Nonlinear stability and statistical mechanics of flow over topography

By GEORGE F. CARNEVALE
Scripps Institution of Oceanography, University of California, San Diego,
La Jolla, CA 92093, USA

AND JORGEN S. FREDERIKSEN
CSIRO, Division of Atmospheric Research, PB 1, Mordialloc, Victoria 3195, Australia

(Received 11 November 1985 and in revised form 23 June 1986)

Inviscid 2D flow (periodic boundaries)

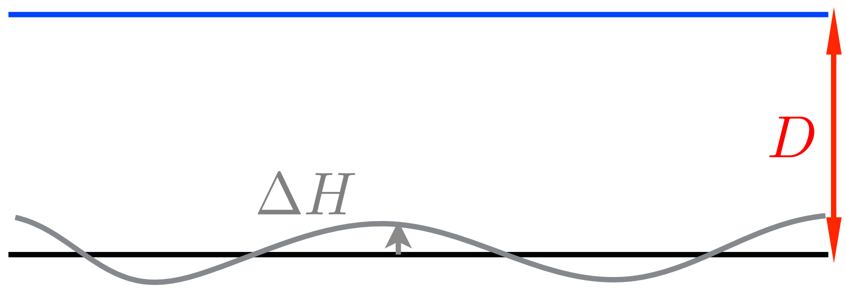
$$\partial_t q + \nabla \cdot (\boldsymbol{v} q) = \partial_t q + J(\psi, q) = \cancel{S} + \cancel{R}$$

advection of potential vorticity

$$\boldsymbol{v} = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x} \right)$$

$$\boxed{\frac{d}{dt} \iint dx dy F(q) = 0}$$

$$q = \nabla^2 \psi + h, \quad h(x, y) = f \frac{\Delta H}{D}$$



Quadratic Invariants

$$\boxed{\partial_t q + \nabla \cdot (\boldsymbol{v} q) = 0}$$

Potential Enstrophy $$Q = \frac{1}{2} \iint q^2 dx dy$$

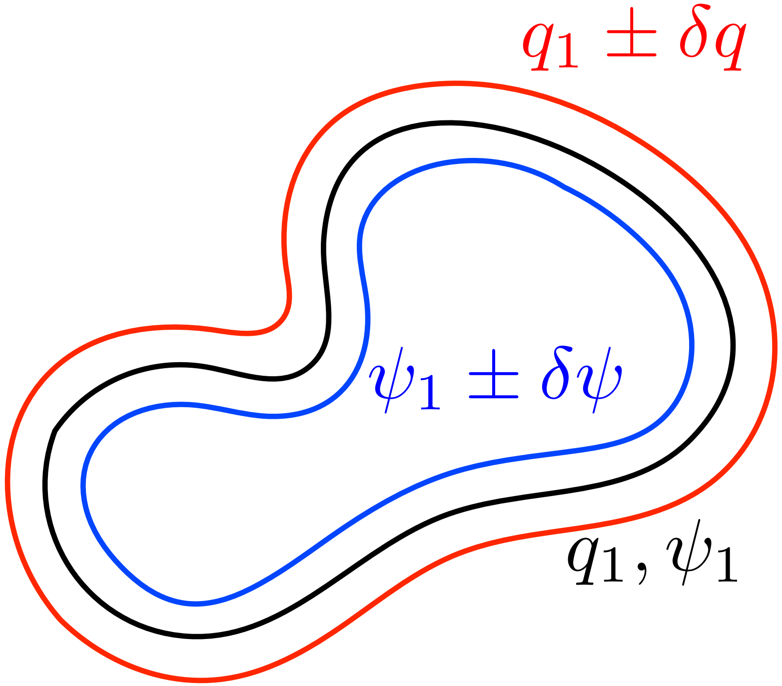
Kinetic Energy $$E = \frac{1}{2} \iint (\nabla \psi)^2 dx dy$$

and many more invariants

$$\frac{d}{dt} \iint dx dy F(q) = 0$$

Section 2: Nonlinear Stability

$$\cancel{\frac{\partial q}{\partial t}} + \hat{z} \cdot (\nabla \psi \times \nabla q) = 0 \longrightarrow \nabla \psi || \nabla q$$



locally $\psi = F'(q)$

non-trivial relationship is

$$\mu \psi = q = \nabla^2 \psi + h$$

$$\psi_k^s = \frac{h_k}{\mu + k^2}$$

Quadratic Invariants (again)

non-zero modes of topography

multiple $\mu' s$ for a given energy (not all necessarily stable)

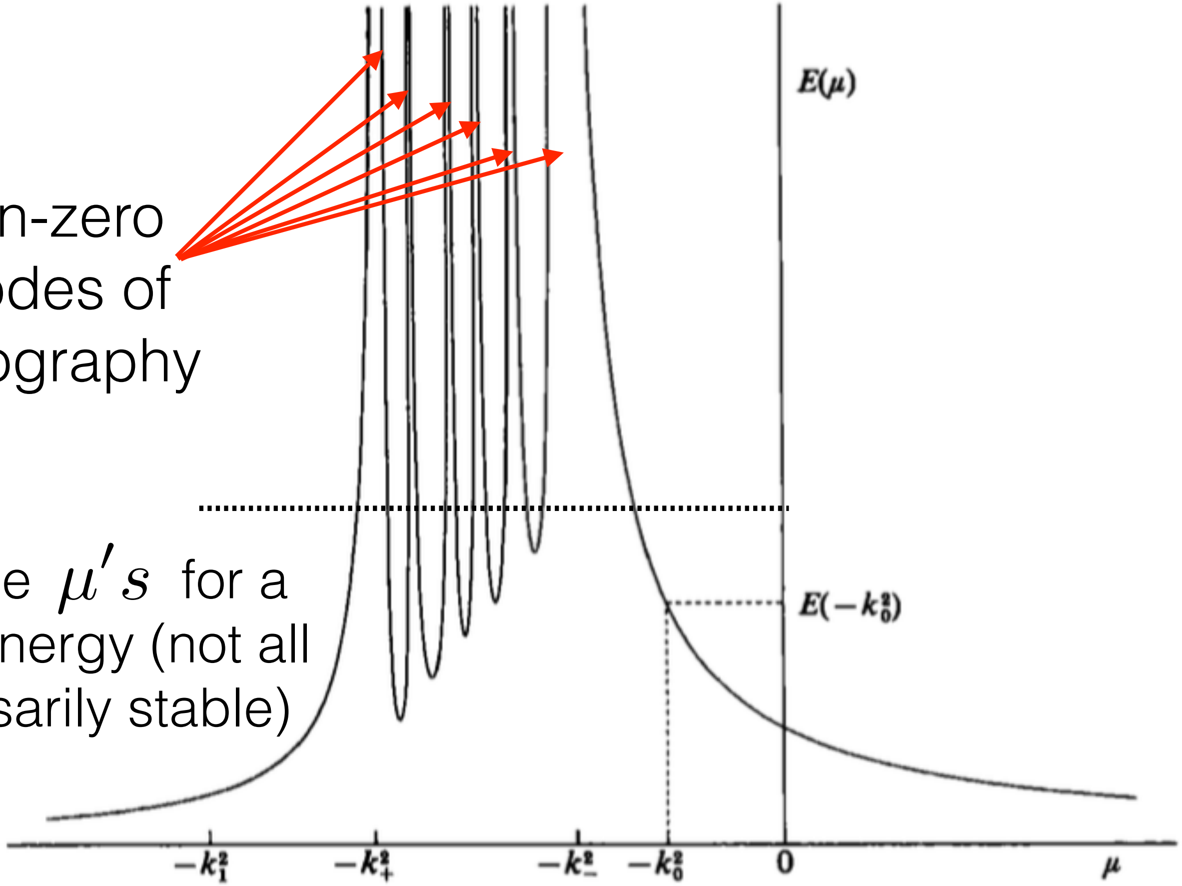


FIGURE 1. Energy in the stationary state $q = \mu\psi$. Schematic sketch of the relation between the total energy and the parameter μ as given by (2.6a). The topography is assumed to have non-zero spectral amplitudes only in the wavenumber range from k_- to k_+. A discrete spectral representation is assumed with resolved wavenumbers only in the range from k_0 to k_1. The solutions ψ^s are nonlinearly stable for all $\mu > -k_0^2$ (also for all $\mu < -k_1^2$ but these do not obtain in the physical limit $k_1 \to \infty$). There is an energy cutoff, $E(-k_0^2)$, above which there can be no physically relevant stable stationary state of this family.

Potential Enstrophy

$$Q^s(\mu) = \frac{1}{2}\sum_k \frac{\mu^2 |h_k|^2}{(\mu + k^2)^2}$$

Kinetic Energy

$$E^s(\mu) = \frac{1}{2}\sum_k \frac{k^2 |h_k|^2}{(\mu + k^2)^2}$$

$$k_0^2 < k^2 < k_1^2$$

$$|h_k|^2 = 0 \text{ for } k_+^2 < k^2 < k_-^2$$

Nonlinear Stability

Perturb the state to determine stability

$$\psi = \psi^s + \delta\psi$$

$$
\begin{aligned}
\overbrace{Q - Q^s + \mu(E - E^s)}^{\delta Q + \mu\delta E} &= \frac{1}{2}\iint dxdy\left[q^2 - (q^s)^2 + \mu(\nabla\psi)^2 - \mu(\nabla\psi^s)^2\right] \\
&= \frac{1}{2}\iint dxdy\left[(\nabla^2(\psi^s + \delta\psi))^2 + 2h\nabla^2\delta\psi - (\nabla^2\psi^s)^2 + \mu(\nabla(\psi^s + \delta\psi))^2 - \mu(\nabla\psi^s)^2\right] \\
&= \frac{1}{2}\iint dxdy\left[2q^s\nabla^2\delta\psi + \mu 2\nabla\delta\psi\cdot\nabla\psi^s + (\nabla^2\delta\psi)^2 + \mu(\nabla\delta\psi)^2\right] \\
&= \frac{1}{2}\iint dxdy\left[(\nabla^2\delta\psi)^2 + \mu(\nabla\delta\psi)^2\right] + \frac{1}{2}\iint dxdy\left[2q^s\nabla^2\delta\psi - 2q^s\nabla^2\delta\psi\right] \\
&\qquad - \frac{1}{2}\oint_C \psi^s(\nabla\delta\psi\cdot\hat{n})dS \\
&= \frac{1}{2}\iint dxdy\left[(\nabla^2\delta\psi)^2 + \mu(\nabla\delta\psi)^2\right] \\
&= \frac{1}{2}\sum_k k^2(k^2 + \mu)|\delta\psi_k|^2
\end{aligned}
$$

for $\begin{matrix}\mu > -k_0^2 \\ \mu < -k_1^2\end{matrix}$ the perturbation doesn't grow or decay

Nonlinear Stability

$$\psi = \psi^s + \delta\psi$$

Perturb the state to determine stability

$$\overbrace{Q - Q^s + \mu(E - E^s)}^{\delta Q + \mu\delta E}$$

$$\boxed{\begin{array}{c}\text{a miracle occurs}\\[1em] = \dfrac{1}{2}\displaystyle\sum_k k^2(k^2 + \mu)|\delta\psi_k|^2\\[1em] \text{single-signed}\end{array}}$$

for $\begin{array}{l}\mu > -k_0^2\\ \mu < -k_1^2\end{array}$ the perturbation doesn't grow or decay in time

we don't know anything about the rest of μ though

Nonlinear Stability

$$\delta(Q+\mu E) = -\sum_k k^2(q_k - \mu\psi_k)\delta\psi_k^*$$

$$\delta^2(Q+\mu E) = \sum_k k^2(k^2+\mu)|\delta\psi_k|^2$$

positive for $\mu > -k_0^2$
minimum enstrophy branch

Recall that since flow is inviscid any perturbation does not decay, regardless of stability

Moving to Section 3

Inviscid statistical equilibrium

a bit of machinery is involved …

probability density depends on invariants of system

$$P(\psi_k) \propto e^{-aE-bQ+\ldots}$$

let's ignore the rest of these $\iint F(q)\,dx\,dy$

$$\langle \psi_k \rangle = \frac{h_k}{(a/b)+k^2}$$

-p is to denote complex conjugate

$$\langle (\zeta_k - \langle \zeta_k \rangle)(\zeta_p - \langle \zeta_p \rangle) \rangle = \frac{k^2}{a+bk^2}\delta_{k,-p}$$

what are a, b?

$$E = \frac{1}{2}\sum_k \frac{1}{a+bk^2} + \frac{k^2|h_k|^2}{((a/b)+k^2)^2}$$

$$Q = \frac{1}{2}\sum_k \frac{k^2}{a+bk^2} + \frac{(a/b)^2|h_k|^2}{((a/b)+k^2)^2}$$

$a + bk^2 > 0$
otherwise
negative energy

for $\mu_{eq} \equiv (a/b)$, $\begin{array}{ll} \mu_{eq} > -k_0^2 & \text{if } b > 0 \\ \mu_{eq} < -k_1^2 & \text{if } b < 0 \end{array}$

Recall from nonlinear stability $\psi_k^s = \dfrac{h_k}{\mu + k^2}$

what does this mean?

Therefore the values of μ^{eq} are restricted to precisely the same range as defined by the stable branches of μ^s derived in the previous section. However, for a given value of E, $\mu^{eq}(E,Q)$ is not the same as $\mu^s(E)$ except when Q is an extremum (i.e. for $Q = Q^s$).

Comparison of E^s given by (2.6) to E^{eq} given by (3.8) shows that for $\mu > -k_0^2$ (the minimum enstrophy branch) we have for equal energies, $E^s = E^{eq}$, that $\mu^s \leqslant \mu^{eq}$ and of course $Q^{eq} \geqslant Q^s = Q_{min}$. Similarly for $\mu \leqslant -k_1^2$ we have $\mu^{eq} \leqslant \mu^s$ and $Q \leqslant Q^s = Q_{max}$. Thus an energy preserving perturbation to an equilibrium state has non-vanishing mean streamfunction. That is, since Q is an extremum in the stable state we have for the perturbed state $Q \neq Q^s$ and $\mu^{eq}(E,Q) \neq \mu^s(E)$. If ensemble averages can be replaced by time averages we then have that the long time average of the perturbed state differs from the stable state which was perturbed. Indeed the long time average of the perturbation cannot vanish. This is the case for the finite resolution model. In contrast, as we shall show in the next section, in the limit of infinite resolution the minimum enstrophy state and the canonical equilibrium are identical.

Infinite Resolution

Reminder

$$E = \frac{1}{2}\sum_k \frac{1}{a+bk^2} + \frac{k^2|h_k|^2}{((a/b)+k^2)^2}$$

$$Q = \frac{1}{2}\sum_k \frac{k^2}{a+bk^2} + \frac{(a/b)^2|h_k|^2}{((a/b)+k^2)^2}$$

$$k_1 \to \infty$$

$$E = \frac{\pi}{2b}\ln\frac{k_1^2+\mu}{k_0^2+\mu} + \frac{1}{2}\iint \mathrm{d}^2k\,\frac{k^2\,|h_k|^2}{(\mu+k^2)^2},$$

$$Q = \frac{\pi}{2b}\left[(k_1^2 - k_0^2) - \mu\,\ln\frac{k_1^2+\mu}{k_0^2+\mu}\right] + \frac{1}{2}\iint \mathrm{d}^2k\,\frac{\mu^2\,|h_k|^2}{(\mu+k^2)^2},$$

Case I. $E \leqslant E^{\mathrm{s}}(-k_0^2)$

$$\mu^{\mathrm{eq}}(E,Q) \to \mu^{\mathrm{s}}(E),$$

$$b \to \frac{\pi k_1^2}{2(Q - Q^{\mathrm{s}}(\mu^{\mathrm{s}}))},$$

Case II. $E \geqslant E^{\mathrm{s}}(-k_0^2)$

$$\mu^{\mathrm{eq}}(E,Q) \to -k_0^2 + k_1^2 \exp\{-2b(E - E^{\mathrm{s}}(-k_0^2))/\pi\},$$

$$b \to \frac{\pi k_1^2}{2}\,[Q - Q^{\mathrm{s}}(-k_0^2) - k_0^2(E - E^{\mathrm{s}}(-k_0^2))]^{-1}.$$

Infinite Resolution (No topography)

Reminder

$$E = \frac{1}{2}\sum_k \frac{1}{a+bk^2} + \frac{k^2|h_k|^2}{((a/b)+k^2)^2}$$

$$Q = \frac{1}{2}\sum_k \frac{k^2}{a+bk^2} + \frac{(a/b)^2|h_k|^2}{((a/b)+k^2)^2}$$

$$k_1 \to \infty$$

$$E = \frac{\pi}{2b}\ln\frac{k_1^2+\mu}{k_0^2+\mu} + \cancel{\frac{1}{2}\iint \mathrm{d}^2k\,\frac{k^2|h_k|^2}{(\mu+k^2)^2}},$$

$$Q = \frac{\pi}{2b}\left[(k_1^2-k_0^2)-\mu\ln\frac{k_1^2+\mu}{k_0^2+\mu}\right] + \cancel{\frac{1}{2}\iint \mathrm{d}^2k\,\frac{\mu^2|h_k|^2}{(\mu+k^2)^2}},$$

Case I. $E \leqslant E^{\mathrm{s}}(-k_0^2)$

maximum energy of minimum enstrophy state

$$\mu^{\mathrm{eq}}(E,Q) \to \mu^{\mathrm{s}}(E),$$

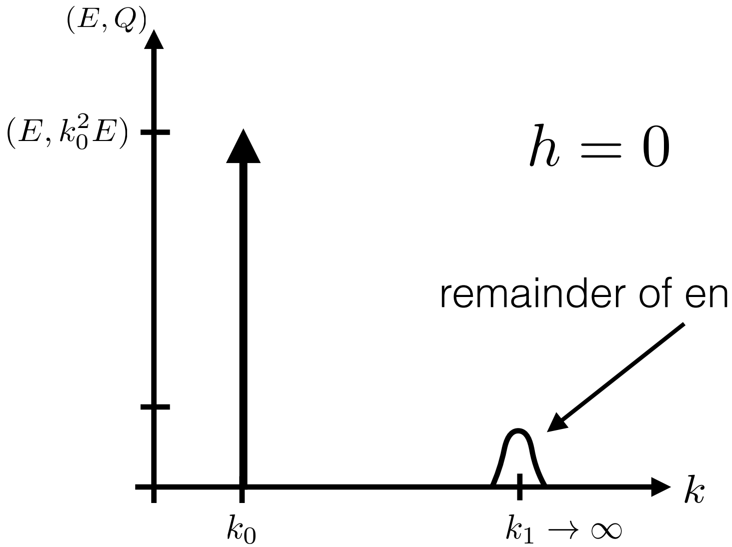
$$b \to \frac{\pi k_1^2}{2(Q-Q^{\mathrm{s}}(\mu^{\mathrm{s}}))},$$

Case II. $E \geqslant E^{\mathrm{s}}(-k_0^2)$

$$\mu^{\mathrm{eq}}(E,Q) \to -k_0^2 + k_1^2\exp\{-2b(E-E^{\mathrm{s}}(-k_0^2))/\pi\},$$

$$b \to \frac{\pi k_1^2}{2}\left[Q-Q^{\mathrm{s}}(-k_0^2)-k_0^2(E-E^{\mathrm{s}}(-k_0^2))\right]^{-1}.$$

Infinite Resolution



142 *F. P. Bretherton and D. B. Haidvogel*

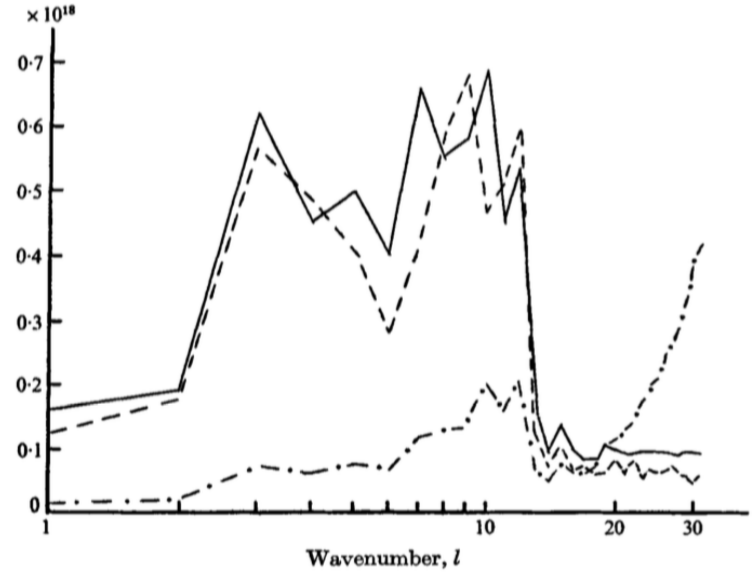
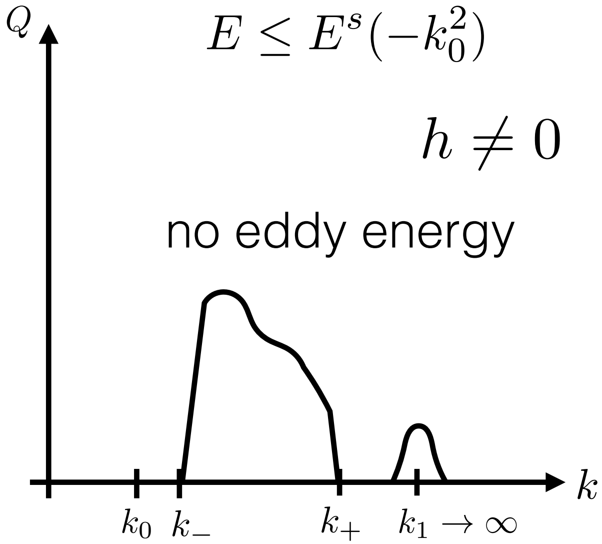
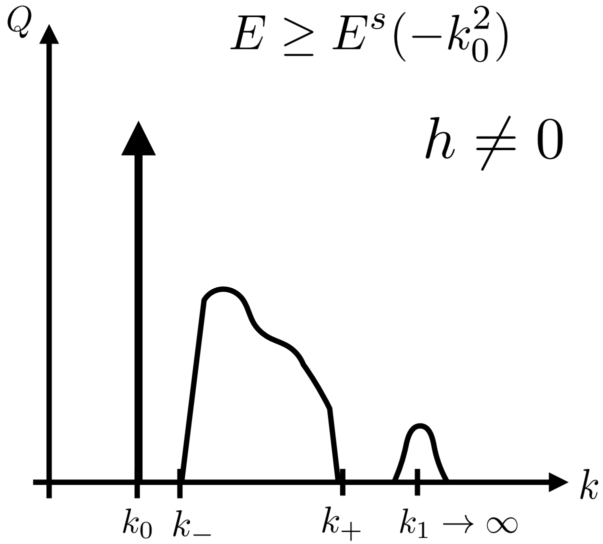


FIGURE 6. One-dimensional spectra for the final state in experiment 1 *vs.* north–south wavenumber. Ordinates: –·–·–, slope spectrum, $10^6 l^3 \Sigma_k |\hat{h}_{kl}|^2$; ———, velocity spectrum predicted from (14), $l^3 \Sigma_k |\hat{h}_{kl}|^2/(\mu+k^2)^2$; – – –, computed velocity spectrum, $l^3 \Sigma_k |\hat{\psi}_{kl}|^2$. Equal areas beneath the curves correspond to equal contributions to the variance.





Beta plane flow

$$\partial_t \zeta + J(\psi, \zeta + h + \beta y) = 0$$

Periodic on ψ, h

Energy is conserved

$$\frac{d}{dt}\frac{1}{2}\iint (\nabla\psi)^2 dx dy = 0$$

Oh No!

Small and large scale potential enstrophy are not conserved

$$\frac{d}{dt}\frac{1}{2}\iint (\zeta + h)^2 \neq 0$$

$$\frac{d}{dt}\frac{1}{2}\iint (\zeta + h + \beta y)^2 \neq 0$$

Beta plane flow

$$\partial_t \zeta + J(\psi, \zeta + h + \beta y) = 0$$

Periodic on ψ, h

Introduce mean eastward flow $\Psi = \psi - Uy$

$$\frac{d}{dt}\frac{1}{2}\iint (\nabla\psi)^2 dxdy = -U \iint h\frac{\partial\psi}{\partial x}dxdy$$

Conserved Quantity $\iint dxdy \left[(\nabla\psi)^2 - \frac{U}{\beta}(\zeta + h)^2 \right]$

$$E_\Psi = \frac{1}{2}U^2 + \iint dxdy (\nabla\psi)^2, \text{ for } \frac{dU}{dt} = \iint h\frac{\partial\psi}{\partial x}dxdy$$

$$Q_\Psi = \beta U + \frac{1}{2}\iint dxdy (\zeta + h)^2$$

let's do the same thing again

without a beta plane

$$\cancel{\frac{\partial q}{\partial t}} + \hat{z} \cdot (\nabla \psi \times \nabla q) = 0 \longrightarrow \nabla \psi || \nabla q$$

possible solution

$$\mu \psi^s = \nabla^2 \psi^s + h$$

with a beta plane

$$\psi \rightarrow \Psi = \psi - Uy$$

$$q \rightarrow q = \zeta + h + \beta y$$

possible solution

$$\mu(\psi^s - U^s y) = \nabla^2 \psi^s + h + \beta y$$

we have parts that are periodic and large scale parts

$$\mu(\psi^s - U^s y) = \nabla^2 \psi^s + h + \beta y$$

$$\mu = -\frac{\beta}{U^s}$$

$$\mu \psi^s = \nabla^2 \psi^s + h$$

$$E^s(\mu) = \frac{1}{2}\frac{\beta^2}{\mu^2} + \frac{1}{2}\sum_k \frac{k^2 |h_k|^2}{(\mu + k^2)^2}$$

$$Q^s(\mu) = -\frac{\beta^2}{\mu} + \frac{1}{2}\sum_k \frac{\mu^2 |h_k|^2}{(\mu + k^2)^2}$$

These formulae may be interpreted as modifications in which a $k = 0$ component has been added to the topography. Thus, since we now have topography at the largest allowed scale there is always an allowed minimum enstrophy solution for any given E; this is made clear in the modified schematic in figure 2, where k_0, k_-, k_+, and k_1 retain there previous definitions referring only to the periodic functions ψ and h. There is a 'spectral gap' between the large-scale, $k = 0$, and the scales with $k \geqslant k_0$.

$$E^s(\mu) = \frac{1}{2}\frac{\beta^2}{\mu^2} + \frac{1}{2}\sum_k \frac{k^2|h_k|^2}{(\mu + k^2)^2}$$

$$Q^s(\mu) = -\frac{\beta^2}{\mu} + \frac{1}{2}\sum_k \frac{\mu^2|h_k|^2}{(\mu + k^2)^2}$$

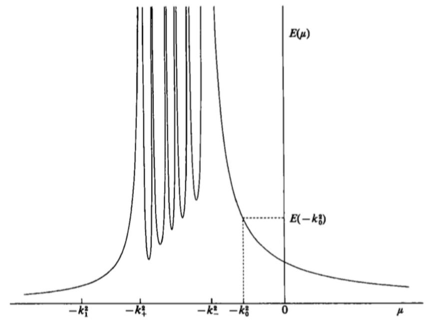


FIGURE 1. Energy in the stationary state $q = \mu\psi$. Schematic sketch of the relation between the total energy and the parameter μ as given by (2.6a). The topography is assumed to have non-zero spectral amplitudes only in the wavenumber range from k_- to k_+. A discrete spectral representation is assumed with resolved wavenumbers only in the range from k_0 to k_1. The solutions ψ^s are nonlinearly stable for all $\mu > -k_0^2$ (also for all $\mu < -k_1^2$ but these do not obtain in the physical limit $k_1 \to \infty$). There is an energy cutoff, $E(-k_0^2)$, above which there can be no physically relevant stable stationary state of this family.

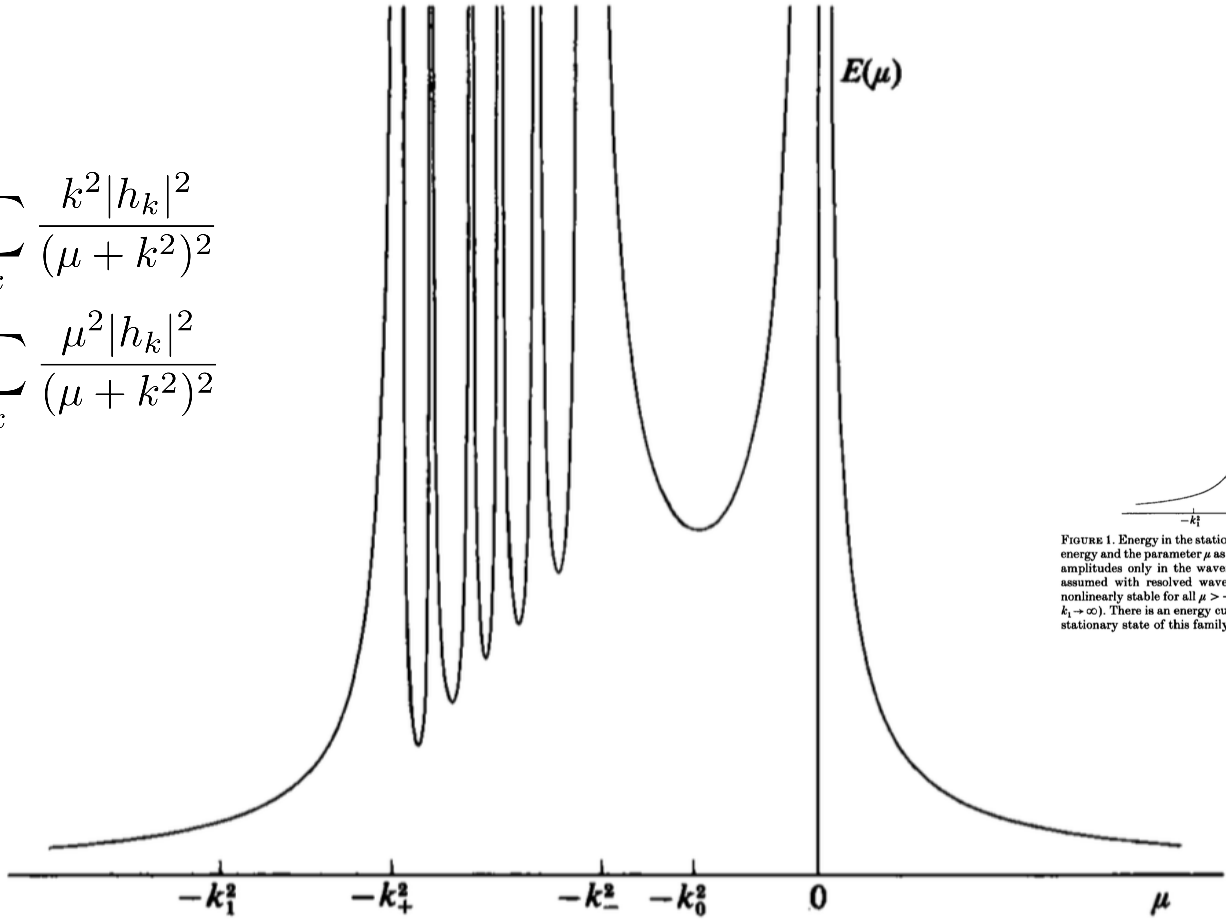


FIGURE 2. Energy in the stationary state $q = \mu\Psi$. As in figure 1 except here a large-scale flow (U) and 'topography' (βy) are included. The range of the physically relevant nonlinearly stable solutions is reduced to $\mu > 0$. There is no energy cutoff, that is, for any given energy there is a corresponding nonlinearly stable state.

Nonlinear Stability (again)

$$\overbrace{Q_{\Psi} - Q^{s} + \mu(E_{\Psi} - E^{s})}^{\delta Q + \mu \delta E} \equiv \tfrac{1}{2}\sum_{\boldsymbol{k}} k^2(k^2+\mu)\,|\delta\psi_{\boldsymbol{k}}|^2 + \tfrac{1}{2}\mu(\delta U)^2, \tag{5.13}$$

$$> 0, \;\; \text{for } \mu > 0$$

$$< 0, \;\; \text{for } \mu < -k_1^2$$

definite if $\mu > 0$ and negative definite if $\mu < -k_1^2$. Thus the stability range is decreased by shifting the largest scale from k_0 to $k = 0$. Note that the positive μ branch corresponds to negative U^s (i.e. westward flow). In the high resolution limit, $k_1 \to \infty$ the stable eastward flow branch no longer exists.

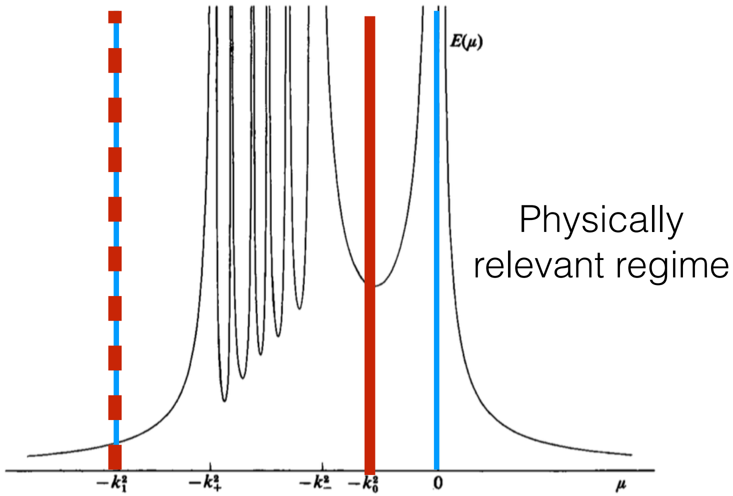


FIGURE 2. Energy in the stationary state $q = \mu\Psi$. As in figure 1 except here a large-scale flow (U) and 'topography' (βy) are included. The range of the physically relevant nonlinearly stable solutions is reduced to $\mu > 0$. There is no energy cutoff, that is, for any given energy there is a corresponding nonlinearly stable state.

Canonical equilibrium (again)

mean and variance of ψ is same

$$P(\{\psi_{\boldsymbol{k}}\}, U) \propto \exp\{-aE_{\Psi} - bQ_{\Psi}\}$$

$$\propto \exp\left\{\tfrac{1}{2}a\left(U + \frac{b}{a}\beta\right)^2\right\} \exp\{-aE_{\psi} - bQ_{\psi}\}. \quad (5.14)$$

$$\langle U \rangle = -\frac{b}{a}\beta = -\frac{\beta}{\mu^{\mathrm{eq}}},$$

where again $\mu^{\mathrm{eq}} = a/b$, and

$$\langle (U - \langle U \rangle)^2 \rangle = \frac{1}{a}.$$

$$a > 0$$

$$a + bk^2 > 0$$

Paradoxically, these results for the statistics of U do not explicitly involve the topography h. In the limit $h \to 0$, (5.15) unambiguously predicts a mean U; however, for h identically zero, U is an invariant with arbitrary value. For the ideal case with h identically zero, the proper distribution allows U to be independently specified unlike (5.14). The point is that the presence of h, no matter how small, breaks the east–west Galilean invariance and U will equilibrate in accordance with (5.16) and (5.17). However, the smaller the value of h, the longer it will take for the equilibration process to approximate the $t = \infty$ statistics.

Canonical equilibrium (again)

$$\langle U \rangle = -\frac{b}{a}\beta = -\frac{\beta}{\mu^{\mathrm{eq}}},$$

where again $\mu^{\mathrm{eq}} = a/b$, and

$$\langle (U - \langle U \rangle)^2 \rangle = \frac{1}{a}.$$

$$a > 0$$

positive eddy energy $\quad a + bk^2 > 0$

for $\mu^{eq} > \quad 0 \quad , \langle U \rangle$ is large amplitude and westward

for $\mu^{eq} < -k_1^2, \langle U \rangle$ is small amplitude and eastward

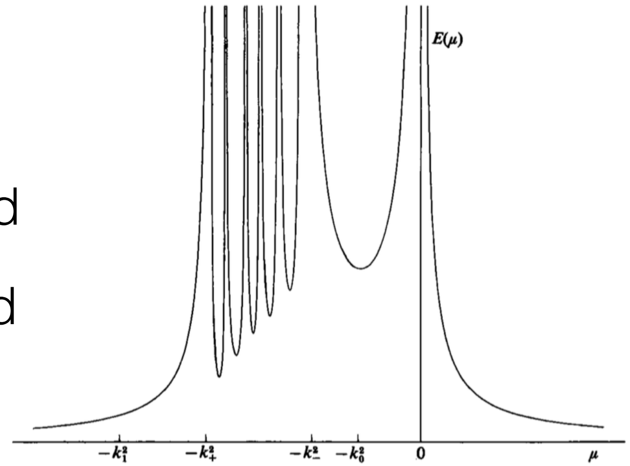


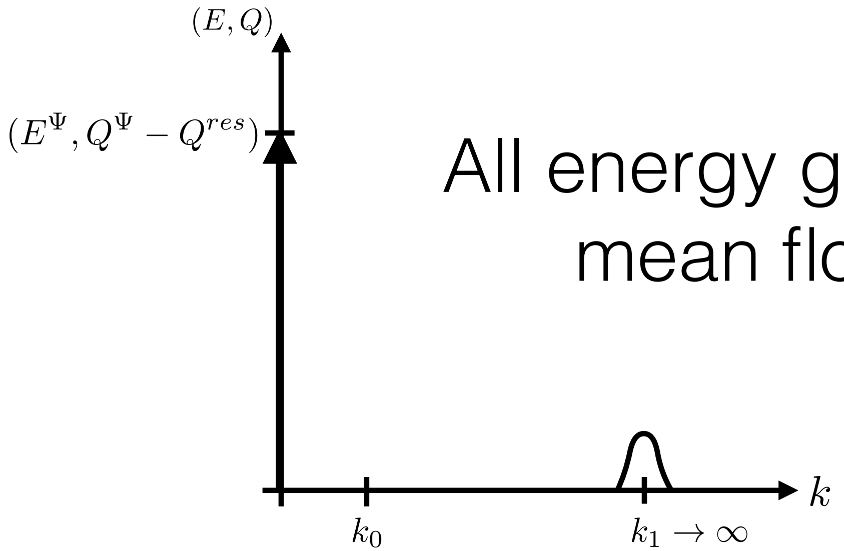
FIGURE 2. Energy in the stationary state $q = \mu\Psi$. As in figure 1 except here a large-scale flow (U) and 'topography' (βy) are included. The range of the physically relevant nonlinearly stable solutions is reduced to $\mu > 0$. There is no energy cutoff, that is, for any given energy there is a corresponding nonlinearly stable state.

Infinite Resolution (beta plane)

For $E = E^{\Psi}, Q = Q^{\psi}$

$$\mu^{\mathrm{eq}}(E, Q) \to \mu^{\mathrm{s}}(E),$$

$$b \to \frac{\pi k_1^2}{2(Q - Q^{\mathrm{s}}(\mu^{\mathrm{s}}))}.$$



All energy goes to mean flow

$$\left.\begin{aligned} \langle U \rangle^{\mathrm{eq}} &\to U^{\mathrm{s}}, \\ \langle (U + \langle U \rangle)^2 \rangle &\to 0, \\ \langle \psi_{\boldsymbol{k}} \rangle &\to \frac{h_{\boldsymbol{k}}}{\mu^{\mathrm{s}} + k^2}, \\ \langle |(\zeta_{\boldsymbol{k}} - \langle \zeta_{\boldsymbol{k}} \rangle)|^2 \rangle &\to 0. \end{aligned}\right\}$$

For the same energy canonical equilibrium is equivalent to nonlinearly stable state

Section 6

Discussion

Is there some scheme that could be used to determine when to include the higher order invariants?

It should be clear that in the calculation of the 'infinite resolution limit' of the energy–enstrophy ensemble, we have assumed a limiting process in which the time is first allowed to go to infinity for a finite resolution. Then, does our result have anything to do with the long-time behaviour of the infinite resolution flow? In

What does conservation of potential vorticity mean?

$$f > 0, h > 0, q = q_0$$

$$q = \nabla^2 \psi + h, \quad h(x, y) = f \frac{\Delta H}{D}$$

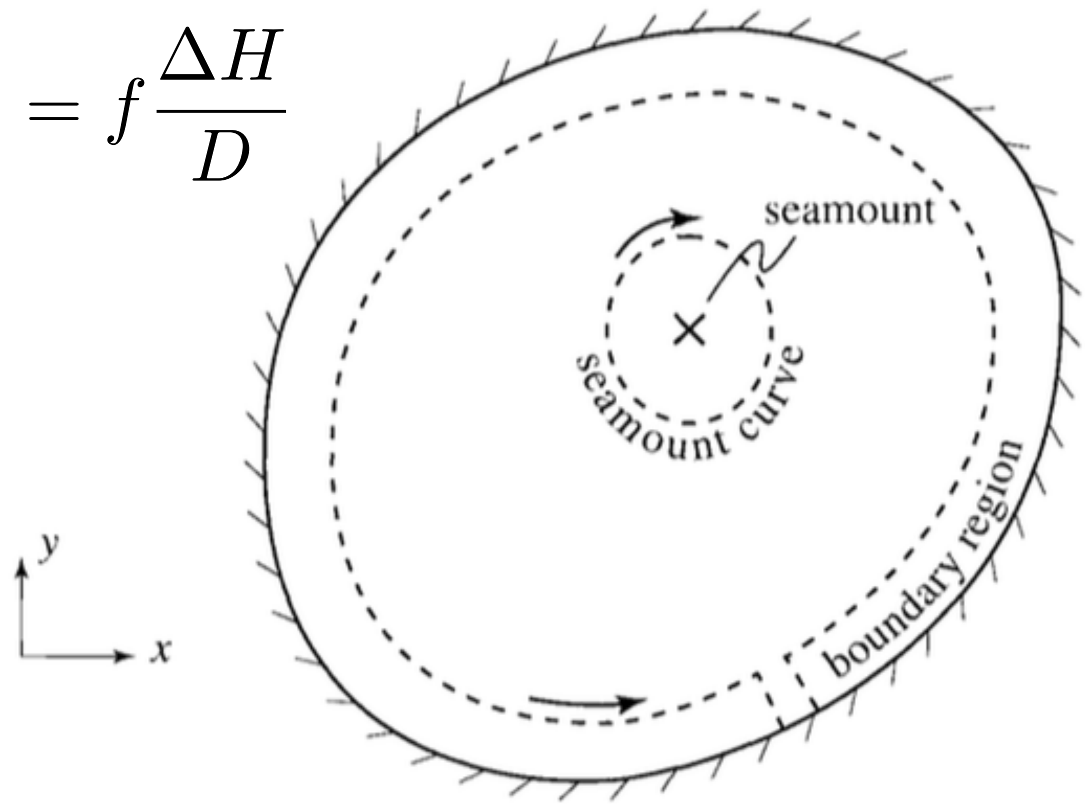


Figure 6.1 By the conservation of potential vorticity on fluid particles, the circulation around a curve encircling a seamount must become negative. Similarly, the flow becomes counterclockwise around the dashed line parallel to the boundary of the basin.

Salmon, R., 'Lectures on Geophysical Fluid Dynamics'

Canonical Equilibrium

Liouville's Theorem

$$\rho_{12} = \rho_1 \rho_2$$

$$\log \rho_{12} = \log \rho_1 + \log \rho_2$$

Higher order moments of potential vorticity

$$\log \rho_a = \alpha_a + \beta E_a(\psi, q) + \gamma Q_a = \alpha_a - aE - bQ + \ldots$$

$$\rho_a = P(\psi_k) \propto e^{-aE-bQ}$$

$$\langle \psi_k \rangle = \frac{h_k}{(a/b) + k^2}$$

$$\langle (\zeta_k - \langle \zeta_k \rangle)(\zeta_p - \langle \zeta_p \rangle) \rangle = \frac{k^2}{a + bk^2} \delta_{k,-p}$$

but what are a and b?

Find a and b from the mean energy and potential enstrophy

$\langle f \rangle^* = \langle f^* \rangle$

$$
\left.\begin{aligned}
E &= \tfrac{1}{2}\sum_{\boldsymbol{k}} k^2 \langle |\psi_{\boldsymbol{k}}|^2 \rangle \\
&= \tfrac{1}{2}\sum_{\boldsymbol{k}} k^2 (\langle |\psi_{\boldsymbol{k}} - \langle \psi_{\boldsymbol{k}} \rangle|^2 \rangle + |\langle \psi_{\boldsymbol{k}} \rangle|^2) \\
&= \tfrac{1}{2}\sum_{\boldsymbol{k}} \frac{1}{a+bk^2} + \frac{k^2 |h_{\boldsymbol{k}}|^2}{((a/b)+k^2)^2}.
\end{aligned}\right\} \tag{3.8a}
$$

$$
\left.\begin{aligned}
Q &= \tfrac{1}{2}\sum_{\boldsymbol{k}} \langle |\zeta_{\boldsymbol{k}} + h_{\boldsymbol{k}}|^2 \rangle \\
&= \tfrac{1}{2}\sum_{\boldsymbol{k}} \langle |\zeta_{\boldsymbol{k}} - \langle \zeta_{\boldsymbol{k}} \rangle|^2 \rangle + |\langle \zeta_{\boldsymbol{k}} \rangle + h_{\boldsymbol{k}}|^2 \\
&= \tfrac{1}{2}\sum_{\boldsymbol{k}} \frac{k^2}{a+bk^2} + \frac{(a/b)^2 |h_{\boldsymbol{k}}|^2}{((a/b)+k^2)^2}.
\end{aligned}\right\} \tag{3.8b}
$$

$a + bk^2 > 0$ otherwise negative energy

Kinetic Energy Invariance

$$\frac{\partial q}{\partial t} + J(\psi, q) = \partial_t \nabla^2 \psi + J(\psi, q) = 0$$

$$\psi \partial_t \nabla^2 \psi + \psi J(\psi, q) = 0$$

$$\iint \left(\psi \partial_t \nabla^2 \psi + \psi J(\psi, q)\right) dxdy = 0 \qquad \iint \psi J(\psi, q) dxdy = 0$$

$$\nabla \cdot (\psi \partial_t \nabla \psi) = \nabla \psi \cdot \partial_t \nabla \psi + \psi \partial_t \nabla^2 \psi$$

$$\oint_c (\psi \partial_t \nabla \psi) \cdot \hat{n} dl - \frac{d}{dt} \frac{1}{2} \iint \nabla \psi \cdot \nabla \psi dxdy = 0$$

$$\psi_c \oint_c (\partial_t \nabla \psi) \cdot \hat{n} dl - \frac{d}{dt} \frac{1}{2} \iint \nabla \psi \cdot \nabla \psi dxdy = 0$$

$$\cancel{\psi_c \iint \frac{\partial q}{\partial t} dxdy} - \frac{d}{dt} \frac{1}{2} \iint \nabla \psi \cdot \nabla \psi dxdy = 0$$