SIO 214A Homework 4 Answers

1.) Answer. To see that $H(x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ is conserved, compute

$$\frac{dH}{dt} = \sum_i \left(\frac{\partial H}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial H}{\partial y_i}\frac{dy_i}{dt} \right) = \sum_i \frac{1}{\Gamma_i}\left(-\frac{\partial H}{\partial x_i}\frac{\partial H}{\partial y_i} + \frac{\partial H}{\partial y_i}\frac{\partial H}{\partial x_i} \right) = 0 \quad (1)$$

The essential thing is that H cannot have any explicit time dependence. For example, if the Γ_i depended on time, H would not be conserved. The other three invariants are easy verified.

2.) Answer. Let (x_1, y_1) be the location of vortex 1, etc. To satisfy the boundary condition, we must have

$$(x_3, y_3) = (x_1, -y_1) \quad (2)$$

and

$$(x_4, y_4) = (x_2, -y_2) \quad (3)$$

at all times. Initially, we are given

$$y_2 = y_1 \quad (4)$$

and by centering the origin of coordinates we can assume

$$x_2 = -x_1 \quad (5)$$

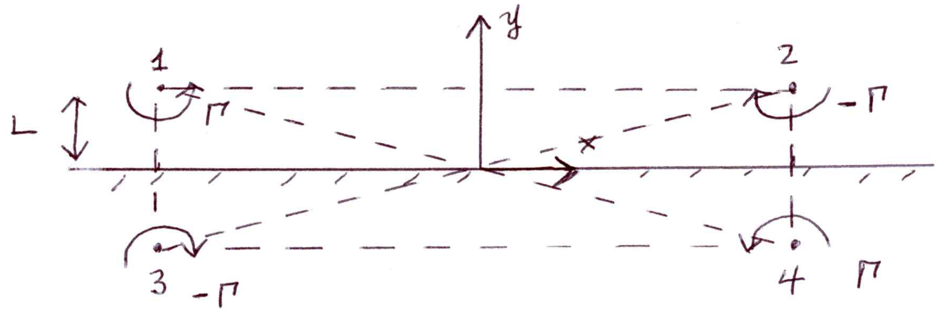
By symmetry, the 6 conditions () will hold at all times. Therefore we can eliminate $x_2, y_2, x_3, y_3, x_4, y_4$ in favor of x_1 and y_1. The conservation laws will allow us to work out the dynamics in terms of these two variables. However, only the conservation law for H will actually be needed. We find that M_x and Ω automatically vanish. For M_y we obtain

$$M_y = \Gamma y_1 + (-\Gamma)y_2 + (-\Gamma)y_3 + \Gamma y_4 = 2\Gamma(y_1 - y_2) \quad (6)$$

which vanishes in the initial state and is therefore always zero. We conclude that

$$y_1 = y_2 \quad (7)$$

at all times. But we knew that already.



The remaining invariant to be considered is the energy H. The sum in H is over the 6 vortex pairs represented by dashed lines in the sketch. Apart from a constant factor, this sum is

$$\ln r_{14} + \ln r_{23} - \ln r_{12} - \ln r_{13} - \ln r_{24} - \ln r_{34} \tag{8}$$

where r_{ij} is the distance between vortex i and vortex j. (The sign in (8) is taken as positive if the two vortices in the pair have the same vorticity, and negative if the vorticities are opposites.) By symmetry,

$$r_{14} = r_{23} \tag{9}$$

$$r_{12} = r_{34} \tag{10}$$

$$r_{13} = r_{24} \tag{11}$$

at all times. Thus (8) becomes

$$2\ln r_{14} - 2\ln r_{12} - 2\ln r_{13} = 2\ln\left(\frac{r_{14}}{r_{12}r_{13}}\right) \tag{12}$$

Since by symmetry

$$r_{14}^2 = 4(x_1^2 + y_1^2), \quad r_{12}^2 = 4x_1^2, \quad r_{13}^2 = 4y_1^2 \tag{13}$$

we finally conclude that

$$\frac{x_1^2 + y_1^2}{x_1^2 y_1^2} = C \tag{14}$$

where C is a constant. By the initial condition that $y_1^2 = L^2$ as $x_1^2 \to \infty$, we find that $C = 1/L^2$. Thus the path of vortex 1 is given by

$$x^2y^2 = L^2(x^2 + y^2) \tag{15}$$

with $x < 0$. The path of vortex 2 obeys the same equation but with $x > 0$. These paths resemble hyperbolas. The closest approach of either vortex to the origin occurs when $x^2 = y^2 = 2L^2$. At the time of closest approach, both vortices are a distance $2L$ from the origin.

It is a bit harder to determine the *time* at which the vortex occupies a particular point along its path, but it is obvious that it can be done: If you know $y = y(x)$ and $dx/dt = f(x, y(x))$, you can separate variables and integrate to find $t = t(x)$.