Two-dimensional turbulence above topography

a 1976 JFM paper by
F. P. Bretherton & D. H. Haidvogel
(as told by Cesar)

Context
MODE experiment
FFT (1965), 2D turbulence studies
Contemporary studies: Rhines (1975, 1977),
Inviscid 2D dynamics
(f-plane barotropic QG dynamics)

\[
\frac{Dq}{Dt} = \partial_t q + J(\psi, q) = 0
\]

\[q = \nabla^2 \psi + h\]

\[u = -\psi_x \quad v = \psi_y\]

\[J(f, g) = f_x g_y - f_y g_x\]

Note \( h = \frac{f_0}{H} \eta_b \)
The enstrophy cascade
2D turbulence

tracer variance forward cascade

BH76 Fig. 1
2D turbulence
tracer variance forward cascade

Initial condition

$q_0 + \delta q$
2D turbulence
tracer variance forward cascade

Conservation of area

Initial condition

After two eddy turnover times

\( q_0 + \delta q \)

\( q_0 \)

\( \delta l \)

\( \delta n = \delta q / |\nabla q| \)

(This comes from a 2D stirring simulation with a velocity field represented by “renovating waves”)

\( q_0 \)
2D turbulence

enstrophy forward cascade

The evolution of the active tracer \( q \) is more complicated because two quadratic quantities must be conserved in the inviscid limit:

(Kinetic) Energy

\[
E = \frac{1}{2} \iint |\nabla \psi|^2 \, dx \, dy = \sum_{|k|} E(|k|)
\]

(Potential) Enstrophy

\[
Q = \frac{1}{2} \iint q^2 \, dx \, dy = \sum_{|k|} |k|^2 E(|k|) + \ldots
\]

\[
q = \nabla^2 \psi + h
\]

\[
|k| = \kappa = (k^2 + l^2)^{1/2}
\]

Small-scale dissipation damps enstrophy much more efficiently.
2D turbulence: enstrophy decay

Time series in an initial value problem with $h=0$

Energy has only a 0.3 \% decay, whereas enstrophy decays by 92\%
A minimum enstrophy principle
What is the flow that minimizes potential enstrophy $Q$ given the energy $E$?

“A simple exercise in calculus of variations” (BH76):

$$\delta Q + \mu \delta E = \iint [q \nabla^2 \delta \psi + \mu (\nabla \psi \cdot \nabla \delta \psi)] \, dx \, dy$$

$$= \iint \nabla^2 (\nabla^2 \psi + h - \mu \psi) \delta \psi \, dx \, dy = 0$$

Assuming periodic BCs:

$$(\mu - \nabla^2) \psi = h$$

$$E = \frac{1}{2} \iint |\nabla \psi|^2 \, dx \, dy \quad Q = \frac{1}{2} \iint q^2 \, dx \, dy$$
What is the flow that minimizes potential enstrophy $Q$ given the energy $E$?

Physical space

$$(\mu - \nabla^2) \psi = h$$

Fourier space

$$\hat{\psi}_0 = \frac{\hat{h}}{k^2 + l^2 + \mu}$$

The Lagrangian multiplier defines a length scale $L_0 = \mu^{-1/2}$

$|k|L_0 << 1 : \psi_0 \approx \mu^{-1} h$ \hspace{1cm} Isobaths are streamlines of the coarse-grained flow

$|k|L_0 >> 1 : q = \nabla^2 \psi_0 + h \approx 0$ \hspace{1cm} The fine-grained PV is homogenized

$|k| = \kappa = (k^2 + l^2)^{1/2}$
What is the flow that minimizes potential enstrophy $Q$ given the energy $E$?

$$\hat{\psi}_0 = \frac{\hat{h}}{k^2 + l^2 + \mu}$$

$\mu$ is determined using the constraint $E$

$$E = \frac{U^2}{2} = \frac{1}{2} \sum \frac{|k|^2}{(|k|^2 + \mu)^2} |\hat{h}|^2(|k|)$$

$|k| = \kappa = (k^2 + l^2)^{1/2}$
An example

“The preferred topography” (BH76)

\[ \hat{h}(|k|) \propto |k|^{-2} \quad |k|_{min} = 1 \ , |k|_{max} = 12 \]

\[ |k| = \kappa = (k^2 + l^2)^{1/2} \quad |k|_{min} = 1 \ , |k|_{max} = 12 \]
A given energy level has multiple solutions but the positive minimizes enstrophy

“It is readily demonstrated that for positive $\mu$ this is in fact a minimum” (BH76)
A given energy level has multiple solutions but the positive minimizes enstrophy.

The second order terms in the variational problem

\[ \delta^2 Q + \mu \delta^2 E = \frac{1}{2} \iint (\nabla^2 \delta \psi)^2 \, dx \, dy + \frac{\mu}{2} \iint |\nabla \delta \psi|^2 \, dx \, dy \]
An example of minimum enstrophy solution

\[ E = 0.05, \quad \mu = 9.22072072 \]

contours: \( h \)

Colors: \( PV \nabla^2 \psi_0 + h \)

Colors: Streamfunction \( \psi_0 \)
An example of minimum enstrophy solution

\[ E = 0.05, \quad \mu = 9.22072072 \]

contours: \( h \)

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Colors: Streamfunction \( \psi_0 \)

Discuss: Why BH76 did not do this!??
I will not discuss.

The role of saddle points

(There a lots of typos in equations 24 through 30 of the paper.)
The role of viscosity
The role of viscosity

\[ \frac{Dq}{Dt} = \nu \nabla^2(\nabla^2 \psi) \]

\[ q = \nabla^2 \psi + h \]

\[ \frac{\partial Q}{\partial t} = -\nu \int \int \nabla q \cdot \nabla (q - h) \, dx \, dy \]

\[ = -\nu \int \int |\nabla q|^2 \, dx \, dy + \nu \int \int \nabla q \cdot \nabla h \, dx \, dy \]

Positive or negative?

If, \( \psi = \psi_0 \) then enstrophy must increase since \( \psi_0 \) is the minimum enstrophy solution.
Closed-basin solution
The circulation on the boundary must be prescribed

If $\psi = 0$ on $\Gamma$ then $C = \int_\Gamma \partial_n \psi ds = \text{constant}$

The minimum enstrophy problem has two constraints

$$\delta \frac{1}{2} \iint (\nabla^2 \psi + h)^2 \, dx \, dy + \mu \delta \frac{1}{2} \iint |\nabla \psi|^2 \, dx \, dy + \lambda \oint \frac{\partial}{\partial n} \delta \psi ds = 0$$

Use integration by parts, e.g.,

$$\iint \nabla^2 \psi \nabla^2 \delta \psi \, dx \, dy = \iint \nabla \cdot [\nabla^2 \psi \nabla \delta \psi] \, dx \, dy - \iint \nabla \nabla^2 \psi \cdot \nabla \delta \psi \, dx \, dy$$

$$= \oint \nabla^2 \psi \frac{\partial}{\partial n} \delta \psi ds \quad = + \oint (\nabla^2 \nabla^2 \psi) \delta \psi \, dx \, dy$$
The circulation on the boundary must be prescribed

If $\psi = 0$ on $\Gamma$ then

$$C = \int_\Gamma \partial_n \psi \, ds = \text{constant}$$

The minimum enstrophy problem has two constraints

$$\nabla^2 (\nabla^2 \psi + h - \mu \psi) = 0 \quad \text{within } S$$

$$\nabla^2 \psi + h + \lambda = 0 \quad \text{on } \Gamma$$

Using $\psi = 0$ on $\Gamma$:

$$\nabla^2 \psi - \mu \psi = -(h + \lambda) \quad \text{within } S$$

$C$ determines the new constant $\lambda$
A numerical experiment
Side remark

CPU 36.4 MHz; 36 MFLOPS

CPU 1.4 GHz; GPU; 100’s GFLOPS to TFLOPS

CDC 7600

iPhone 6

What is the initial energy level and the associated $\mu$?
The flow quickly becomes quasi-steady

\[ J(\psi, q) = 0 \rightarrow q = F(\psi) \]
The $\psi - q$ relationship

Roughly linear…but there are different “regimes”…
But the energy dropped by 58% owing to low resolution...
\[ E = 0.5 \quad |\hat{h}|^2 \propto |k|^{-2} \quad |k|^2 |\psi|^2 (t = 0) \propto |k| \left[ 1 + \left( \frac{k}{6} \right)^4 \right]^{-1} \]

\[ \mu = 2.38288288 \]

Contours represent topography; colors streamfunction or PV

(doubly periodic calculations)
Energy stays nearly constant, enstrophy decays significantly...
Initializing a simulation with the minimum enstrophy solution

Contours represent topography; colors streamfunction or PV

In practice, with viscosity, energy decays super slowly...
Energy decays, enstrophy increases...
The effects of $\beta$
The effects of $\beta$

“In so far the b-slope can be thought of as a Fourier component of zero wavenumber the solution may be instantly obtained”:

$$\psi_0(x, y) = \sum \frac{\hat{h}}{k^2 + l^2 + \mu} + \frac{\beta}{\mu} y$$

Discuss: is this obvious?
The effects of $\beta$

Fofonoff mode type of solution
Numerical experiments

Topography

Final streamfunction

Final PV
The $\psi - q$ relationship
The effects of eddies on the large scale flow
Eddy fluxes

Reynolds decomposition

\[ \psi = \Psi + \psi' \quad q = (\nabla^2 \Psi + \beta y) + (\nabla^2 \psi' + h) \]

..... The eddy PV flux div: \( \nabla \cdot F = J(\psi', q') \)

The PV eqn:

\[ \frac{Dq'}{Dt} = -\beta \psi'_x \]
Eddy fluxes

\[ \frac{Dq'}{Dt} = -\beta \psi'_x \]

\[ q' = -\beta \eta \]

\( \eta \) : northward particle displacement

The single-particle diffusivity is

\[ D = \frac{d}{dt} \frac{1}{2} \eta^2 = \beta^{-2} \frac{d}{dt} \frac{1}{2} q'^2 = \beta^{-1} \overline{v'q'} = -\beta^{-1} (\nabla^2 \psi' + h) \psi'_x \]

\[ \approx -\beta^{-1} \left[ \frac{1}{2} \partial_x (\psi'_x - \psi'_y) + \partial_y \psi'_x \psi'_y + \overline{h\psi'} \right] \approx -\beta^{-1} \overline{h\psi'} \]
Eddy fluxes

\[ D = \frac{d}{dt} \frac{1}{2} \eta^2 = \beta^{-2} \frac{d}{dt} \frac{1}{2} q'^2 = \beta^{-1} \bar{\psi}' q' = -\beta^{-1} (\nabla^2 \bar{\psi}' + h) \bar{\psi}'_x \]

\[ \approx -\beta^{-1} \left[ \frac{1}{2} \partial_x (\psi'^2_x - \psi'^2_y) + \partial_y \bar{\psi}'_x \bar{\psi}'_y + h \bar{\psi}'_x \right] \approx -\beta^{-1} h \bar{\psi}'_x \]

change to spatial average

\[ P^{(x)} = \bar{h} \bar{\psi}'_x \approx -\bar{\psi}' h_x \]

scale separation

Topographic form stress
Eddy fluxes

\[ D = \frac{d}{dt} \frac{1}{2} \overline{\eta'^2} = \beta^{-2} \frac{d}{dt} \frac{1}{2} \overline{q'^2} = \beta^{-1} \overline{v'q'} = -\beta^{-1} (\nabla^2 \psi' + h) \psi'_x \]

\[ \approx -\beta^{-1} \left[ \frac{1}{2} \partial_x (\overline{\psi'_x^2 - \psi'_y^2}) + \partial_y \overline{\psi'_x \psi'_y} + \overline{h \psi'_x} \right] \approx -\beta^{-1} \overline{h \psi'_x} \]

change to spatial average

scale separation

Topographic form stress

Discuss: really?

\[ P^{(x)} = \overline{h \psi'_x} \approx -\overline{\psi' h_x} \]
Summary

• An initially turbulent flow above topography tends to a minimum enstrophy steady solution that is approximately along isobaths.

• On a beta-plane, the minimum enstrophy solution implies an westward interior flow.

• Topographic form stress appears to play a key role in driving the large-scale flow.
Discussion topics
(Or things to think about in the privacy of your own study)

• “The preferred topography”: $|\hat{h}|^2(|k|) \propto |k|^{-2}$

• What is the relevance of freely-decaying solutions to the understanding of real (forced) geophysical flows?

• Differences between closed-basin and doubly-periodic solutions.

• The $h \rightarrow 0$ limit.
The limit $h \to 0$ is a bit murky...

A simple eigenproblem

$$\nabla^2 \psi = \mu \psi$$

The minimum enstrophy solution is

$$\psi_0 = A \cos kx + B \cos ly$$

$$k^2 + l^2 = 1 \quad \rightarrow \quad \mu = -1$$

Discuss: What determines $A$ and $B$?
2D turbulence

An initial value problem (h=0)
Southern Ocean Topography
Drake Passage Topography

Topography $h$

Topographic gradient

$$K = \frac{\lambda}{H} \frac{\nabla h}{|\nabla h|_{rms}}$$

$$\approx 54.24$$

$$H = \langle h \rangle \approx 3716 \text{ m}$$

$\langle . \rangle$ spatial average