout the water column, as to prevent anticyclonic circulation of the northward drifting surface waters.

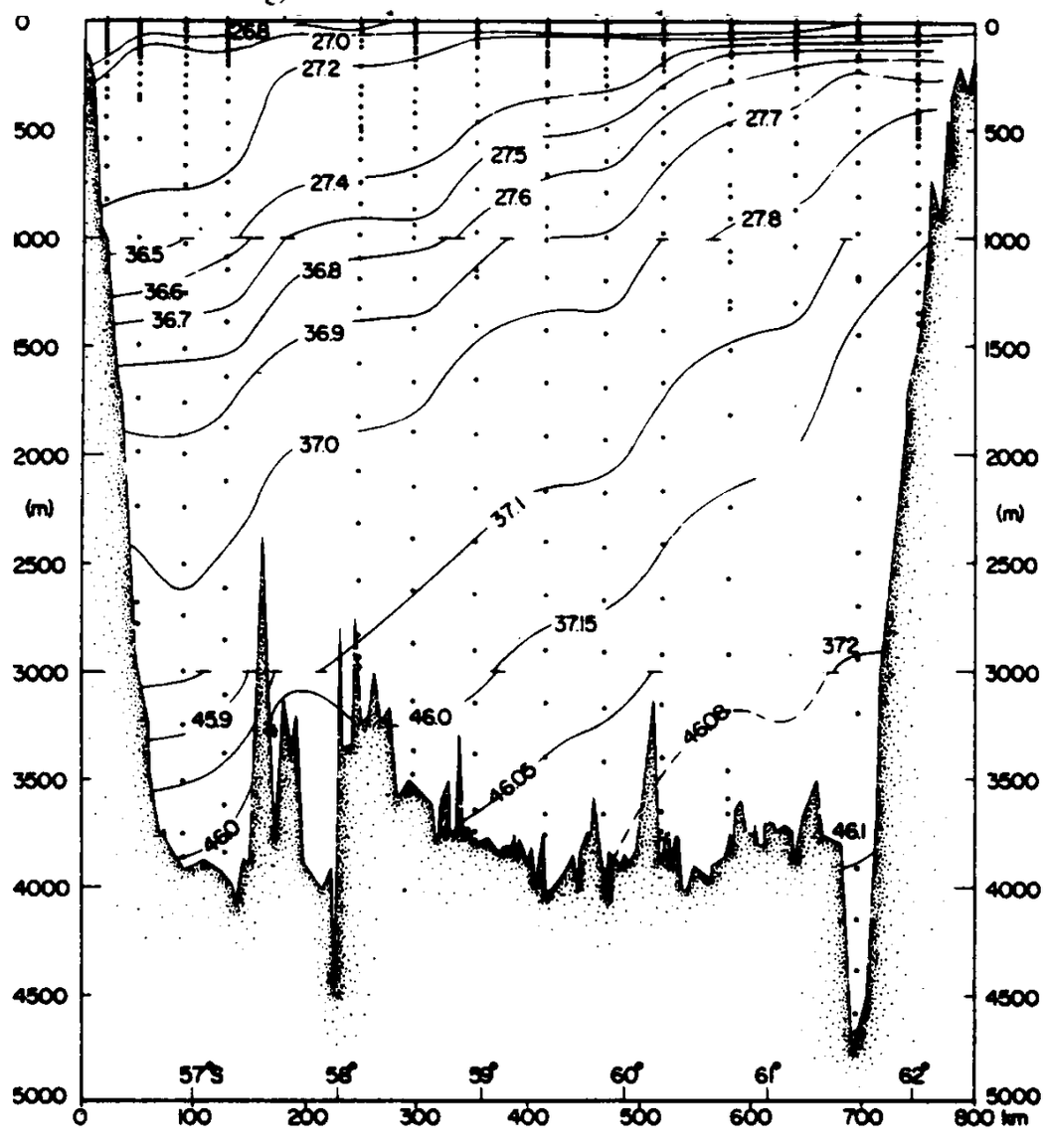
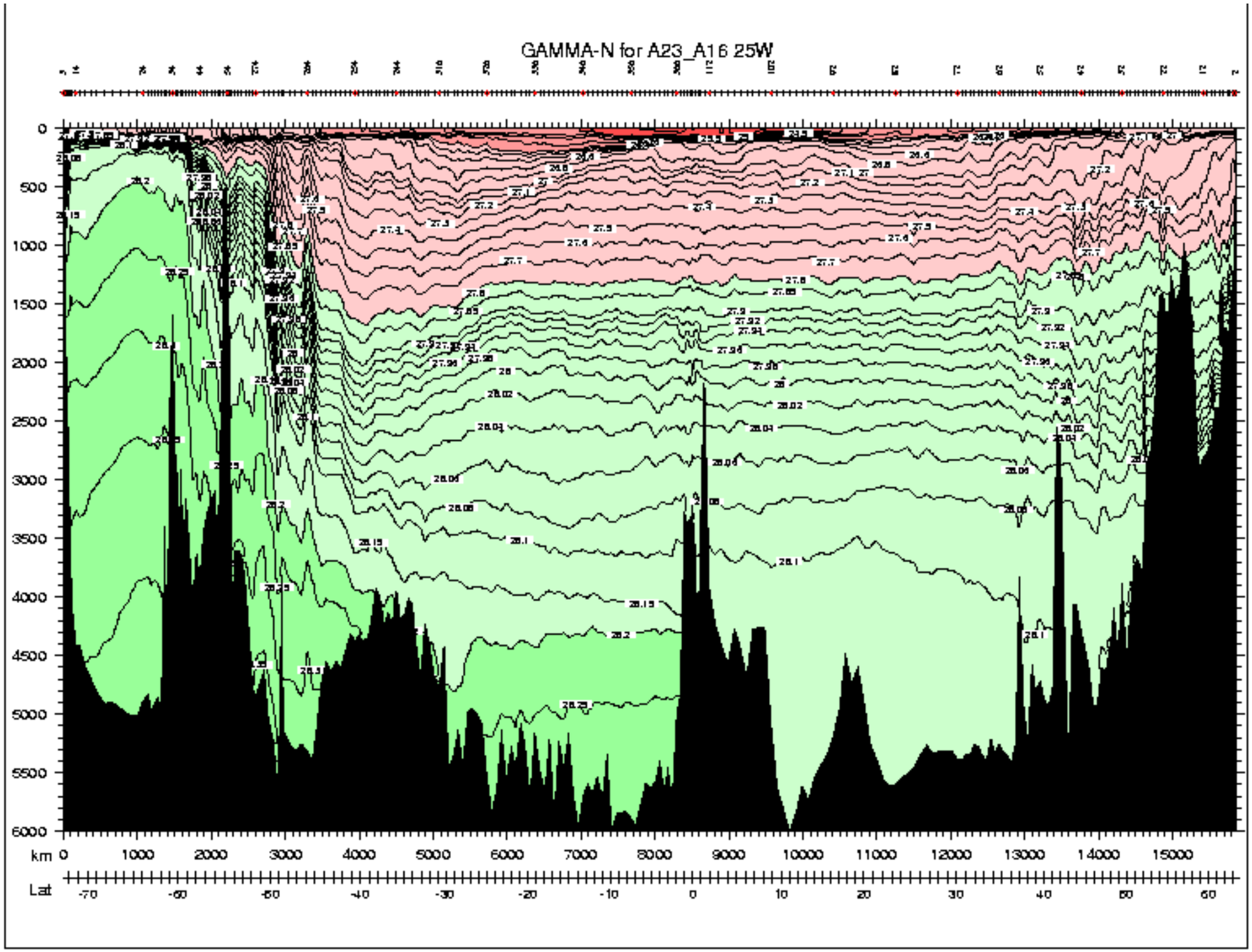
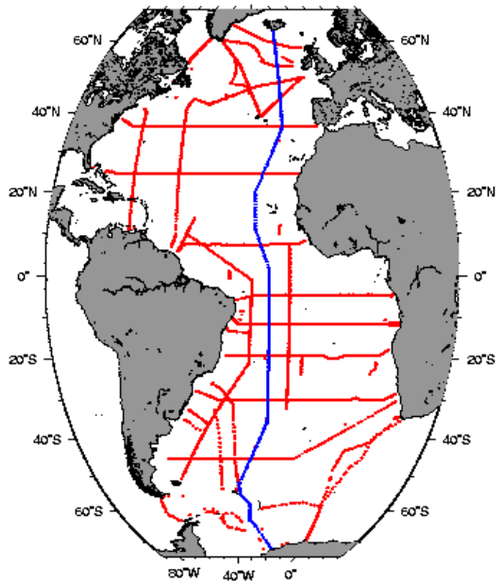


Fig. 2. Section of variability referenced density parameter σ across Drake Passage calculated from *Melville* Section II FDRAKE 75 data. Figure 4 from NOWLIN *et al.* (1977).

from Johnson & Bryden 1989

Southern Ocean section





from Talley's website

Conclusion

The foregoing speculation regarding the dynamics of the AC current would make it differ radically from other major ocean current systems, which are limited essentially to the upper 1,000 meters, and for which the stresses against the lateral boundaries are adequate to balance the wind stress. The difference is due largely to boundaries which prevent the zonal flow of water around the earth. Assume that

CGS units made it difficult for me……

$$1\mathrm{Sverdrup} = 10^6 \mathrm{m}^3\,\mathrm{s}^{-1}$$
$$\approx 10^9 \mathrm{kg}\,\mathrm{s}^{-1}$$
$$= 10^{12} \mathrm{g}\,\mathrm{s}^{-1}$$

$$1\mathrm{dyne}\,\mathrm{cm}^{-2} = 0.1\mathrm{N}\,\mathrm{m}^{-2}$$

M&P say that the eddy viscosity is:

$$A = 10^8 \mathrm{cm}^2\,\mathrm{s}^{-1} = 10^4 \mathrm{m}^2\,\mathrm{s}^{-1}$$

How? Why? Where did that number come from?

of the x-co-ordinate line, but are written out here in view of the frequency of their use. The co-ordinates are

$$\xi_1 = r, \quad \xi_2 = \theta, \quad \text{and} \quad h_1 = 1, \quad h_2 = r,$$

$$\frac{\partial \mathbf{a}}{\partial r} = 0, \quad \frac{\partial \mathbf{a}}{\partial \theta} = \mathbf{b}, \quad \frac{\partial \mathbf{b}}{\partial r} = 0, \quad \frac{\partial \mathbf{b}}{\partial \theta} = -\mathbf{a}.$$

$$\nabla V = \mathbf{a}\frac{\partial V}{\partial r} + \frac{\mathbf{b}}{r}\frac{\partial V}{\partial \theta},$$

$$\mathbf{n}.\nabla \mathbf{F} = \mathbf{a}\left(\mathbf{n}.\nabla F_r - \frac{n_\theta F_\theta}{r}\right) + \mathbf{b}\left(\mathbf{n}.\nabla F_\theta + \frac{n_\theta F_r}{r}\right),$$

$$\nabla.\mathbf{F} = \frac{1}{r}\frac{\partial (rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta},$$

$$\nabla \times \mathbf{F} = \left\{\frac{1}{r}\frac{\partial (rF_\theta)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \theta}\right\}\mathbf{a} \times \mathbf{b},$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2},$$

$$\nabla^2 \mathbf{F} = \mathbf{a}\left(\nabla^2 F_r - \frac{F_r}{r^2} - \frac{2}{r^2}\frac{\partial F_\theta}{\partial \theta}\right) + \mathbf{b}\left(\nabla^2 F_\theta + \frac{2}{r^2}\frac{\partial F_r}{\partial \theta} - \frac{F_\theta}{r^2}\right).$$

Rate-of-strain tensor:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad e_{r\theta} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial \theta}.$$

Equation of motion for an incompressible fluid, with no body force:

$$\frac{\partial u_r}{\partial t} + \left(u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}\right)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right),$$

$$\frac{\partial u_\theta}{\partial t} + \left(u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}\right)u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right).$$

$$\nabla^2 F_r - \frac{F_r}{r^2} = \frac{\partial^2 F_r}{\partial r^2} + \frac{1}{r}\frac{\partial F_r}{\partial r} - \frac{F_r}{r^2}$$

Batchelor's cylindrical coordinate appendix