Eady’s Model

Cesar B. Rocha

PO Theory Seminar

SIO, Fall 2014

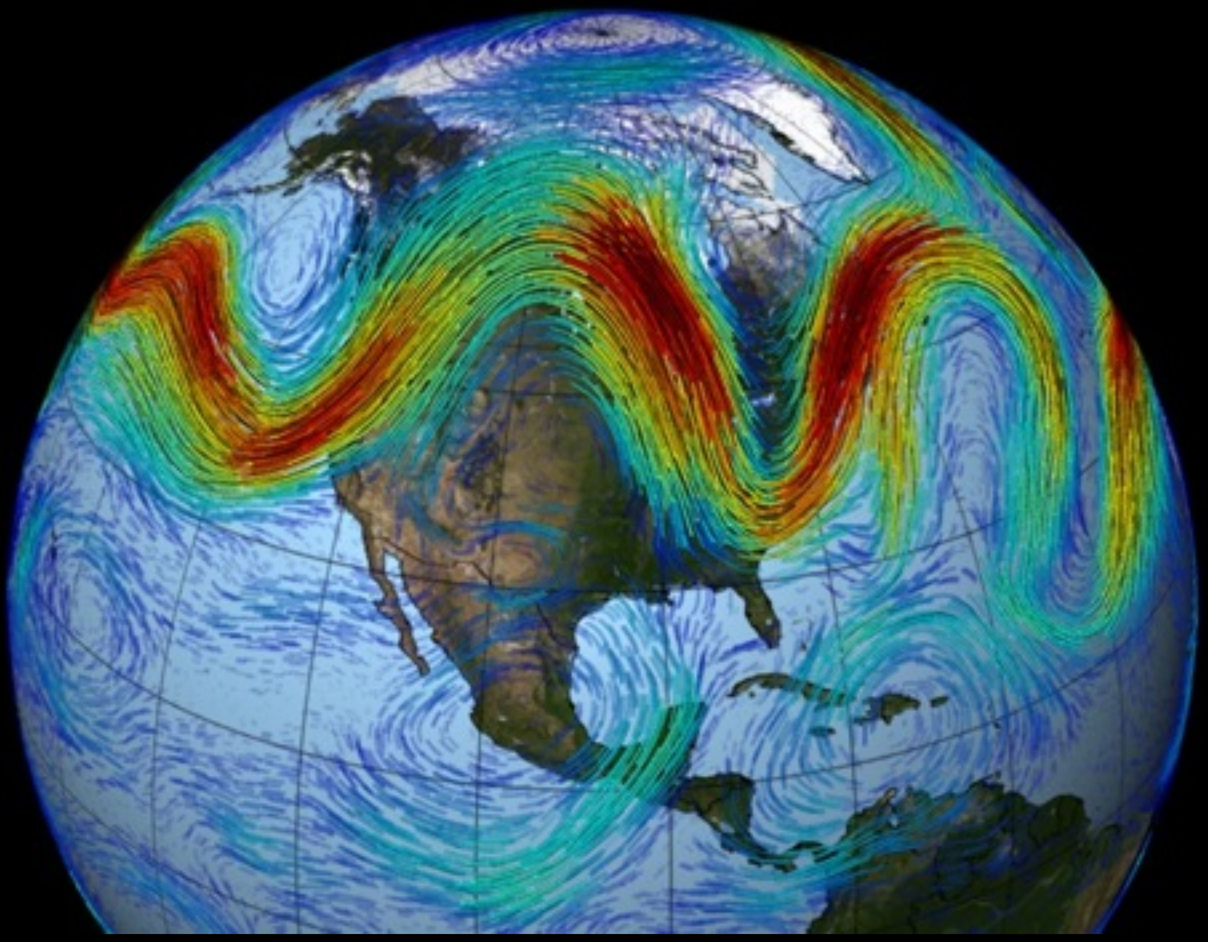


Image Courtesy NASA

Main References

1. Pedlosky, 1987: GFD, 2nd ed., Chapter 7.
2. Vallis, 2006: AOFD, Chapter 6.
3. Eady, 1949: Long Waves and Cyclone Waves, *Tellus, 1.*

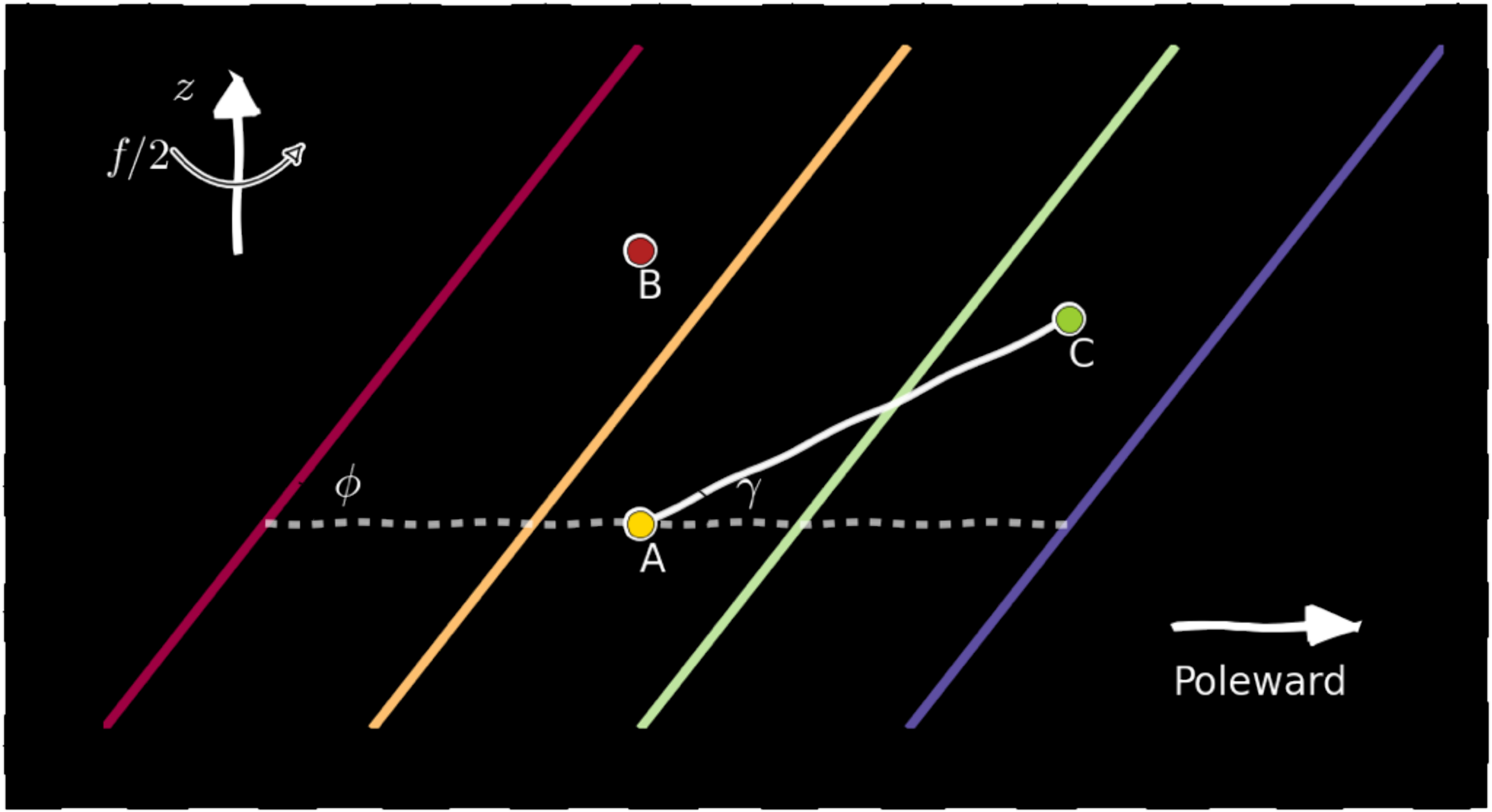
Eady’s triumph

My Goal Today

Show the basic nature of baroclinic instability of large-scale flows

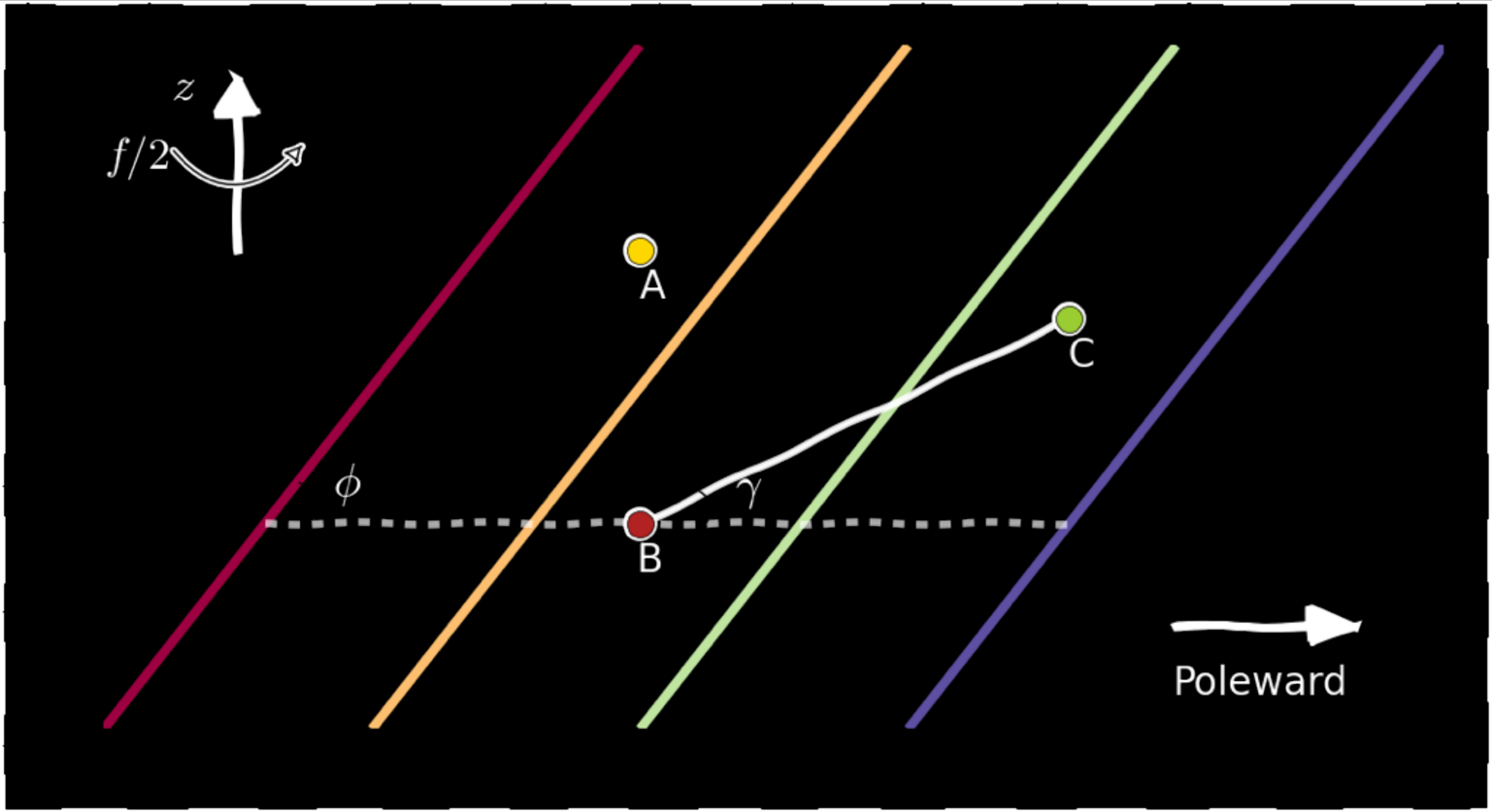
Elements of Baroclinic Instability

The Basic Mechanism



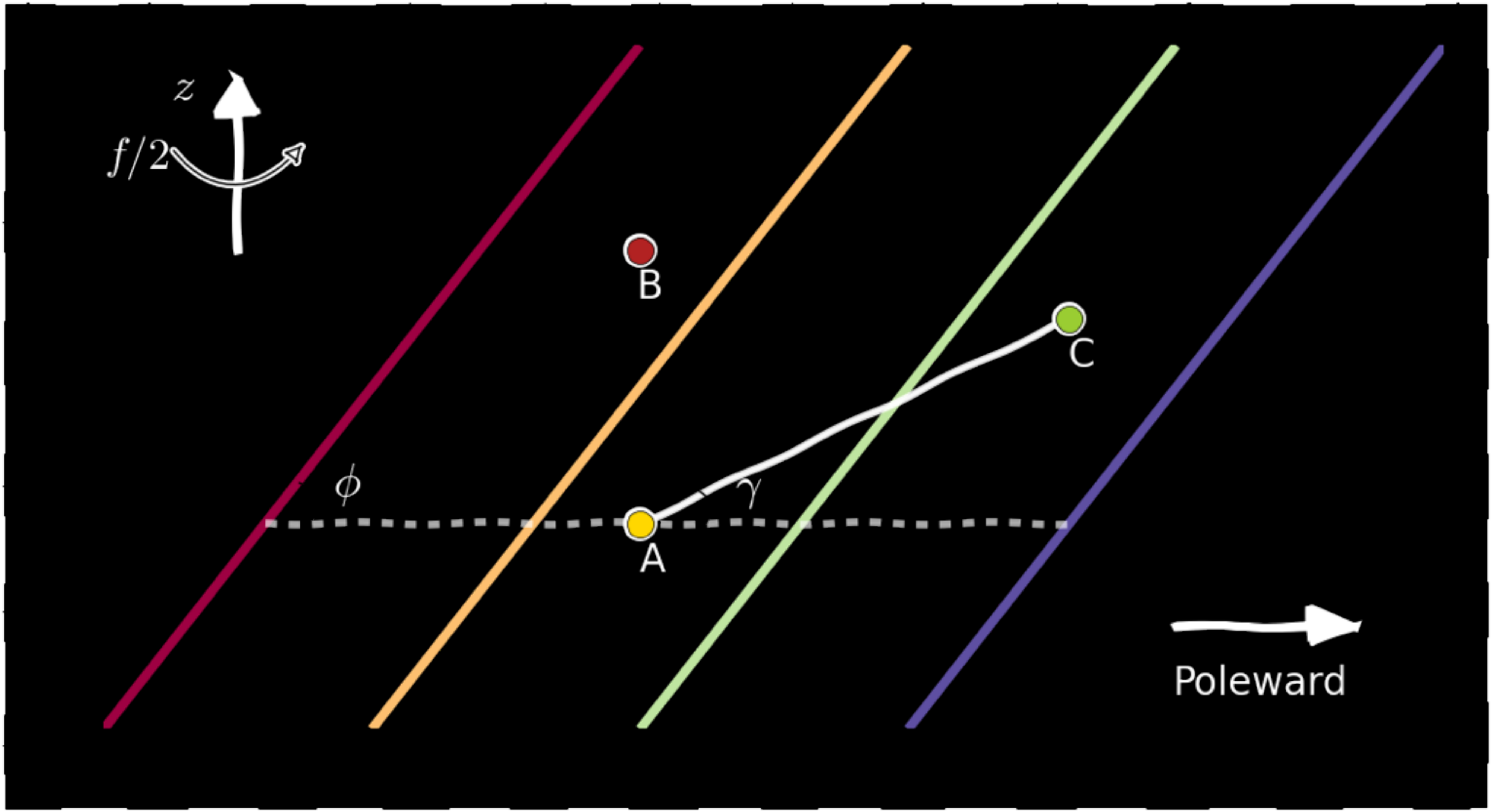
Pedlosky 1987; Vallis 2006

The Basic Mechanism



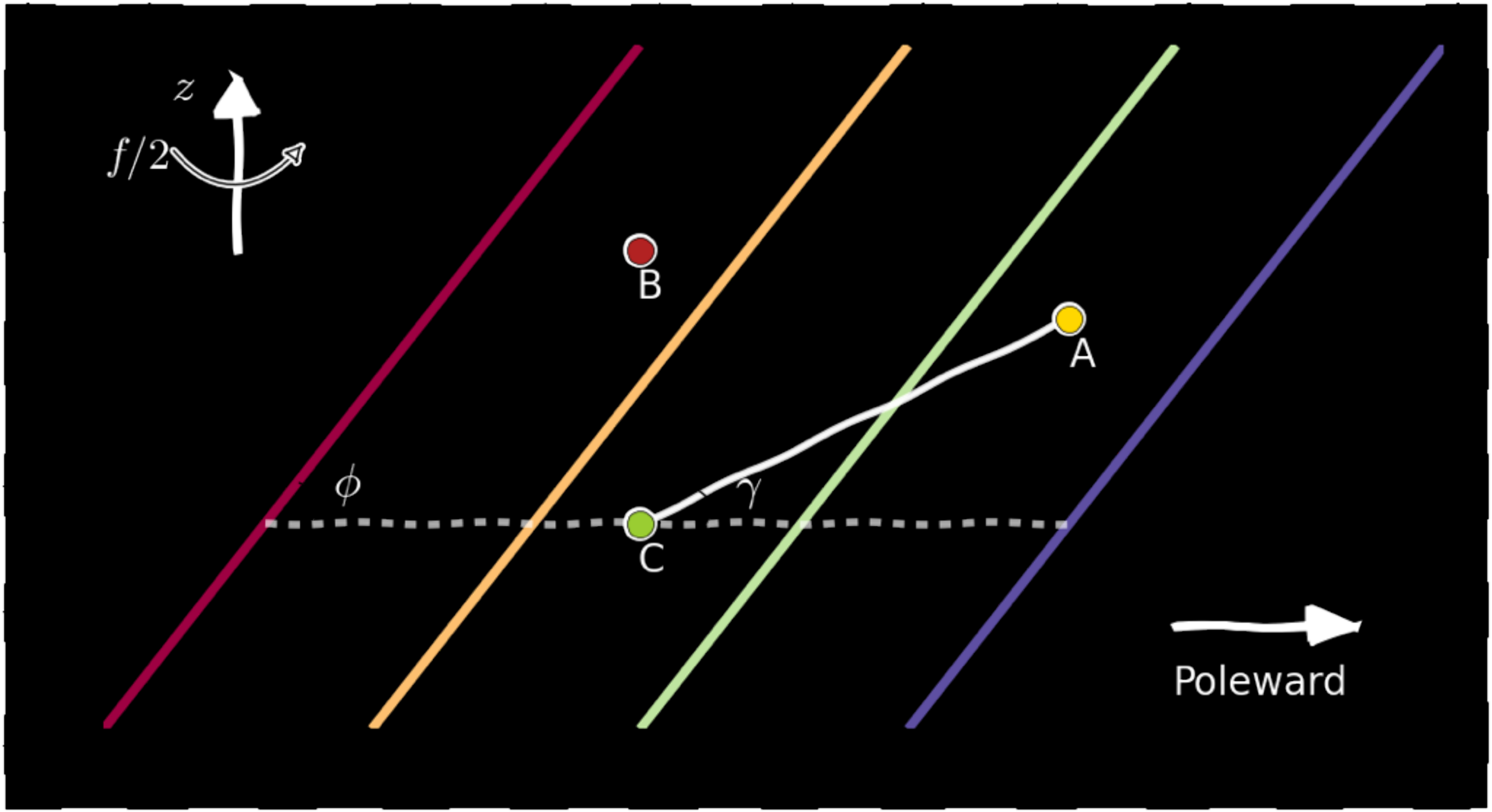
Pedlosky 1987; Vallis 2006

The Basic Mechanism



Pedlosky 1987; Vallis 2006

The Basic Mechanism



Pedlosky 1987; Vallis 2006

QG Equations

$$q_t + J(\psi, q) = 0, \qquad 0 < z < H$$

$$b_t + J(\psi, b) = 0, \qquad z = 0,\ H$$

$$q = \beta y + \nabla^2 \psi + \left[\frac{f_0^2}{N^2(z)} \psi_z\right]_z$$

$$b = f_0 \psi_z$$

Linear Stability Analysis

Linearization

Zonal mean flow

$$U(y,z) = -\Psi_y, \quad f_0 U_z = -B_y$$

Total flow = mean + eddy (infinitesimal)

$$\Psi(y,z) + \psi(x,y,z,t)$$

Linearization

$$q_t - \Psi_y q_x + \psi_x Q_y = 0, \qquad 0 < z < H$$

$$b_t - \Psi_y b_x + \psi_x B_y = 0, \qquad z = 0,\ H$$

$$q(x,y,z,t) = \nabla^2 \psi + \left[\frac{f_0^2}{N^2(z)}\psi_z\right]_z$$

$$Q(y,z) = \beta y + \nabla^2 \Psi \left[\frac{f_0^2}{N^2(z)}\Psi_z\right]_z$$

Normal Modes

$$\psi(x,y,z,t) = \text{Re}\ \phi(y,z)\exp\left[ik(x-ct)\right]$$

$$(U-c)\left\{\phi_{yy} + \left[\frac{f_0^2}{N^2(z)}\phi_z\right] - k^2\phi\right\} + Q_y\phi = 0,\ \ 0 < z < \ H$$

$$(U-c)\phi_z - U_z\phi = 0, \qquad z = 0,\ H$$

Necessary Conditions

$$\int_0^{2L} dy \left\{ \int_0^H \frac{|\phi|^2}{|U-c|^2} Q_y dz + \frac{f_0^2}{N^2(z)} \frac{|\phi|^2}{|U-c|^2} U_z \Big|_0^H \right\} = 0$$

Pedlosky 1987; Vallis 2006

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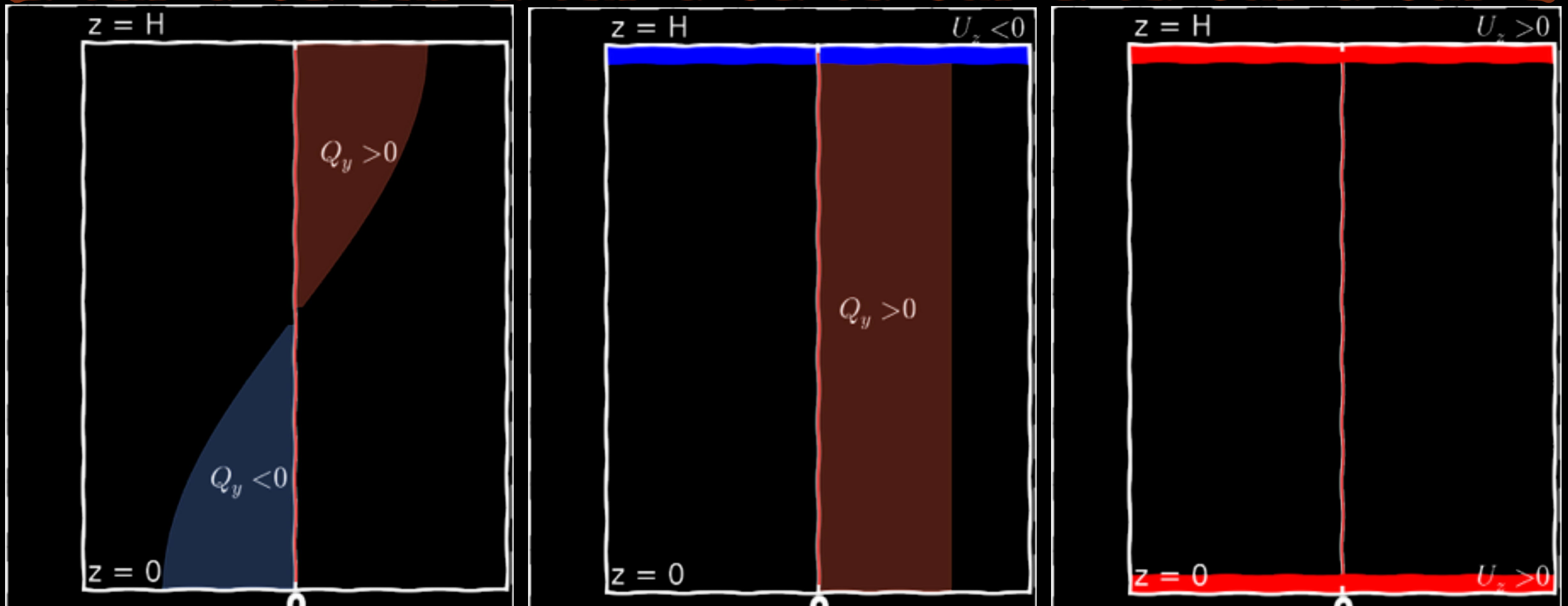
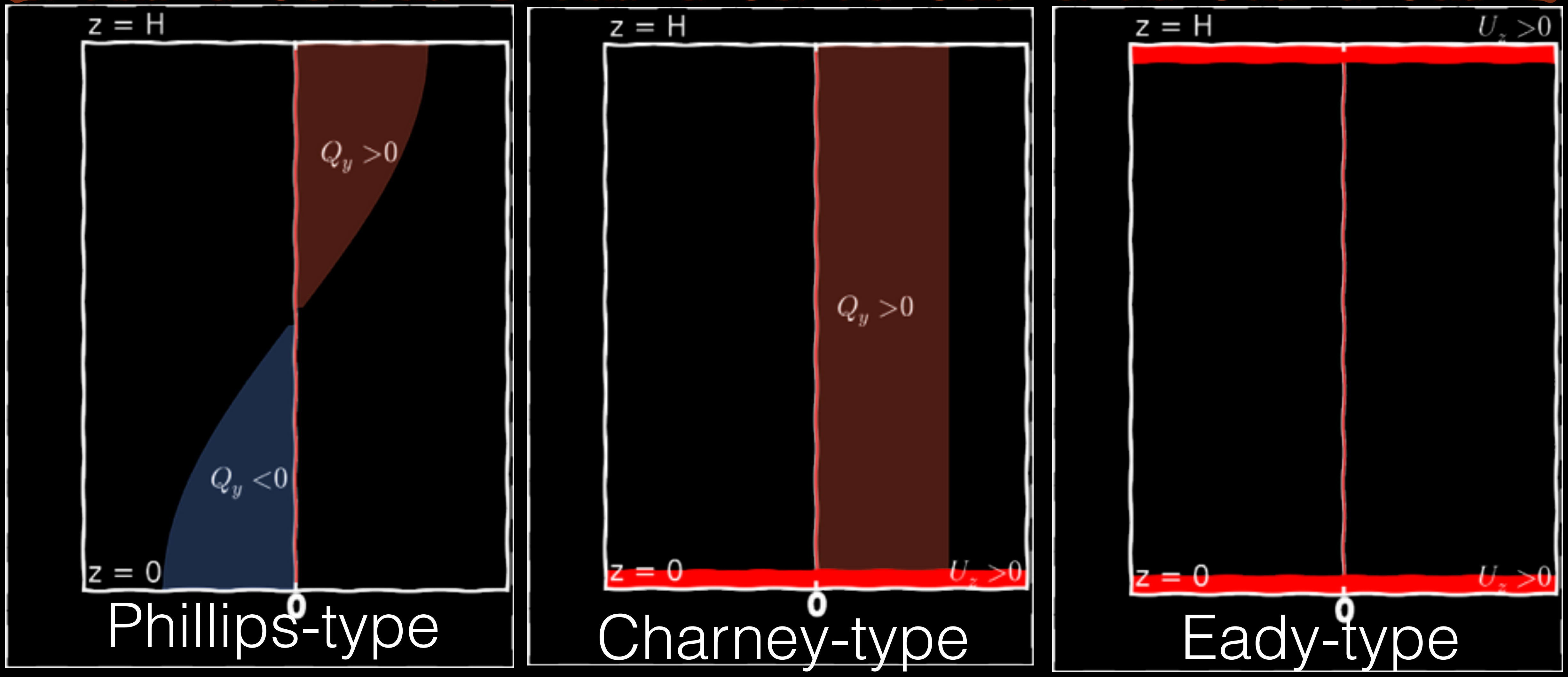


Phillips-type | Charney-type | Eady-type

Pedlosky 1987; Vallis 2006

Necessary Conditions

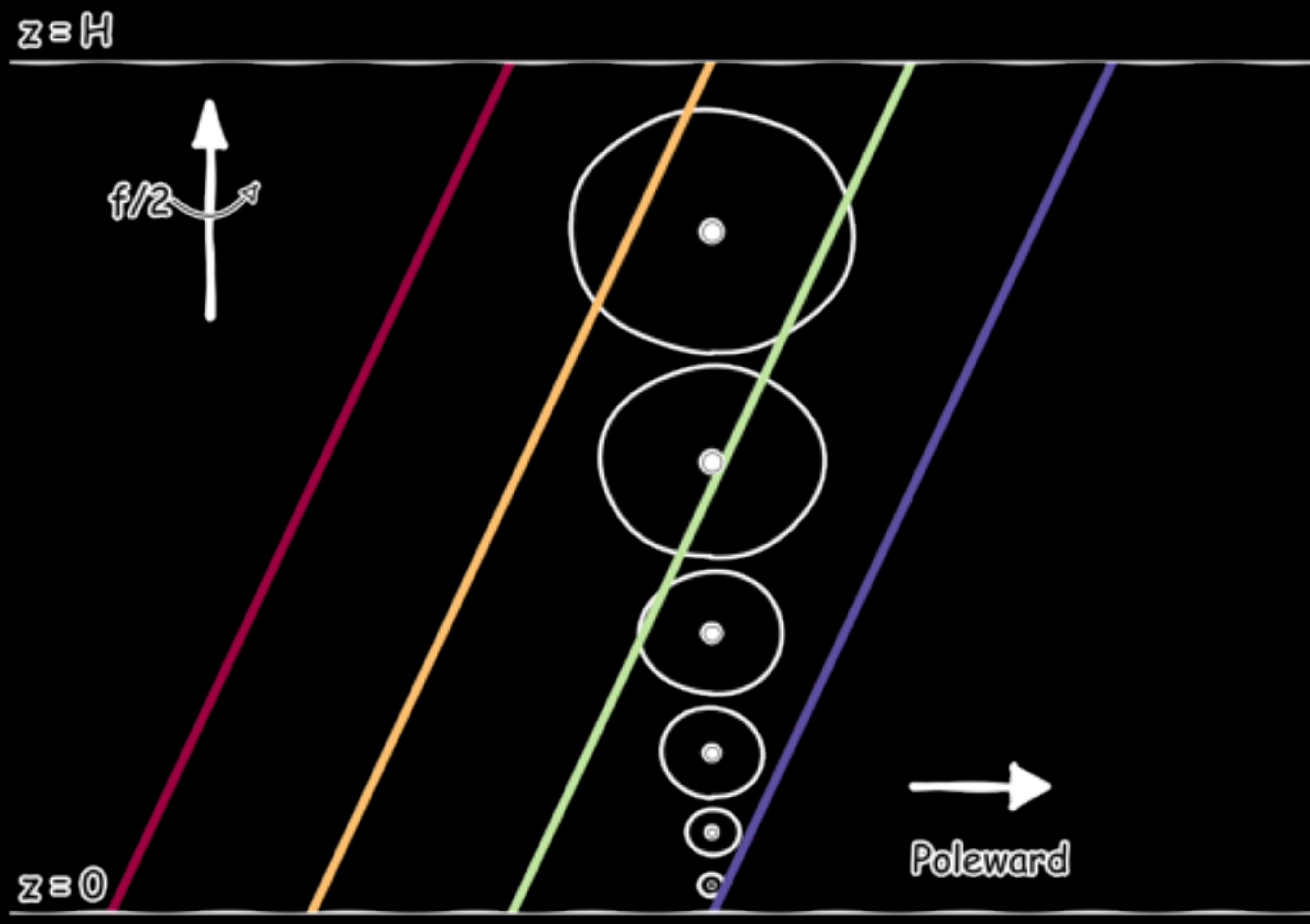
$$\int_0^{2L} dy \left\{ \int_0^H \frac{|\phi|^2}{|U-c|^2} Q_y dz + \frac{f_0^2}{N^2(z)} \frac{|\phi|^2}{|U-c|^2} U_z \Big|_0^H \right\} = 0$$



Pedlosky 1987; Vallis 2006

Eady’s Problem

Eady's Model: Set up



$$U(z) = \Lambda z$$

$$N^2 = \text{const.}$$

$$\beta = 0 \ (f - plane)$$

$$\left.\frac{\partial z}{\partial y}\right)_B = \frac{f_0 \Lambda}{N^2}$$

$$Q = Q_y = 0$$

Normal Mode Equation

$$(U-c)\left\{\phi_{yy}+\left[\frac{f_0^2}{N^2(z)}\phi_z\right]-k^2\phi\right\}+Q_y\phi=0,\ \ 0<z<\ H$$

$$(U-c)\phi_z-U_z\phi=0,\qquad z=0,\ H$$

Normal Mode Equation

$$(U-c)\left\{\phi_{yy}+\left[\frac{f_0^2}{N^2(z)}\phi_z\right]-k^2\phi\right\}+Q_y\phi=0,\ 0<z<\ H$$

$$(U-c)\phi_z-U_z\phi=0,\qquad z=0,\ H$$

$Q_y=0\longrightarrow$ **NO critical layers !**

Normal Mode Equation

$$(\Lambda z - c)\left[\phi_{yy} + \frac{f_0^2}{N^2}\phi_{zz} - k^2\phi\right] = 0, \quad 0 < z < H$$

$$(\Lambda z - c)\phi_z - \Lambda\phi = 0, \quad z = 0,\ H$$

Perturbations confined in a channel of width 2L

$$\phi(y, z) = \cos(l_n y)\varphi(z)$$

$$l_n = (2n+1)\frac{\pi}{2L}, \quad n = 0, 1, 2, \ldots$$

Normal Mode Equation

$$(\Lambda z - c)\left[\varphi_{zz} - \frac{\mu^2}{H^2}\varphi\right] = 0, \quad 0 < z < H$$

$$(\Lambda z - c)\varphi(z) - \Lambda\varphi = 0, \quad z = 0,\ H$$

$$\mu \stackrel{\text{def}}{=} \kappa L_d$$

$$\kappa^2 = k^2 + l_n^2,$$

$$L_d \stackrel{\text{def}}{=} \frac{NH}{f_0}$$

$$\varphi(z) = A\cosh\left(\mu\frac{z}{H}\right) + B\sinh\left(\mu\frac{z}{H}\right)$$

BCs determine the physics

Eigenproblem Solution

$$c = \frac{\Lambda H}{2} \pm \frac{\Lambda H}{\mu} \left\{ \left[\frac{\mu}{2} - \tanh\left(\frac{\mu}{2}\right) \right] \left[\frac{\mu}{2} - \coth\left(\frac{\mu}{2}\right) \right] \right\}^{1/2}$$

Eigenproblem Solution

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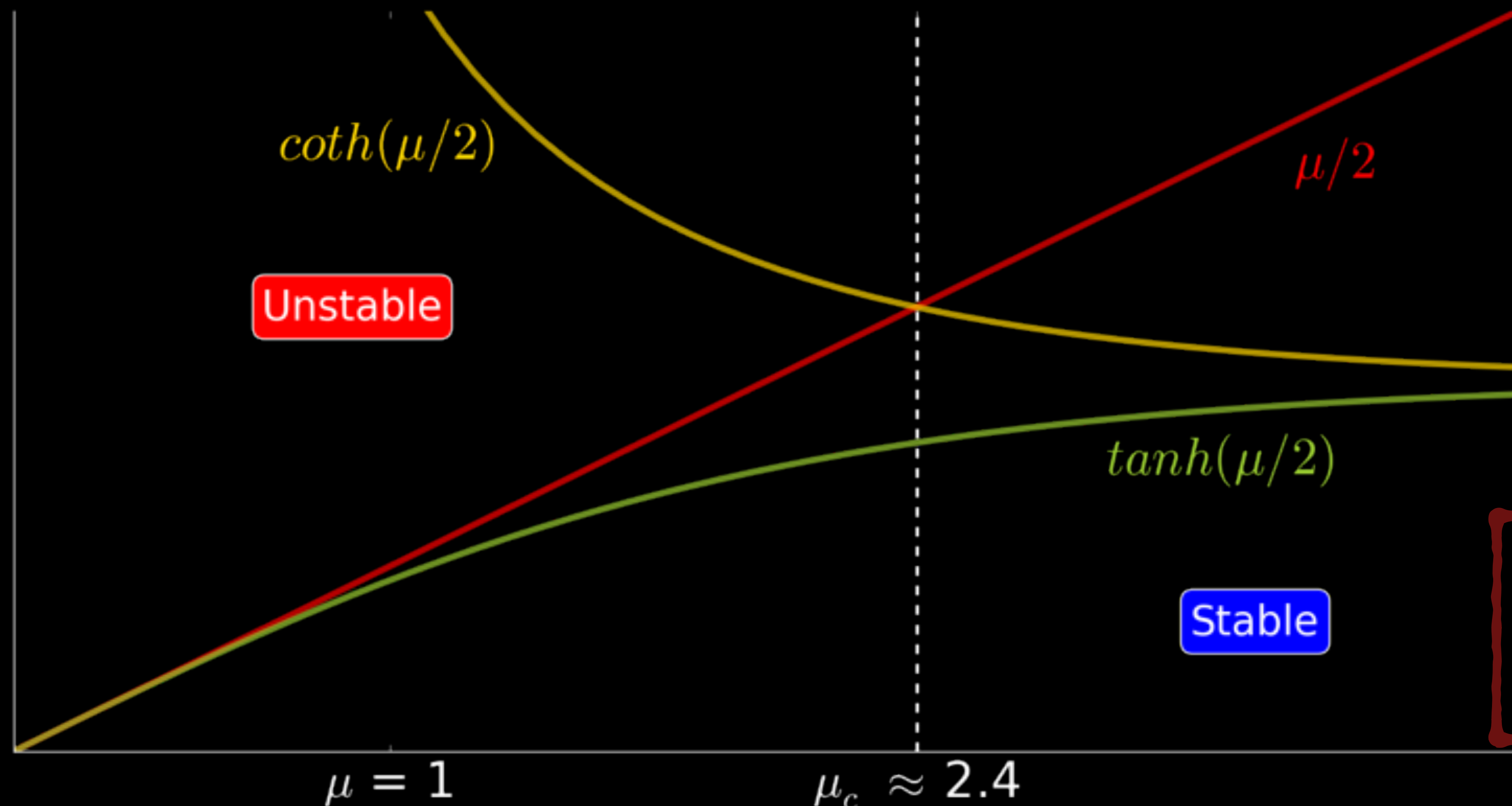
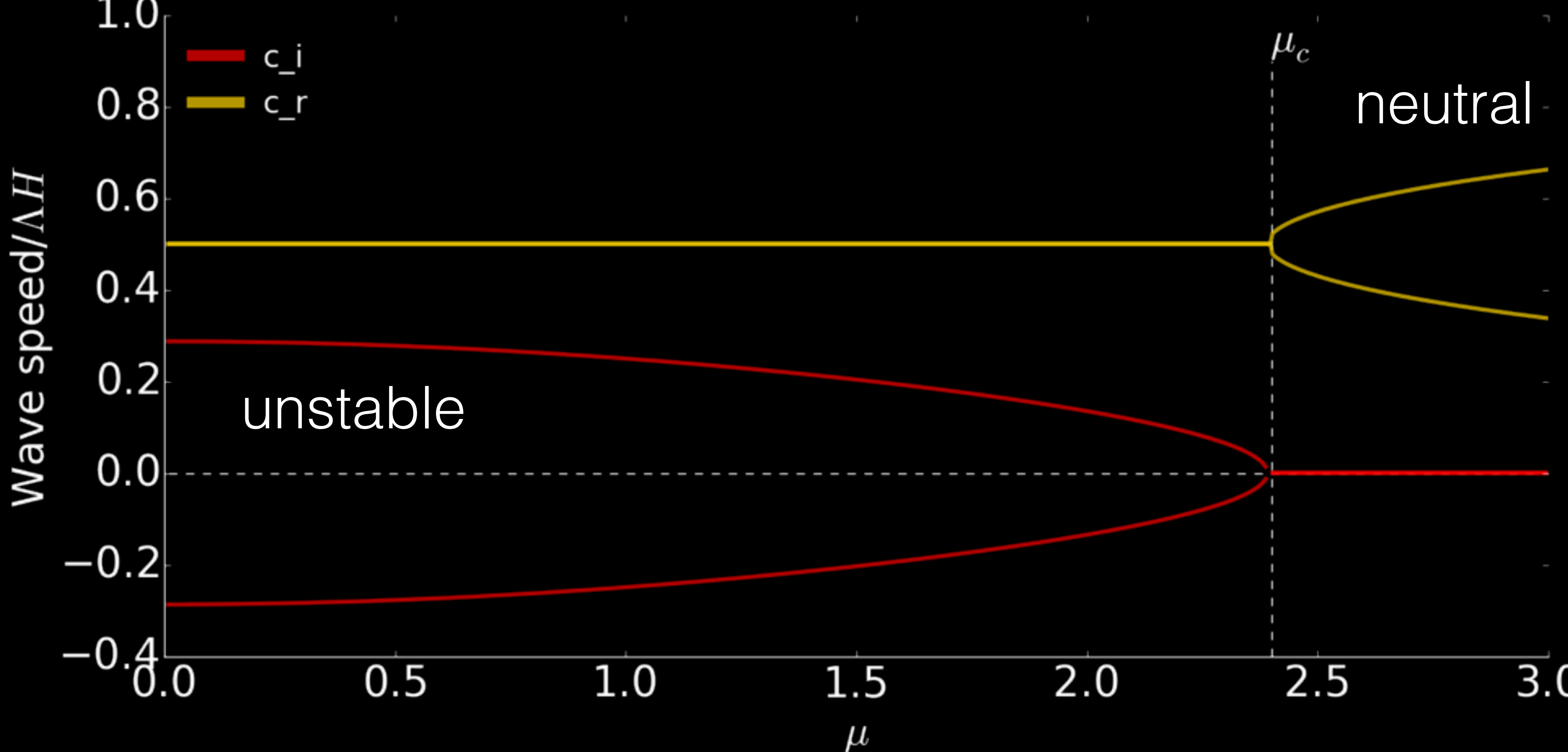


High wavenumber cutoff

$$k_c^2 \approx 2.4^2 - l_n^2$$

$$\frac{\mu_c}{2} = \coth\left(\frac{\mu_c}{2}\right)$$

Wave Speeds



Steering level at z $= H/2$ $(c_r = U)$

$L/L_d = 8$

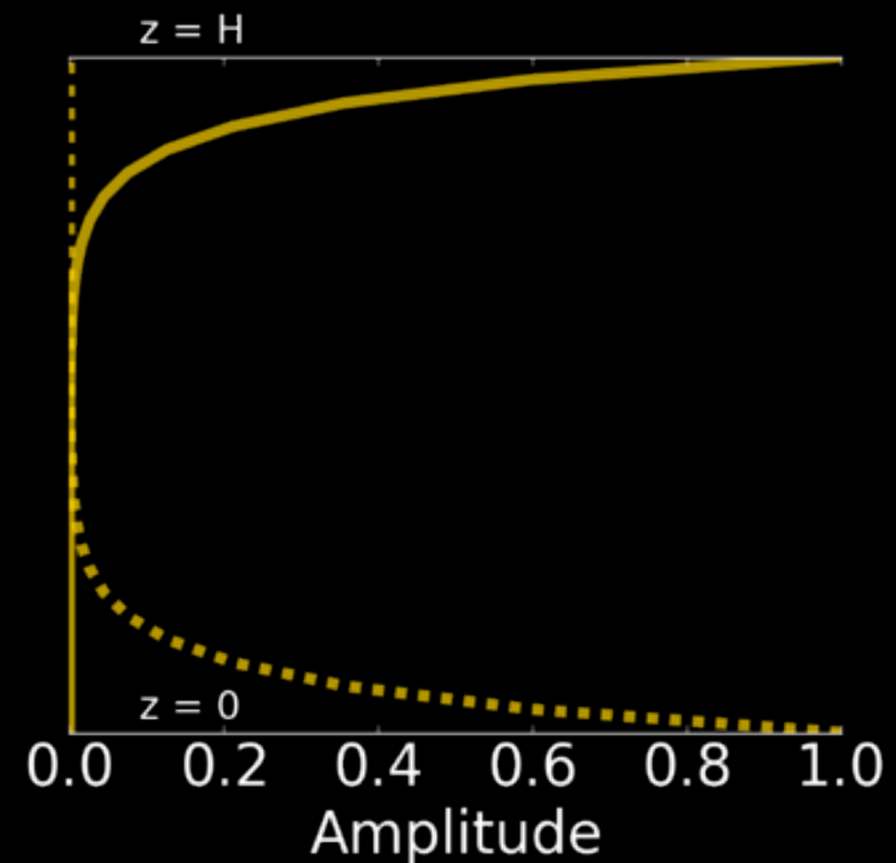
Neutral Modes

$$\lim_{\mu\to\infty} c : \qquad c = 0 \qquad \text{or} \qquad c = \Lambda H$$

$$\varphi(z) \sim \exp\left(-\mu\frac{z}{H}\right) \quad \text{or} \quad \varphi(z) \sim \exp\left(\mu\frac{z-H}{H}\right)$$

Similar to
SQG solutions

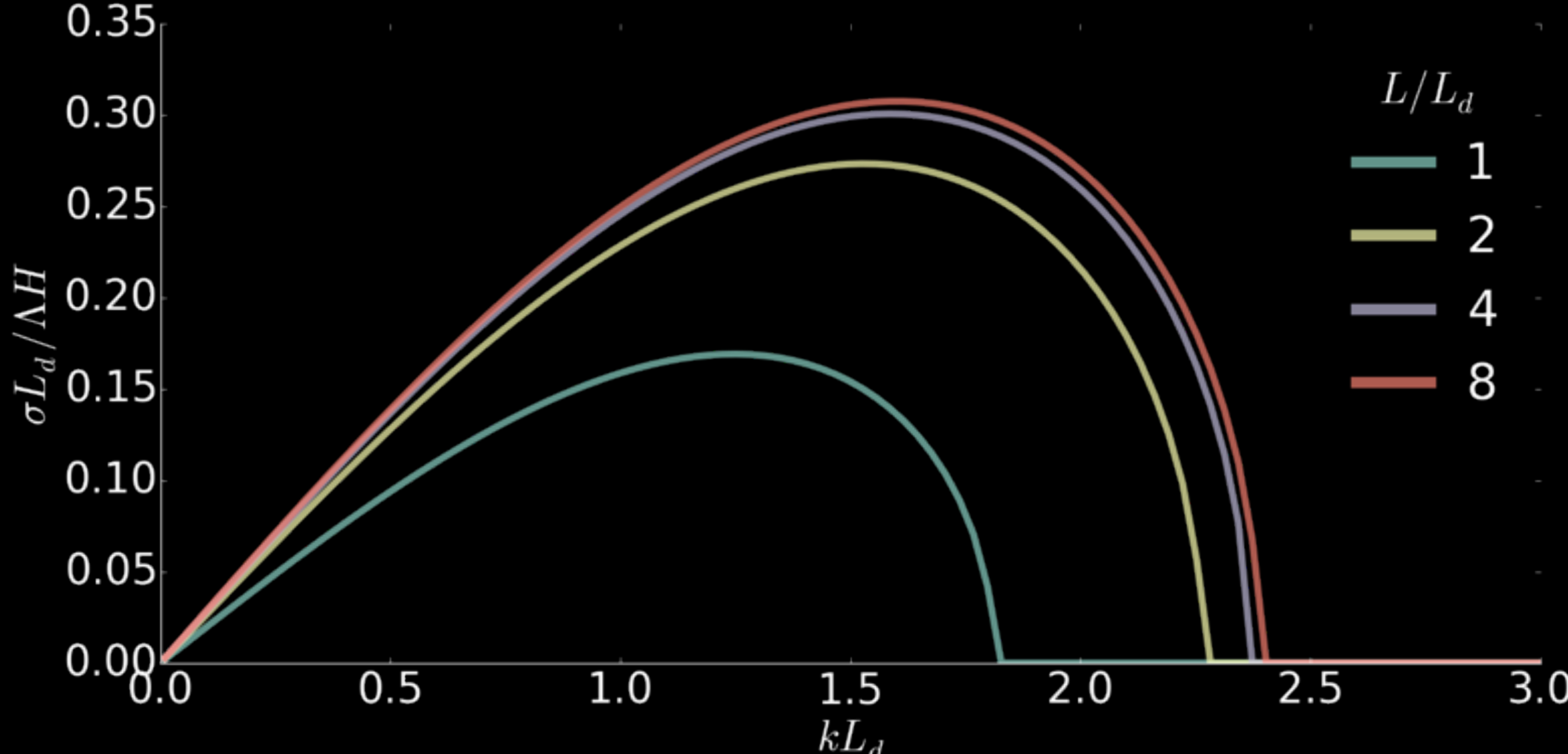
e.g. Tulloch & Smith 2006



Unstable Modes

Growth rate

$$\sigma = kc_i = \frac{k\Lambda H}{\mu}\left\{\left[\frac{\mu}{2} - \tanh\left(\frac{\mu}{2}\right)\right]\left[\coth\left(\frac{\mu}{2}\right) - \frac{\mu}{2}\right]\right\}^{1/2}$$



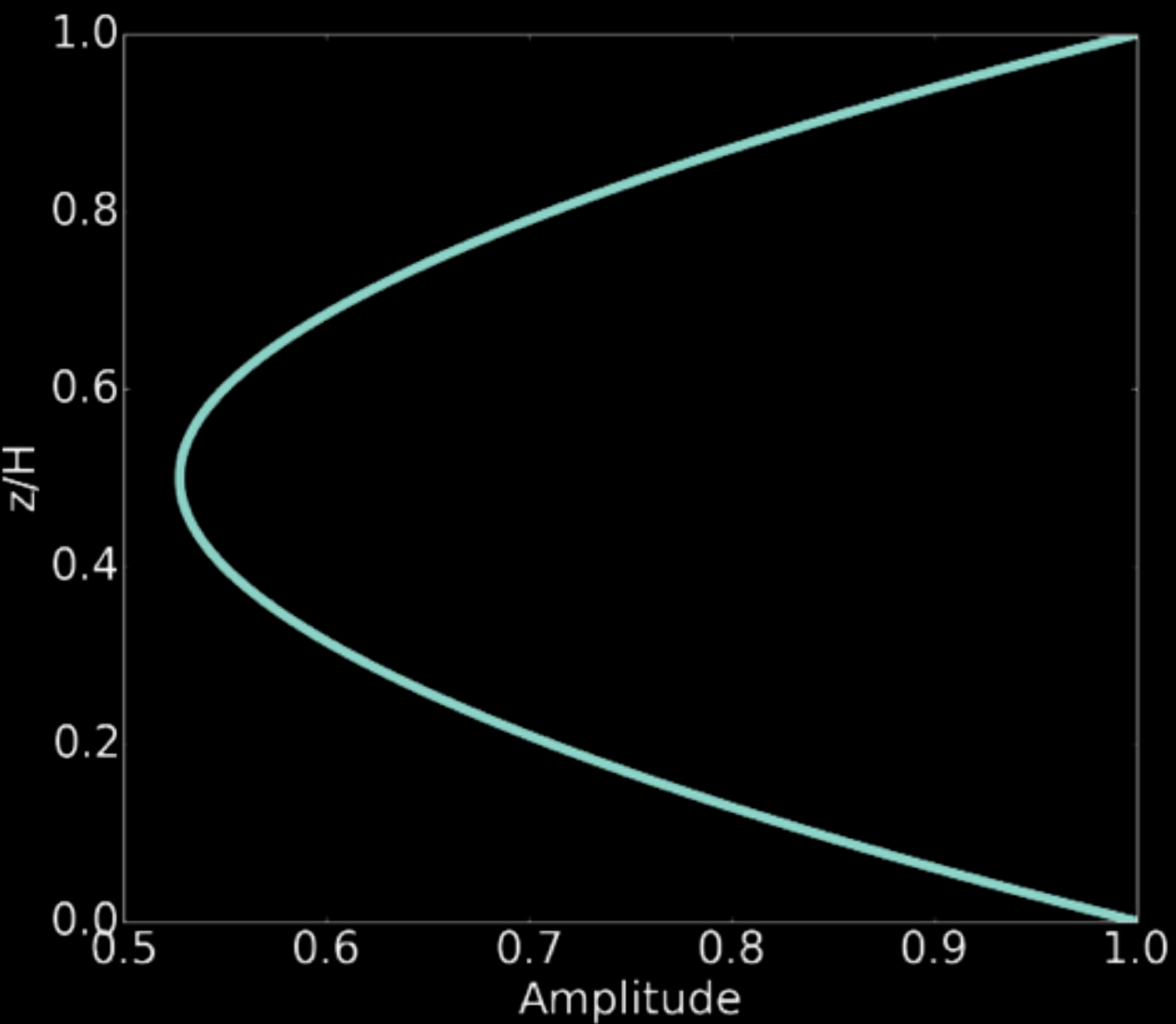
Gravest meridional mode

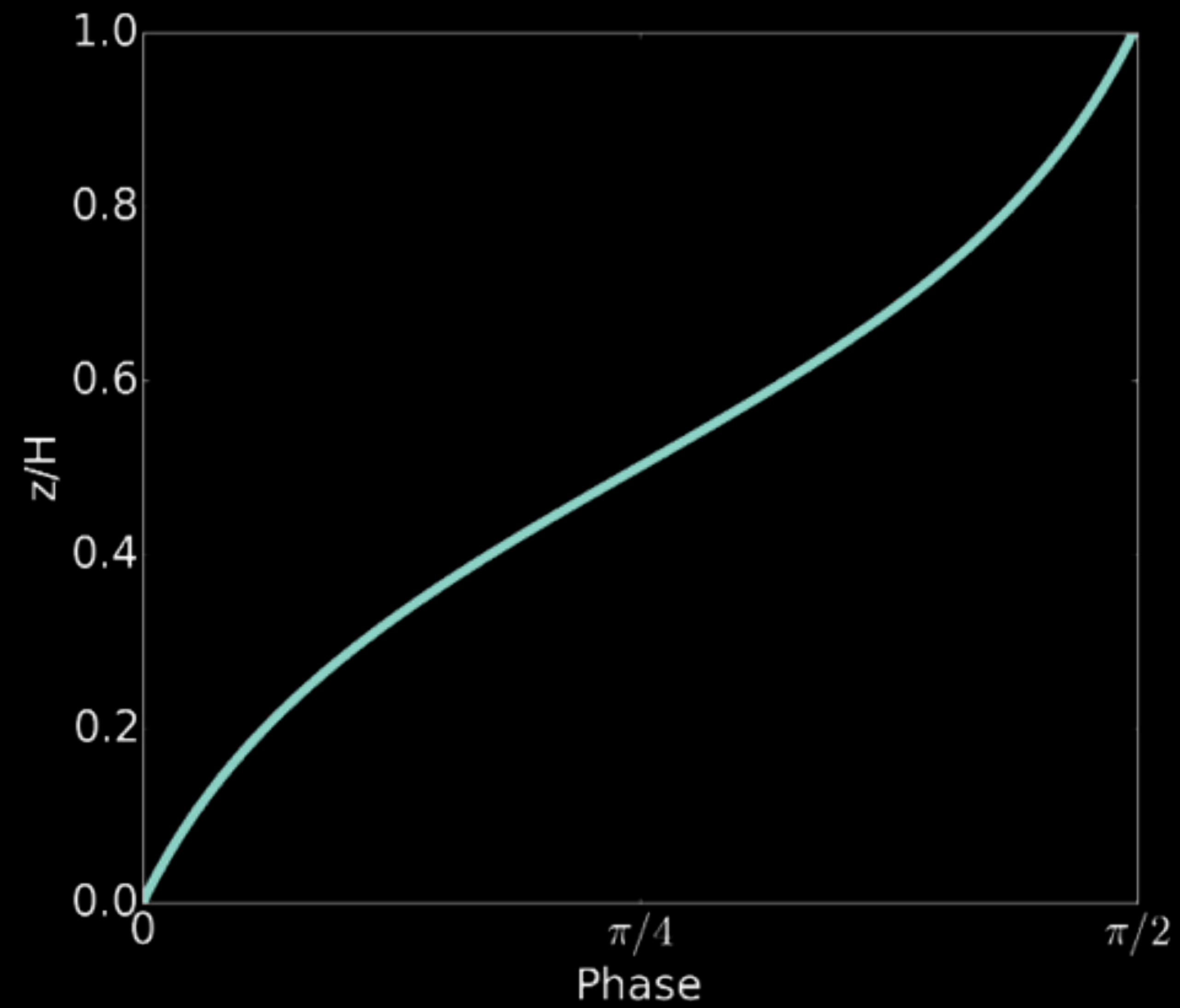
$n = 0$

Vertical Structure

$$\varphi(z) = \cosh(\mu z) - \frac{\sinh(\mu z)}{\mu c}$$

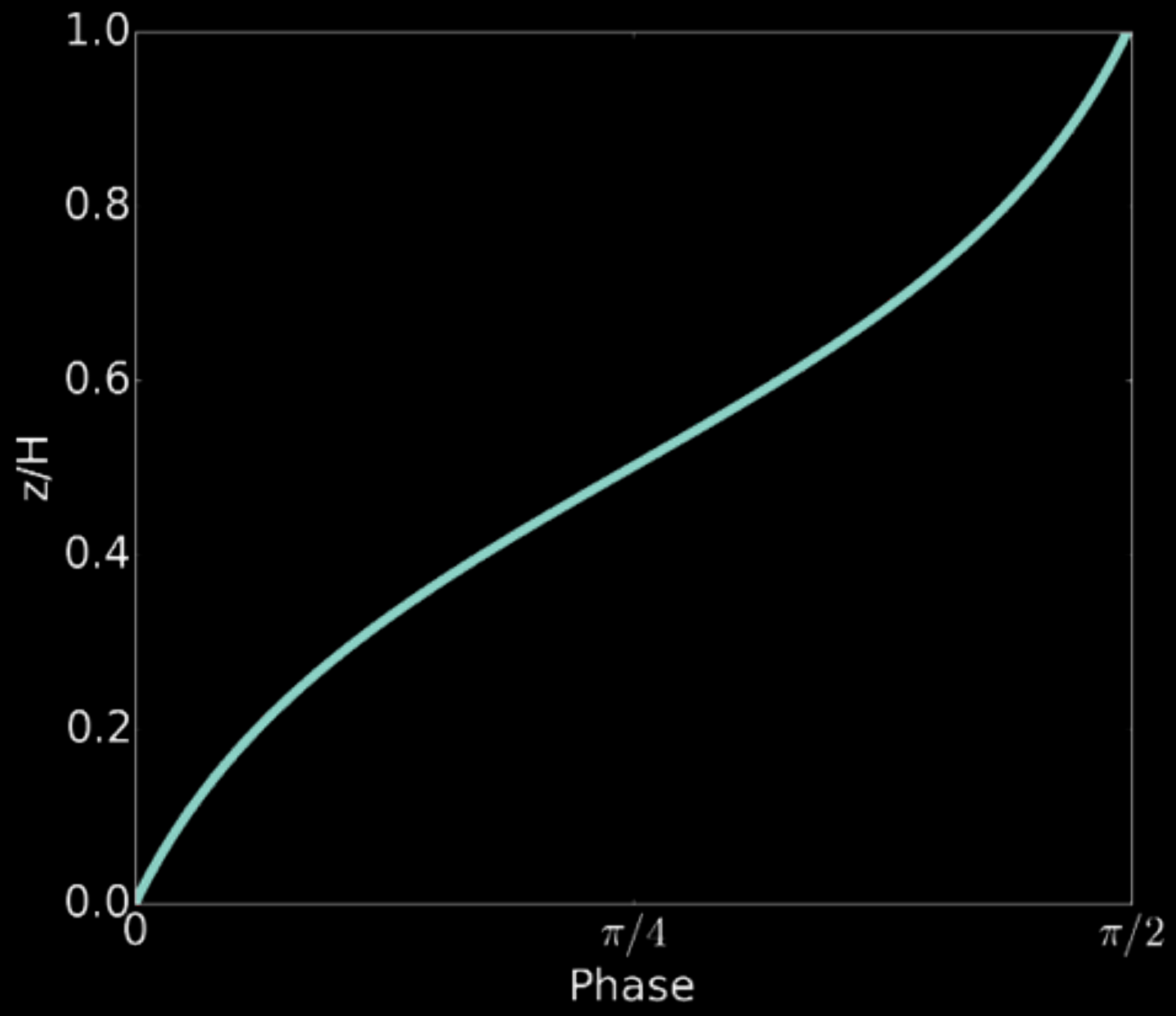
$$n = 0 \quad L/L_d = 8$$
$$kL_d = 1.6$$





Vertical Structure

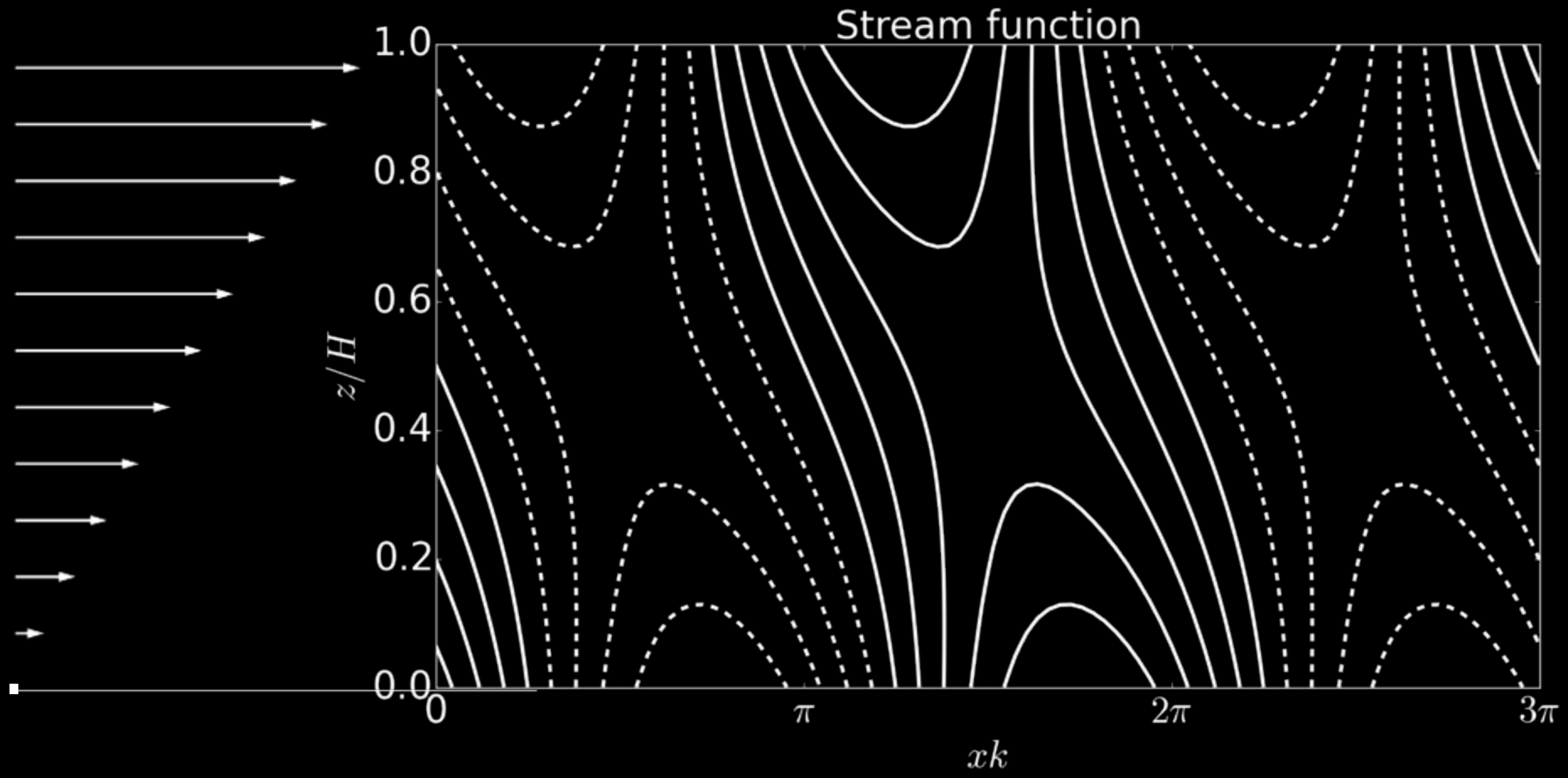
$$\psi \sim \exp i[kx + \mathrm{Phase}(z)] \rightarrow x = -\mathrm{Phase}(z)/k + \mathrm{const.}$$



Perturbations lean against the mean shear

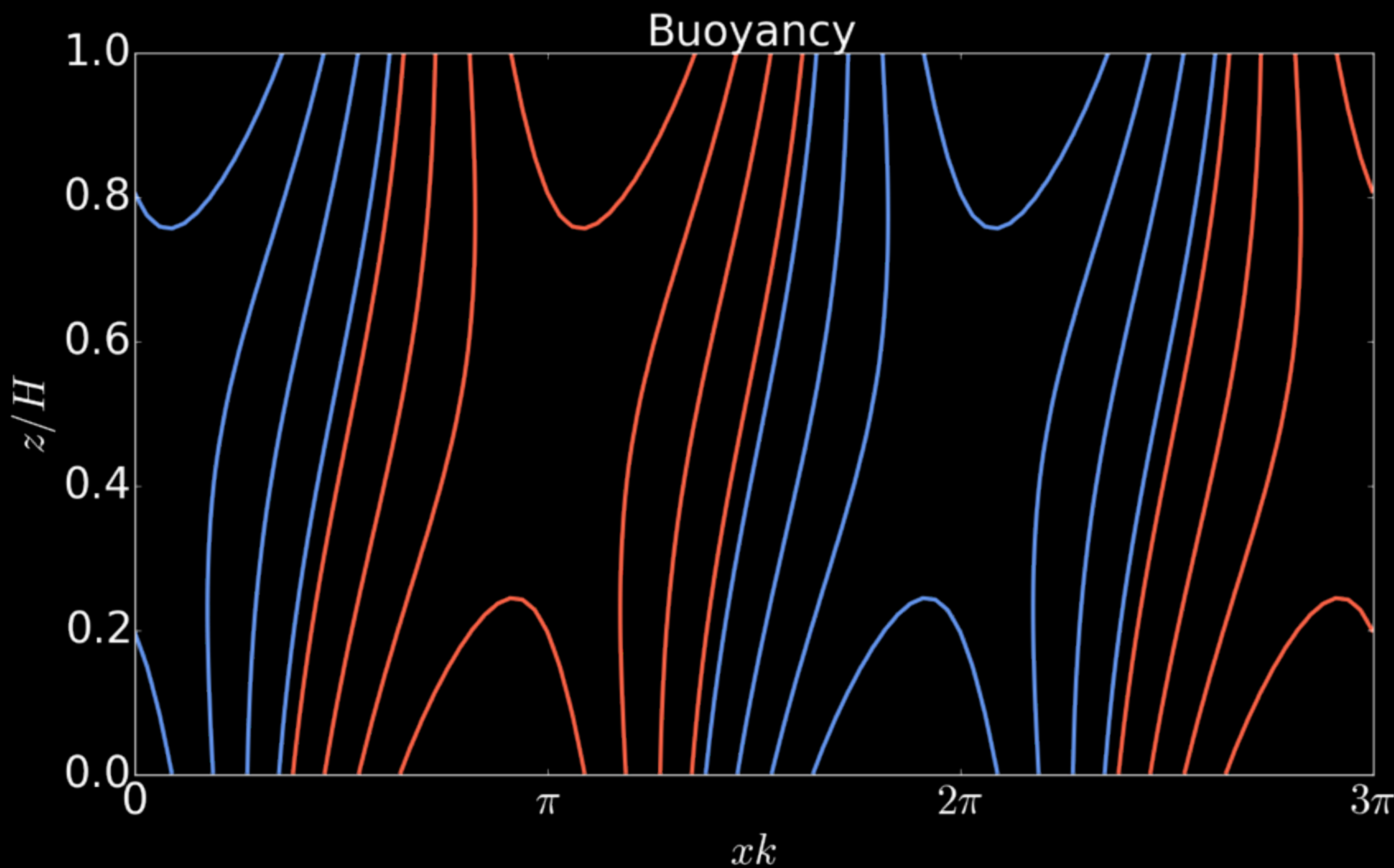
Vertical Structure

$$\psi \sim |\varphi(z)| \cos[kx + \text{Phase}(z) + \text{const.}]$$



Vertical Structure

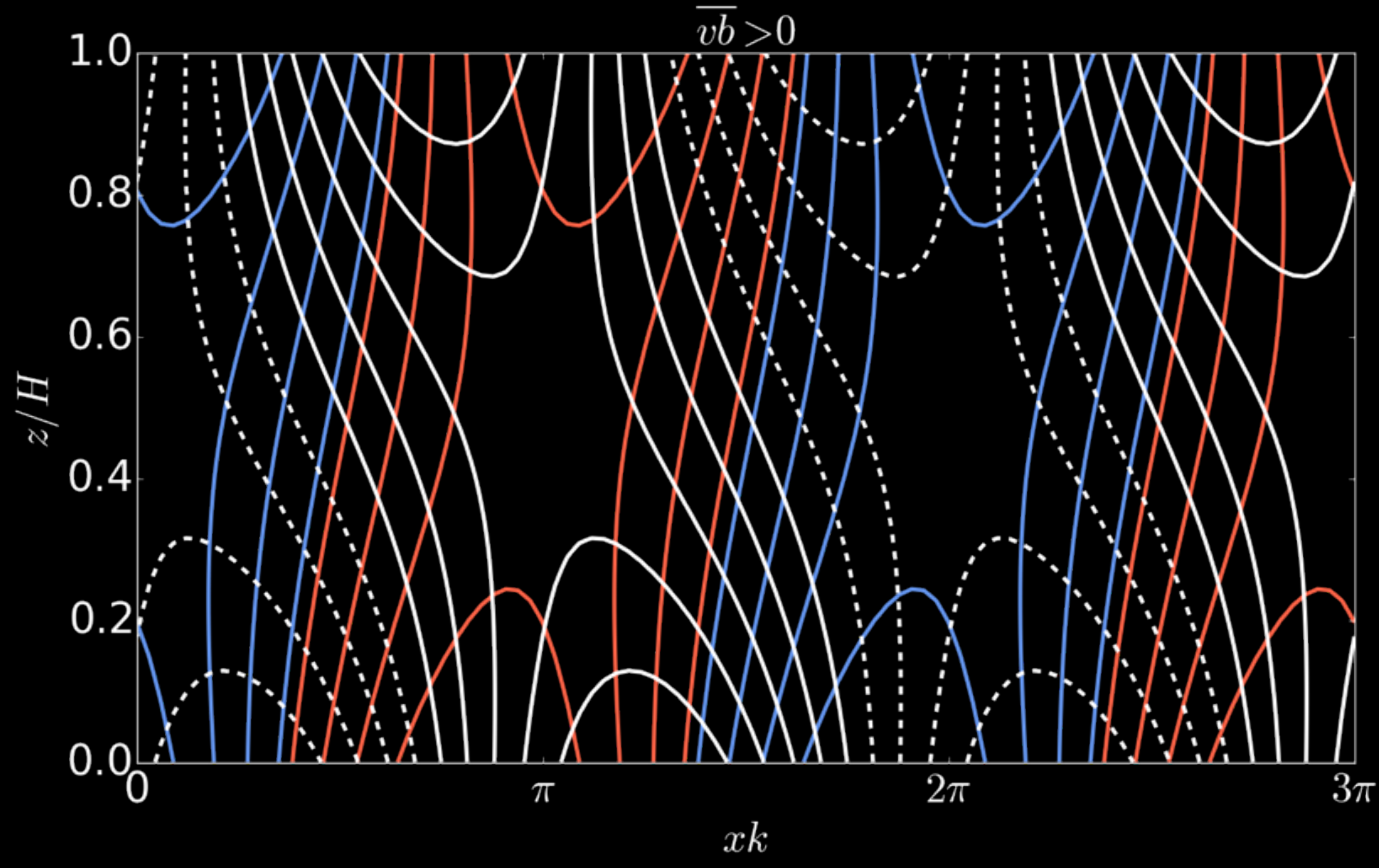
$b \sim \psi_z$



Buoyancy Fluxes

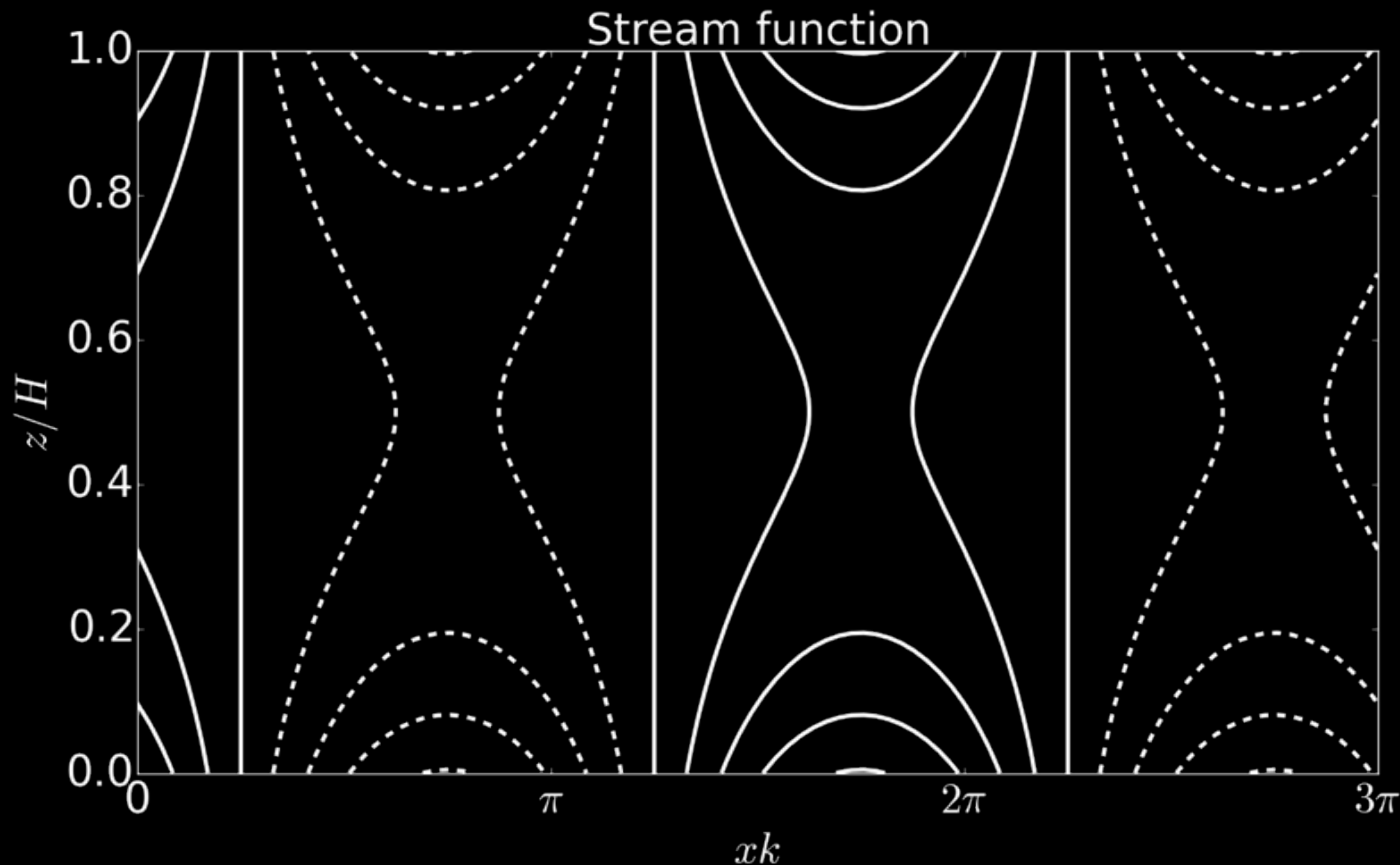
$$\overline{vb} \sim |\varphi(z)|^2 [\mathrm{Phase}(z)]_z > 0$$

Poleward flux



Neutral Modes

$$\psi \sim \varphi(z) cos(kx + \text{const.})$$



Neutral Modes

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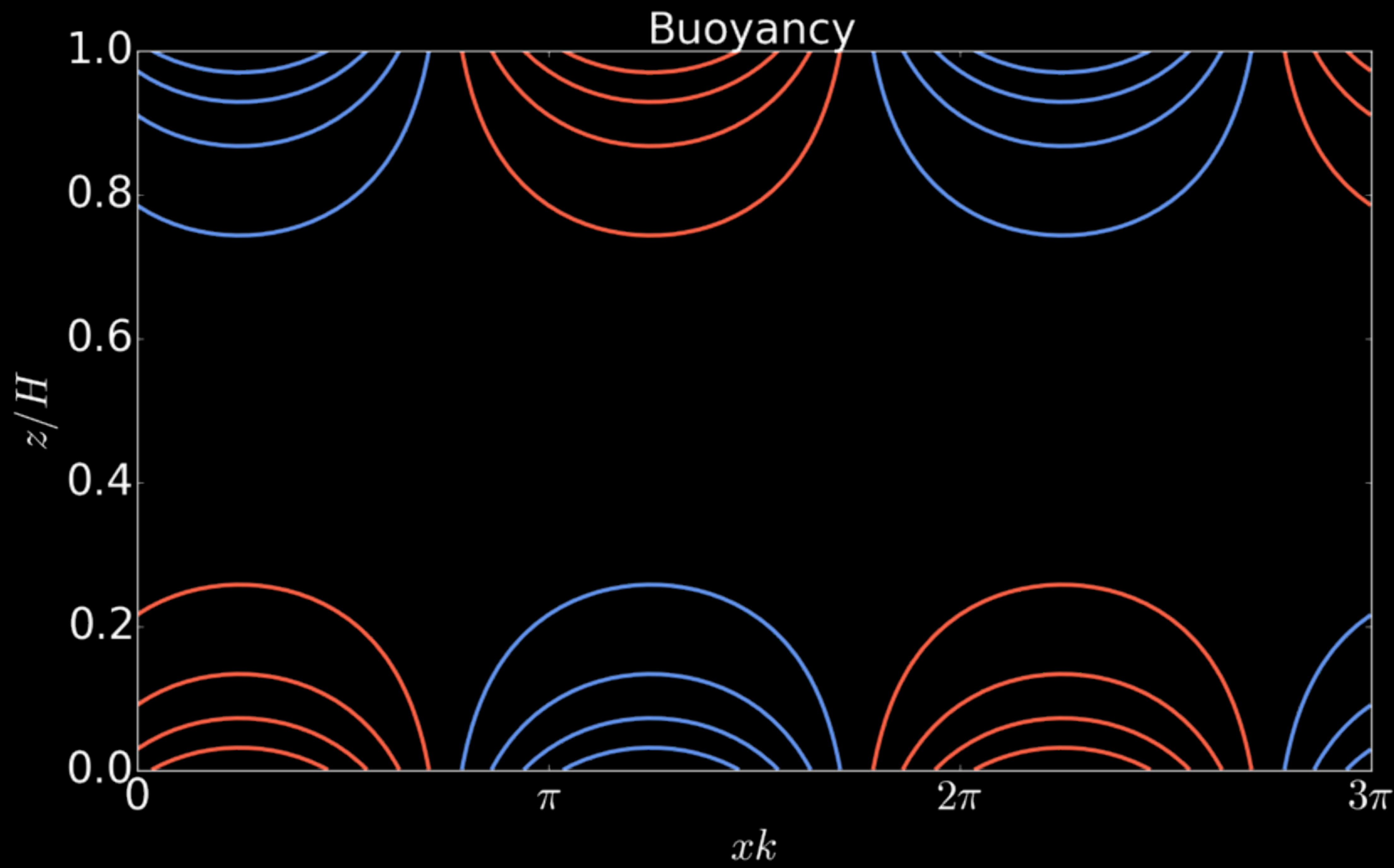
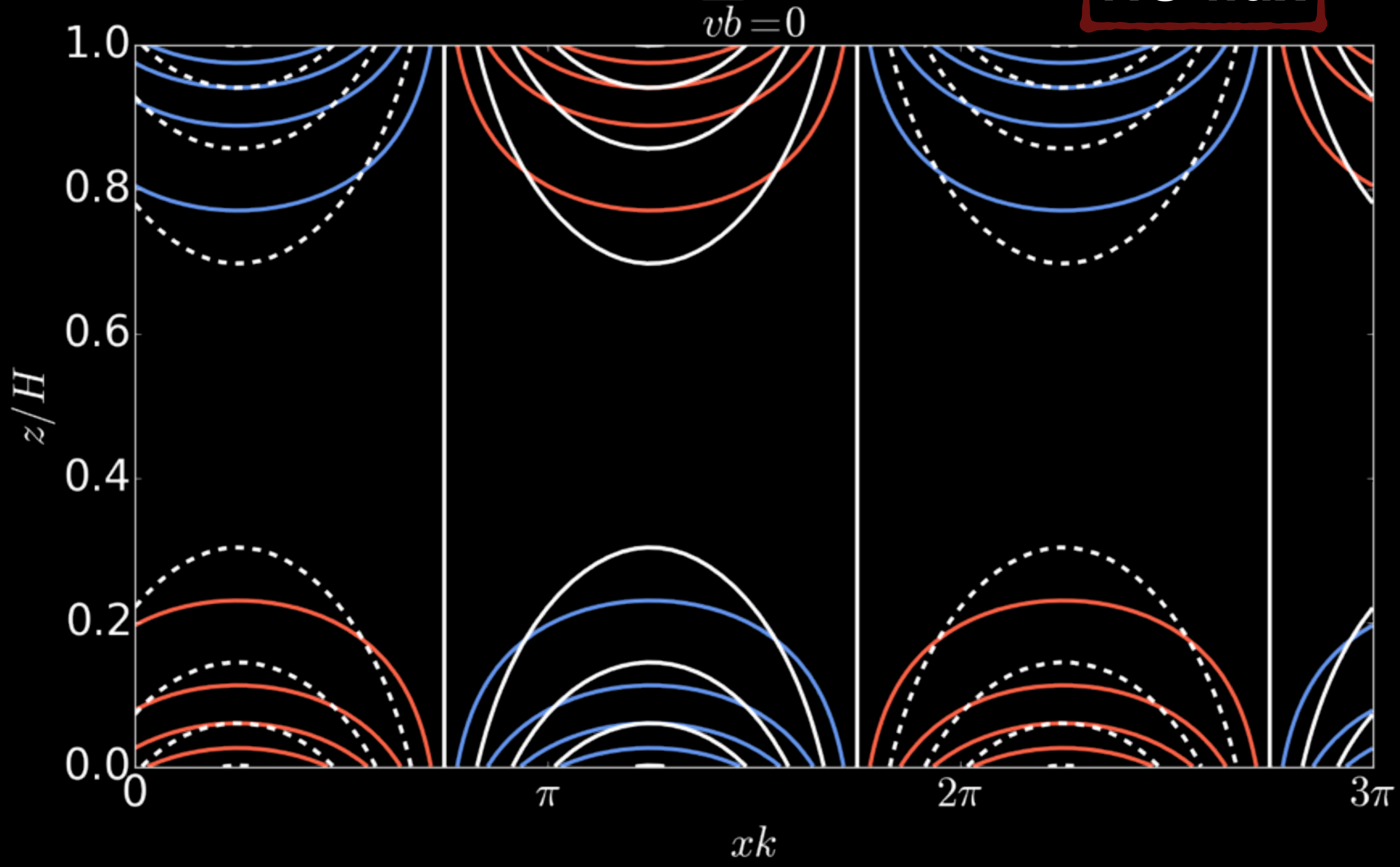


Neutral Modes

NO flux



Some Numbers

$$L_{max} = \frac{2\pi}{k_{max}} \approx 3.9 L_d$$

$$\sigma_{max} \approx 0.3 \frac{U}{L_d}$$

	Atmosphere	Ocean
L_{max}	4000 km	400 km
σ_{max}	0.26 day^{-1}	0.026 day^{-1}
T	4 days	40 days

Vallis 2006

A note on adding β

- Vallis 2006: Numerical solutions
- Lindzen 1994: $$\left[\frac{f_0^2}{N^2(z)}U_z\right]_z = \beta \quad (Q_y = 0)$$

Differential rotation introduces a low wavenumber cutoff

(Similar to two-layer model)

Similarity to Interior Inst.

PV sheets (Bretherton 1966)

$$Q_{upper} = -\frac{f_0^2}{N^2(z)}\frac{\partial \psi}{\partial z}\delta(z - H + \epsilon)$$

$$Q_{lower} = \frac{f_0^2}{N^2(z)}\frac{\partial \psi}{\partial z}\delta(z - \epsilon)$$

$$\int_0^H Q_y dz = \frac{f_0^2}{N^2(z)}\frac{\partial U}{\partial z}\bigg|_{z=H-\epsilon} - \frac{f_0^2}{N^2(z)}\frac{\partial U}{\partial z}\bigg|_{z=\epsilon}$$