SIO 214A Homework 1 Answers

1.) Under the given assumption, the hydrostatic equation reduces to

$$c^2 \frac{d\rho}{dr} = -\rho g \tag{1}$$

with solution

$$\rho(r) = C_0 e^{-gr/c^2} \tag{2}$$

where C_0 is a constant. The constant C_0 is determined by

$$M_A = \int_R^\infty 4\pi r^2 \rho(r) dr \tag{3}$$

where M_A is the total mass of the atmosphere. $H_s = c^2/g$ can be regarded as the depth of the atmosphere in the sense that, according to (2), most of the mass of the atmosphere is contained within a distance H_s of the planet.

In the general case, the hydrostatic equation,

$$c^2 \frac{d\rho}{dr} = -\rho \frac{GM}{r^2} \tag{4}$$

can be solved by separation of variables,

$$\frac{d\rho}{\rho} = -\alpha \frac{dr}{r^2} \tag{5}$$

where

$$\alpha = \frac{GM}{c^2} \tag{6}$$

The solution is

$$\rho(r) = \rho_R e^{\alpha\left(\frac{1}{r} - \frac{1}{R}\right)} \tag{7}$$

where ρ_R is the atmospheric density at $r = R$. Let $r = R + r'$. If $r' << R$ the exponent in (7) takes the form

$$\frac{GM}{c^2 R}\left(\frac{R}{(R+r')} - 1\right) \approx \frac{GM}{c^2 R}\left(1 - \frac{r'}{R} - 1\right) = -\frac{g}{c^2} r' \tag{8}$$

Thus the solution (2) agrees with (7) where $r' << R$ which will be true throughout most of the atmosphere if $H_s << R$. To determine ρ_R in terms of M_A we use (3) again.

2.) For the self-gravitating atmosphere we have

$$F(r) = -\frac{G}{r^2}\int_0^r 4\pi s^2 \rho(s) ds \tag{9}$$

where the integral represents the mass of the atmosphere within the sphere of radius r. Using

$$p(\rho) = K\rho^2, \tag{10}$$

the hydrostatic equation becomes

$$2K\rho\frac{d\rho}{dr} = -\rho\frac{G}{r^2}\int_0^r 4\pi s^2 \rho(s) ds \tag{11}$$

which can be rearranged into the form

$$\beta r^2 \frac{d\rho}{dr} = -\int_0^r s^2 \rho(s) ds \tag{12}$$

where

$$\beta \equiv \frac{K}{2\pi G} \tag{13}$$

taking the derivative of (12) and introducing the new independent variable

$$\xi = \frac{r}{\sqrt{\beta}} \tag{14}$$

we obtain the equation

$$\frac{d}{d\xi}\left(\xi^2 \frac{d\rho}{d\xi}\right) = -\xi^2 \rho(\xi) \tag{15}$$

with solution

$$\rho(\xi) = \rho_0 \frac{\sin\xi}{\xi} \tag{16}$$

where ρ_0 is a constant equal to the density at $r = 0$. Since the density cannot be negative, the atmosphere must end at the first zero-crossing of (16), which is $\xi = \pi$. Thus the atmosphere extends only to

$$r = \sqrt{\frac{\pi K}{2G}} \tag{17}$$

At greater r there is only a vacuum. In this way the self-gravitating solution is dramatically different from the solutions of problem 1, in which $\rho \to 0$ only as $r \to \infty$. Again the central density ρ_0 is determined by the total mass of the atmosphere.