Non-geostropic Baroclinic Stability

Stone, PH. *On Non-Geostrophic Baroclinic Stability*, JAS 1966

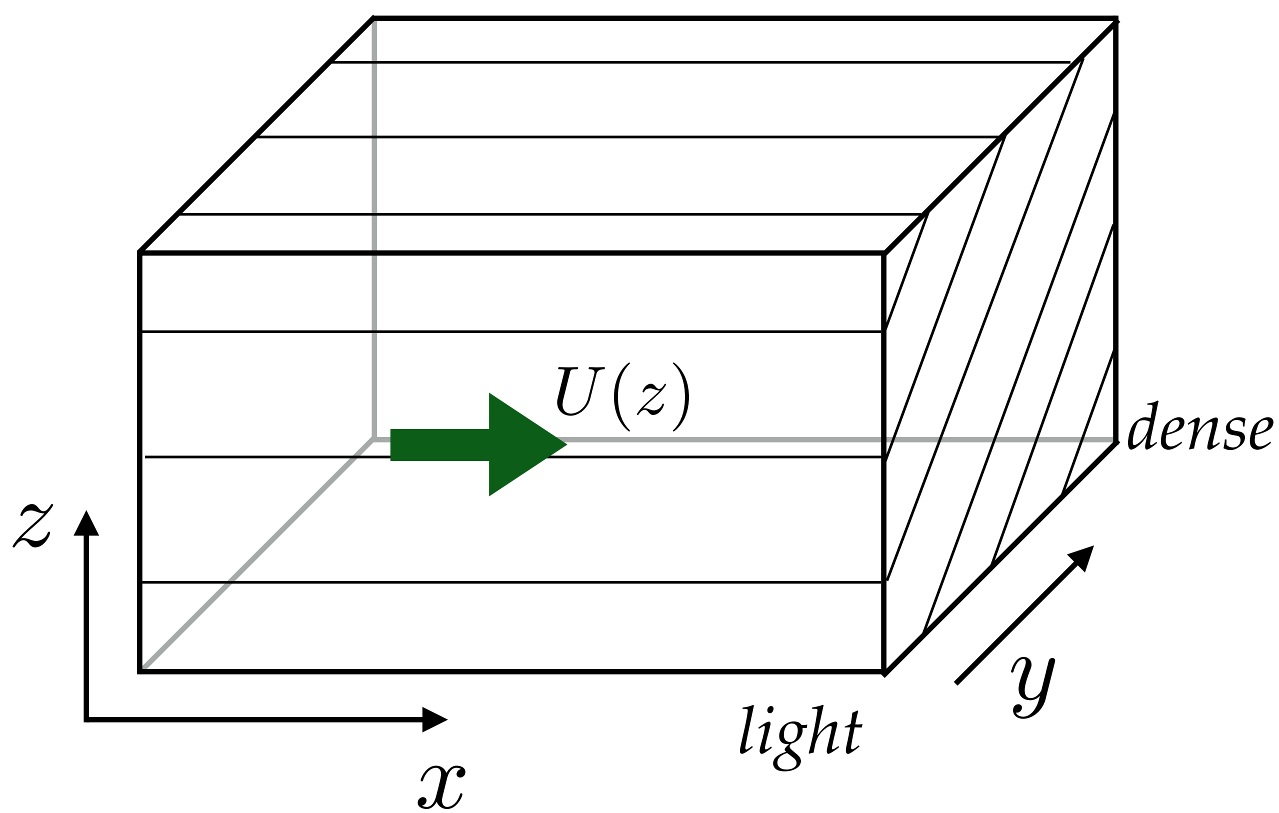
Stone, PH. *On Non-Geostrophic Baroclinic Stability: Part II*, JAS 1966

Haine, TWN. & Marshall, J. *Gravitational, Symmetric and Baroclinic Instability of the Ocean Mixed Layer,* JPO 1998

Holton, *An Introduction to Dynamic Meteorology. Fourth Edition.*

Ruth Musgrave

The background state is in thermal wind balance



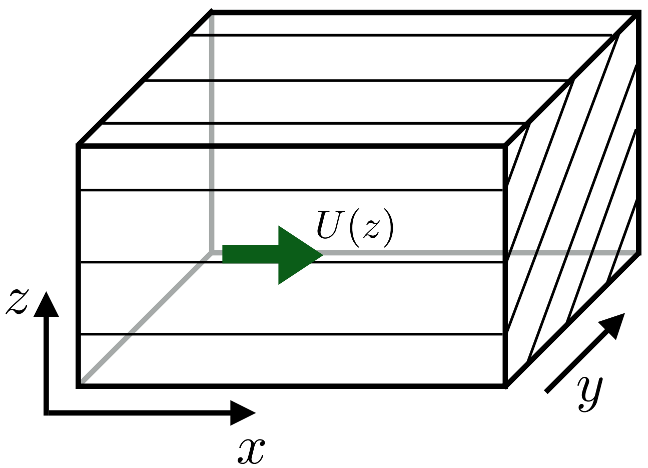
$$N^2 = -\frac{g}{\rho_0}\frac{\partial \bar{\rho}}{\partial z} = \bar{b}_z$$

$$M^2 = -\frac{g}{\rho_0}\frac{\partial \bar{\rho}}{\partial y} = \bar{b}_y$$

$$f\frac{dU}{dz} = \frac{g}{\rho_0}\frac{\partial \bar{\rho}}{\partial y} = -M^2$$

$$U = -\frac{M^2 z}{f}$$

Primitive equations (not QG!), linearized around the background state



Background + perturbations:

$$u' = u + U \quad v' = v \quad w' = w$$

$$b' = b + \bar{b} \quad p' = p + \bar{p}$$

Linearized set of equations:

$$u_t + Uu_x + wU' - fv = -p_x$$

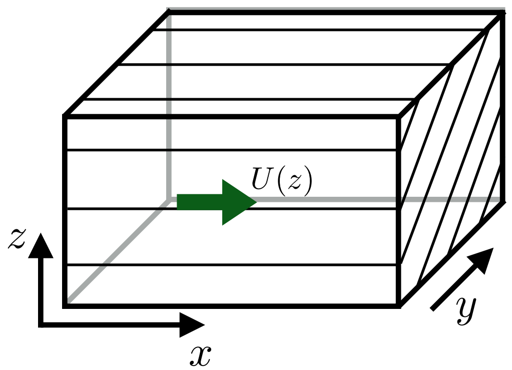
$$v_t + Uv_x + fu = -p_y$$

$$0 = -p_z + b$$

$$b_t + Ub_x + v\bar{b}_y + w\bar{b}_z = 0$$

$$u_x + v_y + w_z = 0$$

Primitive equations (not QG!), linearized around the background state



non-dimensionalize:

$$(x*, y*) = \frac{U}{f}(x, y) \qquad z* = H_0 z$$

$$(u*, v*) = U(u, v) \qquad w* = H_0 f w$$

$$b* = N^2 H_0 b \qquad p* = N^2 H_0^2 p$$

$$t* = \frac{1}{f} t \qquad \sigma* = f\sigma$$

$$H* = \frac{1}{H_0} H$$

Assume a solution like: $e^{i(kx+ly+\sigma t)}$

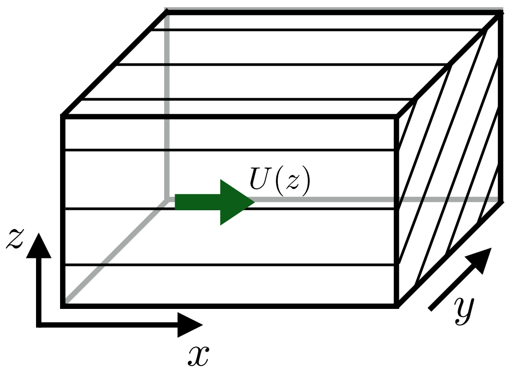
Combine to form eigenvalue equation:

$$[1 - (\sigma + kU(z))^2]w'' - 2\left[\frac{k}{\sigma + kU(z)} - il\right] w' - \left[Ri(k^2 + l^2) - \frac{2ikl}{\sigma + kU(z)}\right] w = 0$$

Boundary conditions $\quad w(0) = w(1) = 0$

$$Ri = \frac{N^2 H_0^2}{U^2} = \frac{N^2 f^2}{M^4}$$

Instabilities of the non-geostropic eigenvalue equation

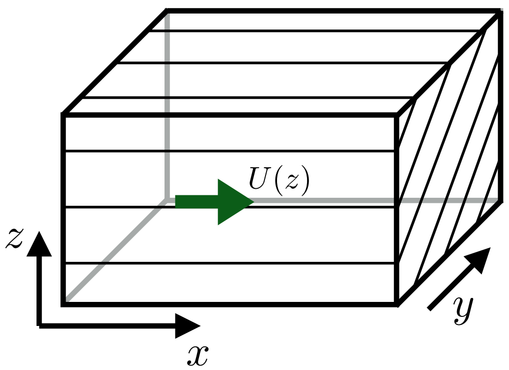


$$[1-(\sigma+kU(z))^2]w'' - 2\left[\frac{k}{\sigma+kU(z)} - il\right]w' - \left[Ri(k^2+l^2) - \frac{2ikl}{\sigma+kU(z)}\right]w = 0$$

Boundary conditions $\quad w(0) = w(1) = 0$

- Gravitational (Ri < 0)
- Kelvin-helmholtz (0 < Ri < 0.25)
- Symmetric (k=0, Ri < 1)
- Baroclinic (l=0, Ri > 1)
- Also allows inertia-gravity waves!

No zonal derivatives: symmetric instability



Let $k = 0$

Eigenvalue equation: $[1 - \sigma^2]w'' + 2ilw' - \mathrm{Ri}\, l^2 w = 0$

Assume solution: $w = w_0 e^{imz}$

$$m^2 + \frac{2l}{1 - \sigma^2} m + \frac{\mathrm{Ri} l^2}{1 - \sigma^2} = 0$$

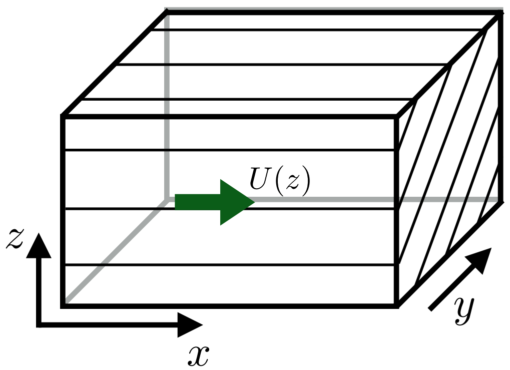
$$m = \frac{l}{\sigma^2 - 1} \pm \sqrt{\frac{l^2}{(\sigma^2 - 1)^2} + \frac{\mathrm{Ri} l^2}{\sigma^2 - 1}}$$

$$w = w_0 (a e^{im_+ z} + b e^{im_- z})$$

Boundary conditions: $w = 0$ at $z = 0, 1$

$$\Longrightarrow m_+ - m_- = 2n\pi \quad \text{and} \quad a = -b$$

Symmetric instability: growth rates



Boundary conditions impose

$$n\pi = \sqrt{\frac{l^2}{(\sigma^2 - 1)^2} + \frac{\text{Ri}l^2}{\sigma^2 - 1}}$$

Growth rate: $$\sigma^2 = 1 + \frac{\text{Ri}l^2}{2n^2\pi^2} \pm \sqrt{\frac{\text{Ri}^2 l^4}{4n^4\pi^4} + \frac{l^2}{n^2\pi^2}}$$

With our assumed solution: $e^{i(kx+ly+\sigma t)}$

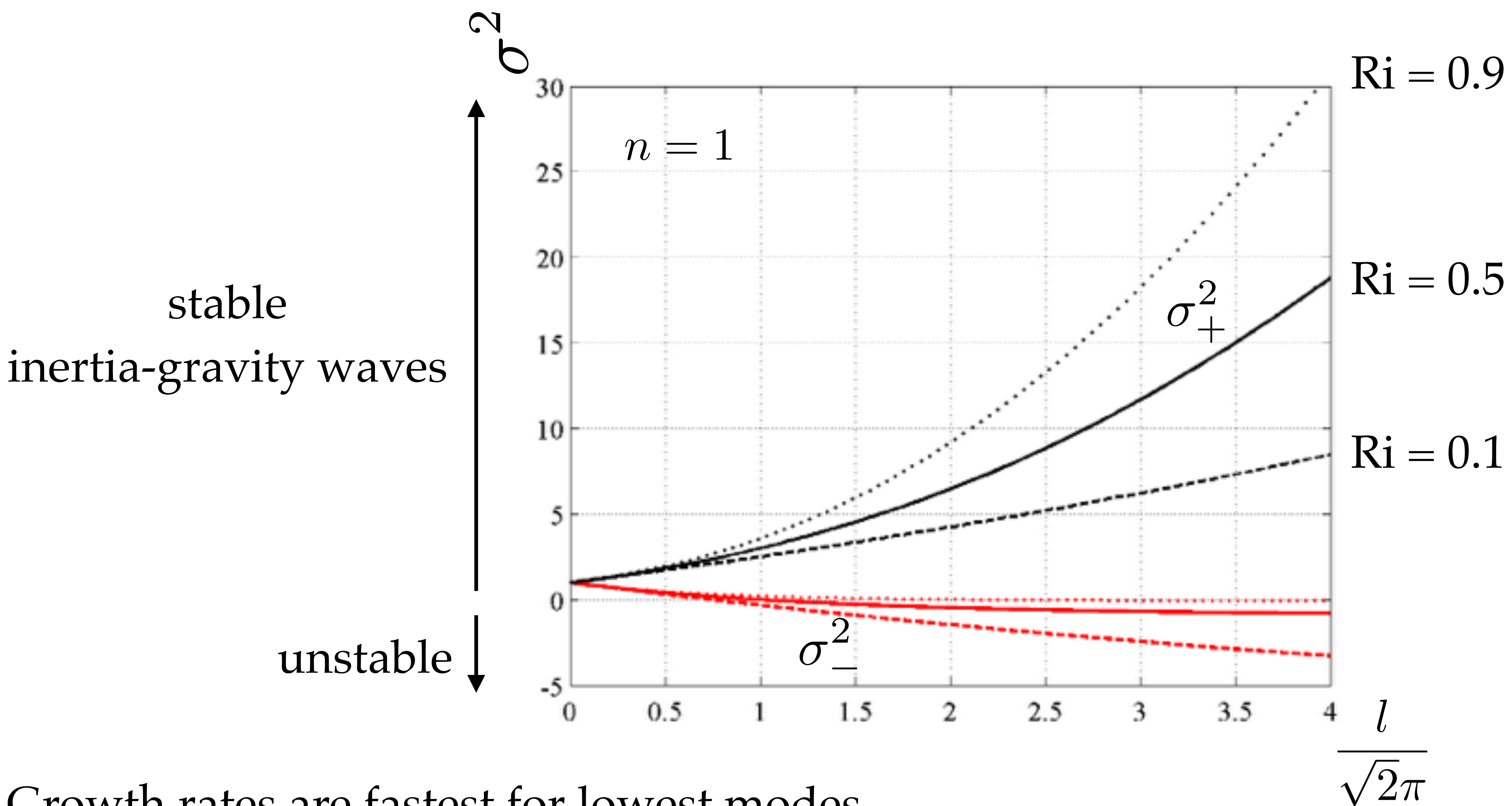
σ_+^2 is always stable

σ_-^2 is sometimes stable

Instability condition: $$\text{Ri} < 1 - \frac{n^2\pi^2}{l^2}$$

There are stable and unstable solutions

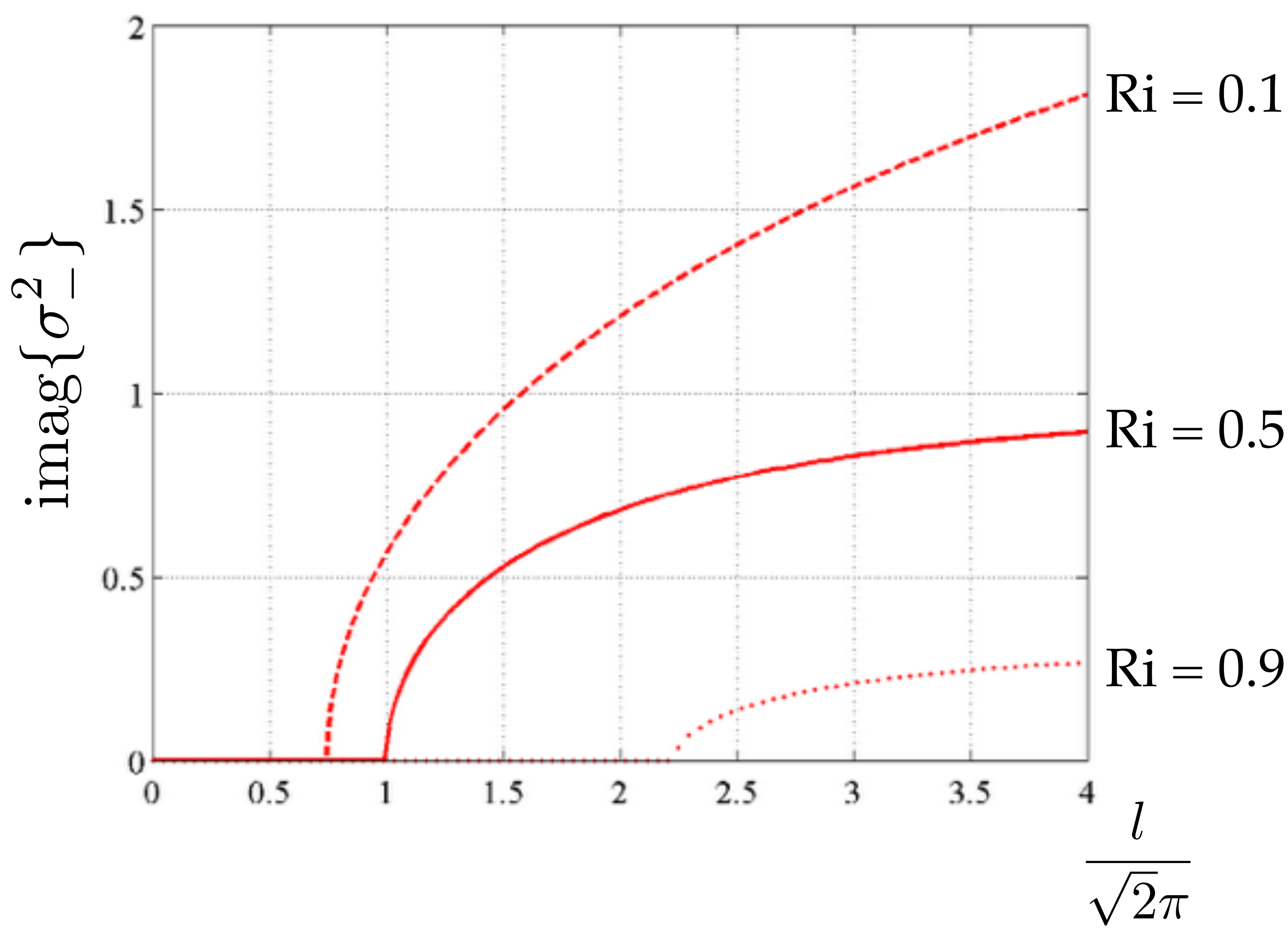
$$\sigma^2 = 1 + \frac{\text{Ri}\,l^2}{2n^2\pi^2} \pm \sqrt{\frac{\text{Ri}^2 l^4}{4n^4\pi^4} + \frac{l^2}{n^2\pi^2}}$$



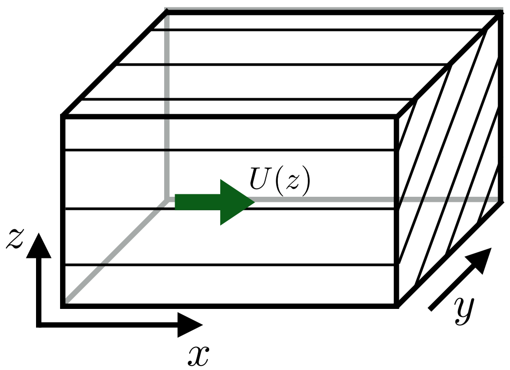
Growth rates are fastest for lowest modes

Symmetric instability: growth rates

$$\sigma_-^2 = 1 + \frac{\mathrm{Ri}\, l^2}{2n^2\pi^2} - \sqrt{\frac{\mathrm{Ri}^2 l^4}{4n^4\pi^4} + \frac{l^2}{n^2\pi^2}}$$

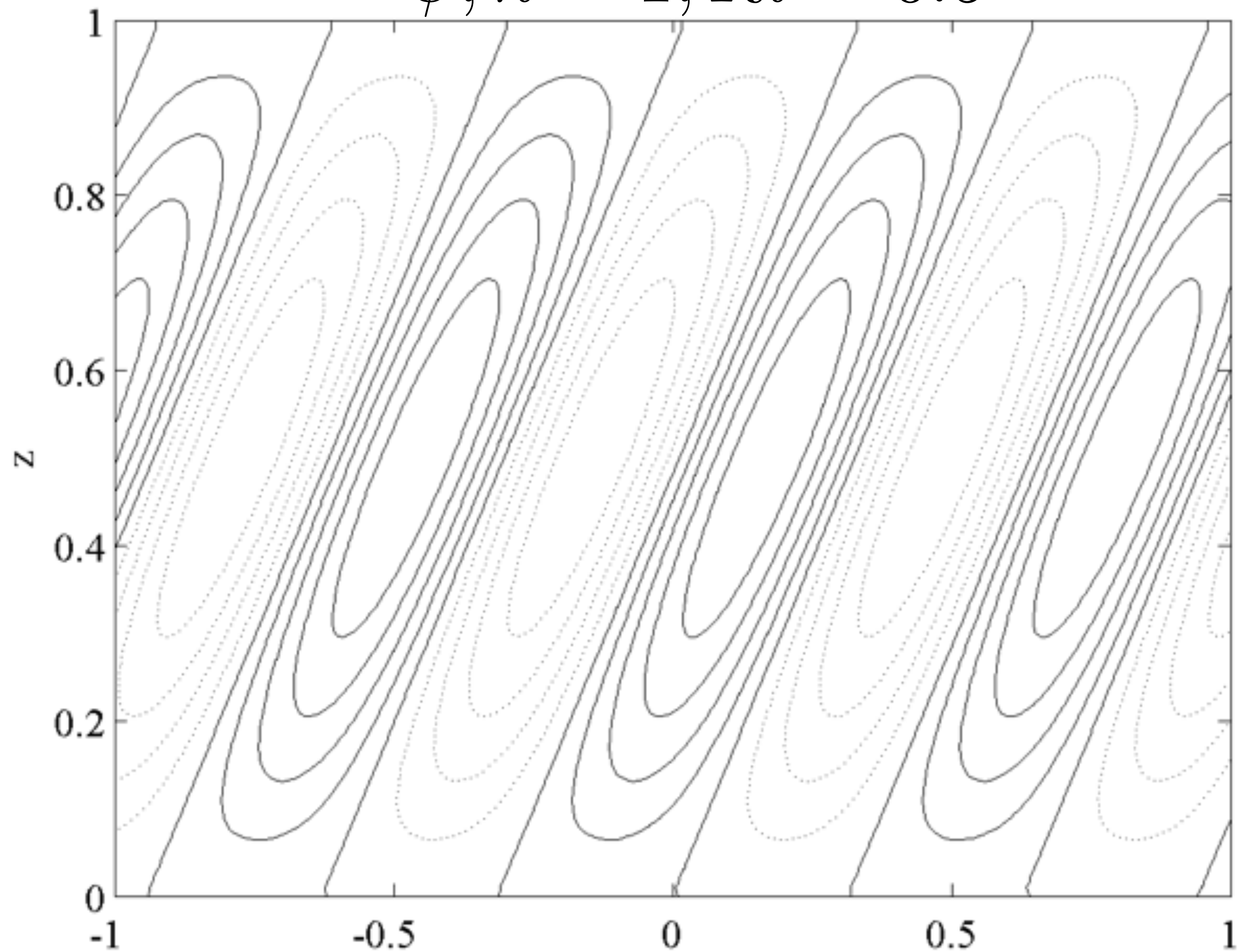


Symmetric instability: eigenmodes



$$\psi = \mathrm{Re}[(e^{im_+z} - e^{im_-z})e^{il0y}]$$

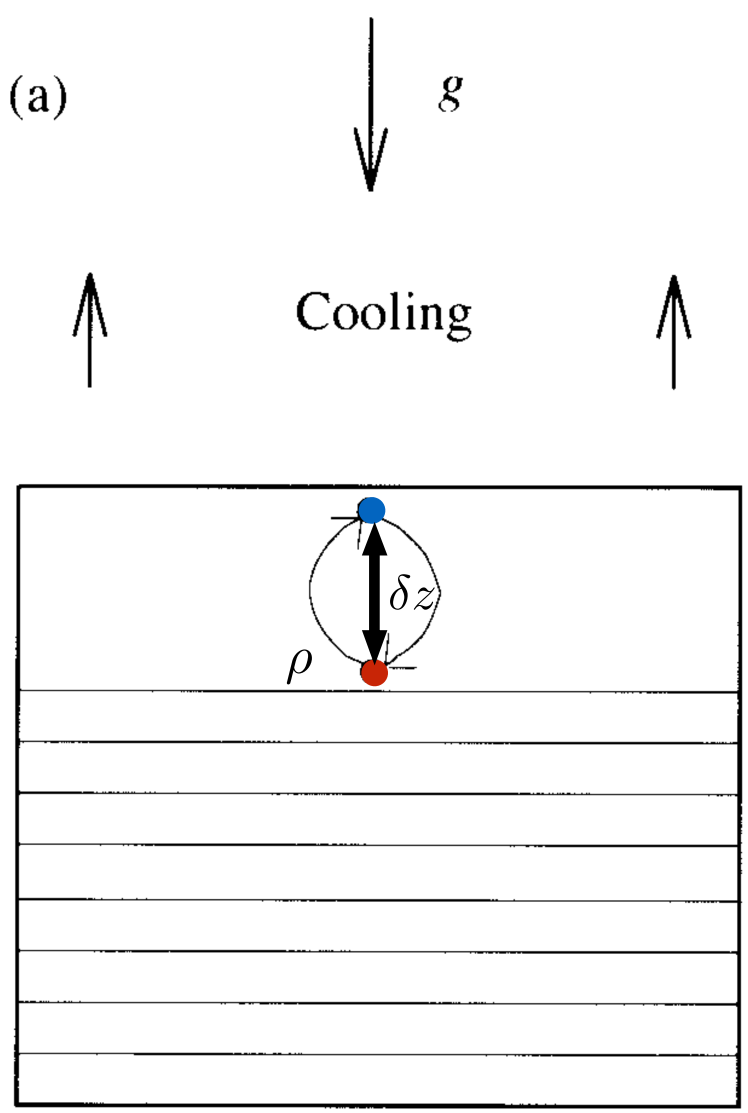
$$\psi, n = 1, Ri = 0.5$$



Instability forms as rolls aligned with the background flow

Symmetric instability:
a mixed gravitational-centrifugal instability

<u>*1. Gravitational instability: source is PE*</u>



$$\frac{\partial p_0}{\partial z} = -\rho_0 g$$

Haine & Marshall, JPO 1998

$$\frac{Dw}{Dt} = \frac{D^2}{Dt^2}(\delta z) = -g - \frac{1}{\rho}\frac{\partial p}{\partial z}$$
$$= g\left(\frac{\rho_0 - \rho}{\rho}\right)$$
$$= -N^2 \delta z$$

$N^2 > 0$ stable

$N^2 = 0$ neutral

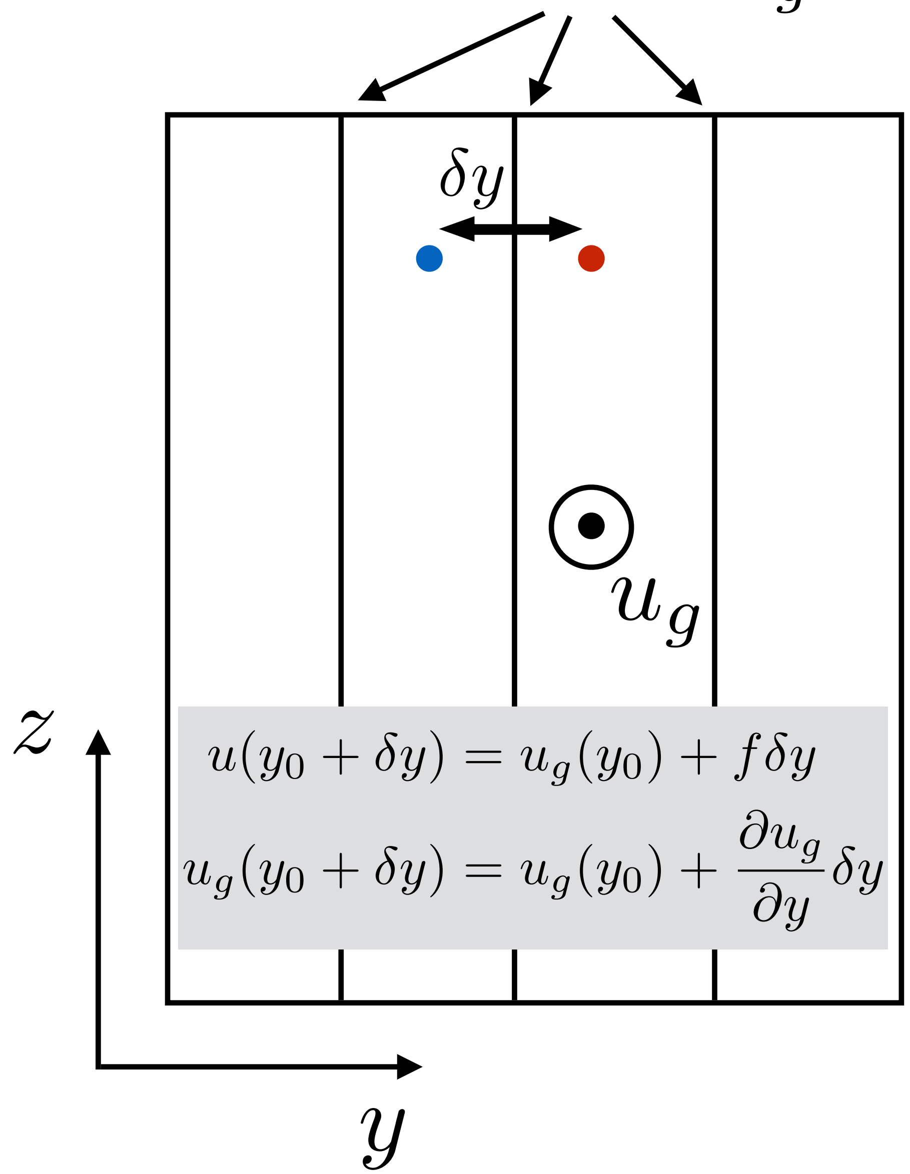
$N^2 < 0$ unstable

following Holton

Symmetric instability:
a mixed gravitational-centrifugal instability

2. Centrifugal instability: source is KE

Lines of constant M for $u_g = u_g(y)$



$$
\begin{aligned}
\frac{Dv}{Dt} = \frac{D^2}{Dt^2}(\delta y) &= f(u_g - u) \\
&= -f\left(f - \frac{\partial u_g}{\partial y}\right)\delta y \\
&= -f\frac{\partial M}{\partial y}\delta y
\end{aligned}
$$

Absolute momentum: $M = fy - u_g$

$$f\frac{\partial M}{\partial y} > 0 \text{ stable}$$

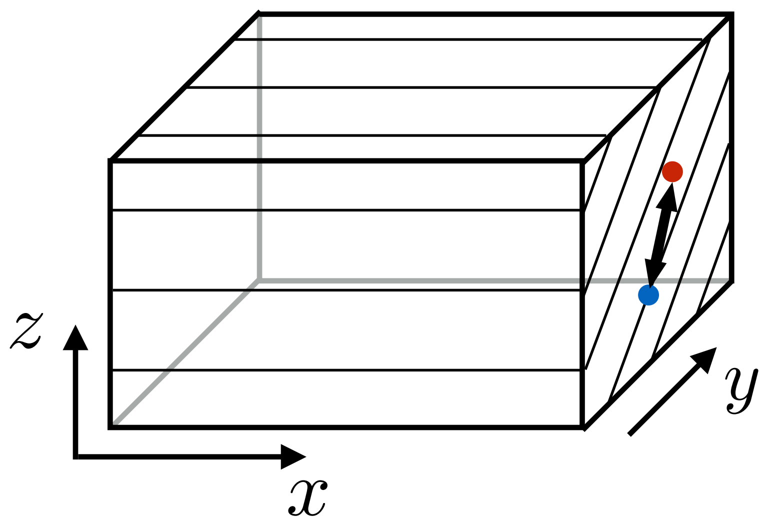
$$f\frac{\partial M}{\partial y} = 0 \text{ neutral}$$

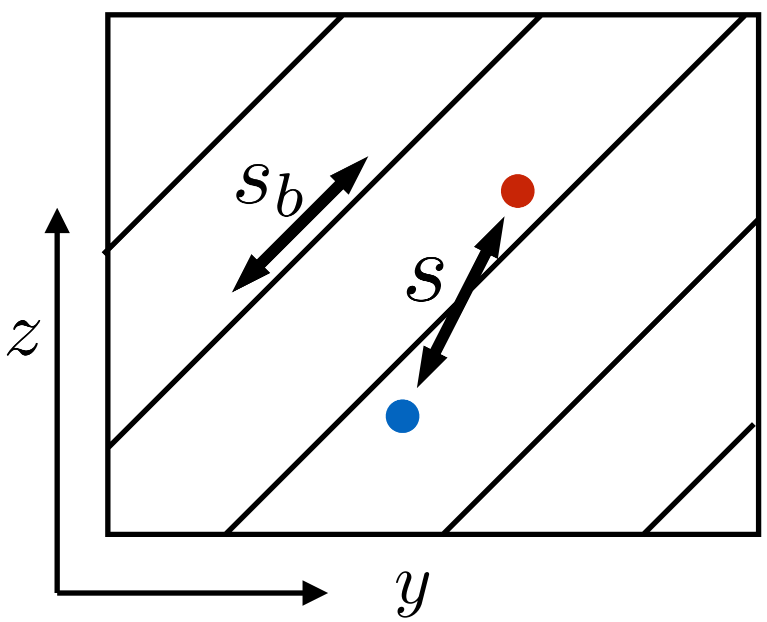
$$f\frac{\partial M}{\partial y} < 0 \text{ unstable}$$

following Holton

Symmetric instability:
a mixed gravitational-centrifugal instability

3. Symmetric instability



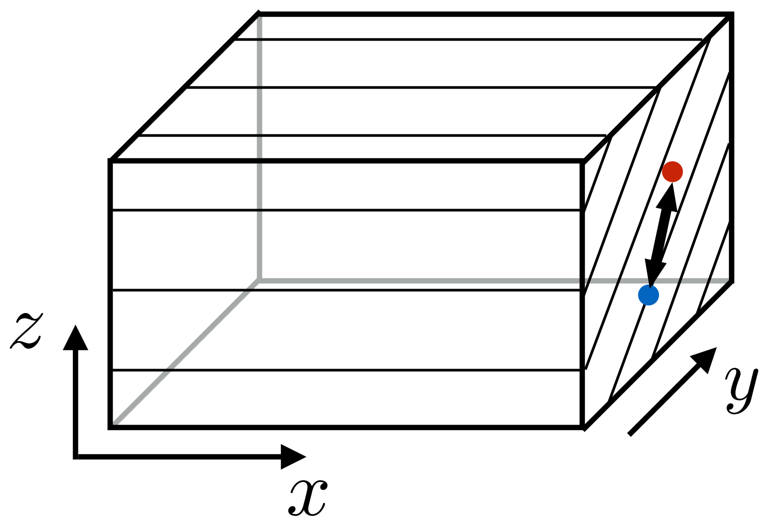


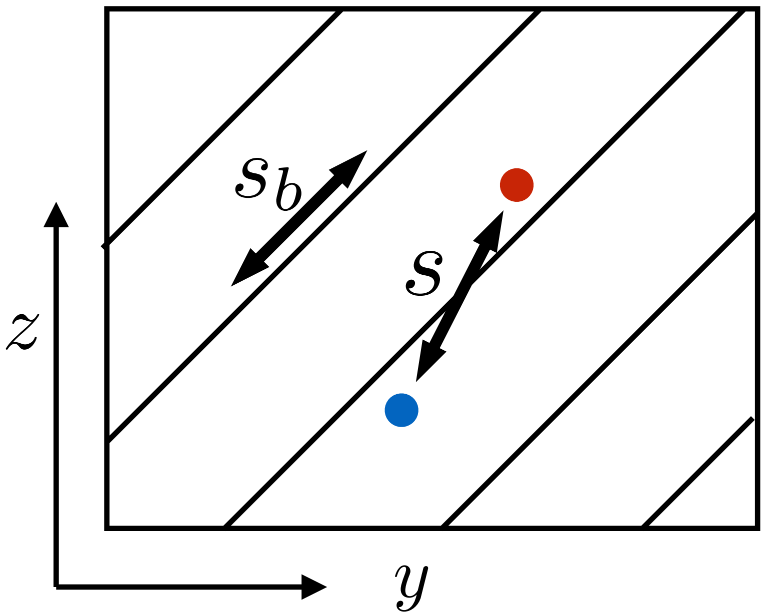
$$s = \frac{z_2 - z_1}{y_2 - y_1}$$

$$s_b = -\frac{M^2}{N^2}$$

Symmetric instability: a mixed gravitational-centrifugal instability

3. Symmetric instability





$$s = \frac{z_2 - z_1}{y_2 - y_1}$$

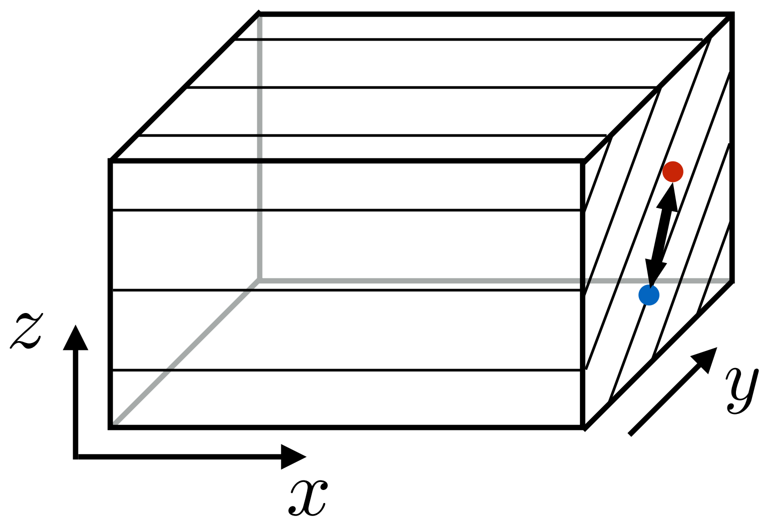
$$s_b = -\frac{M^2}{N^2}$$

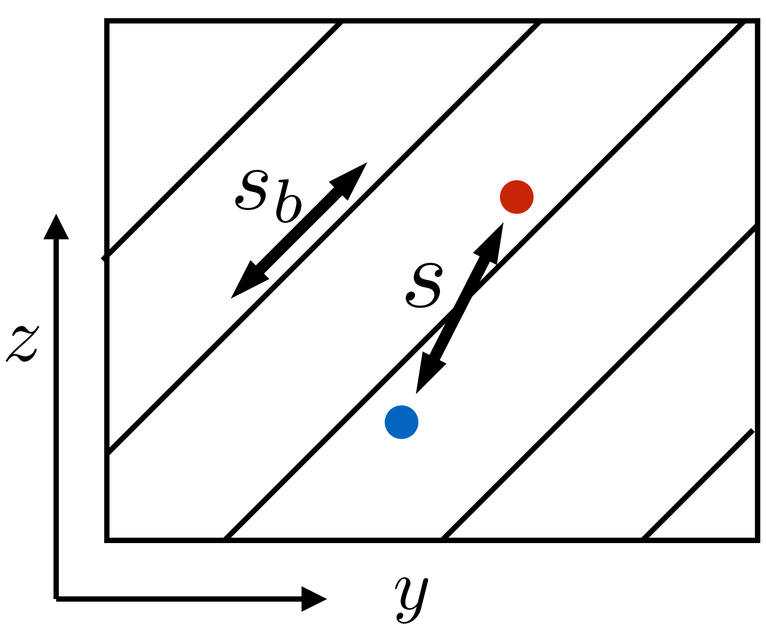
$$\delta b = \bar{b}_y \delta y + \bar{b}_z \delta z$$

$$\begin{aligned} \Delta PE &= \rho_0 (z_2 - z_1)(b_2 - b_1) \\ &= \rho_0 (z_2 - z_1)[M^2 (y_2 - y_1) + N^2 (z_2 - z_1)] \\ &= \rho_0 N^2 \Delta y^2 s (s - s_b) \end{aligned}$$

Symmetric instability: a mixed gravitational-centrifugal instability

3. Symmetric instability





$$s = \frac{z_2 - z_1}{y_2 - y_1}$$

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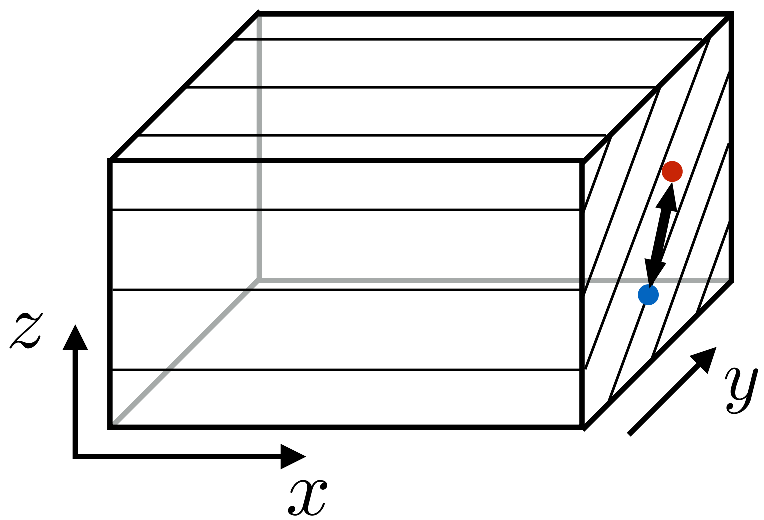
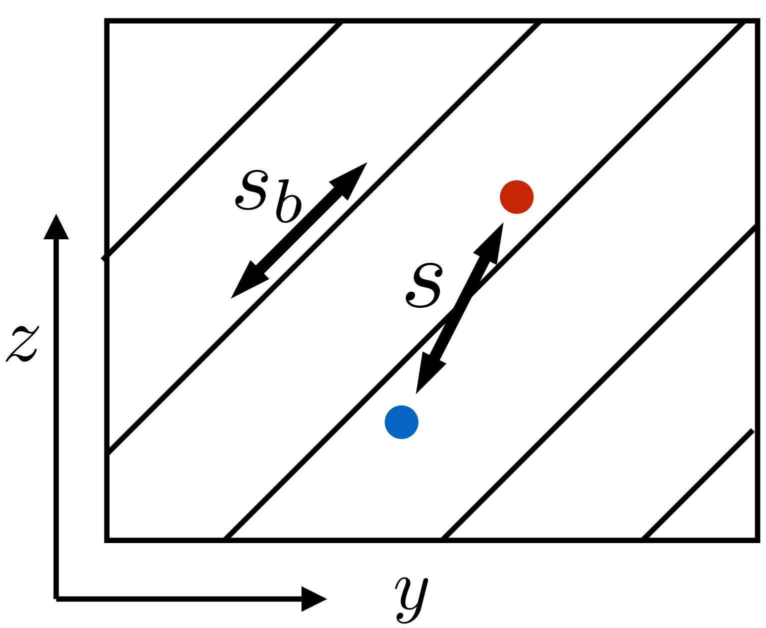
$$\begin{aligned}\Delta KE &= (1/2)\rho_0[\{u_1 + f(y_2 - y_1)\}^2 +\\ &\qquad \{u_2 - f(y_2 - y_1)\}^2 - u_1^2 - u_2^2]\\ &= \rho_0(y_2 - y_1)^2 f\left(f - s\frac{\partial u}{\partial z}\right)\\ &= \rho_0 \Delta y^2[f^2 - N^2 s s_b]\end{aligned}$$

Symmetric instability:
a mixed gravitational-centrifugal instability

3. Symmetric instability





$$s = \frac{z_2 - z_1}{y_2 - y_1}$$

$$s_b = -\frac{M^2}{N^2}$$

$$\begin{aligned}\Delta PE &= \rho_0(z_2 - z_1)(b_2 - b_1) \\ &= \rho_0(z_2 - z_1)[M^2(y_2 - y_1) + N^2(z_2 - z_1)] \\ &= \rho_0 N^2 \Delta y^2 s(s - s_b)\end{aligned}$$

$$\begin{aligned}\Delta KE &= (1/2)\rho_0[\{u_1 + f(y_2 - y_1)\}^2 + \\ &\qquad \{u_2 - f(y_2 - y_1)\}^2 - u_1^2 - u_2^2] \\ &= \rho_0(y_2 - y_1)^2 f\left(f - s\frac{\partial u}{\partial z}\right) \\ &= \rho_0 \Delta y^2[f^2 - N^2 s s_b]\end{aligned}$$

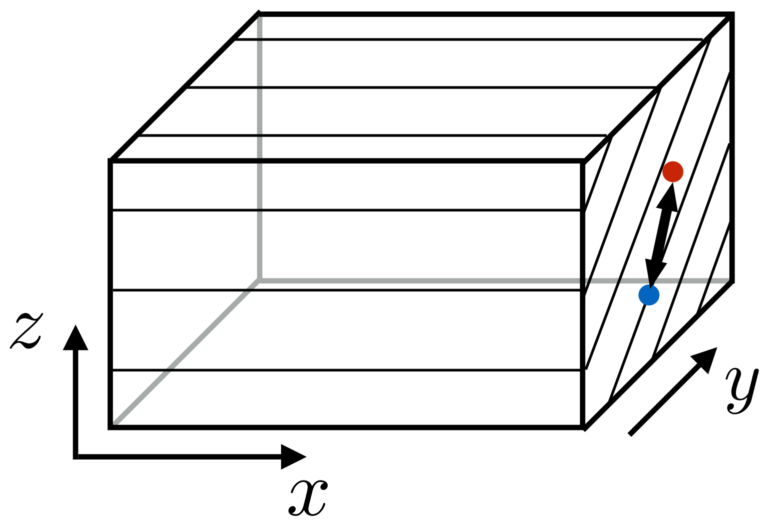
$$\Delta(KE + PE) = \rho_0 \Delta y^2[f^2(1 - 1/Ri) + N^2(s - s_b)^2]$$

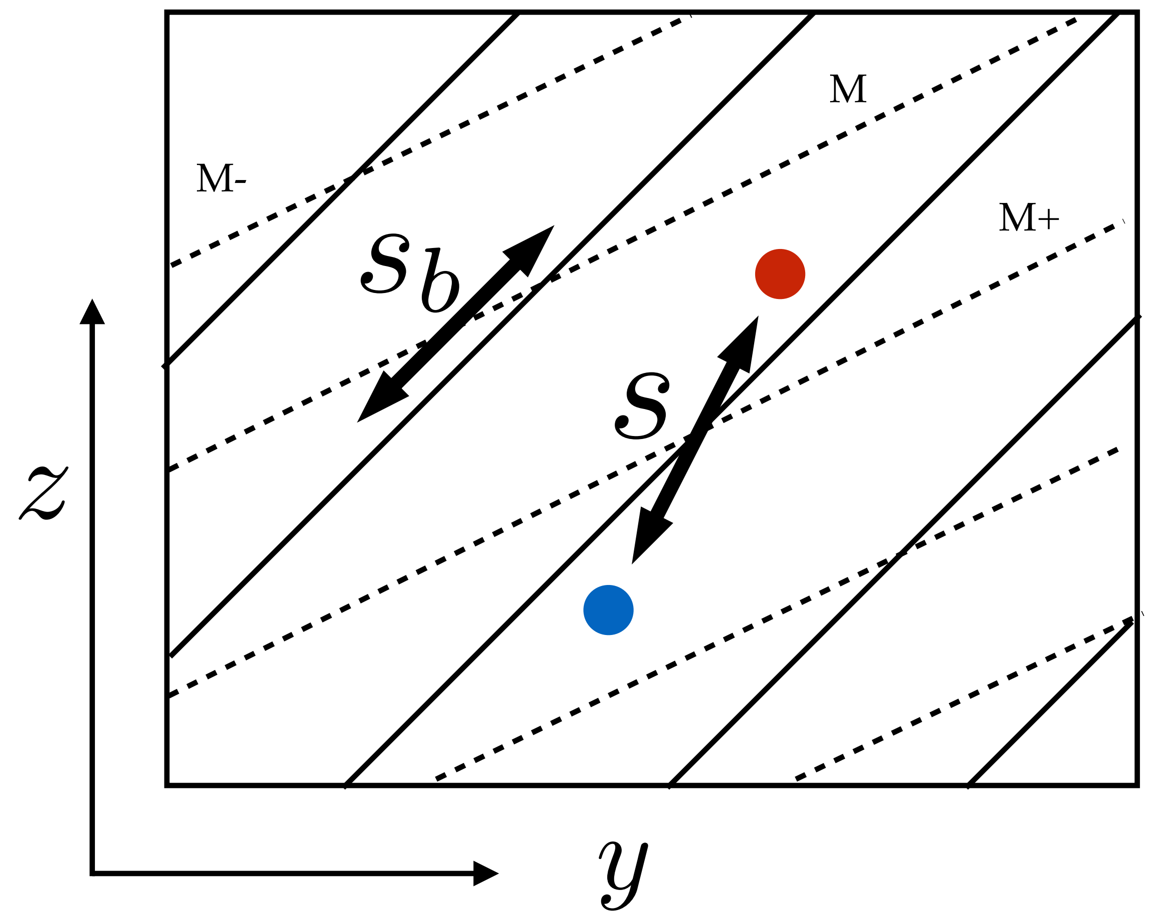
For instability, energy change must be -ve, so $Ri < 1$

Haine & Marshall, JPO 1998

Symmetric instability: a mixed gravitational-centrifugal instability

3. *Symmetric instability*





- Slopes of absolute momentum can be less then isopycnal slopes in regions of weak vertical stratification and strong horizontal stratification.
- Also known as "isentropic inertial instability".

Holton

Non-geostrophic baroclinic instability is qualitatively similar to Eady instability: l=0

$$[1-(\sigma+kU)^2]\partial_{zz}w - \frac{2k}{\sigma+kU}\partial_z w - Rik^2 w = 0$$

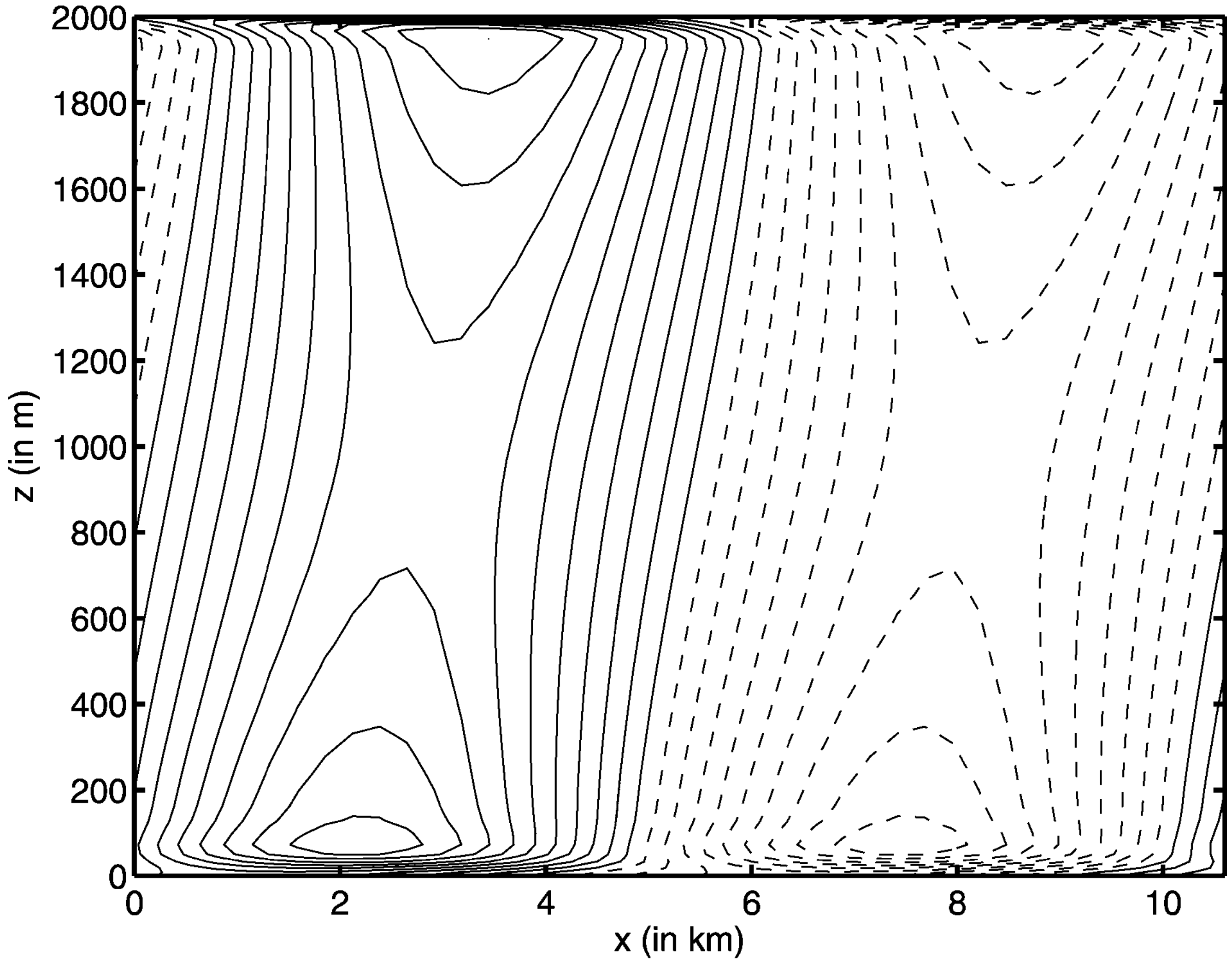


FIG. 4. A snapshot of the temperature perturbation for the fastest-growing mode.

Unstable for Ri > 1

Cessi & Fantini, JPO 2004

Stone: Non-geostrophic stability with perturbations in 2 directions

$$[1-(\sigma+kU(z))^2]w'' - 2\left[\frac{k}{\sigma+kU(z)} - il\right]w' - \left[Ri(k^2+l^2) - \frac{2ikl}{\sigma+kU(z)}\right]w = 0$$

Boundary conditions $\quad w(0) = w(1) = 0$

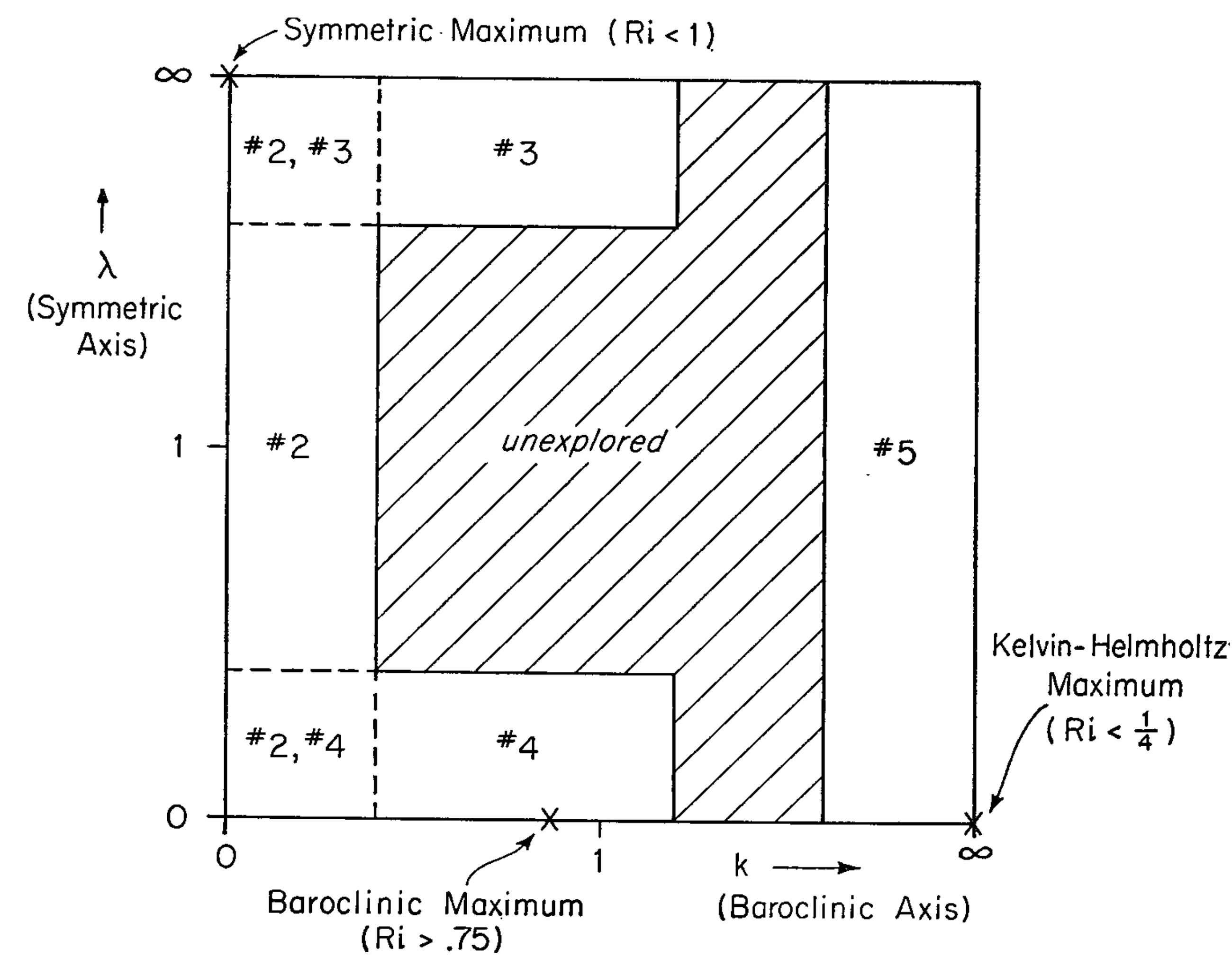
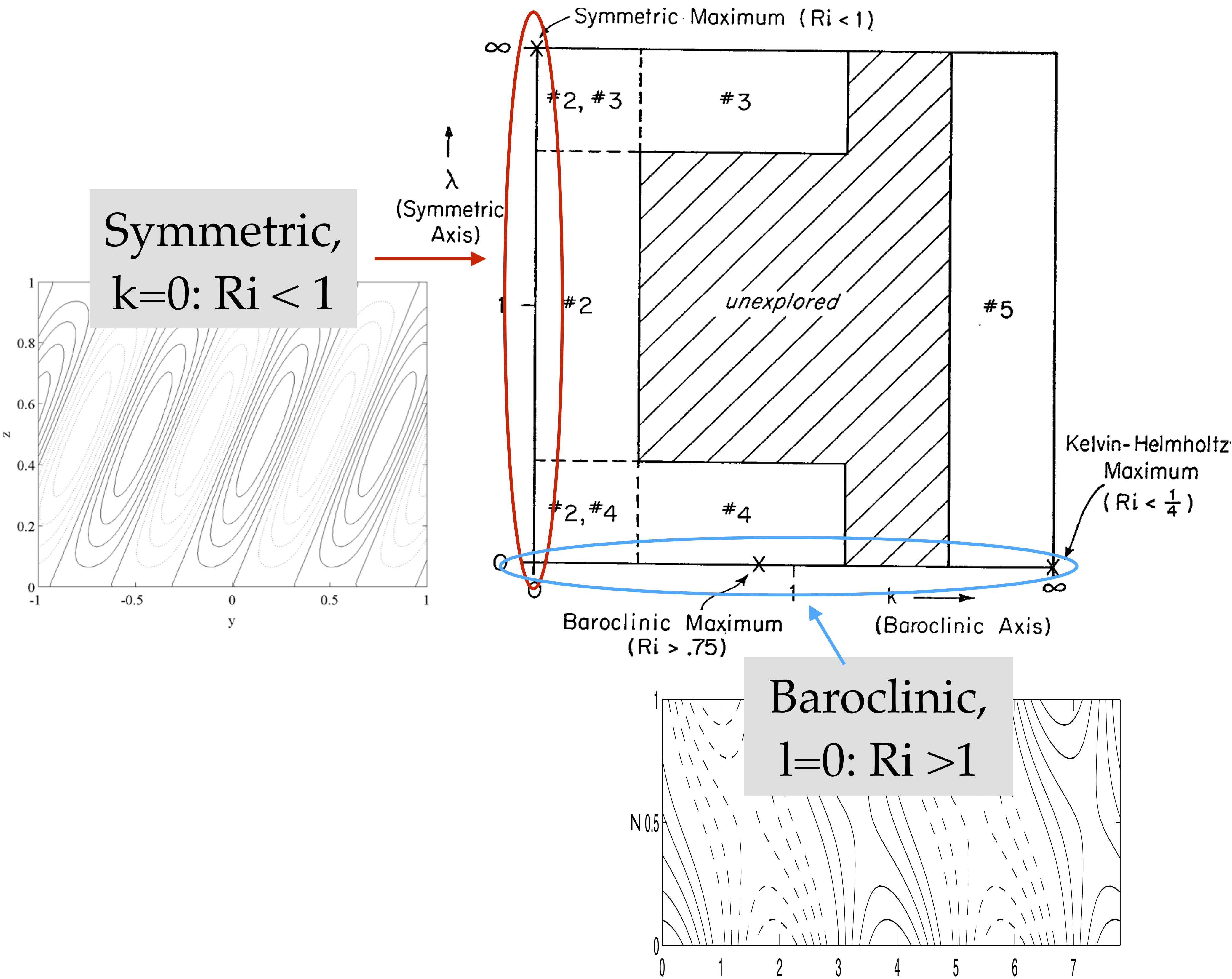


FIG. 5. Schematic diagram of the k—λ plane, showing the various regions where the analyses of Sections 2 through 5 apply. The locations of the three local maxima in the growth rate are indicated schematically by x's, and the range of values of Ri for which these maxima exist are also given.

Stone, JAS 1966

Stone: Non-geostrophic stability with perturbations in 2 directions



Stone, JAS 1966

Non-geostrophic instabilities arising from 2D perturbations

<u>Ri = 2:</u>
~Baroclinic. Max growth rate at l=0. Single unstable mode.

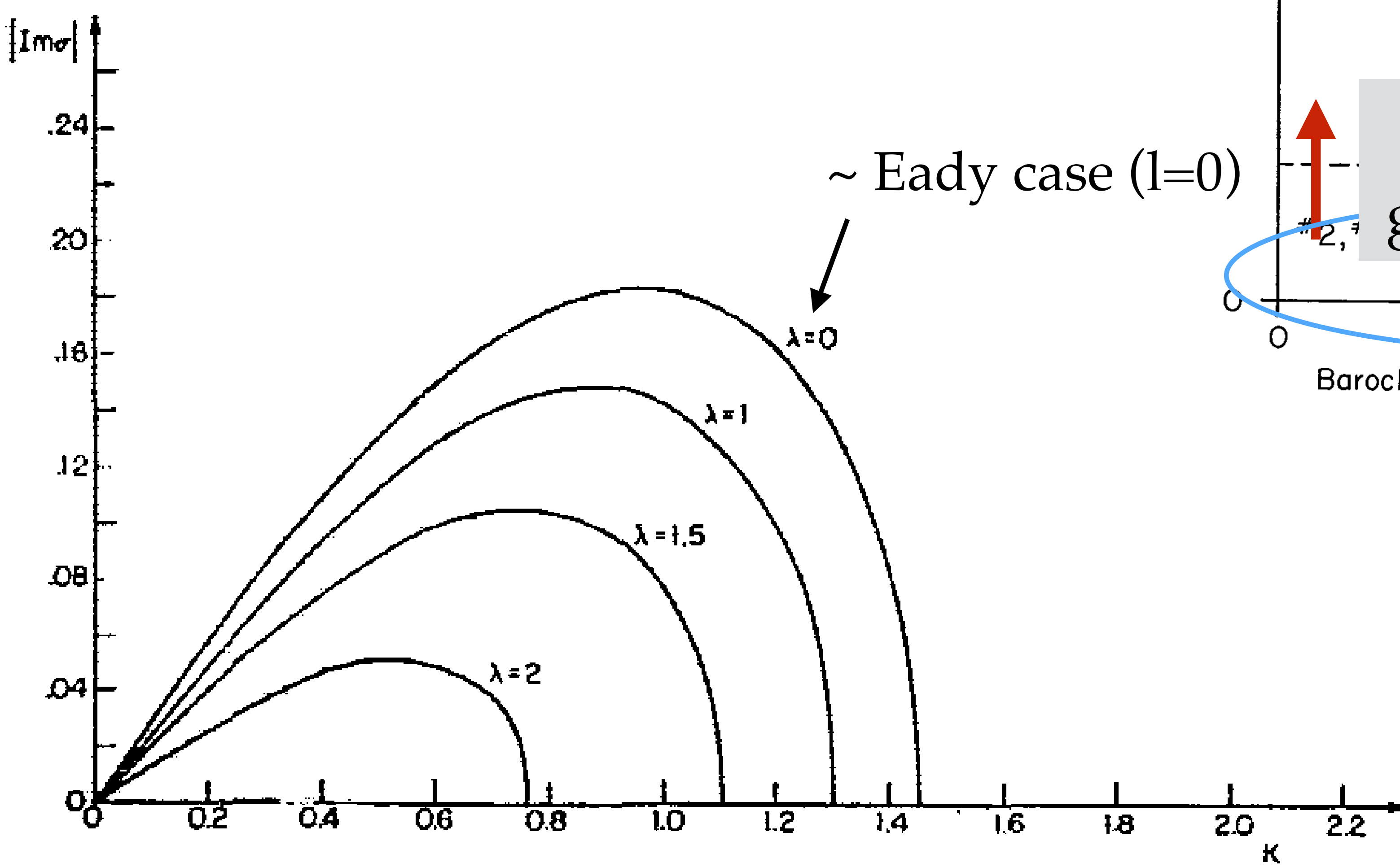
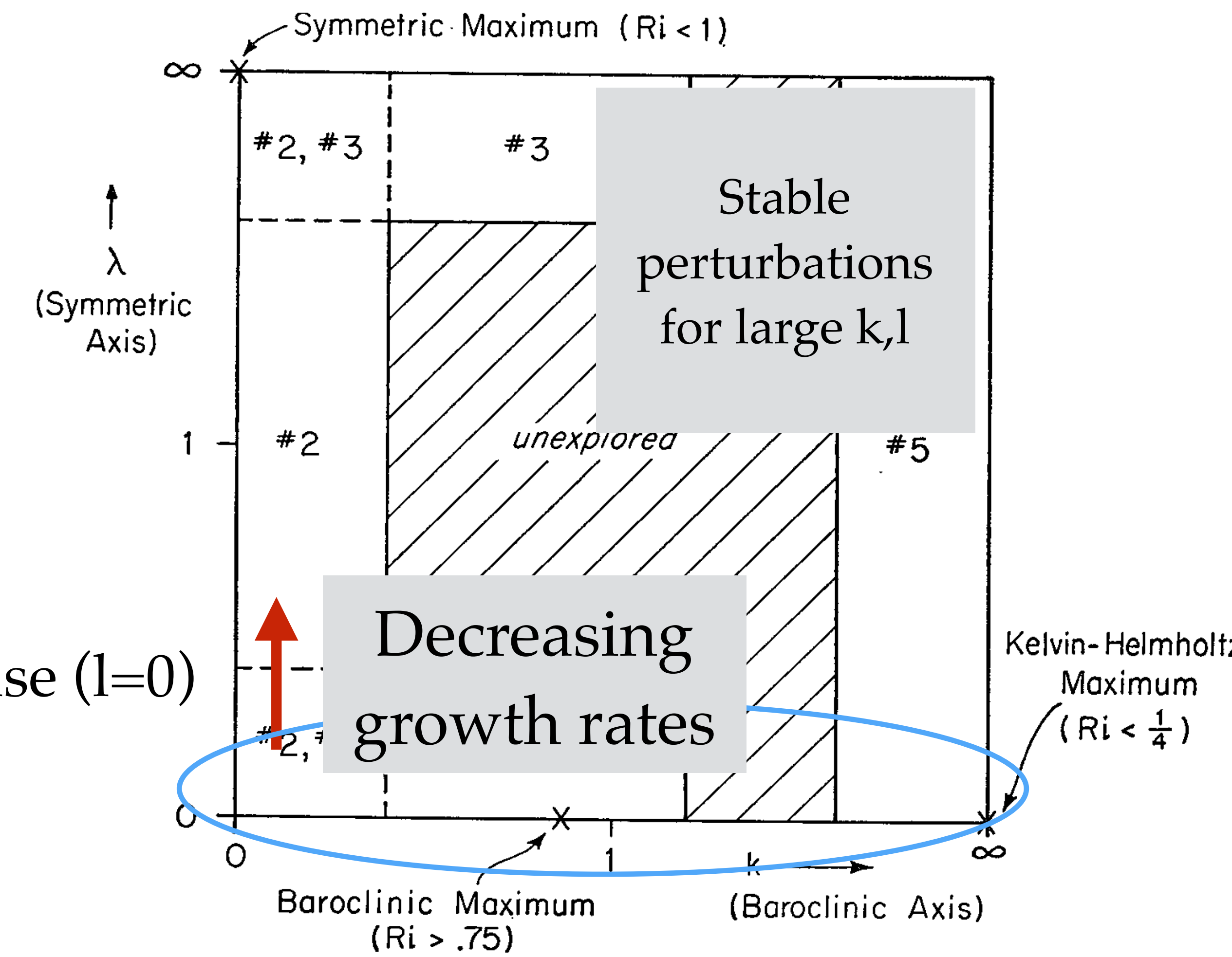


FIG. 1. Growth rates vs k when Ri = 2.



Stone, JAS 1966

Non-geostrophic instabilities arising from 2D perturbations

<u>Ri = 0.92:</u>
More than one vertical mode unstable. Symmetric instability appears.

~Symmetric instability

~Baroclinic instability

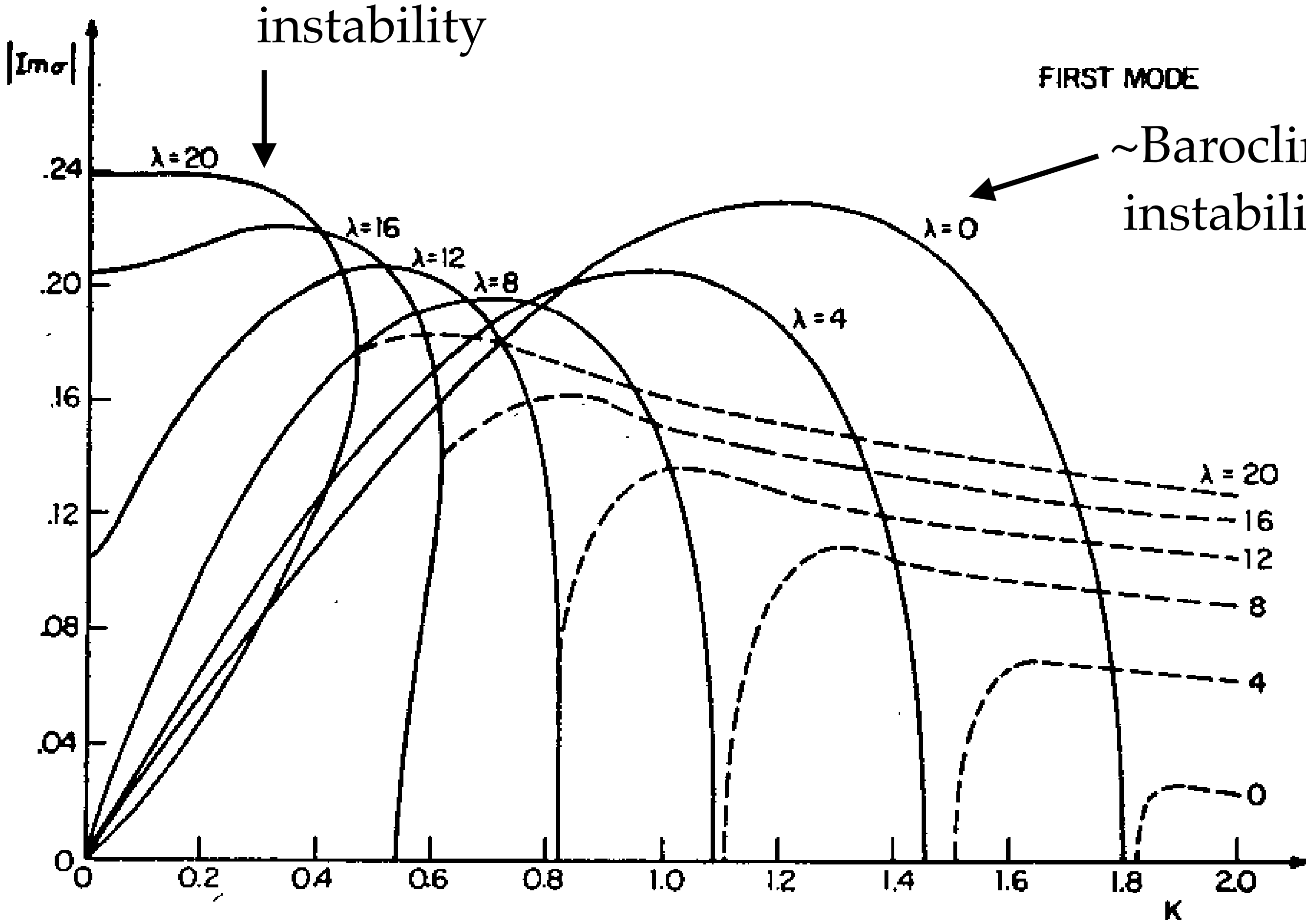
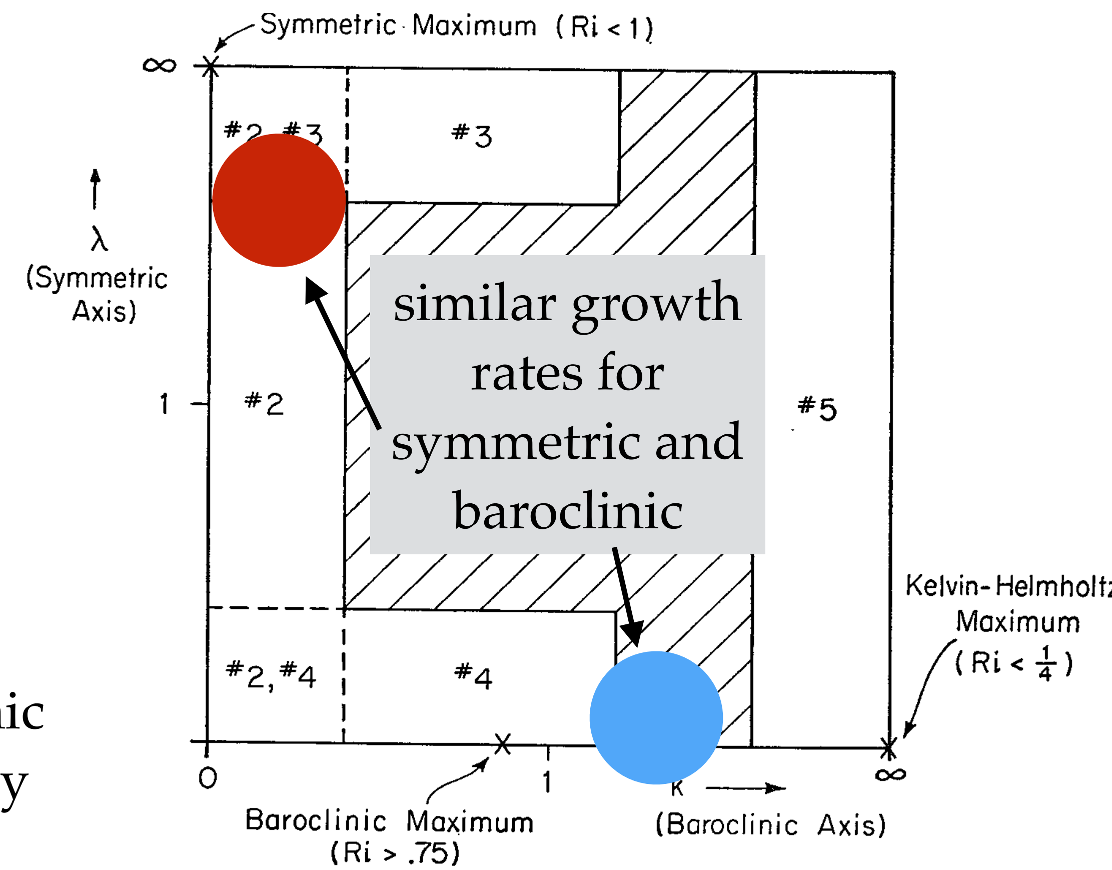


FIG. 3. Growth rates for the most unstable mode when Ri = 0.92.



Secondary unstable modes have smaller growth rates and smaller scales

Stone, JAS 1966

Non-geostrophic instabilities arising from 2D perturbations

<u>Ri = 0.5</u>:
Symmetric has
largest growth rate

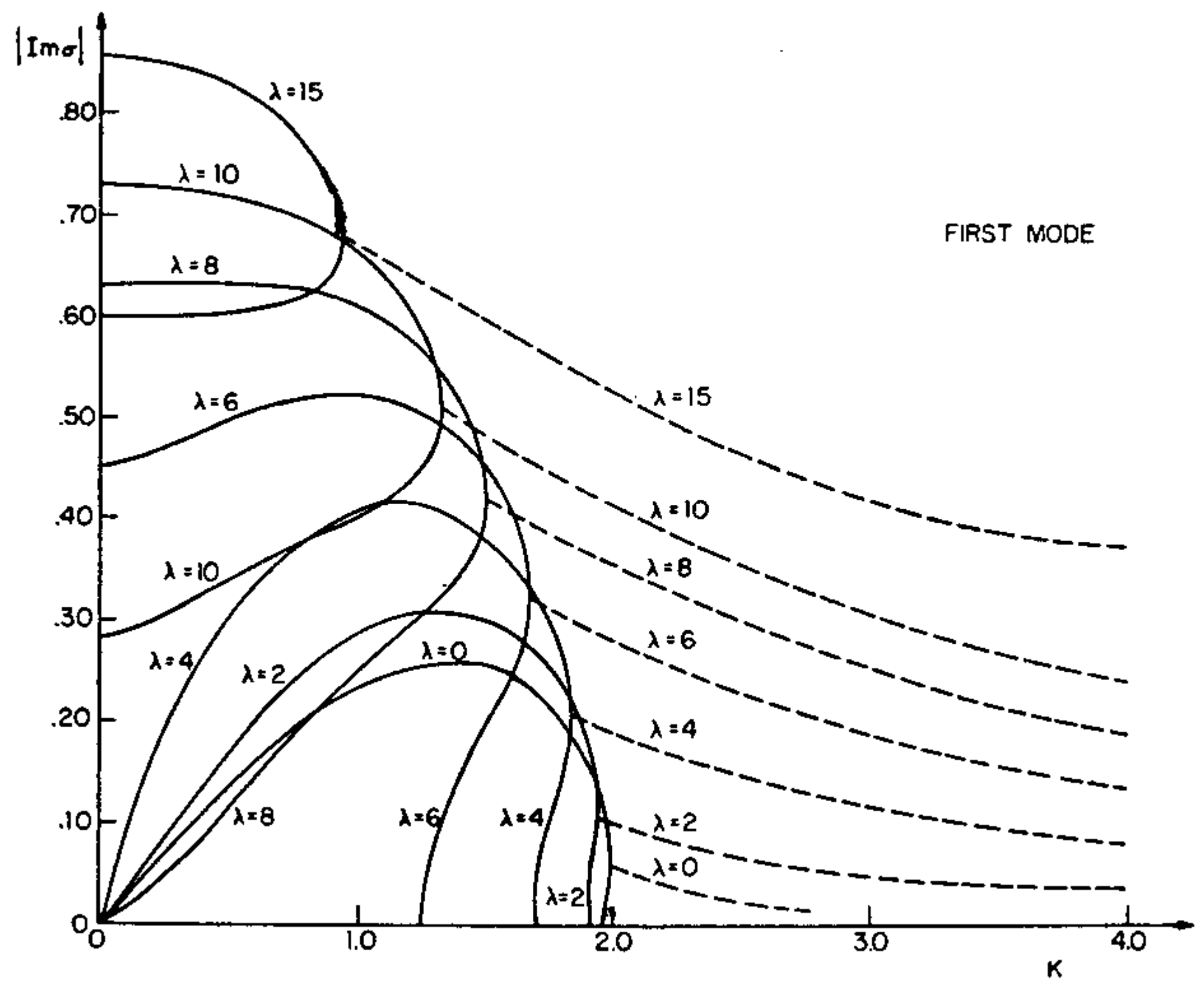
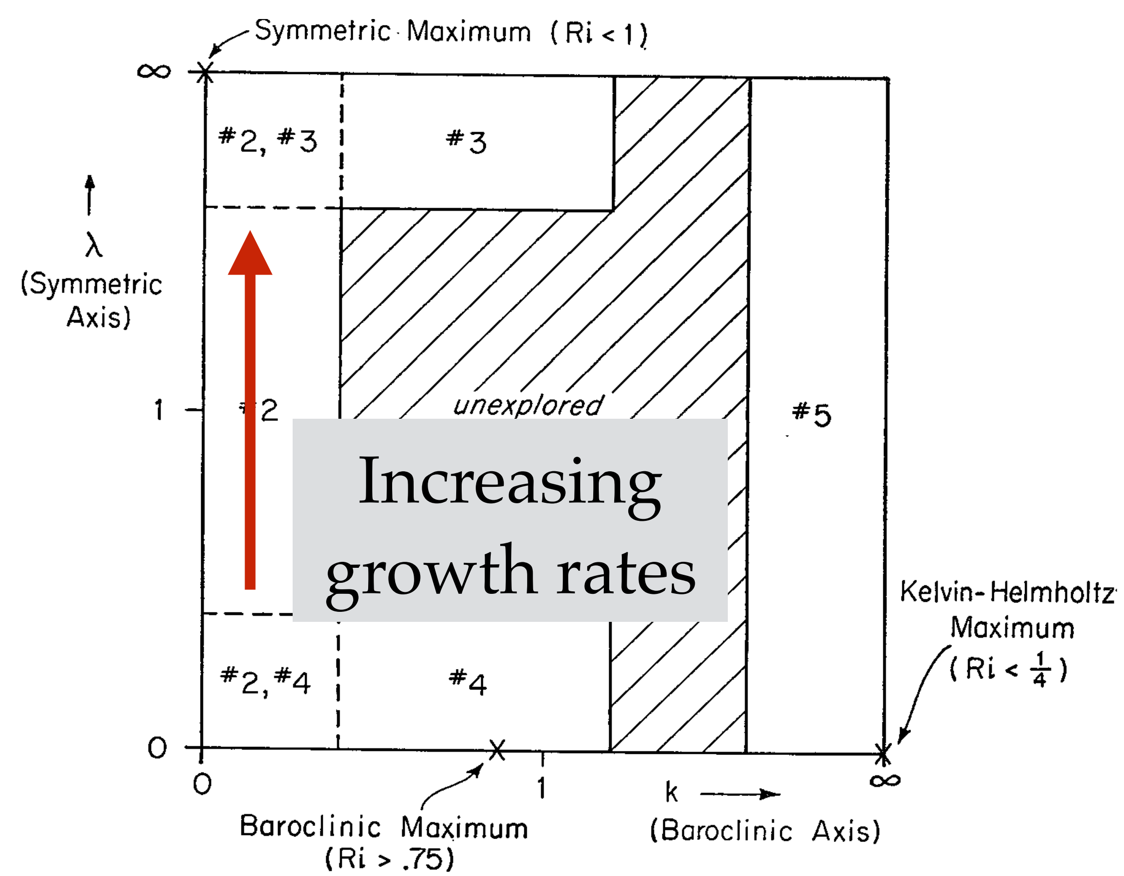


FIG. 4. Growth rates for the most unstable mode when Ri=0.5.



Stone, JAS 1966

Summary

- A flow in thermal wind balance is subject to a range of instabilities when the full primitive equations are considered
  - 0 < Ri < 0.25 -> Kelvin Helmholtz
  - 0.25 < Ri < 0.95 -> Symmetric
  - 0.95 < Ri -> Baroclinic
- Baroclinic instability is not significantly modified in full equations compared to QG for Ri >> 1
- Symmetric instability can be thought of as a combined centrifugal and gravitational instability