Two-dimensional turbulence above topography

a 1976 JFM paper by

F. P. Bretherton & D. H. Haidvogel

(as told by Cesar)

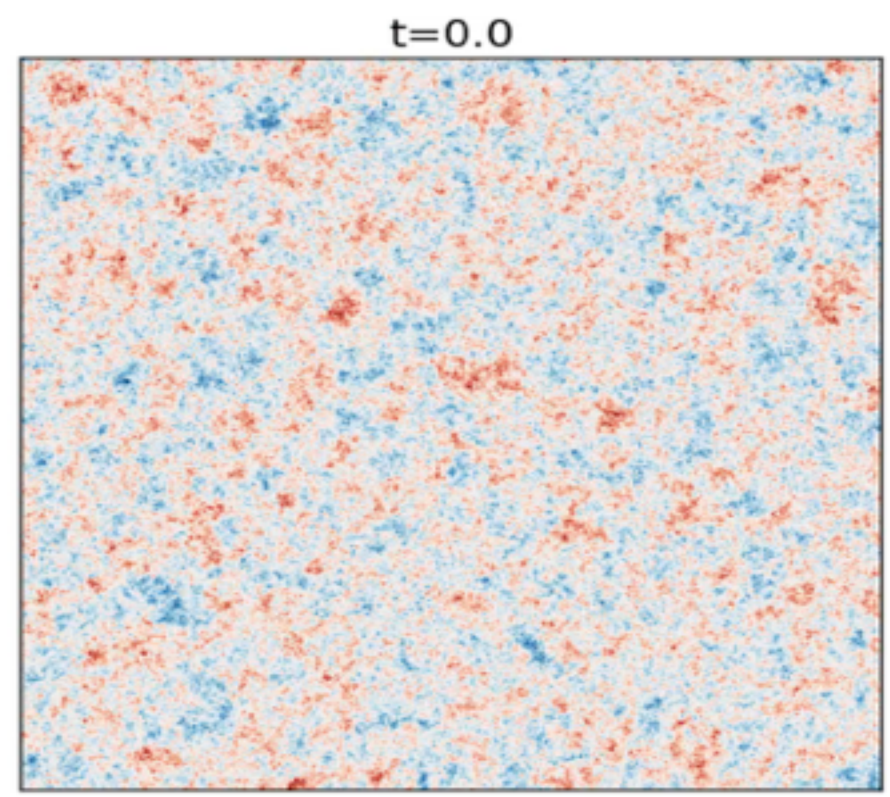
Context

MODE experiment

FFT (1965), 2D turbulence studies

Contemporary studies: Rhines (1975, 1977),

Salmon (1978, 1980), McWilliams (1984) →



Vorticity in a 2048x2048 freely decaying 2D turbulence simulation

h=0

PO Theory Seminar, SIO, Spring 2016

Inviscid 2D dynamics

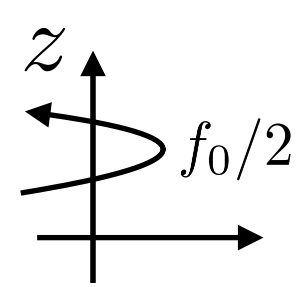
(f-plane barotropic QG dynamics)

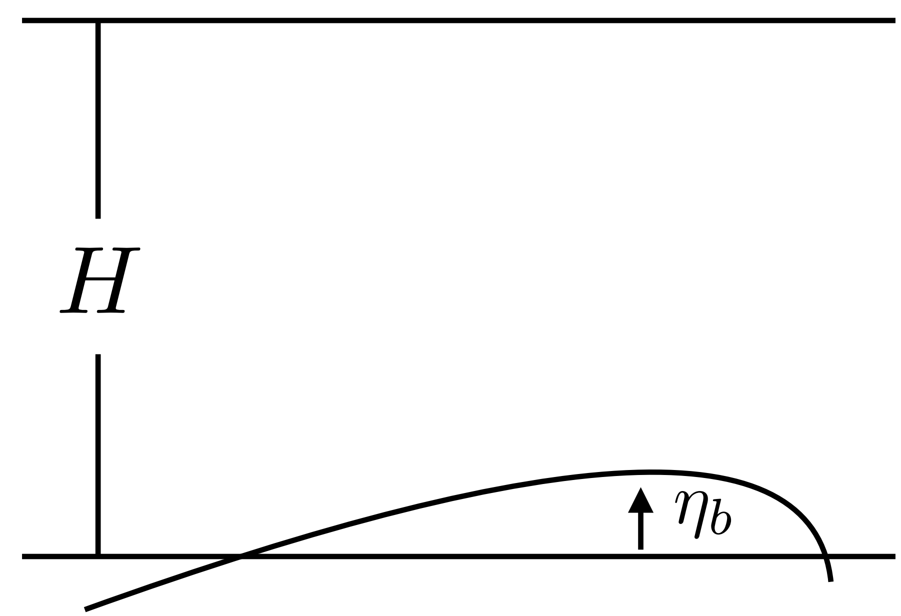
$$\frac{Dq}{Dt} = \partial_t q + \mathsf{J}(\psi, q) = 0$$

$$q = \nabla^2 \psi + h$$

$$u = -\psi_x \qquad v = \psi_y$$

$$\mathsf{J}(f, g) = f_x g_y - f_y g_x$$



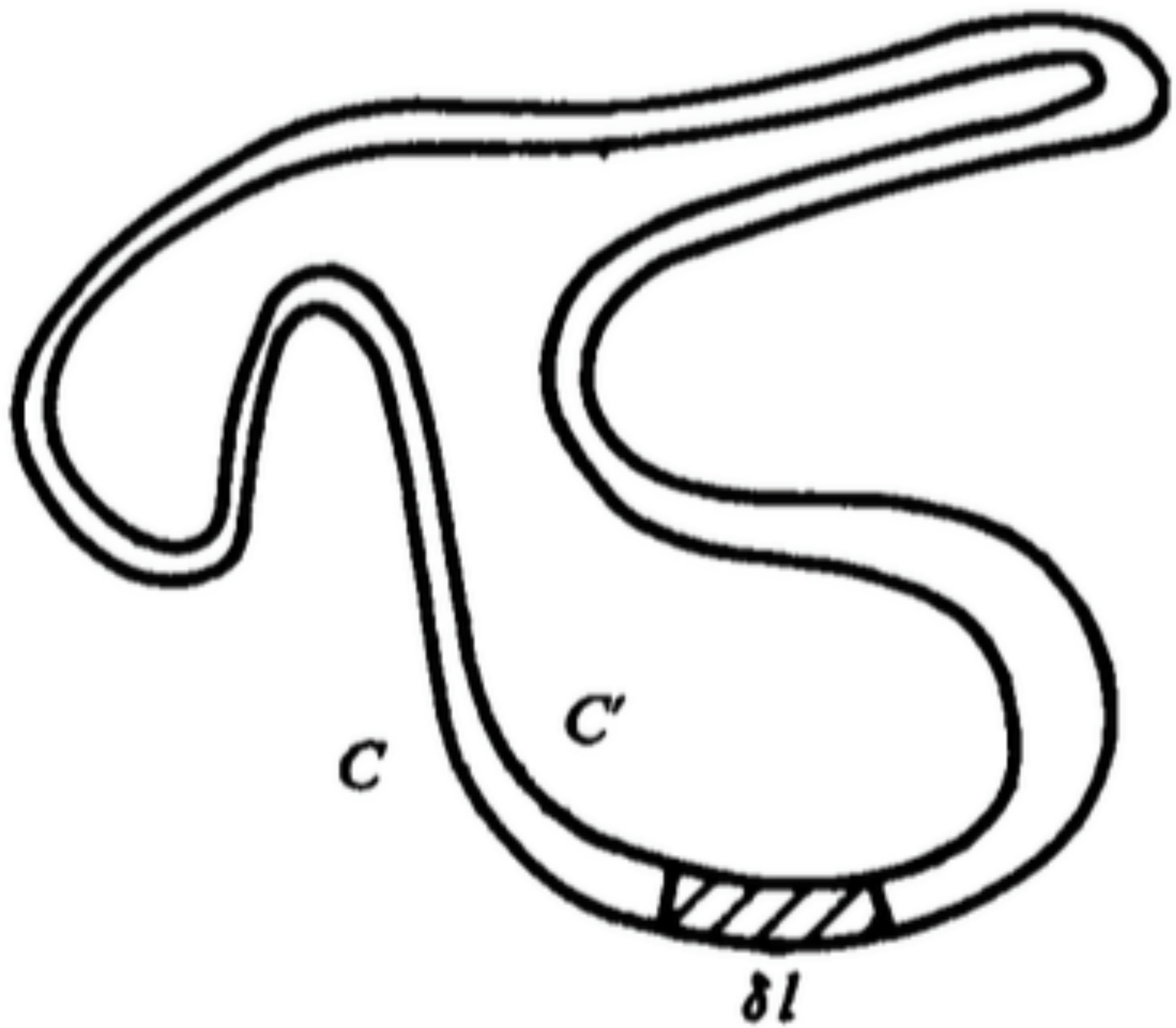


Note $\quad h = \dfrac{f_0}{H}\eta_b$

The enstrophy cascade

2D turbulence

tracer variance forward cascade

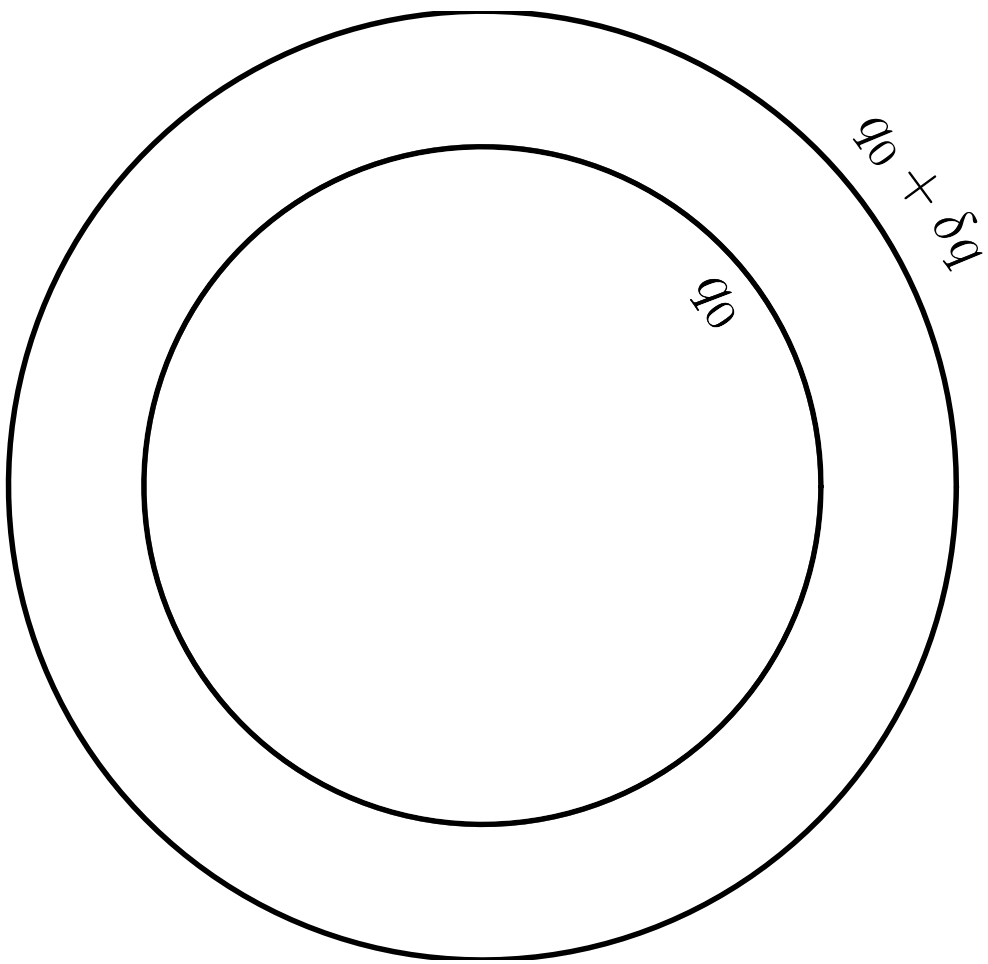


BH76 Fig. 1

2D turbulence

tracer variance forward cascade

Initial condition

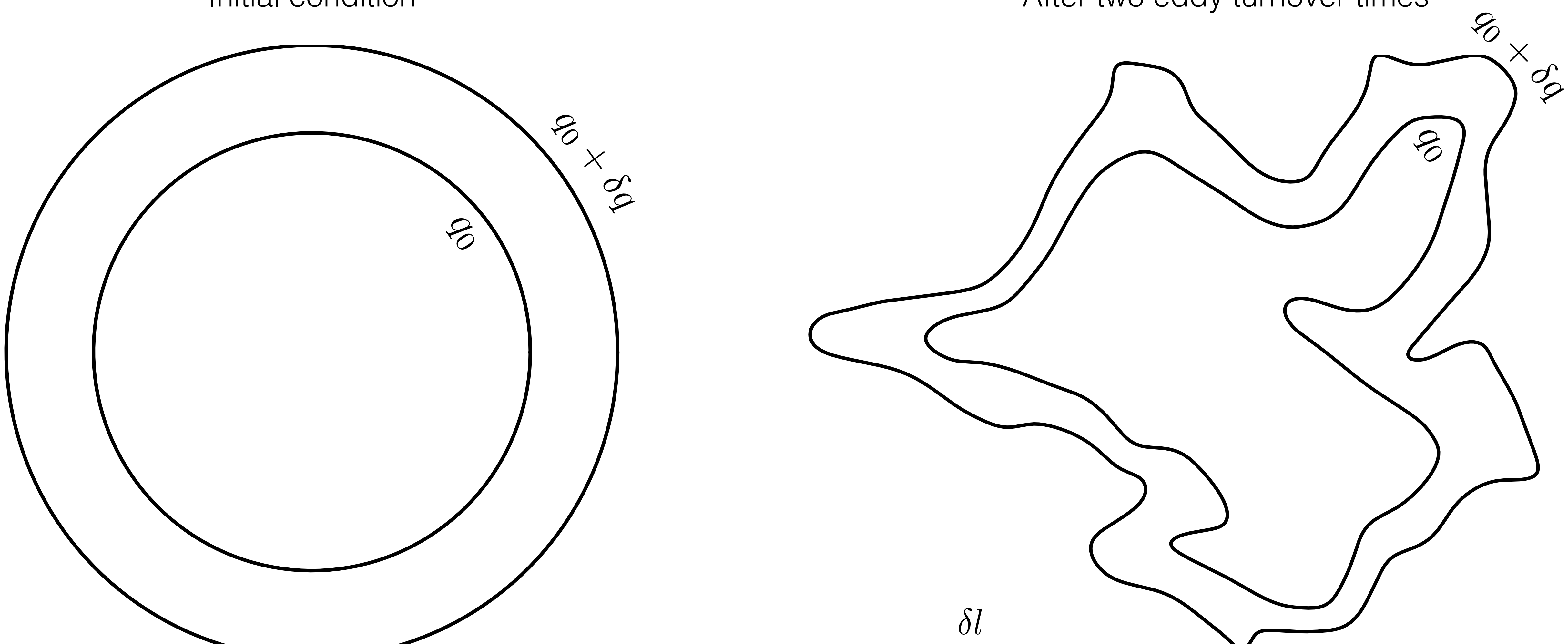


2D turbulence

tracer variance forward cascade

Initial condition

After two eddy turnover times



δl

Conservation of area

$\delta n = \delta q / |\nabla q|$

(This comes from a 2D stirring simulation with a velocity field represented by “renovating waves”)

2D turbulence

enstrophy forward cascade

The evolution of the active tracer q is more complicated because two quadratic quantities must be conserved in the inviscid limit:

(Kinetic) Energy $$E = \frac{1}{2}\iint |\nabla\psi|^2 \mathrm{d}x\,\mathrm{d}y = \sum_{|k|} E(|k|)$$

(Potential) Enstrophy $$Q = \frac{1}{2}\iint q^2 \mathrm{d}x\,\mathrm{d}y = \sum_{|k|} |k|^2 E(|k|) + \ldots$$

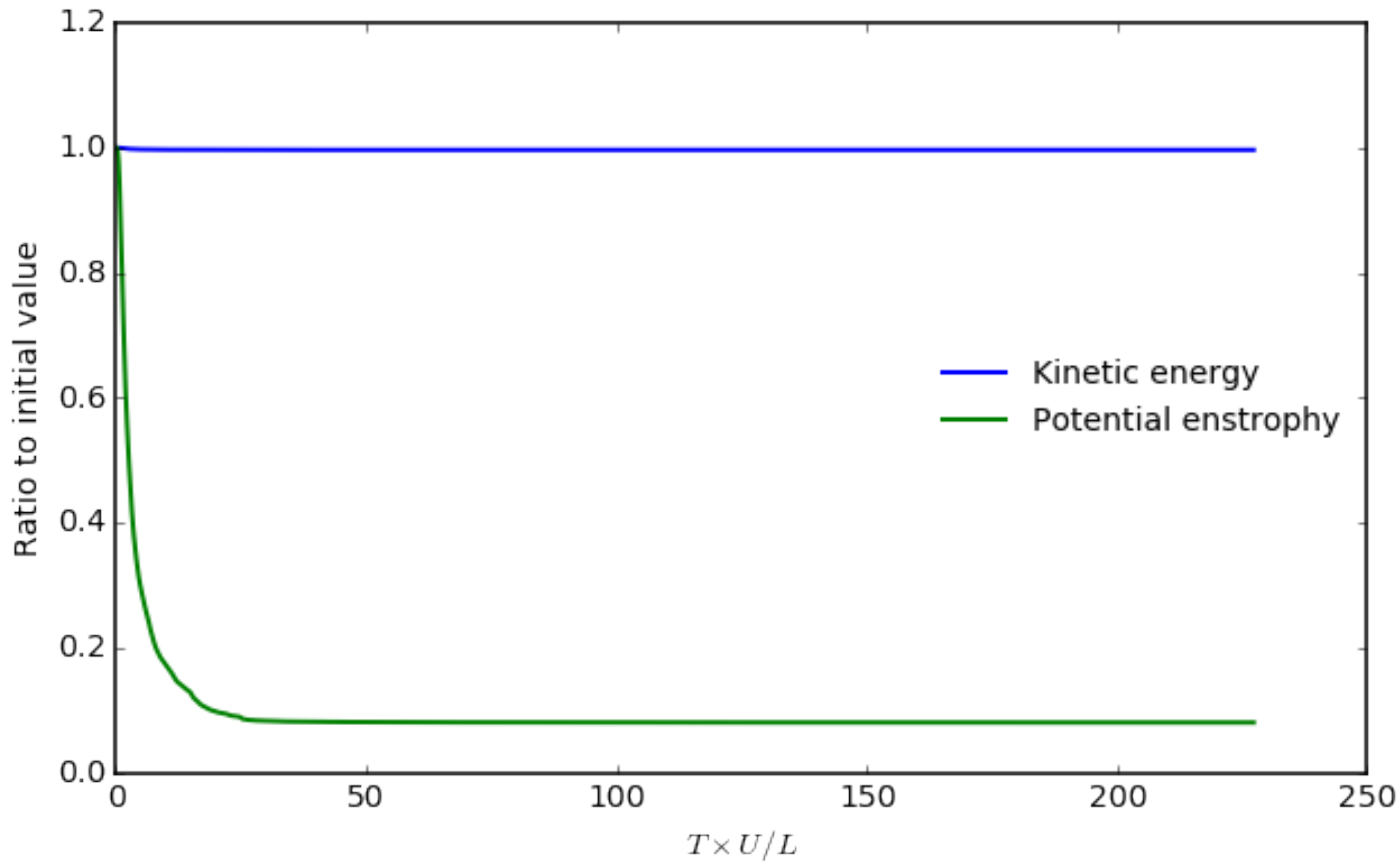
$$q = \nabla^2\psi + h$$

$$|k| = \kappa = (k^2 + l^2)^{1/2}$$

Small-scale dissipation damps enstrophy much more efficiently

2D turbulence: enstrophy decay

Time series in an initial value problem with h=0



Energy has only a 0.3 % decay, whereas enstrophy decays by 92%

A minimum enstrophy principle

What is the flow that minimizes potential enstrophy Q given the energy E?

"A simple exercise in calculus of variations" (BH76):

$$\delta Q + \mu \delta E = \iint \left[q \nabla^2 \delta\psi + \mu (\nabla\psi \cdot \nabla \delta\psi) \right] \, \mathsf{d}x \, \mathsf{d}y$$

$$= \iint \nabla^2 \left(\nabla^2 \psi + h - \mu\psi \right) \delta\psi \, \mathsf{d}x \, \mathsf{d}y = 0$$

Assuming periodic BCs:

$$\left(\mu - \nabla^2 \right) \psi = h$$

$$E = \tfrac{1}{2} \iint |\nabla\psi|^2 \mathsf{d}x \, \mathsf{d}y \qquad Q = \tfrac{1}{2} \iint q^2 \mathsf{d}x \, \mathsf{d}y$$

What is the flow that minimizes potential enstrophy Q given the energy E?

Physical space

$$\left(\mu - \nabla^2\right)\psi = h$$

Fourier space

$$\hat{\psi}_0 = \frac{\hat{h}}{k^2 + l^2 + \mu}$$

The Lagrangian multiplier defines a length scale $L_0 = \mu^{-1/2}$

$|k|L_0 << 1 \quad : \quad \psi_0 \approx \mu^{-1} h$ Isobaths are streamlines of the coarse-grained flow

$|k|L_0 >> 1 \quad : \quad q = \nabla^2 \psi_0 + h \approx 0$ The fine-grained PV is homogenized

$$|k| = \kappa = (k^2 + l^2)^{1/2}$$

What is the flow that minimizes potential enstrophy Q given the energy E?

$$\hat{\psi}_0 = \frac{\hat{h}}{k^2 + l^2 + \mu}$$

μ is determined using the constraint E

$$E = \frac{\overline{U^2}}{2} = \tfrac{1}{2} \sum \frac{|k|^2}{(|k|^2 + \mu)^2} |\hat{h}|^2(|k|)$$

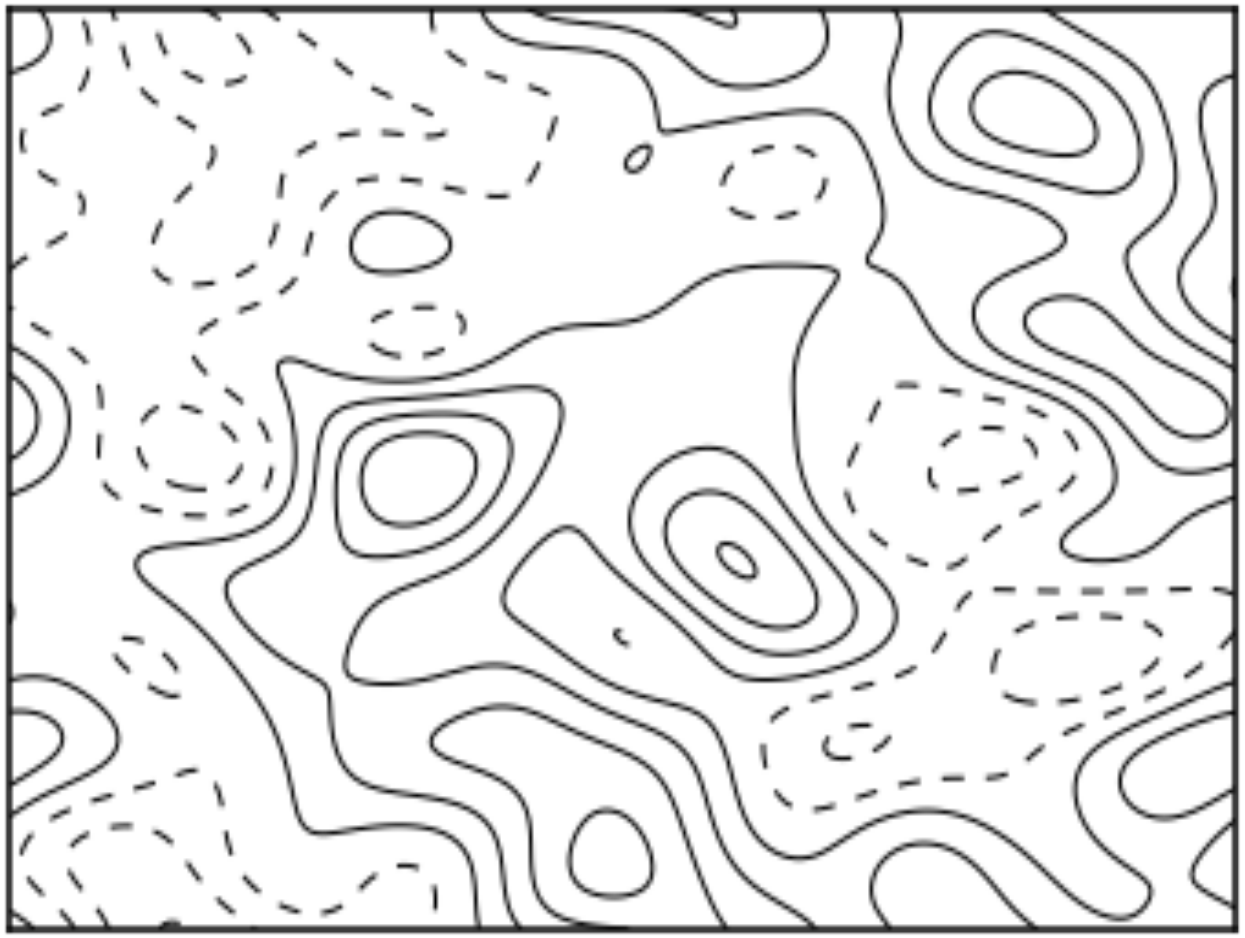
prescribed

$$|k| = \kappa = (k^2 + l^2)^{1/2}$$

An example

“The preferred topography” (BH76)

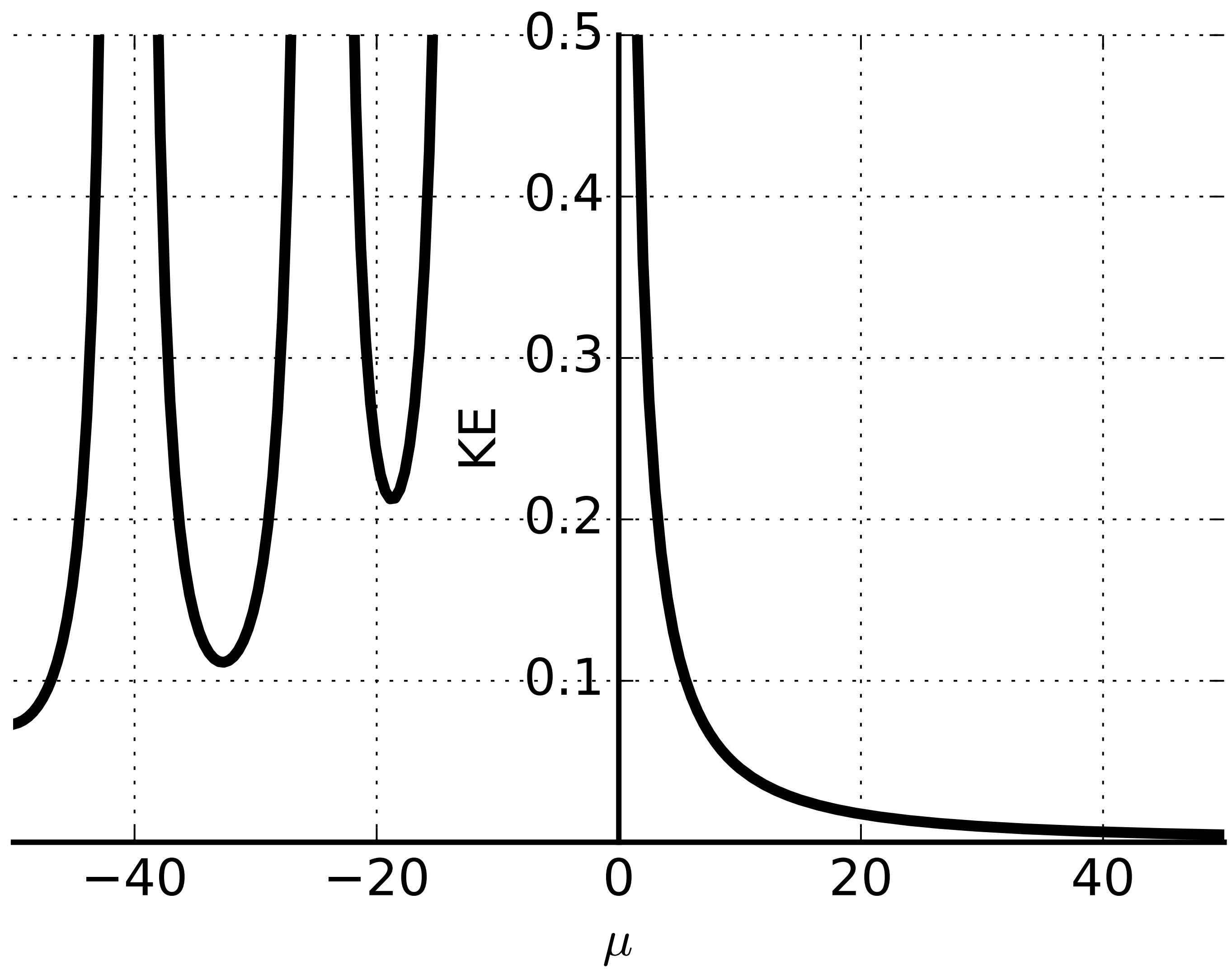
$$\hat{h}(|k|) \propto |k|^{-2} \qquad |k|_{min} = 1 \ , |k|_{max} = 12$$



$$|k| = \kappa = (k^2 + l^2)^{1/2}$$

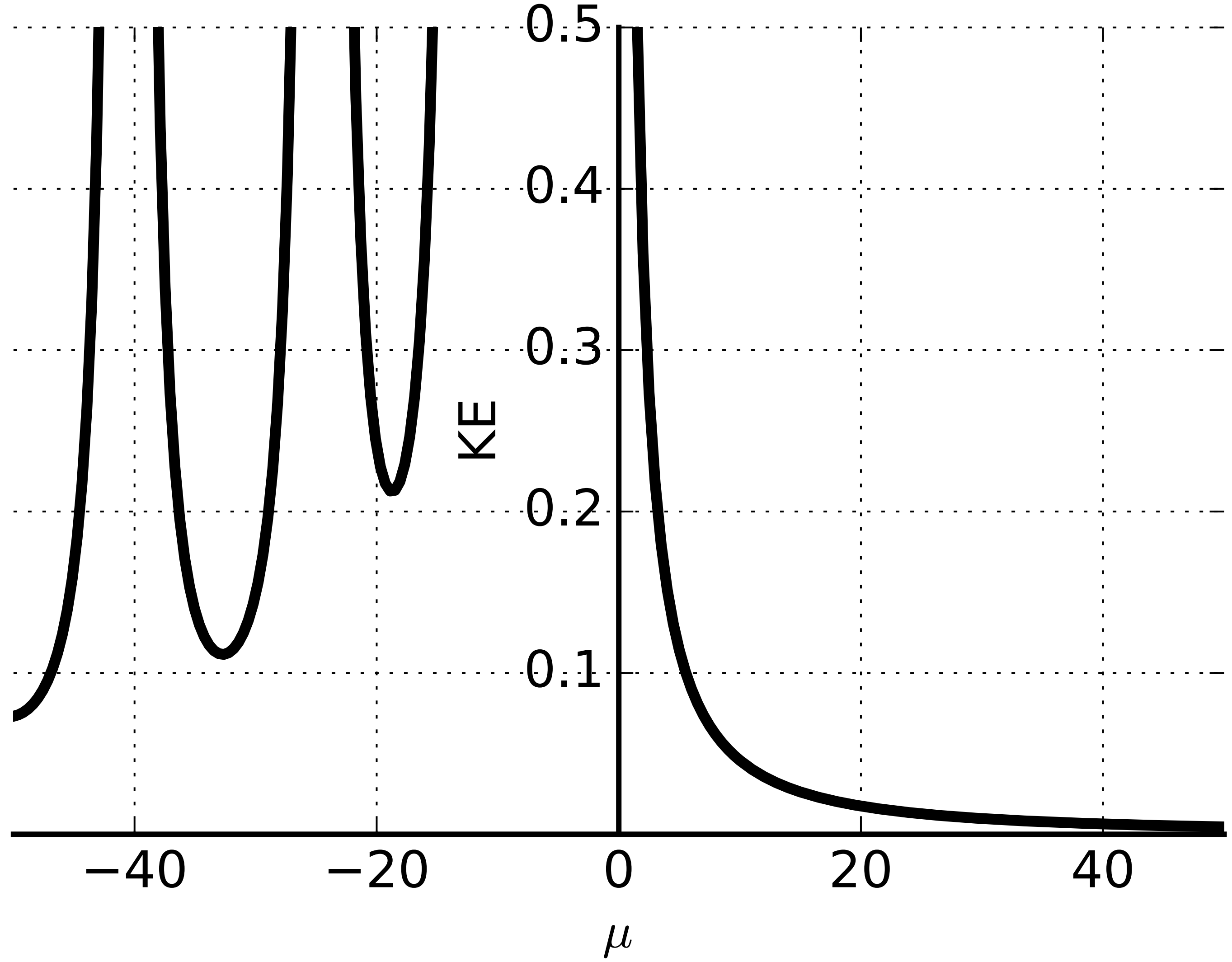
$$|k|_{min} = 1 \ , |k|_{max} = 12$$

A given energy level has multiple solutions but the positive minimizes enstrophy



"It is readily demonstrated that for positive μ this is in fact a minimum" (BH76)

A given energy level has multiple solutions but the positive minimizes enstrophy



The second order terms in the variational problem

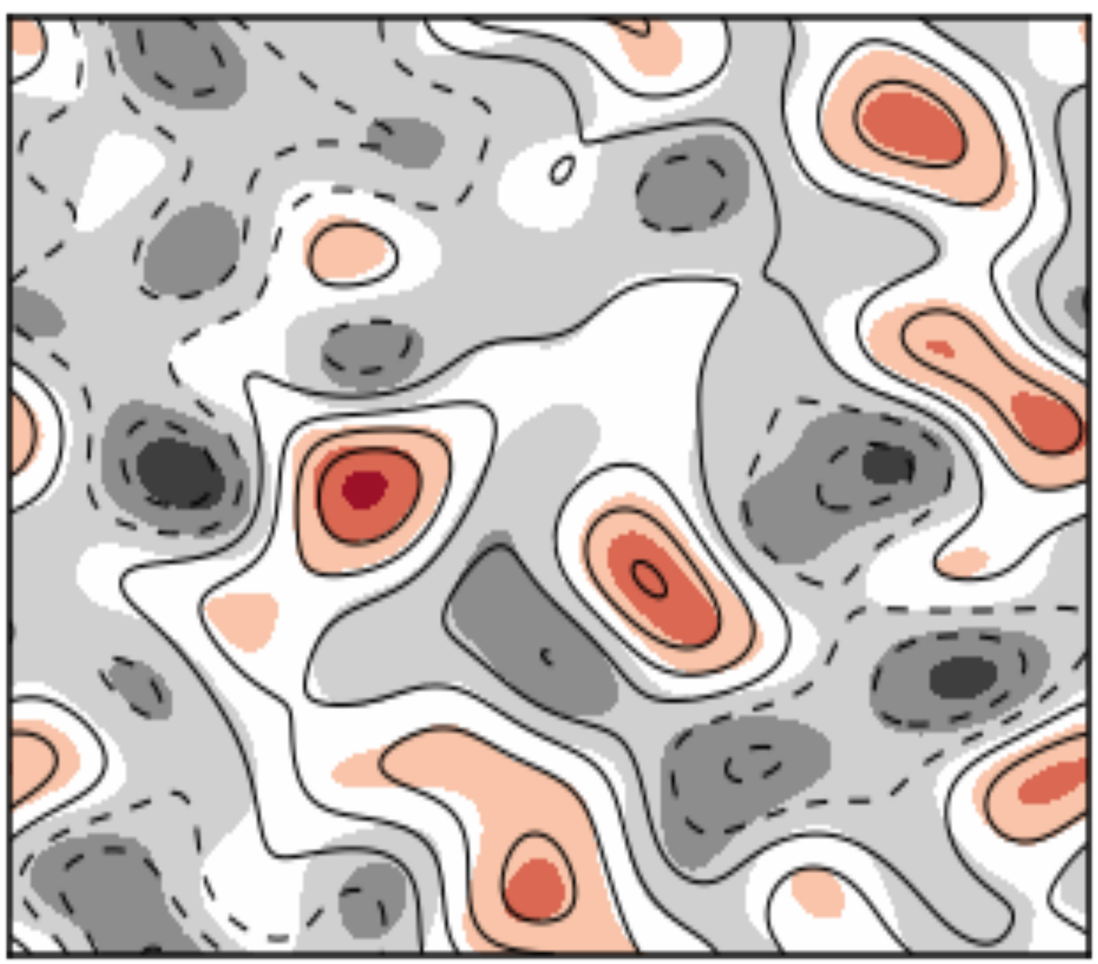
$$\delta^2 Q + \mu\delta^2 E = \frac{1}{2}\iint (\nabla^2\delta\psi)^2 \mathrm{d}x\,\mathrm{d}y + \frac{\mu}{2}\iint |\nabla\delta\psi|^2 \mathrm{d}x\,\mathrm{d}y$$

An example of minimum enstrophy solution

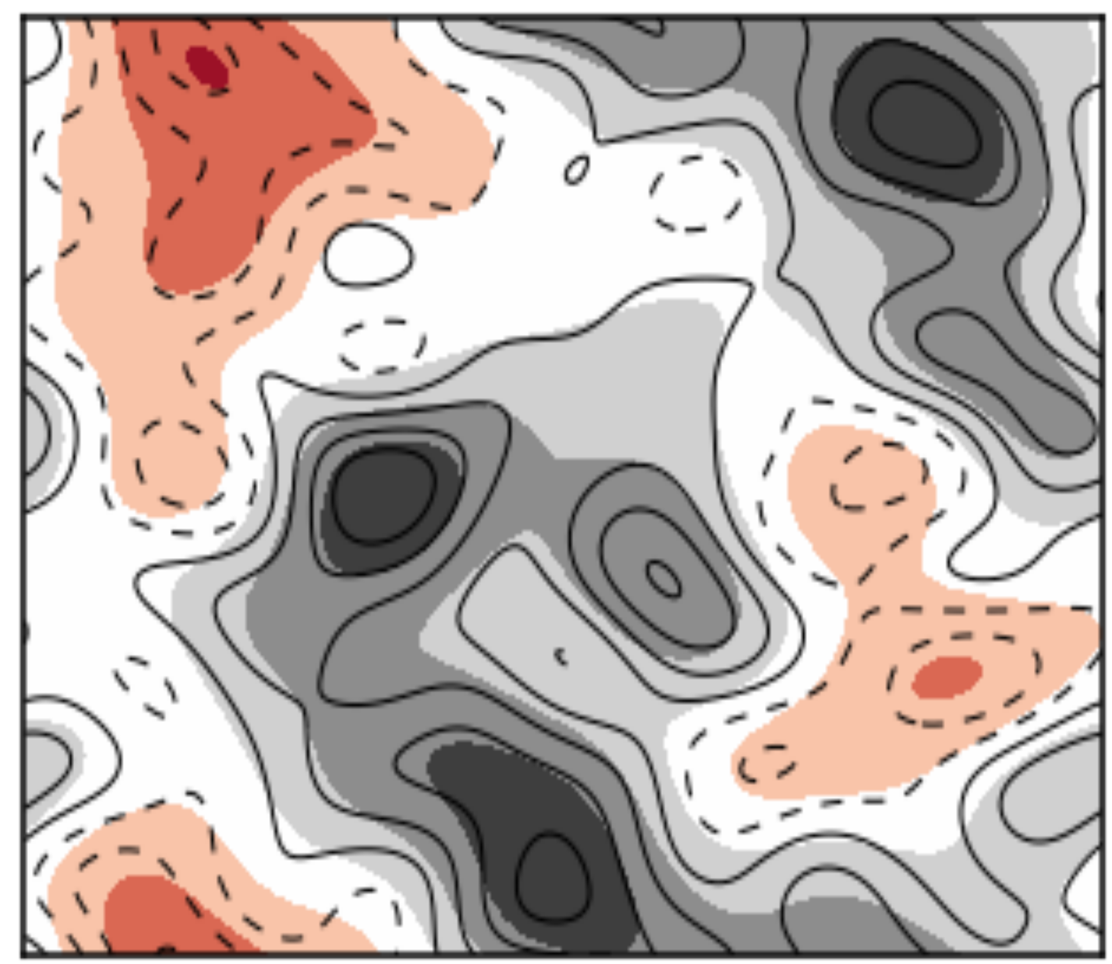
$$E = 0.05\,, \qquad \mu = 9.22072072$$

contours: h

Colors: PV $\nabla^2\psi_0 + h$



Colors: Streamfunction ψ_0



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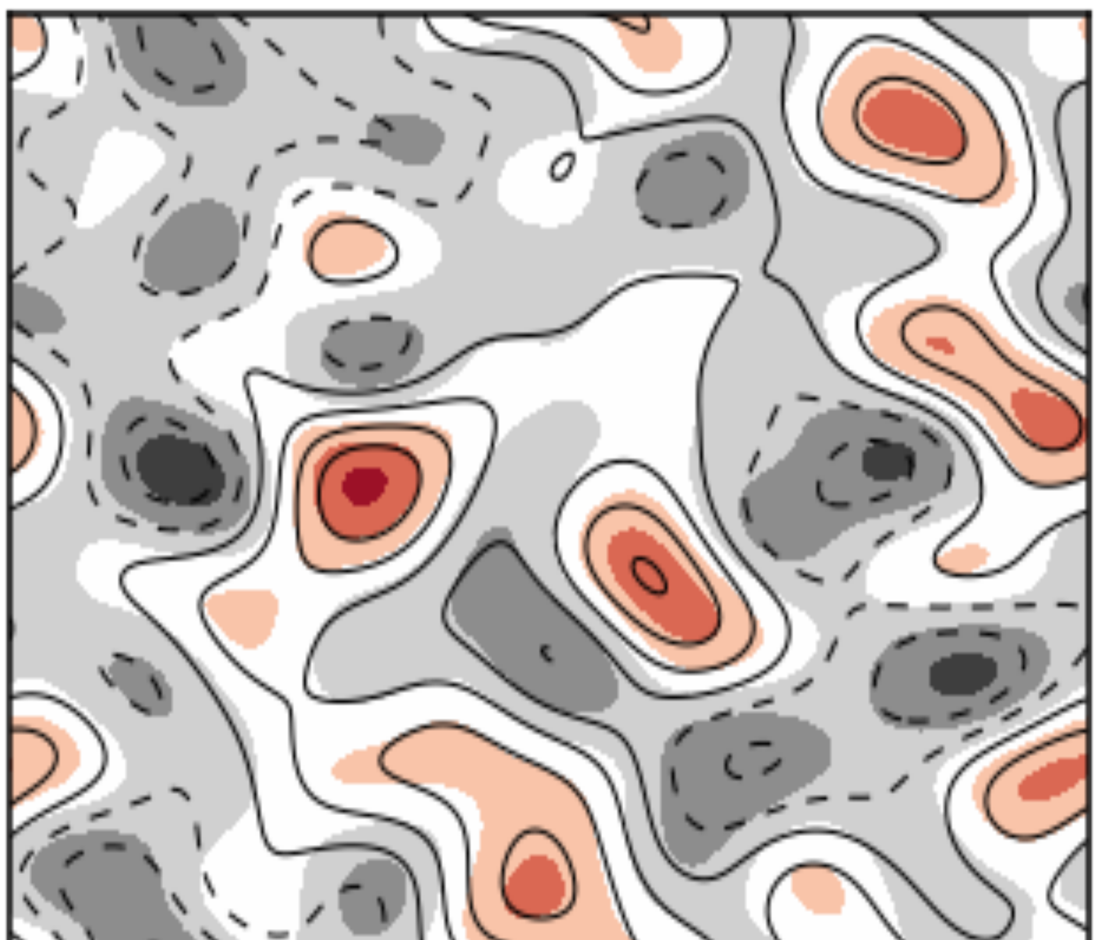

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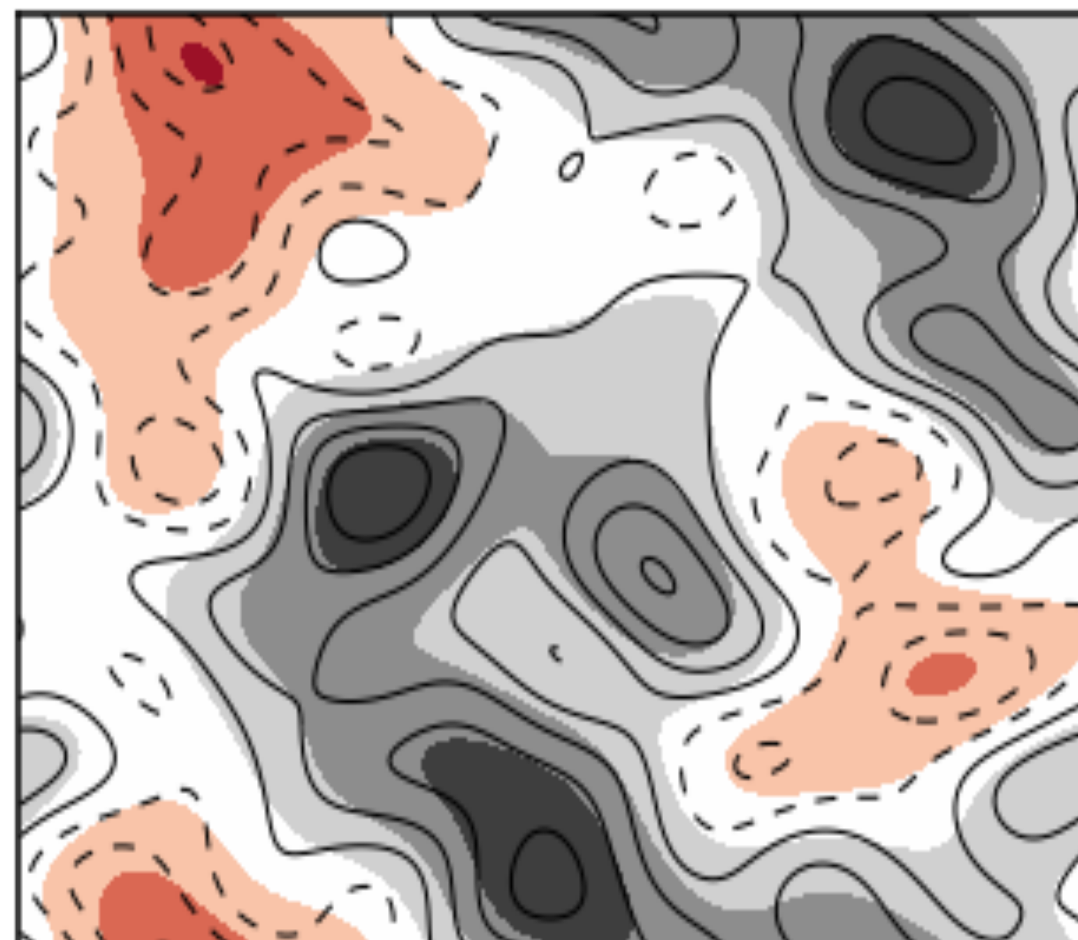


Discuss: Why BH76 did not do this!?

I will not discuss.

The role of saddle points

(There a lots of typos in equations 24 through 30 of the paper.)

The role of viscosity

The role of viscosity

$$\frac{Dq}{Dt} = \nu \nabla^2 (\nabla^2 \psi)$$

$$q = \nabla^2 \psi + h$$

$$\begin{aligned}\frac{\partial Q}{\partial t} &= -\nu \iint \nabla q \cdot \nabla (q - h) \mathsf{d}x \mathsf{d}y \\ &= -\nu \iint |\nabla q|^2 \mathsf{d}x \mathsf{d}y + \nu \iint \nabla q \cdot \nabla h \, \mathsf{d}x \mathsf{d}y\end{aligned}$$

↑ Positive or negative?

If, $\psi = \psi_0$ then enstrophy must increase since ψ_0 is the minimum enstrophy solution.

Closed-basin solution

The circulation on the boundary must be prescribed

If $\quad \psi = 0 \text{ on } \Gamma \quad$ then $\quad C = \oint_\Gamma \partial_n \psi \, \mathrm{d}s = \text{ constant}$

The minimum enstrophy problem has two constraints

$$\delta \tfrac{1}{2} \iint (\nabla^2 \psi + h)^2 \mathrm{d}x \mathrm{d}y + \mu \delta \tfrac{1}{2} \iint |\nabla \psi|^2 \mathrm{d}x \mathrm{d}y + \lambda \oint \frac{\partial}{\partial n} \delta \psi \mathrm{d}s = 0$$

Use integration by parts, e.g.,

$$\iint \nabla^2 \psi \nabla^2 \delta \psi \, \mathrm{d}x \mathrm{d}y = \underbrace{\iint \nabla \cdot [\nabla^2 \psi \nabla \delta \psi] \, \mathrm{d}x \mathrm{d}y}_{= \oint \nabla^2 \psi \frac{\partial}{\partial n} \delta \psi \mathrm{d}s} \underbrace{- \iint \nabla \nabla^2 \psi \cdot \nabla \delta \psi \, \mathrm{d}x \mathrm{d}y}_{= + \iint (\nabla^2 \nabla^2 \psi) \delta \psi \, \mathrm{d}x \mathrm{d}y}$$

The circulation on the boundary must be prescribed

If $\psi = 0 \text{ on } \Gamma$ then $C = \oint_\Gamma \partial_n \psi \mathsf{d}s = \text{ constant}$

The minimum enstrophy problem has two constraints

$$\nabla^2(\nabla^2\psi + h - \mu\psi) = 0 \qquad \text{within S}$$

$$\nabla^2\psi + h + \lambda = 0 \qquad \text{on} \quad \Gamma$$

Using $\psi = 0$ on Γ :

$$\nabla^2\psi - \mu\psi = -(h + \lambda) \qquad \text{within S}$$

C determines the new constant λ

A numerical experiment

Side remark

CPU 36.4 MHz; 36 MFLOPS

CDC 7600

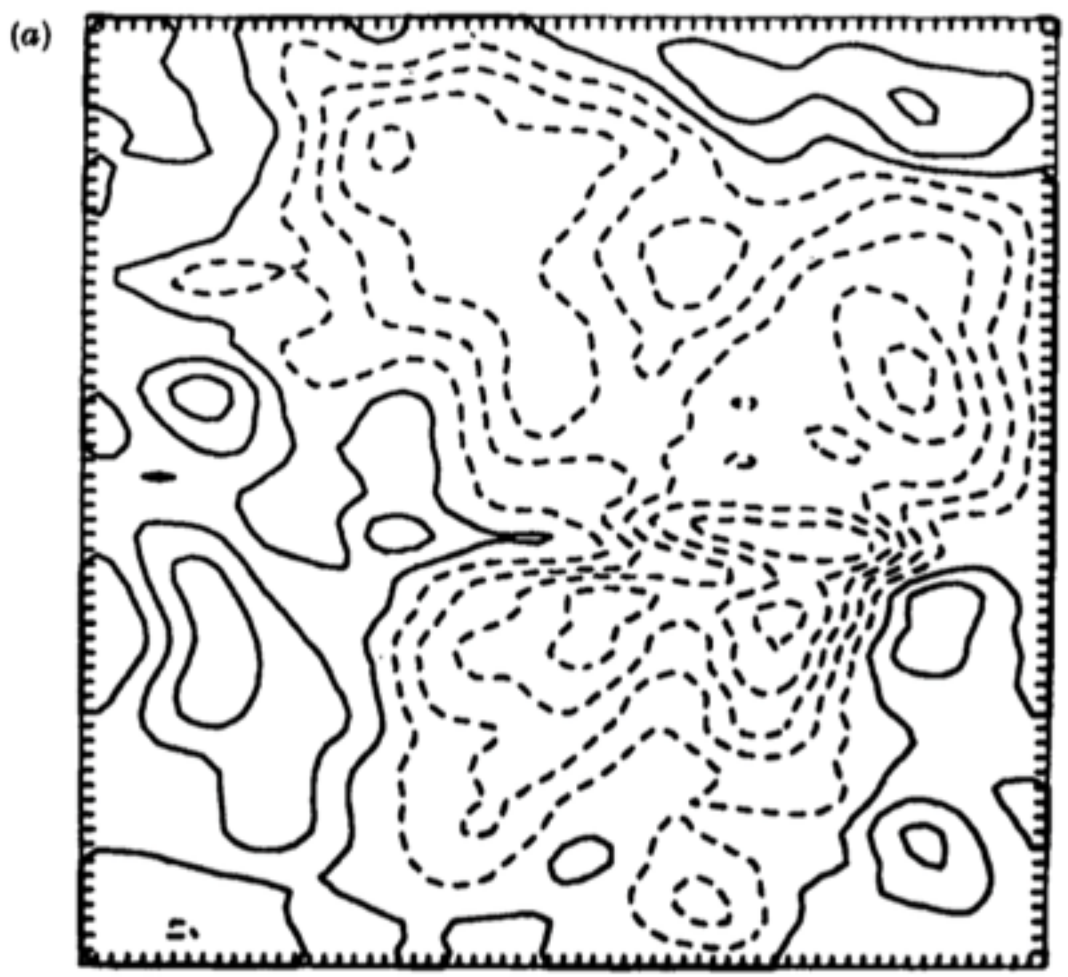
CPU 1.4 GHz; GPU;
100’s GFLOPS to TFLOPS

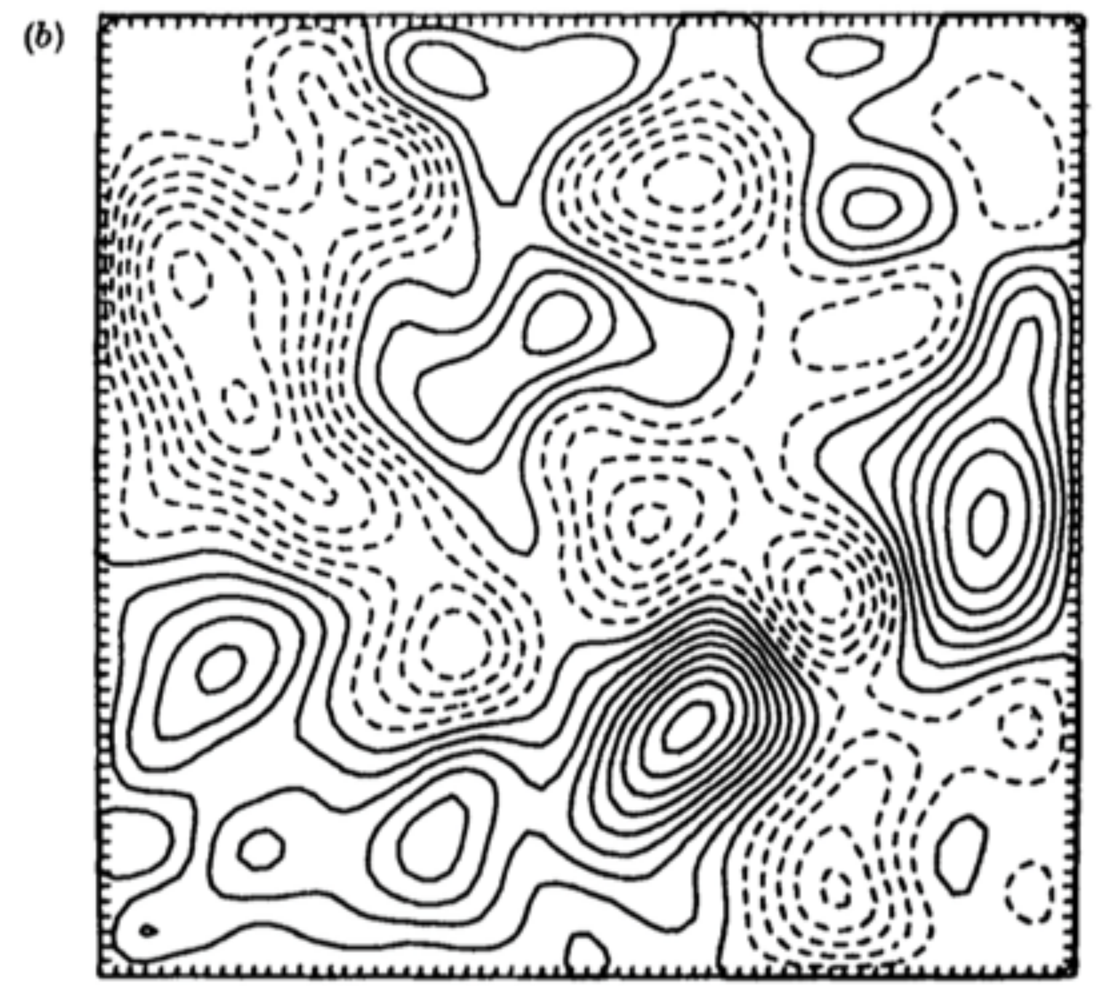
IPhone 6

Source: Wikipedia

Topography

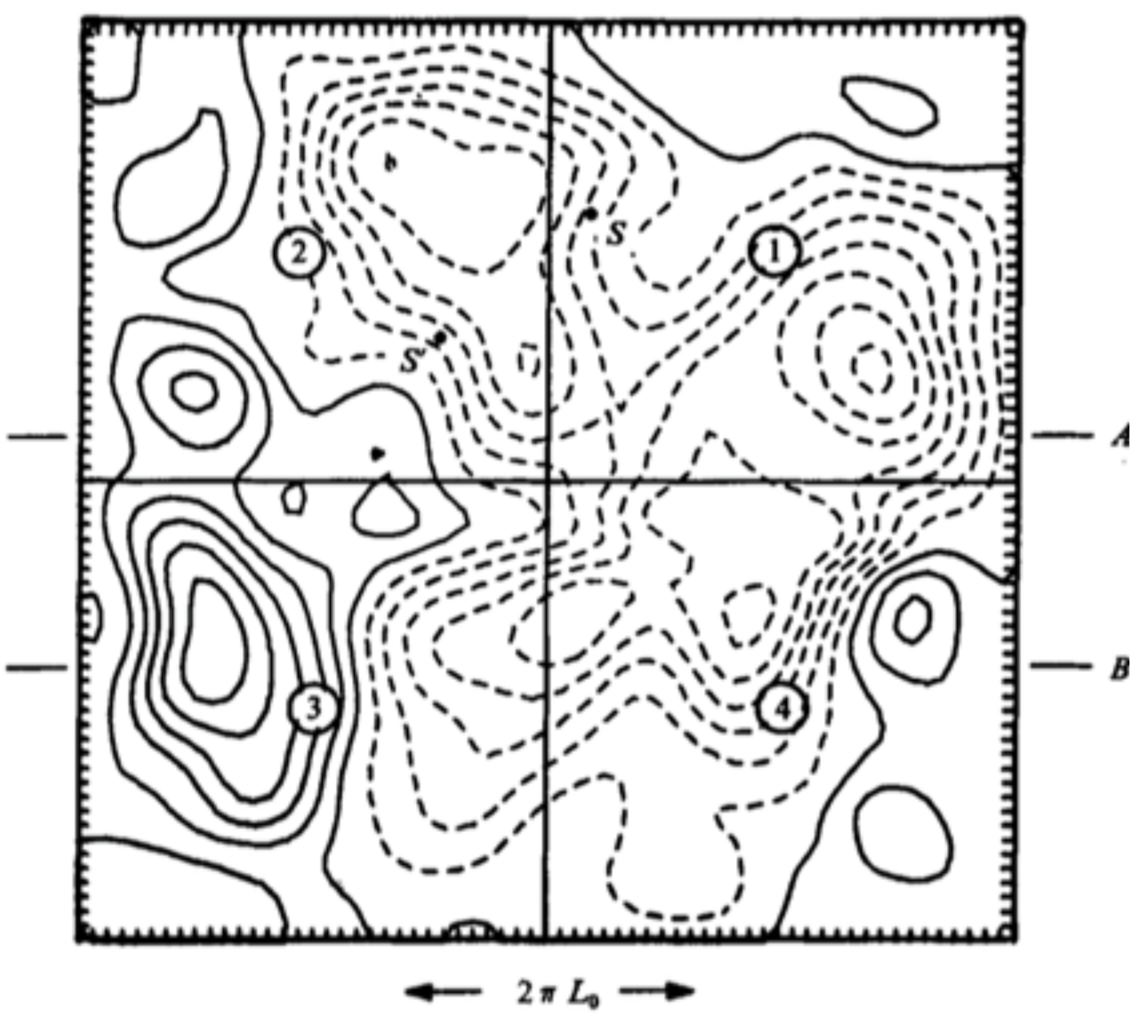
Initial streamfunction



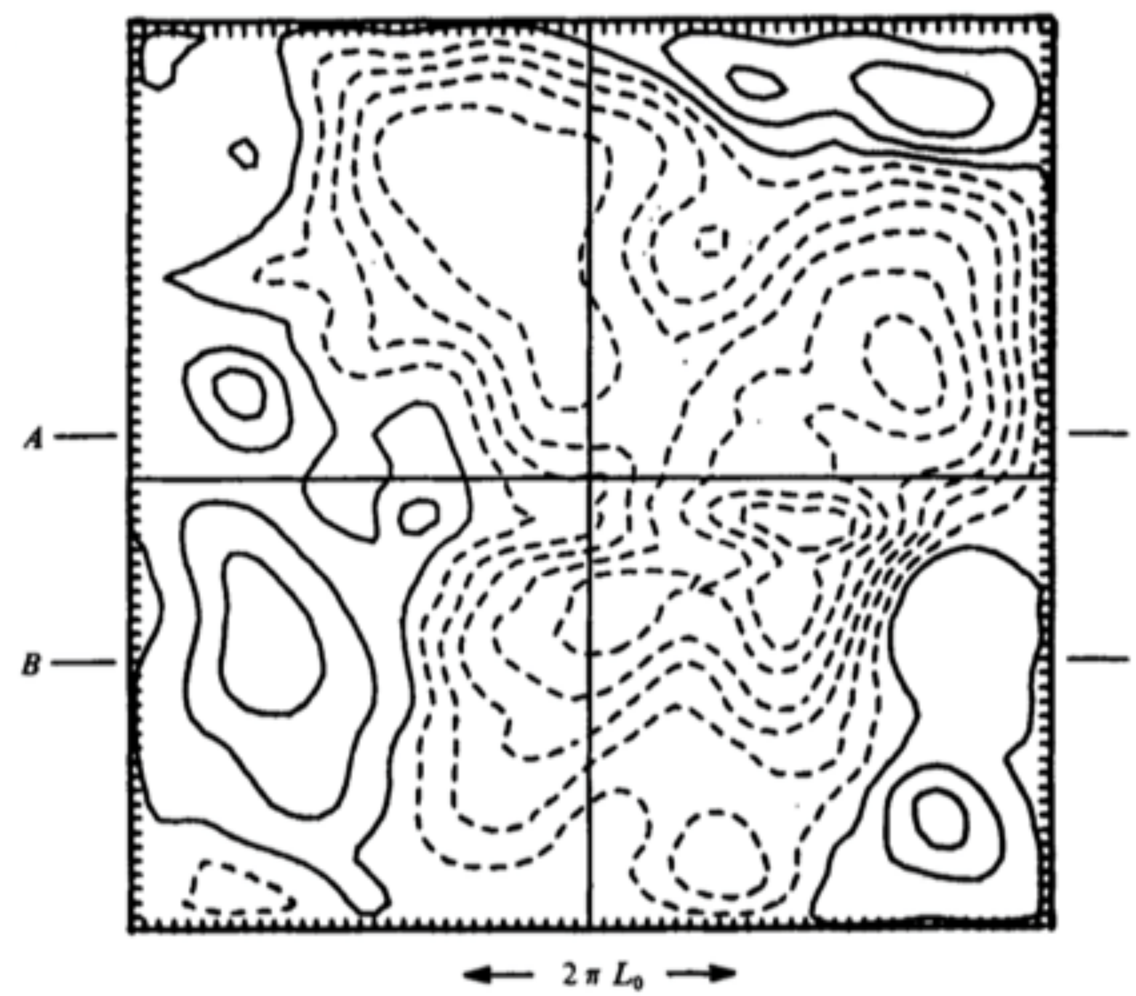


What is the initial energy level and the associated μ ?

Final streamfunction

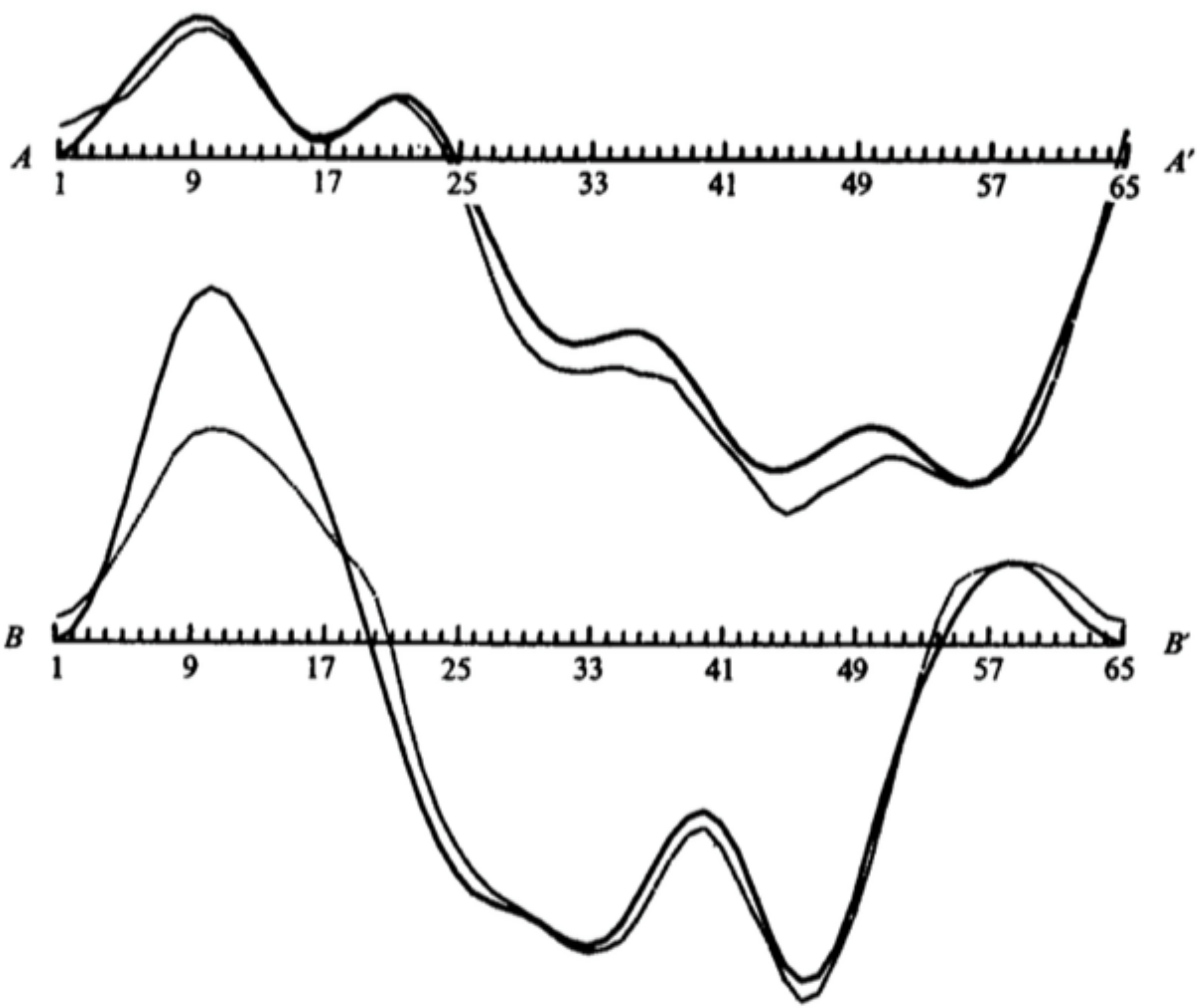


Final PV



The flow quickly becomes quasi-steady

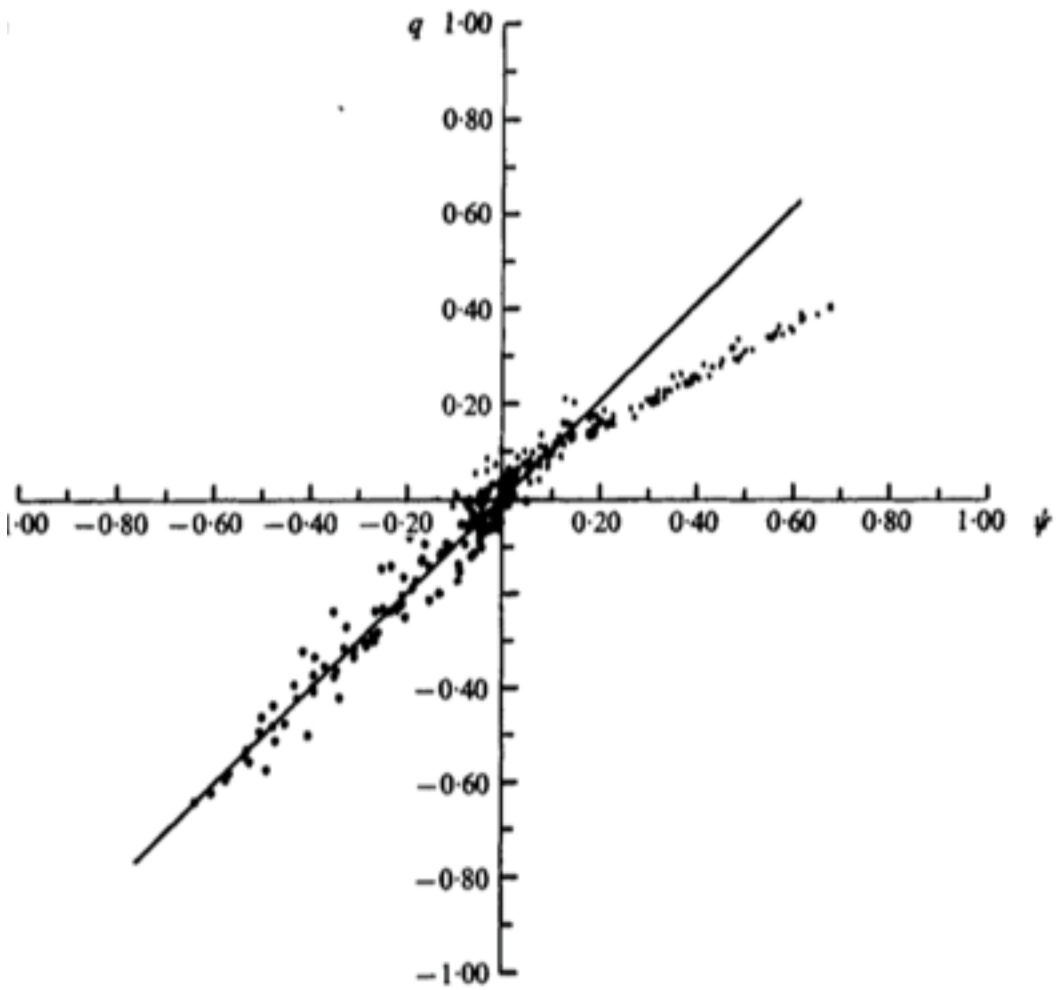
$$\mathsf{J}(\psi, q) = 0 \rightarrow q = F(\psi)$$

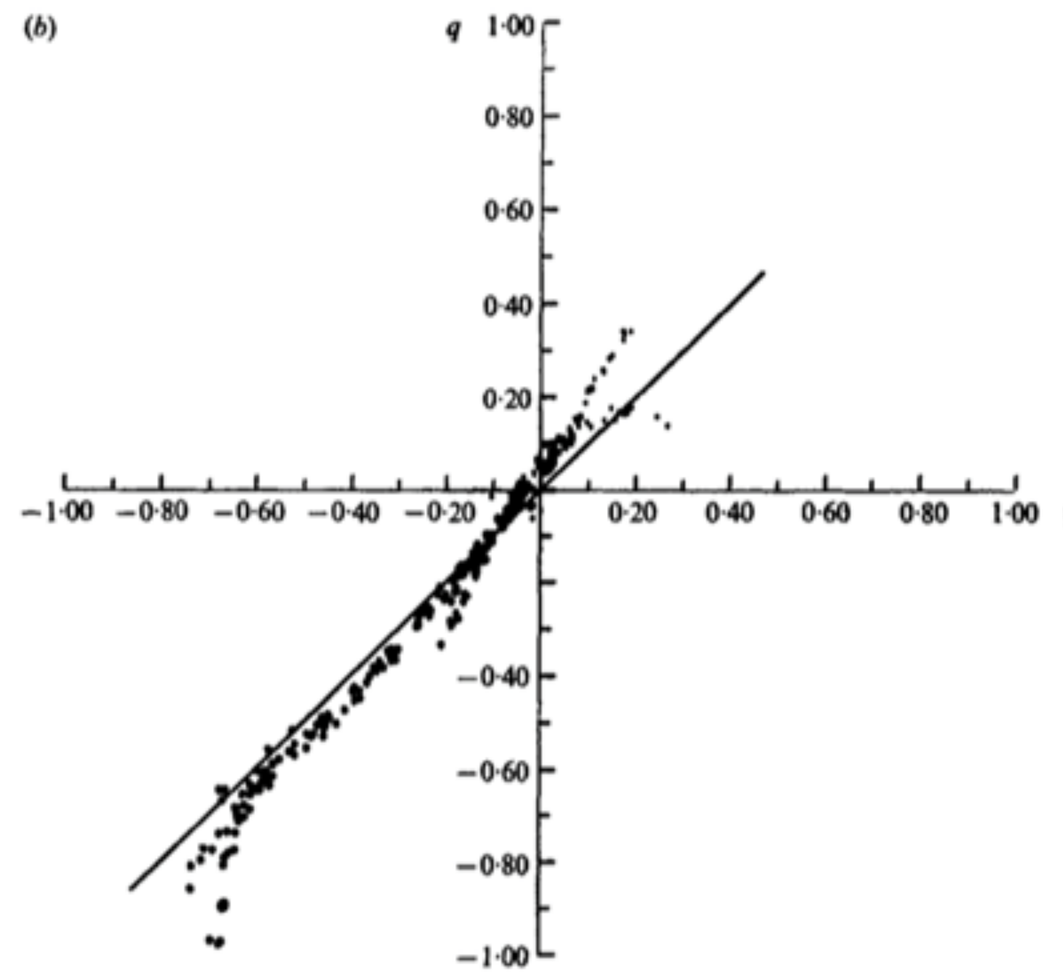


Streamfunction

PV

The $\psi - q$ relationship



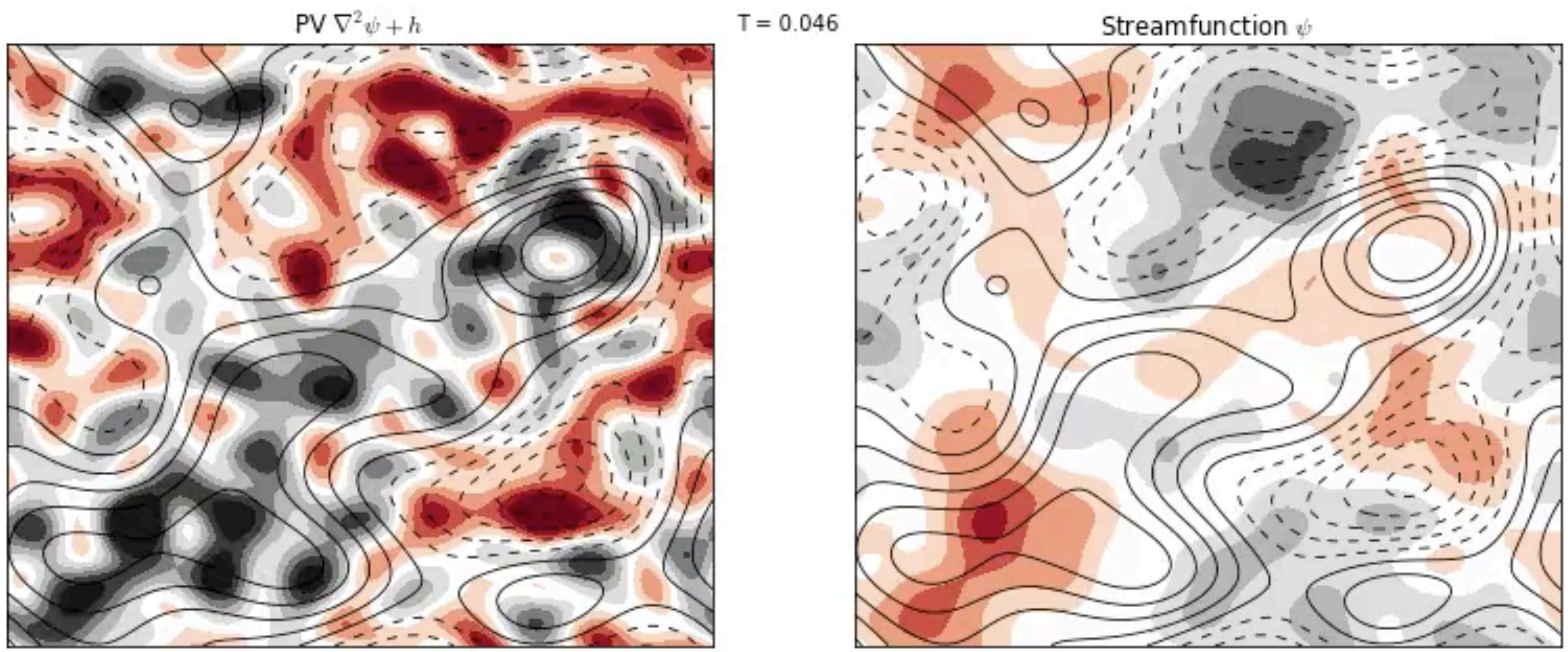


Roughly linear…but there are different “regimes”…

But the energy dropped by 58% owing to low resolution…

$$E = 0.5 \quad |\hat{h}|^2 \propto |k|^{-2} \quad |k|^2|\psi|^2(t=0) \propto |k| \left[1 + \left(\frac{k}{6}\right)^4\right]^{-1}$$

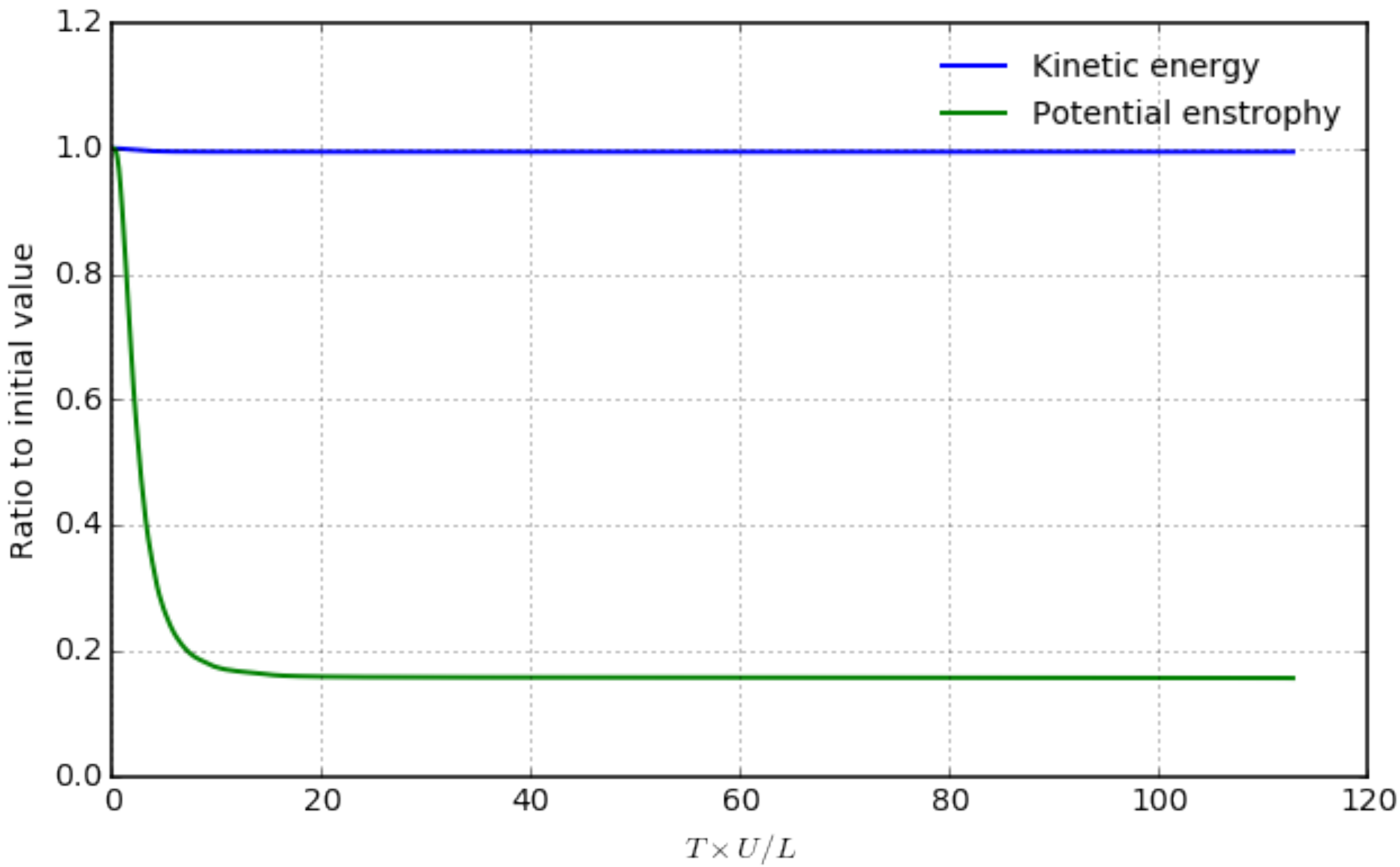
$$\mu = 2.38288288$$



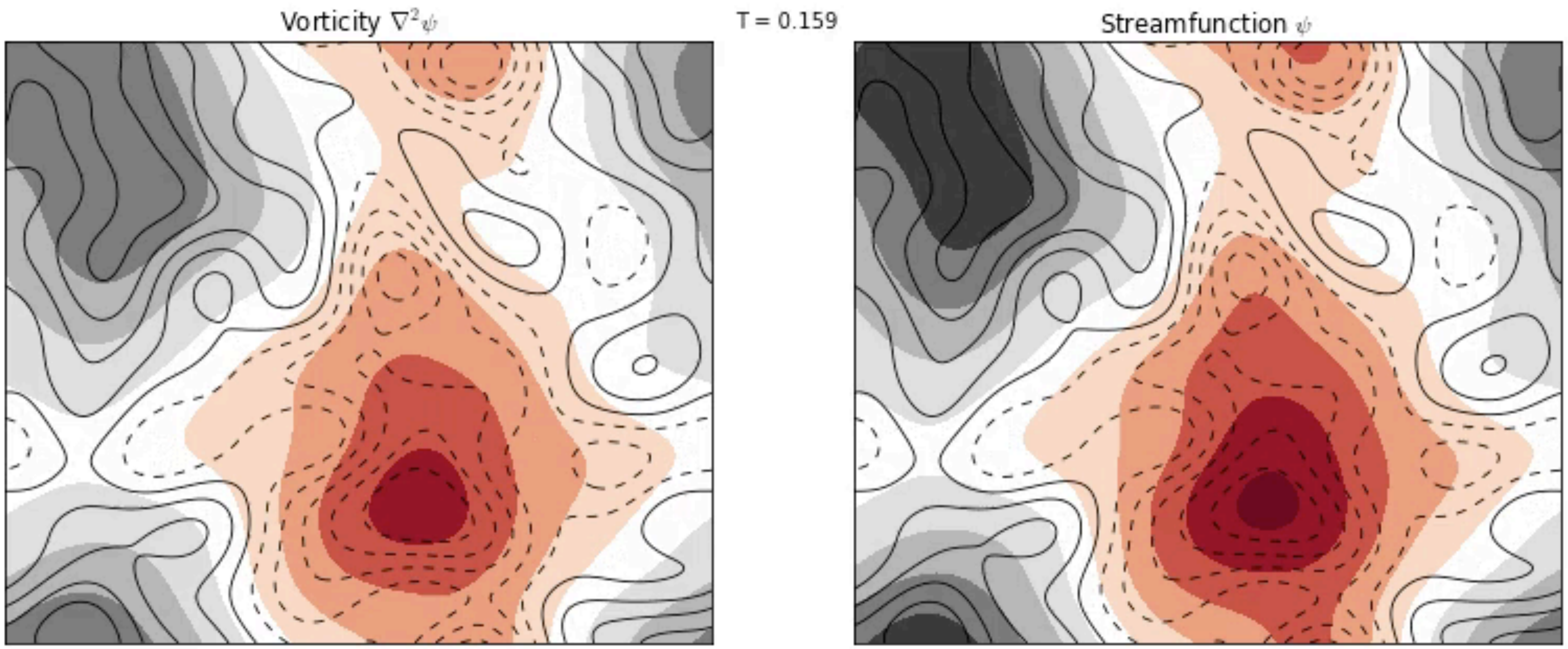
Contours represent topography; colors streamfunction or PV

(doubly periodic calculations)

Energy stays nearly constant, enstrophy decays significantly…



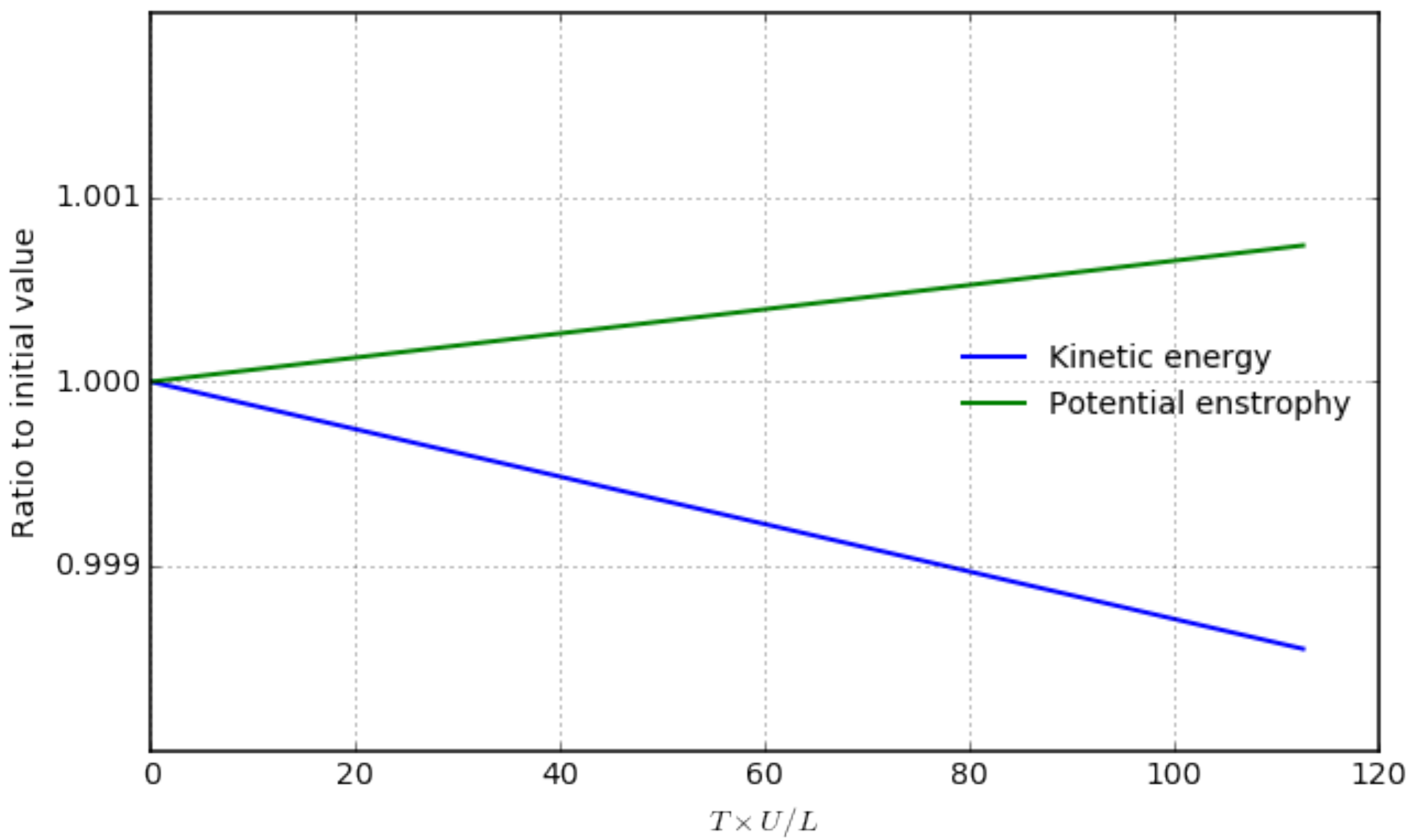
Initializing a simulation with the minimum enstrophy solution



Contours represent topography; colors streamfunction or PV

In practice, with viscosity, energy decays super slowly…

Energy decays, enstrophy increases…



The effects of β

The effects of β

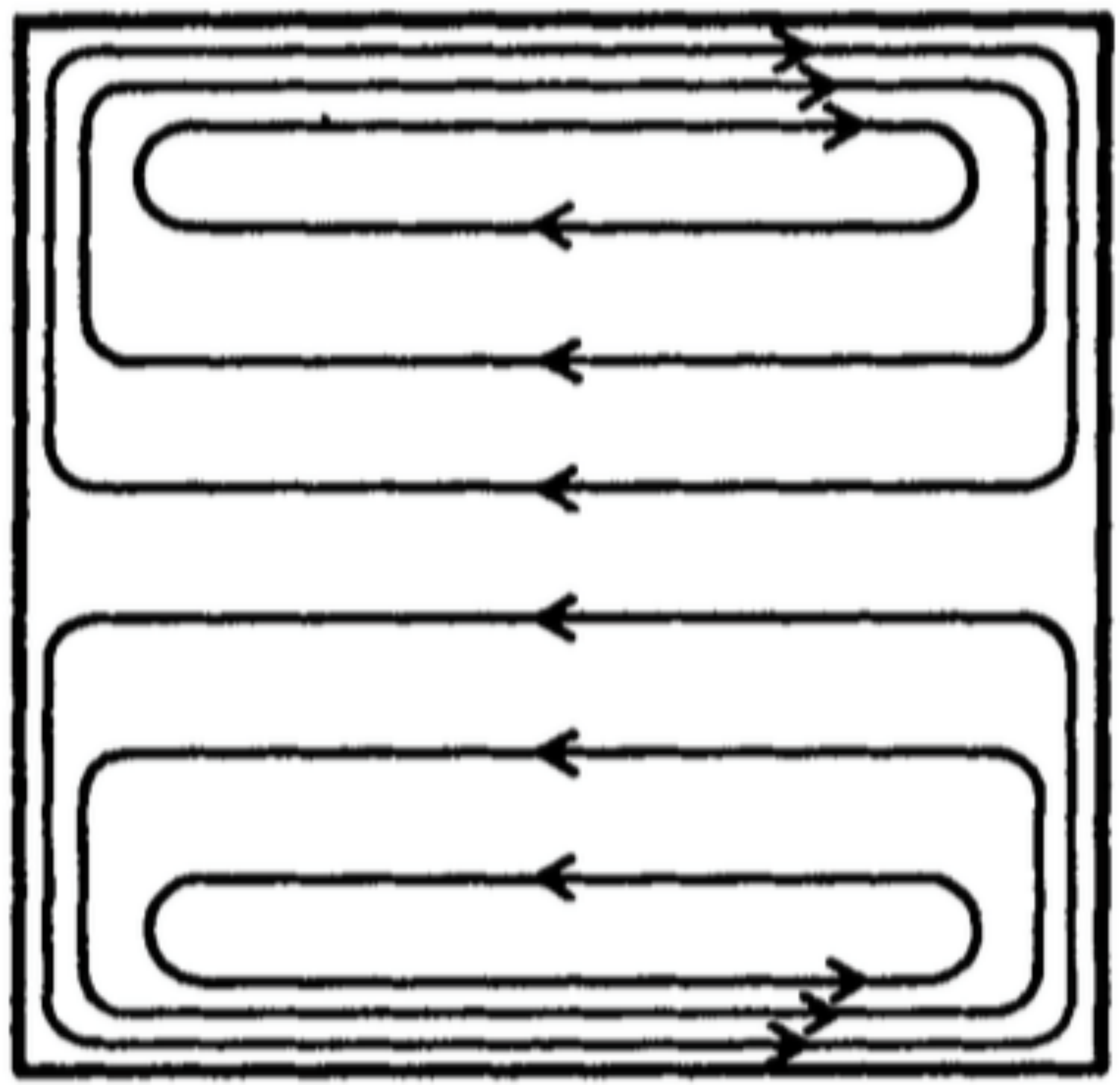
"In so far the the b-slope can be thought of as a Fourier component of zero wavenumber the solution may be instantly obtained":

$$\psi_0(x,y) = \sum \frac{\hat{h}}{k^2 + l^2 + \mu} + \frac{\beta}{\mu} y$$

Discuss: is this obvious?

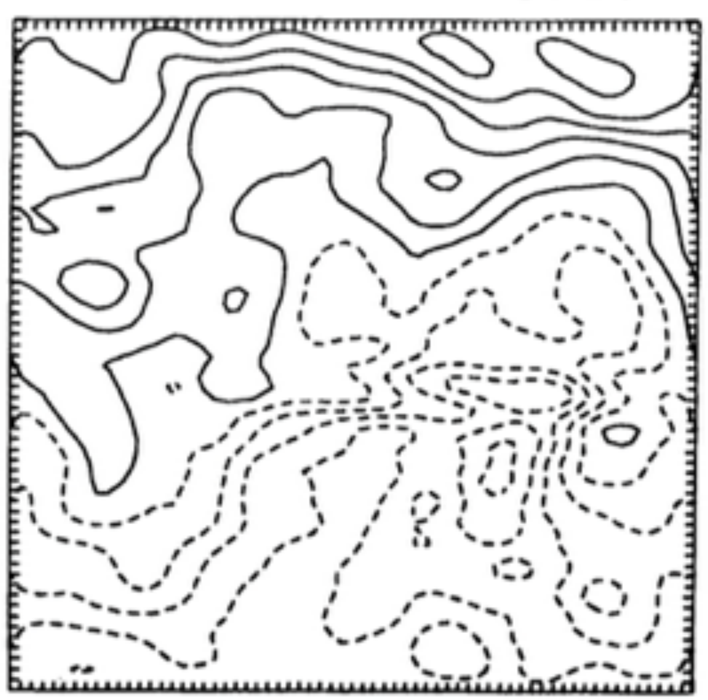
The effects of β

Fofonoff mode type of solution

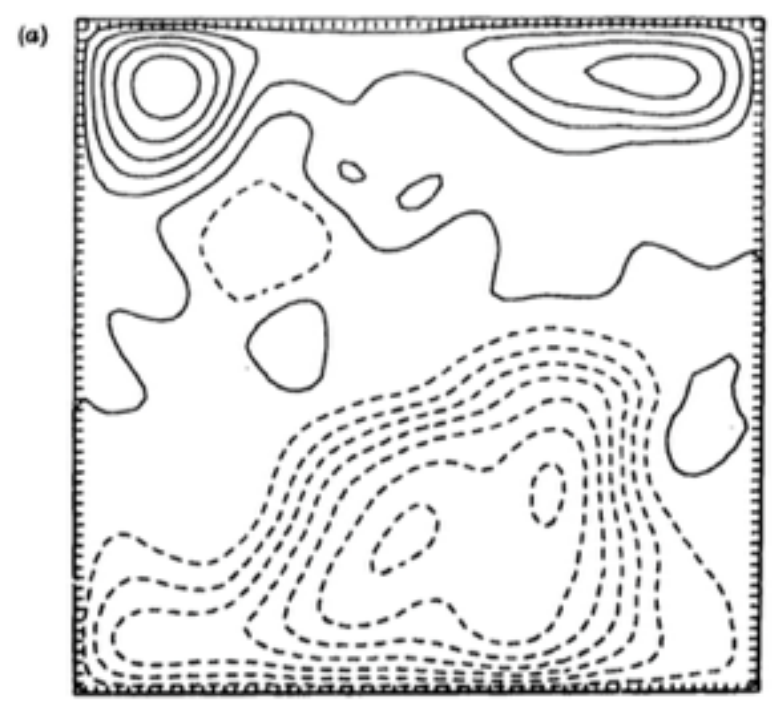


Numerical experiments

Topography

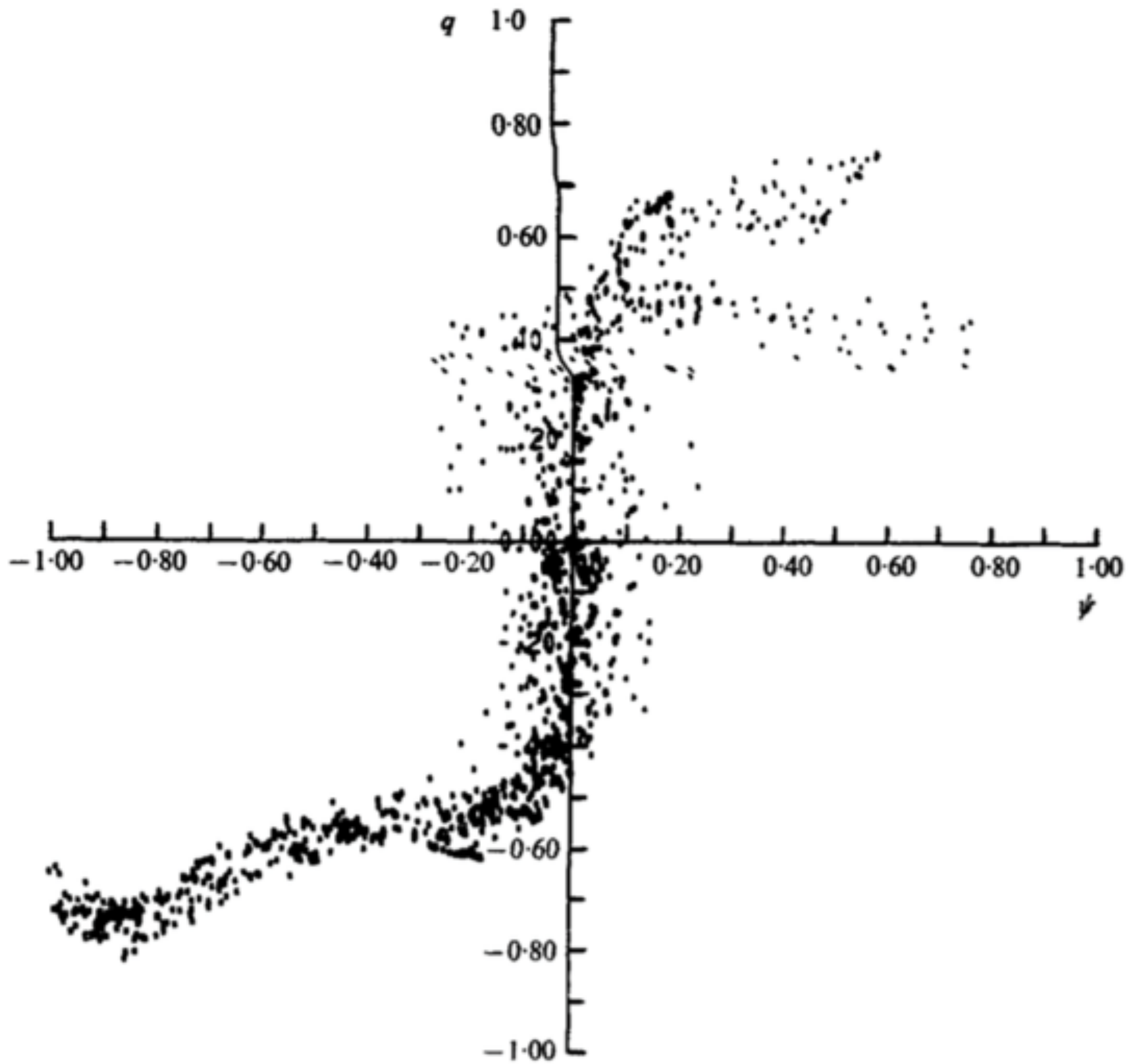


Final streamfunction



Final PV

The $\psi - q$ relationship



The effects of eddies on the large scale flow

Eddy fluxes

Reynolds decomposition

$$\psi = \Psi + \psi' \qquad q = (\nabla^2\Psi + \beta y) + \underbrace{(\nabla^2\psi' + h)}_{q'}$$

…. The eddy PV flux div: $\nabla \cdot \mathsf{F} = \overline{\mathsf{J}(\psi', q')}$

The PV eqn:

$$\frac{Dq'}{Dt} = -\beta\psi'_x$$

Eddy fluxes

$$\frac{Dq'}{Dt} = -\beta \psi'_x$$

$$q' = -\beta \eta$$

η : northward particle displacement

The single-particle diffusivity is

$$D = \frac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\overline{\eta^2} = \beta^{-2}\frac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\overline{q'^2} = \beta^{-1}\overline{v'q'} = -\beta^{-1}\overline{(\nabla^2\psi' + h)\psi'_x}$$

$$\approx -\beta^{-1}\left[\tfrac{1}{2}\partial_x\overline{(\psi'^2_x - \psi'^2_y)} + \partial_y\overline{\psi'_x\psi'_y} + \overline{h\psi'_x}\right] \approx -\beta^{-1}\overline{h\psi'_x}$$

Eddy fluxes

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change to spatial average

scale separation

$$P^{(x)} = \overline{h\psi'_x} \approx -\overline{\psi' h_x}$$

Topographic
form stress

Eddy fluxes

$$D = \frac{\mathsf{d}}{\mathsf{d}t}\tfrac{1}{2}\overline{\eta^2} = \beta^{-2}\frac{\mathsf{d}}{\mathsf{d}t}\tfrac{1}{2}\overline{q'^2} = \beta^{-1}\overline{v'q'} = -\beta^{-1}\overline{(\nabla^2\psi' + h)\psi'_x}$$

$$\approx -\beta^{-1}\left[\tfrac{1}{2}\partial_x\overline{(\psi'^2_x - \psi'^2_y)} + \partial_y\overline{\psi'_x\psi'_y} + \overline{h\psi'_x}\right] \approx -\beta^{-1}\overline{h\psi'_x}$$

change to spatial average

scale separation

$$P^{(x)} = \overline{h\psi'_x} \approx -\overline{\psi' h_x}$$

Topographic form stress

Discuss: really?

Summary

- An initially turbulent flow above topography tends to a minimum enstrophy steady solution that is approximately along isobaths.
- On a beta-plane, the minimum enstrophy solution implies an westward interior flow.
- Topographic form stress appears to play a key role in driving the large-scale flow.

Discussion topics

(Or things to think about in the privacy of your own study)

- "The preferred topography": $|\hat{h}|^2(|k|) \propto |k|^{-2}$
- What is the relevance of freely-decaying solutions to the understanding of real (forced) geophysical flows?
- Differences between closed-basin and doubly-periodic solutions.
- The $h \to 0$ limit.

The limit h -> 0 is a bit murky…

A simple eigenproblem

$$\nabla^2 \psi = \mu \psi$$

The minimum enstrophy solution is

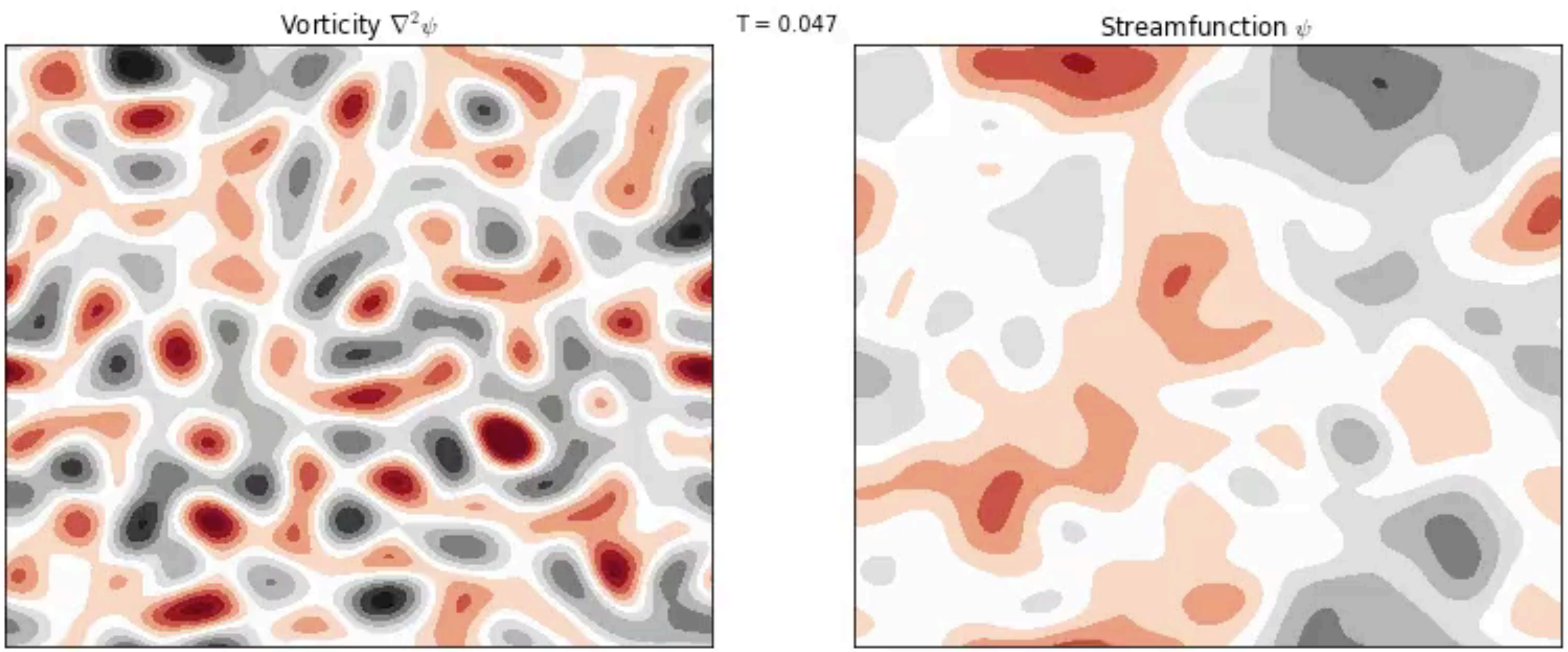
$$\psi_0 = A \cos kx + B \cos ly$$

$$k^2 + l^2 = 1 \quad \longrightarrow \quad \mu = -1$$

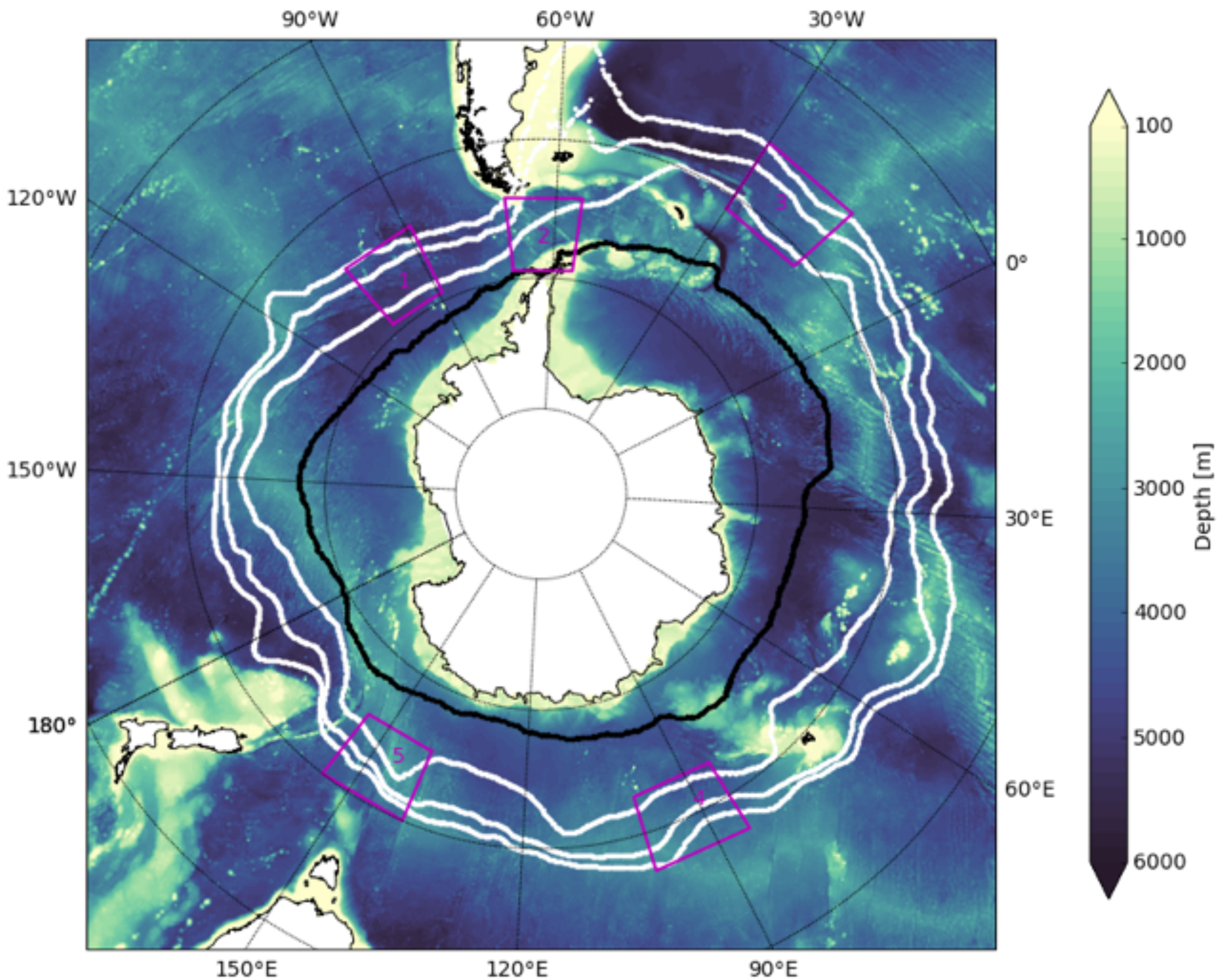
Discuss: What determines A and B?

2D turbulence

An initial value problem (h=0)

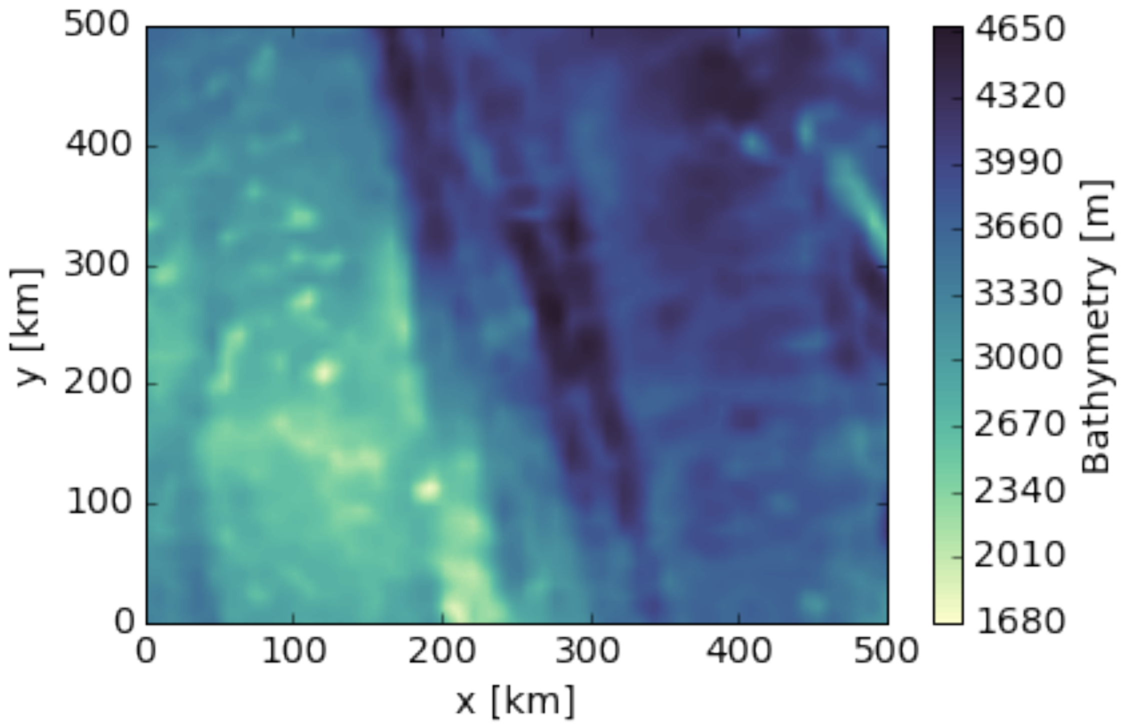


Southern Ocean Topography

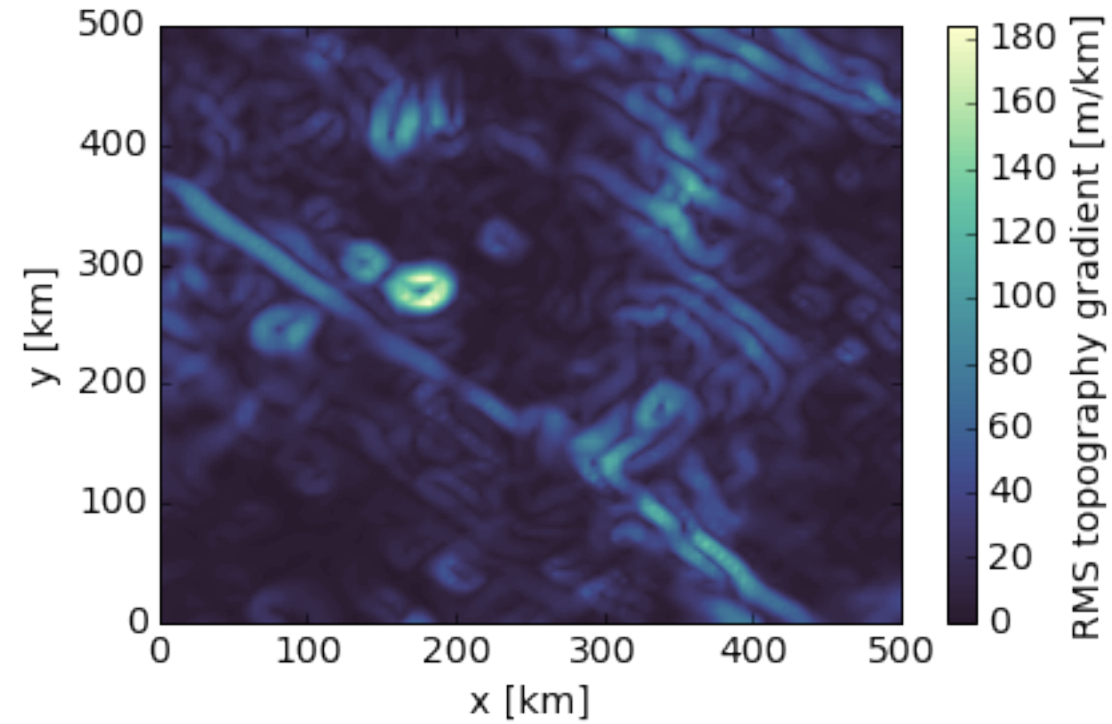


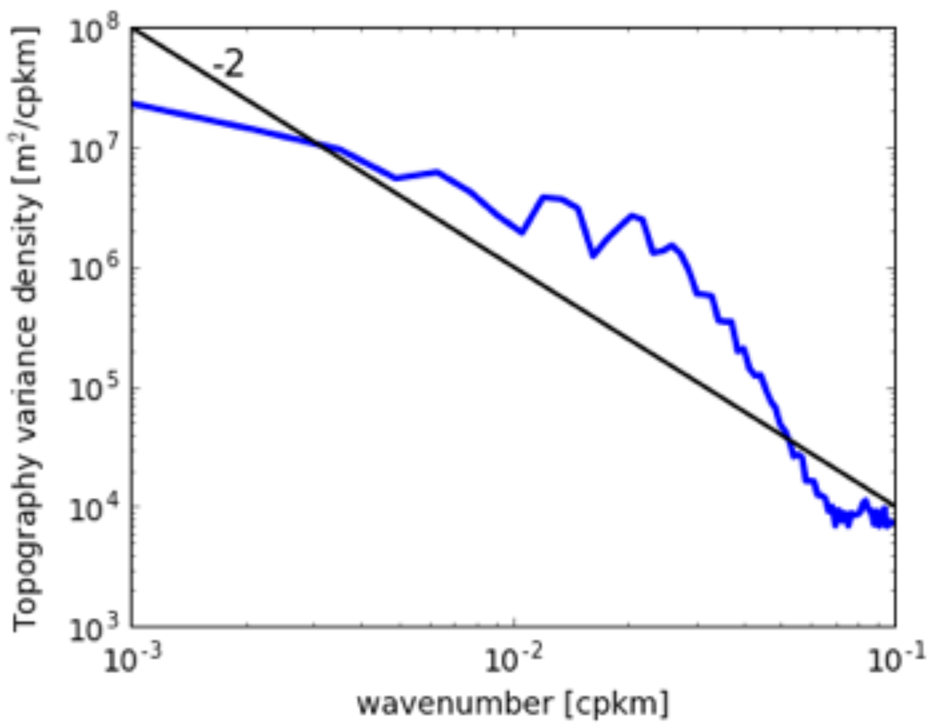
Drake Passage Topography

Topography h



Topographic gradient





$$K = \beta_{topo}/\beta = \frac{\frac{|f_0|}{H}|\nabla h|_{rms}}{\beta}$$

$$\approx 54.24$$

$$H = \langle h \rangle \approx 3716 \text{ m}$$

$\langle \cdot \rangle$ spatial average