4
Vorticity and Turbulence

Turbulence is an immense and controversial subject. The next three chapters present some ideas from turbulence theory that seem relevant to flow in the oceans and atmosphere. In this chapter, we examine the connections between vorticity and turbulence.

1. *The vorticity equation*

From ocean models that omit inertia, we turn to flows in which the inertia is a dominating factor. Vorticity is of central importance, and, in the case of three-dimensional motion, we must take its vector character fully into account. We begin with the equations

$$
\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + 2\,\boldsymbol{\Omega}\times\mathbf{v} = -\frac{1}{\rho}\nabla p - \nabla\Phi(\mathbf{x}) \\
&p = p(\rho,\eta) \\
&\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \\
&\frac{\partial \eta}{\partial t} + (\mathbf{v}\cdot\nabla)\eta = 0
\end{aligned}
\qquad (1.1)
$$

for a perfect fluid in rotating coordinates. Here, $\Phi(\mathbf{x})$ is the potential for external forces, η is the specific entropy, and the other symbols have their usual meanings. By the general vector identity,

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}), \qquad (1.2)$$

we have

$$\nabla(\mathbf{v}\cdot\mathbf{v}) = 2(\mathbf{v}\cdot\nabla)\mathbf{v} + 2\,\mathbf{v}\times\boldsymbol{\omega}, \qquad (1.3)$$

where

$$\boldsymbol{\omega} \equiv \nabla\times\mathbf{v} \qquad (1.4)$$

is the *vorticity*. Thus, we can rewrite the momentum equation (1.1a) in the form

$$\frac{\partial \mathbf{v}}{\partial t} + (\boldsymbol{\omega} + 2\,\boldsymbol{\Omega})\times\mathbf{v} = -\nabla P + p\nabla\left(\frac{1}{\rho}\right), \qquad (1.5)$$

where

$$P \equiv \frac{p}{\rho} + \tfrac{1}{2}\mathbf{v}\cdot\mathbf{v} + \Phi\,. \qquad (1.6)$$

Introducing the *absolute* velocity and vorticity,

$$\mathbf{v}_a \equiv \mathbf{v} + \boldsymbol{\Omega}\times\mathbf{r} \quad \text{and} \quad \boldsymbol{\omega}_a \equiv \nabla\times\mathbf{v}_a = \boldsymbol{\omega} + 2\boldsymbol{\Omega} \qquad (1.7)$$

(respectively) in the *nonrotating* coordinate system, we can write (1.5) more compactly as

$$\frac{\partial\mathbf{v}}{\partial t} + \boldsymbol{\omega}_a\times\mathbf{v} = -\nabla P + p\nabla\left(\frac{1}{\rho}\right). \qquad (1.8)$$

We form the vorticity equation by taking the curl of (1.8). By another general vector identity,

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}\,, \qquad (1.9)$$

we have

$$\nabla\times(\boldsymbol{\omega}_a\times\mathbf{v}) = \boldsymbol{\omega}_a\,(\nabla\cdot\mathbf{v}) + 0 + (\mathbf{v}\cdot\nabla)\boldsymbol{\omega}_a - (\boldsymbol{\omega}_a\cdot\nabla)\mathbf{v}, \qquad (1.10)$$

(since the divergence of a curl always vanishes). Thus the curl of (1.8) is

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\boldsymbol{\omega}_a + \boldsymbol{\omega}_a\,(\nabla\cdot\mathbf{v}) = (\boldsymbol{\omega}_a\cdot\nabla)\mathbf{v} + \nabla p\times\nabla\frac{1}{\rho} \qquad (1.11)$$

Then, eliminating $\nabla\cdot\mathbf{v}$ between (1.11) and the continuity equation (1.1c), we finally obtain

$$\frac{D}{Dt}\,(\boldsymbol{\omega}_a/\rho) = [(\boldsymbol{\omega}_a/\rho)\cdot\nabla]\,\mathbf{v} + \frac{1}{\rho}\nabla p\times\nabla\frac{1}{\rho} \qquad (1.12)$$

Eqn. (1.12) is the general vorticity equation for a perfect fluid. In the special case of *homentropic flow*, in which the pressure depends only on the density, $p=p(\rho)$, the last term in (1.12) vanishes, and (1.12) reduces to

$$\frac{D\mathbf{w}}{Dt} = (\mathbf{w}\cdot\nabla)\mathbf{v}, \qquad (p = p(\rho))\,, \qquad (1.13)$$

where

$$\mathbf{w} \equiv \boldsymbol{\omega}_a/\rho \tag{1.14}$$

is the ratio of the absolute vorticity to the density.[1] In the very special case of a constant-density fluid, (1.13) reduces to

$$D\boldsymbol{\omega}_a/Dt = (\boldsymbol{\omega}_a\cdot\nabla)\,\mathbf{v}, \qquad (\rho{=}const.) \tag{1.15}$$

2. *Ertel's theorem*

According to the general vorticity equation (1.12), namely,

$$\frac{D\mathbf{w}}{Dt} = (\mathbf{w}\cdot\nabla)\mathbf{v} + \frac{\nabla\rho\times\nabla p}{\rho^3}, \tag{2.1}$$

the quotient $\mathbf{w}{=}\boldsymbol{\omega}_a/\rho$ is conserved on fluid particles *except* for the terms on the right-hand side of (2.1). We shall see that the first of these terms, $(\mathbf{w}\cdot\nabla)\mathbf{v}$, represents the *tilting* and *stretching* of $\mathbf{w}$. The last term in (2.1) represents *pressure-torque*. The pressure-torque vanishes if the fluid is homentropic. We consider that case first.

If the fluid is homentropic, then (2.1) reduces to (1.13). To understand (1.13), let

$$\delta\mathbf{r}(t) = \mathbf{r}_2(t) - \mathbf{r}_1(t) \tag{2.2}$$

be the infinitesimal displacement between two moving fluid particles with position vectors $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$. Then

$$\frac{d}{dt}\delta\mathbf{r}(t) = \frac{d}{dt}\mathbf{r}_2(t) - \frac{d}{dt}\mathbf{r}_1(t). \tag{2.3}$$

If $\delta\mathbf{r}$ is small, a Taylor-expansion of (2.3) yields

$$\frac{d}{dt}\delta r_i(t) = v_i(\mathbf{r}_1 + \delta\mathbf{r}) - v_i(\mathbf{r}_1) = \frac{\partial v_i}{\partial x_j}\delta r_j, \tag{2.4}$$

where the subscripts denote components, and *repeated* subscripts denote summation from 1 to 3. By comparing (2.4) to (1.13) in the form

$$\frac{Dw_i}{Dt} = \frac{\partial v_i}{\partial x_j} w_j, \tag{2.5}$$

we see that the vector field $\mathbf{w}{=}\boldsymbol{\omega}_a/\rho$ *obeys the same equation as a field of infinitesimal displacement vectors between fluid particles*. We say that $\boldsymbol{\omega}_a/\rho$ is *frozen into the fluid*. However, since the velocity field is continuous, the distortion experienced by $\boldsymbol{\omega}_a/\rho$ is *continuous*. $\boldsymbol{\omega}_a/\rho$ can never be torn apart. Its topology is preserved despite distortion.

These properties, so evident from the analogy between $\boldsymbol{\omega}_a/\rho$ and $\delta\mathbf{r}$, lie at the heart of the many vorticity theorems in fluid mechanics.

The most important of these is Ertel's theorem. Still considering the case of homentropic flow, let $\theta(\mathbf{x},t)$ be any scalar conserved on fluid particles,

$$\frac{D\theta}{Dt}=0\,. \tag{2.6}$$

The scalar θ need not have physical significance; it could be an arbitrarily defined *passive tracer*. Let $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ be defined as before. Then (2.6) implies that

$$\frac{d}{dt}\theta(\mathbf{r}_1(t),t)-\frac{d}{dt}\theta(\mathbf{r}_2(t),t)=0\,. \tag{2.7}$$

If the distance between the two fluid particles is infinitesimal, then (2.7) becomes

$$\frac{d}{dt}\left[\frac{\partial\theta}{\partial x_j}\delta r_j\right]=0\,. \tag{2.8}$$

Now let

$$\delta r_i(0)=\gamma\, w_i(\mathbf{r}_1(0),0) \tag{2.9}$$

where $\mathbf{w}(\mathbf{r}_1,0)$ is the initial $\mathbf{w}$ at the location $\mathbf{r}_1$, and γ is an infinitesimal constant with appropriate dimensions. In other words, choose the two fluid particles to lie infinitesimally far apart along a line *parallel to the vorticity*. Then, since $\mathbf{w}$ and $\delta\mathbf{r}$ obey the same equation,

$$\delta r_i(t)=\gamma\, w_i(\mathbf{r}_1,t) \tag{2.10}$$

at later times t. Since γ is a constant, it follows from (2.8) and (2.10) that

$$\frac{D}{Dt}\left[\frac{\partial\theta}{\partial x_j}w_j\right]=\frac{D}{Dt}[\nabla\theta\cdot\mathbf{w}]=0 \qquad \text{(homentropic flow)}. \tag{2.11}$$

Eqn. (2.11) is *Ertel's theorem* for homentropic fluid. According to (2.11), homentropic flow conserves $\nabla\theta\cdot\boldsymbol{\omega}_a/\rho$ on fluid particles, where θ is *any* conserved scalar satisfying (2.6). This derivation shows that (2.11) rests on nothing besides the *frozen-in* nature of the field $\boldsymbol{\omega}_a/\rho$.

In the general case of *non-homentropic flow*, (2.11) generalizes easily to

$$\frac{D}{Dt}[\nabla\theta\cdot\mathbf{w}]=\frac{\nabla\theta\cdot(\nabla\rho\times\nabla p)}{\rho^3} \tag{2.12}$$

and $\nabla\theta\cdot\mathbf{w}$ is *not* conserved on fluid particles. The right-hand side of (2.12) arises from the pressure-torque in (2.1). However, if we choose the scalar θ to be (any function of) the entropy η (which satisfies (2.6)), then the right-hand side of (2.12) vanishes (because $p=p(\rho,\eta)$), and (2.12) reduces to

$$\frac{DQ}{Dt} = 0, \tag{2.13}$$

where

$$Q \equiv (\boldsymbol{\omega}_a \cdot \nabla\eta) / \rho \tag{2.14}$$

is the *potential vorticity*. Eqn. (2.13), also called Ertel's theorem, is the most general statement of potential vorticity conservation. The potential vorticity laws obtained in previous chapters (from various approximations to (2.1)) can all be be viewed as approximations to (2.13-14).

Of course, we can prove all these results directly from (1.1) by pedestrian mathematical manipulations, but that makes it harder to appreciate their physical significance.

3. *A deeper look at potential vorticity*

Again assume that the fluid is homentropic. Let $\theta_1(\mathbf{x},t)$, $\theta_2(\mathbf{x},t)$, and $\theta_3(\mathbf{x},t)$ be any three *independent* (but otherwise arbitrary) scalars satisfying

$$\frac{D\theta_1}{Dt} = 0, \qquad \frac{D\theta_2}{Dt} = 0, \qquad \frac{D\theta_3}{Dt} = 0. \tag{3.1}$$

By *independent* we mean that $\nabla\theta_1$, $\nabla\theta_2$, and $\nabla\theta_3$ everywhere point in different directions. It is easy to see that if the θ_i are *initially* independent, then they remain so by (3.1). (One possible choice for the θ_i would be the initial Cartesian components of the fluid particles.) Since the fluid is homentropic, Ertel's theorem (2.11) tells us that

$$\frac{DQ_1}{Dt} = 0, \qquad \frac{DQ_2}{Dt} = 0, \qquad \frac{DQ_3}{Dt} = 0, \tag{3.2}$$

where

$$Q_1 = \mathbf{w}\cdot\nabla\theta_1, \qquad Q_2 = \mathbf{w}\cdot\nabla\theta_2, \qquad Q_3 = \mathbf{w}\cdot\nabla\theta_3 \tag{3.3}$$

are the potential vorticities corresponding to the θ_i.

Since the θ_i are independent, we can regard them as *curvilinear coordinates* in *xyz*-space. By (3.1), these curvilinear coordinates are also *Lagrangian coordinates,* because the surfaces of constant θ_i move with the fluid. We regard the vectors $\nabla\theta_1, \nabla\theta_2$, and $\nabla\theta_3$ as *basis vectors* attached to the Lagrangian coordinates. As the fluid moves, these basis vectors tilt and stretch with the flow. By (3.3), the conserved Q_i are just the dot-products (3.3) of **w** with these moving basis vectors. The dot-products are conserved because the tilting and stretching terms on the right-hand side of (1.13), which destroy the conservation of **w**, are taken into account by the motion of the basis vectors $\nabla\theta_i$.[2]

Now let $\mathbf{A}=(A_1, A_2, A_3)$ be the components of the (absolute) velocity $\mathbf{v}_a$ with respect to these same basis vectors. That is, let

$$\mathbf{v}_a = A_1\nabla\theta_1 + A_2\nabla\theta_2 + A_3\nabla\theta_3 . \tag{3.4}$$

We shall show that, with a very weak further restriction on the choice of θ_i,

$$\mathbf{Q} \equiv (Q_1, Q_2, Q_3) = \nabla_\theta \times \mathbf{A} , \tag{3.5}$$

where

$$\nabla_\theta \equiv \left(\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}, \frac{\partial}{\partial\theta_3} \right) \tag{3.6}$$

is the gradient operator in the Lagrangian coordinates. That is, the conserved potential vorticity **Q** is the curl of the absolute velocity $\mathbf{v}_a$ in Lagrangian coordinates. Then Ertel's theorem (3.2) can be written in the suggestive form

$$\frac{D}{Dt}(\nabla_\theta \times \mathbf{A}) = 0 . \tag{3.7}$$

Hence, *the potential vorticity* (3.5) *is just ordinary vorticity measured in Lagrangian coordinates*. If the fluid is homentropic, then (3.7) implies that the potential vorticity is simply a static vector field,

$$\nabla_\theta \times \mathbf{A} = \mathbf{F}(\theta_1, \theta_2, \theta_3) , \tag{3.8}$$

in $\theta_1, \theta_2, \theta_3$-space, where **F** is determined by the initial conditions. A translation of (3.8) into conventional notation yields what some writers call *Cauchy's solution of the vorticity equation*.

To show that (3.5) agrees with (3.3), we suppress the subscript a on $\boldsymbol{\omega}_a$ and $\mathbf{v}_a$, and compute

$$Q_r = (\boldsymbol{\omega}\cdot\nabla\theta_r)/\rho = \frac{1}{\rho}\varepsilon_{ijk}\frac{\partial v_k}{\partial x_j}\frac{\partial\theta_r}{\partial x_i} = \frac{1}{\rho}\varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(A_s \frac{\partial\theta_s}{\partial x_k} \right)\frac{\partial\theta_r}{\partial x_i} = \frac{1}{\rho}\varepsilon_{ijk}\frac{\partial\theta_r}{\partial x_i}\frac{\partial A_s}{\partial x_j}\frac{\partial\theta_s}{\partial x_k} . \tag{3.9}$$

Thus

$$Q_r = \frac{1}{\rho}\frac{\partial(\theta_r, A_s, \theta_s)}{\partial(x,y,z)} = \frac{1}{\rho}\frac{\partial(\theta_1,\theta_2,\theta_3)}{\partial(x,y,z)}\frac{\partial(\theta_r, A_s, \theta_s)}{\partial(\theta_1,\theta_2,\theta_3)}$$
$$= \frac{1}{\rho}\frac{\partial(\theta_1,\theta_2,\theta_3)}{\partial(x,y,z)}\varepsilon_{ijk}\frac{\partial\theta_r}{\partial\theta_i}\frac{\partial A_s}{\partial\theta_j}\frac{\partial\theta_s}{\partial\theta_k} = \frac{1}{\rho}\frac{\partial(\theta_1,\theta_2,\theta_3)}{\partial(x,y,z)}\varepsilon_{rjs}\frac{\partial A_s}{\partial\theta_j} \tag{3.10}$$

That is,

$$Q_r = \frac{1}{\rho}\frac{\partial(\theta_1,\theta_2,\theta_3)}{\partial(x,y,z)}\left[\nabla_\theta \times \mathbf{A}\right]_r . \tag{3.11}$$

If the Lagrangian coordinates are *mass-labeling coordinates* in the sense of Chapter 1, that is, if

$$d\theta_1\, d\theta_2\, d\theta_3 = d(mass), \tag{3.12}$$

then (3.11) reduces to (3.5). (In Chapter 1 we used the symbols *a,b,c* to denote mass-labelling coordinates, and $\partial/\partial\tau$ to denote D/Dt.)

In general *non-homentropic* flow, the pressure-torque on the right-hand side of (2.1) destroys two of the three components of the conservation law (3.7). In that case, it is convenient to take the entropy η as one of the Lagrangian coordinates. Then, since the pressure-torque in (2.1) has no component in the direction of $\nabla\eta$, the η-component of (3.7) survives,

$$\frac{D}{Dt}\left[(\nabla_\theta \times \mathbf{A})\cdot\nabla_\theta\eta\right] = 0 . \tag{3.13}$$

By steps similar to those in (3.9) and (3.10), we can show that the conserved quantity in (3.13) is the general potential vorticity (2.14).

Although (3.13) contains only one-third of the dynamical information in (3.7), it is — in strongly stratified flow — a much more useful equation. In unstratified ($\nabla\eta$=0) flow, the θ_i-surfaces typically become very convoluted, and the simplicity of the Lagrangian equation (3.7) is offset by the complexity of transforming this result back into *xyz*-coordinates. However, in strongly stratified flow, the gravitational restoring forces resist the folding of isentropic surfaces, rendering the single equation (3.13) much more useful. Moreover, if the fluid is rapidly rotating, then (3.13) controls the nearly geostrophic part of the motion (as we have seen in Chapter 2).

4. *Alternative statements of the vorticity law*

As we have seen, the quotient $\boldsymbol{\omega}_a/\rho$ is conserved on fluid particles except for the effects of tilting, stretching and pressure-torque. However, the effects of tilting and stretching can be absorbed into a Lagrangian description of the motion. Then only pressure-torque stands in the way of conservation. In this section, we examine alternative (and more conventional) ways of saying these same things.

First, consider the *circulation*

$$C \equiv \oint \mathbf{v} \cdot d\mathbf{r} \,, \tag{4.1}$$

where the integration is around a closed *material* loop of fluid particles, that is, around a loop that always contains the same fluid particles. By Stokes's theorem

$$C = \iint \boldsymbol{\omega} \cdot \mathbf{n} \, dA \,, \tag{4.2}$$

where **n** is the normal to an arbitrary surface containing the loop. If the fluid is rotating, we also define the circulation relative to the *inertial* reference frame,

$$C_a \equiv \oint \mathbf{v}_a \cdot d\mathbf{r} = \iint (\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \mathbf{n} \, dA \,. \tag{4.3}$$

Now, by the momentum equation (1.1a) for a rotating fluid,

$$\begin{aligned}
\frac{dC_a}{dt} &= \oint \frac{D}{Dt}\left[(\mathbf{v} + \Omega \times \mathbf{r}) \cdot d\mathbf{r} \right] \\
&= \oint \left[\left(\frac{D\mathbf{v}}{Dt} + \Omega \times \mathbf{v} \right) \cdot d\mathbf{r} + (\mathbf{v} + \Omega \times \mathbf{r}) \cdot d\mathbf{v} \right] \\
&= \oint \left(\frac{D\mathbf{v}}{Dt} + 2\Omega \times \mathbf{v} \right) \cdot d\mathbf{r} \\
&= \oint \left(-\frac{1}{\rho} \nabla p - \nabla \Phi \right) \cdot d\mathbf{r} \\
&= -\oint \frac{dp}{\rho}
\end{aligned} \tag{4.4}$$

If the fluid is homentropic, then $p = p(\rho)$ and the right-hand side of (4.4) vanishes; the circulation (4.3) is conserved.

We see that the effects of vortex tilting and stretching are *built in* to the definition of circulation. The material loop of fluid particles tilts and stretches with the motion of the fluid. Only the pressure-torque, represented by the last term in (4.4), causes the circulation to change. And, as anticipated by our discussion of Ertel's theorem, even the pressure-torque does not *entirely* destroy the conservation of circulation; the circulation is conserved if we choose the material loop of fluid particles to lie entirely within a surface of constant entropy.

The concept of *vortex tubes* offers another way to describe the *frozen-in* evolution of the vorticity field. Suppose that the fluid is *nonrotating*. (The extension to rotating coordinates is easy.) At a fixed time *t*, choose an arbitrary closed loop within the fluid, and consider the tube formed by its indefinite extension in the direction of the vorticity

$\boldsymbol{\omega}$. Refer to Figure 4.1. At the same fixed time, consider two loops, L_1 and L_2, around the vortex tube. Since $\boldsymbol{\omega}$ is everywhere tangent to the sides of the vortex tube, and

$$\nabla \cdot \boldsymbol{\omega} = 0, \tag{4.5}$$

the divergence theorem tells us that

$$\iint \boldsymbol{\omega} \cdot \mathbf{n}_1 \, dA_1 = \iint \boldsymbol{\omega} \cdot \mathbf{n}_2 \, dA_2 \,, \tag{4.6}$$

where $\mathbf{n}_i$ are the unit normals to surfaces containing the loops L_i, and dA_i are the corresponding area elements. Thus, the *strength* of the vortex tube is the same at every cross-section.

Now suppose that the fluid is *homentropic*. Then the vortex tube is a *material* volume that moves with the fluid particles composing it; by the analogy between $\boldsymbol{\omega}/\rho$ and the infinitesimal displacement vector between fluid particles on the surface of the vortex tube, $\boldsymbol{\omega}$ *remains* tangent to the *moving* surface of the vortex tube. Hence the strength of the vortex tube *remains* uniform along the tube. Moreover, the circulation theorem tells us that

$$\frac{d}{dt} \iint \boldsymbol{\omega} \cdot \mathbf{n} \, dA = 0, \tag{4.7}$$

so that the strength of the vortex tube is also constant *in time*. The vortex tube can stretch, increasing its vorticity, but the cross-sectional area then experiences a compensating decrease. Once again, the effects of tilting and stretching have been built into a definition in order to produce a conservation law.

We can think of any homentropic flow as a (generally complicated) tangle of vortex tubes. (Think of a big pile of spaghetti, with each noodle a closed loop.) As the flow evolves, these vortex tubes experience a continuous distortion, but (in the absence of friction) their strength *and their topology* are obviously preserved.

Helicity is a vorticity invariant that reflects the topology. Let V be a closed *material* volume of homentropic fluid whose surface is (and remains) everywhere tangent to the vorticity $\boldsymbol{\omega}$. That is, let V be a collection of closed vortex tubes. Then the *helicity*,

$$H(t) \equiv \iiint_V \mathbf{v} \cdot \boldsymbol{\omega} \, dV \tag{4.8}$$

is conserved,

$$\frac{dH}{dt} = 0 \,. \tag{4.9}$$

This follows from

$$
\begin{aligned}
\frac{DH}{Dt} &= \frac{D}{Dt}\iiint_V \mathbf{v}\cdot(\boldsymbol{\omega}/\rho)\,\rho\,dV \\
&=\iiint_V \frac{D}{Dt}[\,\mathbf{v}\cdot(\boldsymbol{\omega}/\rho)\,]\,\rho\,dV \\
&=\iiint_V \left[\frac{D}{Dt}\mathbf{v}\cdot(\boldsymbol{\omega}/\rho) + \mathbf{v}\cdot\frac{D}{Dt}(\boldsymbol{\omega}/\rho)\right]\rho\,dV \\
&=\iiint_V [\,-\nabla P\cdot\boldsymbol{\omega} + \mathbf{v}\cdot(\boldsymbol{\omega}\cdot\nabla)\,\mathbf{v}\,]\,dV \\
&=\iiint_V [\,-\nabla\cdot(P\boldsymbol{\omega}) + \tfrac{1}{2}\nabla\cdot(\,\boldsymbol{\omega}\,\mathbf{v}\cdot\mathbf{v})\,]\,dV = 0,
\end{aligned}
\tag{4.10}
$$

where $dP=dp/\rho + d\Phi$. The last line vanishes because $\boldsymbol{\omega}$ is tangent to the surface of V.

The helicity H turns out to be a measure of the *knotted-ness* of the material volume of vortex tubes.[3] Consider, for example, two thin vortex tubes (represented abstractly as lines) with volumes V_1 and V_2, that are linked together as shown in Figure 4.2. The vortex lines within each tube are simple, parallel (*i.e.* untwisted) closed curves. The arrows point along the tubes in the direction of the vorticity, $\mathbf{n}_i$ are unit normals to surfaces S_i containing the axes of the tubes, and the vorticity outside the tubes is assumed to vanish. By definition,

$$
H = \iiint \mathbf{v}\cdot\boldsymbol{\omega}\,dV_1 + \iiint \mathbf{v}\cdot\boldsymbol{\omega}\,dV_2. \tag{4.11}
$$

But

$$
\begin{aligned}
\iiint \mathbf{v}\cdot\boldsymbol{\omega}\,dV_1 &= \oint dr_1 \iint dA_1\,\mathbf{v}\cdot\boldsymbol{\omega} \\
&= \oint dr_1\,\mathbf{v}\cdot\iint dA_1\,\boldsymbol{\omega} = \oint d\mathbf{r}_1\cdot\mathbf{v}\iint dA_1\,\omega = \kappa_1 \oint d\mathbf{r}_1\cdot\mathbf{v}
\end{aligned}
\tag{4.12}
$$

where $\omega=|\boldsymbol{\omega}|$ and

$$
\kappa_1 \equiv \iint \omega\,dA_1 \tag{4.13}
$$

is the (constant) strength of vortex tube 1. On the other hand,

$$
\oint d\mathbf{r}_1\cdot\mathbf{v} = \iint \boldsymbol{\omega}\cdot\mathbf{n}_1\,dS_1 = \kappa_2\,, \tag{4.14}
$$

where κ_2 is the strength of vortex tube 2. Thus, (4.12) becomes

$$
\iiint \mathbf{v}\cdot\boldsymbol{\omega}\,dV_1 = \kappa_1\,\kappa_2. \tag{4.15}
$$

By similar steps,

$$\iiint \mathbf{v} \cdot \boldsymbol{\omega} \, dV_2 = \kappa_2 \, \kappa_1 . \tag{4.16}$$

Hence the helicity (4.11) is

$$H = 2\kappa_1 \, \kappa_2 . \tag{4.17}$$

If the vortex tubes were *not* linked, we would find that H=0. If the vorticity in one of the tubes were reversed, then H would change sign. If both tubes were reversed, then H would be unchanged, but the resulting configuration is simply a rotated version of the sketch in Figure 4.2.

Again we emphasize that all these vorticity laws are direct consequences of the frozen-in nature of vorticity in the case of homentropic flow. Ertel's theorem, which amounts to a transformation of the vorticity equation into Lagrangian coordinates, is the most illuminating of these vorticity laws, but helicity conservation is perhaps the most exotic. However, helicity conservation applies only to material volumes of closed vortex tubes, and thus excludes those portions of the fluid whose vortex tubes terminate at boundaries. Moreover, although there is a helicity invariant corresponding to every subvolume of closed vortex tubes, it is easy to imagine a very complicated vorticity distribution in which a single vortex line passes arbitrarily close to every point in the fluid. Then the only subdomain of closed vortex tubes is the *whole* fluid, and (because vortex tube linkages with opposite signs produce cancelling contributions to the helicity) the single helicity invariant cannot tell us very much about the topology of the vorticity field.

5. *Turbulence*

Every aspect of turbulence is controversial. Even the *definition* of fluid turbulence is a subject of disagreement. However, nearly everyone would agree with some elements of the following *description*:

(1.) Turbulence is associated with vorticity. In any case, the existence of vorticity is surely a prerequisite for turbulence in the sense that irrotational flow is smooth and steady to the extent that the boundary conditions permit.[4]

(2.) Turbulent flow has a very *complex* structure, involving a broad range of space- and time-scales.

(3.) Turbulent flow fields exhibit a high degree of *apparent randomness* and *disorder*. However, close inspection often reveals the presence of orderly embedded flow structures (sometimes called *coherent structures*).

(4.) Turbulent flows are three-dimensional (unless *constrained* to be two-dimensional by strong rotation or stratification), and have a high rate of viscous energy *dissipation*.

(5.) Advected tracers are *rapidly mixed* by turbulent flow.

(6.) Turbulent flow fields often exhibit high levels of *intermittency*. (Roughly speaking, a flow is intermittent if its variability is dominated by infrequent large events.)

However, one further property of turbulence seems to be more fundamental than all of these others, because it largely explains why turbulence demands a statistical treatment.

This property has been variously called *instability*, *unpredictability*, or *lack of bounded sensitivity*. In more fashionable terms, *turbulence is chaotic*.

To understand what this means, consider two turbulent flows, both obeying the Navier-Stokes equations (say), but beginning from slightly different initial conditions. Experience shows that no matter how small the initial difference, the two flows will rapidly diverge, and will soon be as different from each other as if the initial difference had been 100%.

This instability property has practical consequences. Imagine a laboratory experiment with a turbulent fluid, in which the experimenter measures some arbitrary flow quantity $V(t)$ as a function of time. For example, $V(t)$ could be the temperature or velocity at a fixed point in the flow. Refer to Figure 4.3. The experimenter is interested in $V(t_1)$, the value at time t_1. To be sure of his result, he repeats the experiment, arranging the apparatus and initial conditions to be as nearly the same as possible. But no matter how hard he tries, the new value of $V(t_1)$ is always discouragingly different from the original measurement. The experimenter is finally satisfied to repeat the experiment a great many times, and to compute the probability distribution of $V(t_1)$. He becomes a statistician. Because of the instability property, he reasons, only statistics are of value in predicting the outcome of future experiments.

The question arises: Can the statistics be found without actually performing all of the experiments? That is, can the statistical averages of turbulent flow be calculated from physical law, without first solving the equations (either experimentally or with a big computer) and then averaging the results of many solutions? Many people regard this unanswered question as the central problem of turbulence.

The most direct approach to the prediction of statistics is to average the equations of motion, thereby obtaining evolution equations for the averages. Unfortunately, as explained in Chapter 1, direct averaging leads to an unclosed hierarchy of statistical moment equations, in which the equation for the time derivative of the n-th moment always involves the $(n+1)$-th moment. These moment equations cannot be solved without making additional hypotheses to close them. We set aside this *closure problem* until Chapter 5, and thus temporarily abandon any hope of obtaining a complete statistical description of turbulent flow. However, we find that many of the important *qualitative* properties of turbulence can perhaps be understood on the basis of relatively simple ideas, many of which involve vorticity.

6. *Kolmogorov's Theory*

Now we consider constant-density flow governed by the Navier-Stokes equations,

$$\begin{aligned} &\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j} \\ &\frac{\partial v_i}{\partial x_i} = 0 \end{aligned} \tag{6.1}$$

Once again, the summation convention applies to repeated subscripts. First we review elementary properties of (6.1). Then we examine the most famous (but still very controversial!) theory of three-dimensional turbulence.

In principle, it is always possible to rewrite (6.1) as a single, prognostic equation in the velocity $\mathbf{v}=(v_1,v_2,v_3)$. This converts the pressure term in (6.1a) into a nonlinear term like the advection term. To see this, we take the divergence of (6.1a) to obtain an elliptic equation,

$$\frac{\partial v_j}{\partial x_i}\frac{\partial v_i}{\partial x_j} = -\frac{\partial^2 p}{\partial x_i \partial x_i} = -\nabla^2 p\,, \tag{6.2}$$

for the pressure p. Given the velocity field $\mathbf{v}(\mathbf{x})$ and the appropriate boundary condition, we can solve (6.2) for $p(\mathbf{x})$. For fluid inside a rigid container, the appropriate boundary condition is $\mathbf{v}=0$. The boundary condition $\mathbf{v}=0$ implies that

$$0 = -\frac{\partial p}{\partial x_n} + \nu \frac{\partial^2 v_n}{\partial x_n^{\;2}} \qquad \text{(no summation on } n\text{)} \tag{6.3}$$

on the boundary, where n denotes the direction normal to the boundary. Eqn. (6.2) and the Neumann boundary condition (6.3) determine the pressure throughout the flow. Only in simple geometry (like that considered below) is it possible to solve (6.2-3) explicitly, but (at least in principle) it is clearly always possible to replace the pressure term in (6.1a) by a quadratic expression in the velocity.

Next we consider the energy equation obtained by contracting the momentum equation (6.1a) with v_i, namely

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}v_i v_i\right) + \frac{\partial}{\partial x_j}\left(\tfrac{1}{2}v_i v_i v_j\right) = -\frac{\partial}{\partial x_i}\left(v_i p\right) + \nu v_i \frac{\partial^2 v_i}{\partial x_j \partial x_j}\,. \tag{6.4}$$

Integrating (6.4) over the whole domain inside the rigid boundary, and using the boundary condition $\mathbf{v}=0$, we obtain

$$\frac{d}{dt}\iiint d\mathbf{x}\,\tfrac{1}{2}v_i v_i = \nu \iiint d\mathbf{x}\left[\frac{\partial}{\partial x_j}\left(v_i \frac{\partial v_i}{\partial x_j}\right) - \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\right] = -\nu \iiint d\mathbf{x}\left[\frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\right]. \tag{6.5}$$

Thus, neither the advection term nor the pressure term affects the total energy, but the viscous term *always* causes energy to decrease.

If the flow is *spatially unbounded*, then it is illuminating to examine the Fourier transforms of these equations. Let

$$v_i(\mathbf{x},t) = \iiint d\mathbf{k}\; u_i(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}\,. \tag{6.6}$$

Since $\mathbf{v}(\mathbf{x},t)$ is real, its Fourier transform $\mathbf{u}(\mathbf{k},t)$ is conjugate symmetric,

$$u_i(\mathbf{k}) = u_i(-\mathbf{k})^* . \tag{6.7}$$

By Fourier's theorem,

$$u_i(\mathbf{k},t) = \frac{1}{(2\pi)^3} \iiint d\mathbf{x}\; v_i(\mathbf{x},t) e^{-i\mathbf{k}\cdot\mathbf{x}} . \tag{6.8}$$

Similarly, let

$$p(\mathbf{x},t) = \iiint d\mathbf{k}\; p(\mathbf{k},t) e^{i\mathbf{k}\cdot\mathbf{x}} . \tag{6.9}$$

Then the elliptic equation (6.2) for the pressure becomes

$$-\iiint d\mathbf{m} \iiint d\mathbf{n}\; m_i u_j(\mathbf{m}) n_j u_i(\mathbf{n})\; e^{i(\mathbf{m}\cdot\mathbf{x}+\mathbf{n}\cdot\mathbf{x})} = -\iiint d\mathbf{m} \left(-m^2\right) p(\mathbf{m}) e^{i\mathbf{m}\cdot\mathbf{x}} \tag{6.10}$$

where $m=|\mathbf{m}|$, etc. Multiplying (6.10) by $e^{-i\,\mathbf{k}\cdot\mathbf{x}}$ and integrating over all $\mathbf{x}$, we obtain the Fourier tranform of (6.2) in the form

$$-\iiint d\mathbf{m} \iiint d\mathbf{n}\; m_i n_j\; u_j(\mathbf{m}) u_i(\mathbf{n}) \delta(\mathbf{m}+\mathbf{n}-\mathbf{k}) = k^2 p(\mathbf{k}) . \tag{6.11}$$

Then, using (6.11), and proceeding in a similar manner, we obtain the Fourier transform of the momentum equation (6.1a) in the form

$$\begin{aligned} &\frac{\partial}{\partial t} u_i(\mathbf{k},t) + i \iiint d\mathbf{m} \iiint d\mathbf{n}\; n_j u_j(\mathbf{m}) u_i(\mathbf{n})\, \delta(\mathbf{m}+\mathbf{n}-\mathbf{k}) \\ &\quad = i \iiint d\mathbf{m} \iiint d\mathbf{n} \frac{k_i m_r n_j}{k^2} u_j(\mathbf{m}) u_r(\mathbf{n})\, \delta(\mathbf{m}+\mathbf{n}-\mathbf{k}) - \nu k^2 u_i(\mathbf{k}) \end{aligned} \tag{6.12}$$

More concisely,

$$\frac{\partial}{\partial t} u_i(\mathbf{k},t) = \iiint d\mathbf{m} \iiint d\mathbf{n}\;\; A_{ijr}(\mathbf{m},\mathbf{n},\mathbf{k})\, u_j(\mathbf{m}) u_r(\mathbf{n}) \delta(\mathbf{m}+\mathbf{n}-\mathbf{k}) - \nu k^2 u_i(\mathbf{k}) , \tag{6.13}$$

where $A_{ijr}(\mathbf{m},\mathbf{n},\mathbf{k})$ is the *coupling coefficient* between $\mathbf{u}_i(\mathbf{k})$, $\mathbf{u}_j(\mathbf{m})$, and $\mathbf{u}_r(\mathbf{n})$. The nonlinear term on the left-hand side of (6.12) represents the advection of momentum. The nonlinear term on the right-hand side of (6.12) represents the effect of pressure. Thus the A_{ijr}-term in (6.13) represents both pressure and advection.

If pressure and advection were absent, (6.13) would be a linear equation,

$$\frac{\partial}{\partial t} u_i(\mathbf{k},t) = -\nu k^2 u_i(\mathbf{k}) , \tag{6.14}$$

in which the various wavenumbers are *uncoupled*. The solution of (6.14) is

$$u_i(\mathbf{k},t) = u_i(\mathbf{k},0)\exp\left(-\nu k^2 t\right). \tag{6.15}$$

According to (6.15), the velocity in wavenumber **k** decays exponentially, at a rate that increases with increasing wavenumber magnitude k. Thus, viscosity damps the smallest spatial scales fastest.

The nonlinear terms in (6.13) are much more complicated than the viscous-decay term and take the form of *triad interactions* that couple together wavenumbers satisfying the *selection rule* **m**+**n**=**k**. As we already know, these triad interactions do not change the total energy,

$$\iiint d\mathbf{x}\ \tfrac{1}{2} v_i v_i = 4\pi^3 \iiint d\mathbf{k}\ u_i(\mathbf{k}) u_i(\mathbf{k})^* \equiv \int_0^\infty dk\ E(k) \cdot \iiint d\mathbf{x}, \tag{6.16}$$

but they do transfer energy between wavenumbers satisfying the selection rule.[5] The last equality in (6.16) defines the *energy spectrum* $E(k)$.

Now consider the following situation: an initially quiescent fluid, in a container of size L, is stirred by some external agency at lengthscales comparable to L. Suppose that this stirring force is nonzero only for $L^{-1}<k<K_F$. Then after a very short time the spectrum is strongly excited only on $k<K_F$ (Figure 4.4a). At these small wavenumbers, the viscous dissipation is negligible, but the nonlinear terms can transfer energy to higher wavenumbers via the triad interactions. For example, two wavenumbers **m** and **n** with magnitudes $m,n<K_F$ can transfer energy into **k** with $k<2K_F$. After this has occurred, the energy spectrum is excited on $k<2K_F$. Applying this idea again and again, we form the picture in Figure 4.4b. When the energy reaches very high wavenumbers, the viscosity finally becomes important, and an equilibrium is established in which $E(k)$ (or, more precisely, its statistical average) reaches a steady state.

As this equilibrium develops, there is no fundamental reason why very *nonlocal* triad interactions, linking wavenumbers of very different sizes, could not become important, as shown in Figure 4.4c. *Suppose, however, that they don't*. This is reasonable if the individual wavenumbers represent *eddies*, and if only eddies of comparable size exchange energy efficiently. Then the equilibrium resembles Figure 4.4b, and is called a *turbulent cascade* of energy. (A *cascade* is a waterfall consisting of many small steps.) Actually, eddies with very different sizes *do* interact strongly, but their interaction takes the form of large eddies sweeping small eddies from one place to another *without significantly distorting them*. Without distortion, there is no real energy transfer between the eddies.

There are many reasons why the assumption of a turbulent cascade might not be correct. However, Kolmogorov (1941) proposed a bold theory (now often called K41) based squarely upon it.[6] He reasoned that the shape of the energy spectrum $E(k)$ at a wavenumber k many cascade-steps above K_F should be insensitive to the precise nature of the stirring. On these large k, the energy spectrum ought to depend only on the wavenumber magnitude k, the molecular viscosity ν, and the rate at which energy moves rightward (that is, toward higher k) through the spectrum. The latter rate is equal to ε, the rate of energy dissipation per unit volume. Refer to Figure 4.5. On the *inertial range* between K_F and K_D, the wavenumber at which viscous dissipation first becomes important, the spectrum $E(k)$ should depend only on ε and k. The dimensions of these quantities are

$$[E(k)] = L^3T^{-2}, \qquad [\varepsilon] = L^2T^{-3}, \qquad [k] = L^{-1}, \qquad [\nu] = L^2T^{-1}. \tag{6.17}$$

Hence, dimensional analysis tells us that

$$E(k) = C\varepsilon^{2/3}k^{-5/3} \qquad \text{and} \qquad K_D = O\left(\varepsilon^{1/4}\nu^{-3/4}\right), \tag{6.18}$$

where C is Kolmogorov's universal constant.

Observed spectra often agree with (6.18a) and suggest that $C \approx 1.5$. Nevertheless, considerable uncertainty surrounds this whole subject. It is now generally agreed that Kolmogorov's theory cannot *in principle* be exactly right, and experimental measurements of higher statistical moments, which are also predicted by the complete Kolmogorov theory, do not support the theory. We consider these points in the next section.

7. *Intermittency and the beta-model*

In a famous footnote to his book on fluid mechanics, L. D. Landau noted an important inconsistency in K41. This led to a revision of the theory, but most people feel that it also destroyed any hope that the theory can be exactly right.[7] Landau's objection is neither the only, nor perhaps even the most serious objection to K41. However, it has helped theorists to better appreciate the enormous assumptions underlying Kolmogorov's theory.

The essence of Landau's objection is that K41 cannot apply to a collection of flows with different dissipation rates ε. First consider two completely separate flows, denoted by the subscripts 1 and 2. The first flow is vigorously stirred, so that ε_1 is large. The second flow is only moderately stirred, so that ε_2 is small. If both flows are turbulent, then according to K41,

$$E_1(k) = C\varepsilon_1^{2/3}k^{-5/3} \qquad \text{and} \qquad E_2(k) = C\varepsilon_2^{2/3}k^{-5/3}. \tag{7.1}$$

Next, consider the *composite* system consisting of these two *separate* flows. If the two flows have equal volumes, then the dissipation rate and the energy spectrum of the composite system are given by

$$\varepsilon = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2) \qquad \text{and} \qquad E(k) = \tfrac{1}{2}\left(E_1(k) + E_2(k)\right). \tag{7.2}$$

But then

$$E(k) \neq C\varepsilon^{2/3}k^{-5/3}. \tag{7.3}$$

That is, the composite system cannot obey K41, essentially because the average of a two-thirds power is not the power of the average.

So far there is no problem, because the composite flow is not a single flow, and hence there is no reason why K41 should apply to it. But suppose that the subscripts 1 and 2 refer not to separate flows, but to large regions of the *same* flow with locally different dissipation rates. We conclude uncomfortably that K41 cannot apply to the whole flow if it is also *locally* correct. In particular, K41 should fail in cases where the dissipation rate ε *averaged over length-scales characteristic of the inertial range* fluctuates.

The *beta-model* is a schematic model that clarifies this argument and suggests the nature of the correction to K41.[8] Consider a turbulent flow in a container of size L_0. The fluid is stirred on scales comparable to L_0, and the energy is subsequently transferred to smaller spatial scales via the nonlinear terms in the momentum equations. Again we suppose this transfer to be a series of cascade steps from scale L_0 to L_1=L_0/2 to L_2=L_1/2, and so on. (The factor of 1/2 is inessential; any other fraction will work.) The n-th cascade step corresponds to eddy-size

$$L_n = \frac{L_0}{2^n} \equiv k_n^{-1}. \tag{7.4}$$

We also define:

V_n, the characteristic velocity *change* across eddies of size L_n

ε_n, the rate at which energy passes through the n-th cascade step; and

$$E_n = \int_{k_n}^{k_{n+1}} E(k)\,dk.$$

E_n is the energy (per unit volume of the *whole flow*) contained in eddies of size L_n.

Now, the cascade can proceed in two ways, as shown in Figure 4.6. At each cascade step, the eddies created can fill the whole space uniformly, or they can fill only a fraction β of the available space and be correspondingly stronger. We shall see that the K41 theory corresponds to the first alternative (β=1).

For general β, the total energy in eddies of size L_n is the energy within the eddies themselves, V_n^2, times the fraction β^n of the total volume occupied by these eddies. That is,

$$E_n \sim \beta^n V_n^2, \tag{7.5}$$

where the symbol ~ denotes *very rough equality*. This energy moves through the n-th cascade step in a *turn-over time*

$$T_n \sim \frac{L_n}{V_n} = \frac{1}{k_n V_n}, \tag{7.6}$$

so that

$$\varepsilon_n \sim \frac{E_n}{T_n} \sim \beta^n V_n^3 k_n. \tag{7.7}$$

If the turbulence is stationary, then ε_n must be independent of n; otherwise the energy would pile up at some intermediate wavenumber. But if

$$\varepsilon_n = \varepsilon, \tag{7.8}$$

then (7.5) and (7.7) imply that

$$E_n \sim \varepsilon^{2/3} k_n^{-2/3} \beta^{n/3}. \tag{7.9}$$

Since

$$E_n = \int_{k_n}^{k_{n+1}} E(k)dk = \int_{\ln k_n}^{\ln k_{n+1}} kE(k)d(\ln k) \quad \propto k_n E(k_n), \tag{7.10}$$

(7.9) corresponds to the spectrum

$$E(k_n) \quad \sim \varepsilon^{2/3} k_n^{-5/3} \beta^{n/3}. \tag{7.11}$$

According to (7.11), the energy spectrum at wavenumber k_n is smaller than that predicted in K41 by the factor $\beta^{n/3}$ (where β<1 if the cascade is not space-filling). Let

$$\beta = \frac{1}{2^s} \qquad (s \geq 0) \tag{7.12}$$

be the definition of s. The intermittency of the turbulence increases with s. Then since

$$\beta^n = \frac{1}{(2^n)^s} = \left(\frac{L_n}{L_0}\right)^s = \left(\frac{k_0}{k_n}\right)^s. \tag{7.13}$$

(7.11) becomes

$$E(k) \sim k_0^{s/3} \varepsilon^{2/3} k^{-(5+s)/3}. \tag{7.14}$$

Again, (7.14) reduces to K41 in the case of a space-filling cascade (β=1, s=0). However, for intermittent (s>0) turbulence, the spectrum falls off more steeply. Physically, the steeper fall-off occurs because spatial concentration of the eddies makes them more intense, and thus shortens the *residence time* (7.6) for energy at each cascade step.

Observations support the prediction of K41 (with $s=0$) for the spectrum,[9] but suggest increasing disagreement with K41 as higher-order moments are considered.[10] Recall that the spectrum is the Fourier transform (with respect to r) of

$$F(r) \equiv \langle v_i(\mathbf{x}+\mathbf{r}) v_i(\mathbf{x}) \rangle, \tag{7.15}$$

where we assume that the flow is statistically homogeneous and isotropic. As an example of a higher-order statistic, consider the *structure function*

$$F_p(r) \equiv \left\langle \left| \mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x}) \right|^p \right\rangle. \tag{7.16}$$

If r^{-1} lies within the inertial range, then, by dimensional analysis, K41 predicts that

$$\left\langle \left| \mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x}) \right|^p \right\rangle = C_p (\varepsilon r)^{p/3}, \tag{7.17}$$

where C_p is a universal constant. On the other hand,

$$G_p(r) \equiv \left\langle \left| \mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x}) \right|^2 \right\rangle^{p/2} = D_p (\varepsilon r)^{p/3}, \tag{7.18}$$

where D_p is another universal constant. Thus, because (7.17) and (7.18) have the same dimensions, their ratio

$$R_p \equiv \frac{F_p(r)}{G_p(r)} = \frac{\left\langle \left| \mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x}) \right|^p \right\rangle}{\left\langle \left| \mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x}) \right|^2 \right\rangle^{p/2}} = \frac{C_p}{D_p} \tag{7.19}$$

must be a universal constant, independent of ε and r. When $p=4$, the quantity (7.19) is called *kurtosis*.

Observations suggest that (7.19) increases with p and with r^{-1}. Since R_p measures spatial intermittency (more sensitively for larger p), these observations suggest a spatial intermittency that increases with decreasing eddy size. This contradicts K41, and it suggests that the eddies of decreasing size are indeed confined to a decreasing fraction of the fluid volume, as in the beta-model.

The beta-model predicts that

$$F_p\left(\frac{1}{k_n}\right) \sim \beta^n (V_n)^p \tag{7.20}$$

and

$$G_p\left(\frac{1}{k_n}\right) \sim \left(\beta^n V_n^{\ 2}\right)^{p/2}. \tag{7.21}$$

Hence the beta-model prediction for (7.19) is

$$R_p \sim \beta^{n(1-p/2)} \sim \left(\frac{k_n}{k_0}\right)^{-s+sp/2}. \tag{7.22}$$

That is,

$$R_p(r) \sim \left(\frac{r}{L_0}\right)^{s-sp/2}. \tag{7.23}$$

If the cascade is space-filling (s=0), then (7.23) is independent of r, in agreement with K41. However, for s>0 and p >2, $R_p(r)$ increases with decreasing r.

8. *Two-dimensional turbulence*

Again we consider constant-density flow governed by the Navier-Stokes equations. Now, however, we suppose that the flow is *two-dimensional*,

$$\mathbf{v} = \left(u(x,y), v(x,y), 0\right), \tag{8.1}$$

so that the vorticity equation,

$$\frac{D}{Dt}\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\, \mathbf{v} + \nu\, \nabla^2 \boldsymbol{\omega}, \tag{8.2}$$

reduces to

$$\frac{D\omega}{Dt} = \nu \nabla^2 \omega. \tag{8.3}$$

Here,

$$\boldsymbol{\omega} = \omega\, \mathbf{k} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}, \tag{8.4}$$

and **k** is the unit vector in the z-direction. Thus, apart from the effects of viscosity, the (vertical component of) vorticity is conserved on fluid particles. In particular, the effects of vortex stretching and tilting are absent in two dimensional flow. As we shall see, this causes two-dimensional turbulence to behave completely differently from three-dimensional turbulence.

Since the flow is nondivergent,

$$u=-\frac{\partial\psi}{\partial y},\qquad v=+\frac{\partial\psi}{\partial x}, \tag{8.5}$$

for some $\psi(x,y,t)$. Then

$$\omega=\nabla^2\psi \tag{8.6}$$

and (8.3) can be written as an equation in a single dependent variable,

$$\frac{\partial}{\partial t}\nabla^2\psi+J\left(\psi,\nabla^2\psi\right)=\nu\nabla^4\psi,\qquad J(A,B)\equiv\frac{\partial(A,B)}{\partial(x,y)}. \tag{8.7}$$

One often hears that two-dimensional turbulence does not really exist, because two-dimensional Navier-Stokes turbulence is always unstable with respect to three-dimensional motions. While this is probably true, we recognize (8.7) as the simplest case of the quasigeostrophic equation (for a single layer with constant Coriolis parameter f and no bottom topography), and we recall (from Chapter 2) that, although f does not even appear in (8.7), it is responsible for the *validity* of (8.7): Low-frequency motions of a *rotating*, constant-density fluid can remain two-dimensional. However, the real importance of two-dimensional turbulence theory to geophysical fluid dynamics lies in the fact that the theory covers the quasigeostrophic generalizations of (8.7). These are the subject of Chapter 6.

In this section, we concentrate on properties of the solutions to (8.7) with vanishing viscosity. Our conclusions illuminate the role of the nonlinear terms in (8.7). In the following section, we re-admit the viscosity and address the statistical equilibrium of two-dimensional flows with forcing and dissipation.

If $\nu=0$, motion governed by (8.7) conserves (twice) the energy,

$$E\equiv\iint d\mathbf{x}\,\nabla\psi\cdot\nabla\psi, \tag{8.8}$$

and every quantity of the form

$$\iint d\mathbf{x}\;F\left(\nabla^2\psi\right), \tag{8.9}$$

where F is an arbitrary function. The quantity (8.9) is conserved because the vorticity $\nabla^2\psi$ is conserved on fluid particles, and because the velocity field is nondivergent. The conservation law (8.9) has no analogue in three-dimensional turbulence, where stretching and tilting can change the vorticity on fluid particles. The *enstrophy,*

$$Z\equiv\iint d\mathbf{x}\left(\nabla^2\psi\right)^2, \tag{8.10}$$

is an important case of (8.9).

To investigate the consequences of the conservation of E and Z in inviscid two-dimensional turbulence, we first consider spatially unbounded flow with Fourier transform

$$\psi(x,y,t) = \iint d\mathbf{x}\ \psi(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \psi(\mathbf{k},t) = \psi(-\mathbf{k},t)^*. \tag{8.11}$$

However, our most important results also apply to infinitely-periodic flow and to bounded flow. Substituting (8.11) into (8.8) and (8.10), we see that the energy,

$$E = (2\pi)^2 \iint d\mathbf{k}\ k^2\ |\psi(\mathbf{k},t)|^2 \equiv \int_0^\infty dk\, E(k), \tag{8.12}$$

and enstrophy,

$$Z = \int_0^\infty dk\ k^2\ E(k), \tag{8.13}$$

are the zeroth and second moments of the energy spectrum $E(k)$.

Now suppose that $\nu=0$ and that the energy is initially concentrated at some wavenumber k_1. If the energy subsequently spreads to both higher and lower wavenumbers, then more energy must move toward the lower wavenumbers than toward higher wavenumbers, in order to conserve both (8.12) and (8.13). The transfer of energy from small to large scales of motion is the opposite of the transfer usually observed in three-dimensional turbulence, and has sometimes been called *negative eddy viscosity*.[11]

Suppose that the energy originally at k_1 subsequently flows into the two wavenumbers $k_0=k_1/2$ and $k_2=2k_1$. By conservation of energy,

$$E_0 + E_2 = E_1, \tag{8.14}$$

and by conservation of enstrophy,

$$\left(\frac{k_1}{2}\right)^2 E_0 + (2k_1)^2 E_2 = k_1{}^2 E_1. \tag{8.15}$$

It follows that

$$E_0 = \tfrac{4}{5} E_1 \qquad \text{and} \qquad E_2 = \tfrac{1}{5} E_1, \tag{8.16}$$

so that 80% of the energy ends up in the lower wavenumber. However, since the enstrophy in this wavenumber is

$$Z_0 = \left(\frac{k_1}{2}\right)^2 E_0 = \left(\frac{k_1}{2}\right)^2 \tfrac{4}{5} E_1 = \tfrac{1}{5}\left(k_1^{\,2} E_1\right) = \tfrac{1}{5} Z_1 \,, \tag{8.17}$$

it contains only 20% of the enstrophy. The other 80% of the enstrophy ends up in the higher wavenumber k_2.

A more convincing proof that energy and enstrophy move in opposite directions through the spectrum proceeds as follows. Still assuming $\nu = 0$, we consider the expression

$$\frac{d}{dt}\int (k - k_1)^2 E(k)dk \,, \tag{8.18}$$

which is positive if the energy initially concentrated at wavenumber k_1 subsequently spreads out. But

$$\frac{d}{dt}\int (k - k_1)^2 E(k)dk = \frac{d}{dt}\left[\int k^2 E\, dk - 2k_1 \int kE\, dk + k_1^{\,2} \int E\, dk\right] = -2k_1 \frac{d}{dt}\int kE\, dk \,, \tag{8.19}$$

because the energy (8.12) and enstrophy (8.13) are conserved. It follows that

$$\frac{d}{dt}\left[\frac{\int kE(k)\, dk}{\int E(k)\, dk}\right] < 0 \,. \tag{8.20}$$

The quotient in (8.20) is a logical definition of the wavenumber characterizing the energy-containing scales of the motion. Thus energy moves toward lower wavenumbers.

By similar reasoning,

$$\frac{d}{dt}\int \left(k^2 - k_1^{\,2}\right)^2 E(k)dk = \frac{d}{dt}\left[\int k^4 E\, dk - 2k_1^{\,2} \int k^2 E\, dk + k_1^{\,4} \int E\, dk\right] = \frac{d}{dt}\int k^4 E\, dk \tag{8.21}$$

should be positive. If we let

$$Z(k) \equiv k^2 E(k) \tag{8.22}$$

be the enstrophy spectrum, then positive (8.21) implies that

$$\frac{d}{dt}\left[\frac{\int k^2\, Z(k)dk}{\int Z(k)dk}\right] > 0 \,. \tag{8.23}$$

The quotient in (8.23) is a logical definition of the (squared) wavenumber characterizing the enstrophy-containing scales of the motion. Thus enstrophy moves toward higher wavenumbers.

Since

$$\int k^2 Z(k)dk = \iint d\mathbf{x}\,(\nabla\omega \cdot \nabla\omega)\,, \tag{8.24}$$

the increase (8.23) in the enstrophy-weighted wavenumber is associated with an increase in the mean squared-gradient of vorticity, sometimes called *palinstrophy*. This leads to a *physical-space* picture of the process by which enstrophy moves to higher wavenumbers. Consider two nearby (*i.e.* nearly parallel) lines of constant vorticity in inviscid two-dimensional flow, as shown in Figure 4.7. If $\nu=0$, so that the vorticity is conserved on fluid particles, then these lines are also *material lines*. That is, they always contain the same fluid particles. It is plausible that, *on average*, these material lines of constant vorticity get longer as time increases. That is, their constituent fluid particles move further apart. But if these material lines get longer, they must also get closer together, because the area between the lines is constant in incompressible flow. However, the vorticity gradient is inversely proportional to the distance between constant-vorticity lines. Therefore, if the constant-vorticity lines get longer, then the magnitude of the vorticity gradient, and hence (8.24), must increase.[12]

What makes us think that the constant-vorticity lines get longer? We can show that the *average* length of material lines increases if the velocity field is *statistically isotropic*.[13] Consider two neighboring fluid particles with position vectors $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, and let $\mathbf{r}_i(t)=\mathbf{r}_i(0)+\Delta\mathbf{r}_i(t)$. The initial particle locations $\mathbf{r}_i(0)$ are *given* and *nonrandom* (*i.e.* statistically sharp), but the particle locations subsequently acquire a statistical distribution that depends on the statistics of the velocity field. The mean square separation between the fluid particles at time *t* is

$$\begin{aligned}&\left\langle|\mathbf{r}_1(t)-\mathbf{r}_2(t)|^2\right\rangle = \\ &|\mathbf{r}_1(0)-\mathbf{r}_2(0)|^2 + 2[\mathbf{r}_1(0)-\mathbf{r}_2(0)]\cdot\langle\Delta\mathbf{r}_1(t)-\Delta\mathbf{r}_2(t)\rangle + \left\langle|\Delta\mathbf{r}_1(t)-\Delta\mathbf{r}_2(t)|^2\right\rangle\end{aligned} \tag{8.25}$$

But since

$$< \Delta\mathbf{r}_i(t) >= 0 \tag{8.26}$$

in isotropic flow, (8.25) implies that

$$\left\langle|\mathbf{r}_1(t)-\mathbf{r}_2(t)|^2\right\rangle \geq |\mathbf{r}_1(0)-\mathbf{r}_2(0)|^2\,. \tag{8.27}$$

Unfortunately, this proof does *not,* strictly speaking, apply to the case of particles on a line of constant vorticity, because we have assumed that the initial locations of the fluid particles are uncorrelated with the fluid velocity. Hence (8.27) contributes plausibility, but no rigor, to the picture of palinstrophy increase sketched above.

Now we pause to make a very important point. Although the arguments of this section utilize *exact* conservation laws, our final conclusions are essentially *statistical*, because they also depend on assumptions about the *average behavior* of the flow. Without such assumptions, it would be impossible to prove that (for example) the enstrophy moves to higher wavenumbers in the flow. The reason for this is that *inviscid* mechanics is *time-reversible*: For every inviscid flow in which enstrophy moves to smaller scales of motion, there is an inviscid flow in which exactly the opposite occurs (namely, the first flow *running backwards in time*). Thus, every example provides its own counter-example! Our statistical hypotheses amount to statements that the example is *more likely* than the counter-example. These hypotheses frequently enter as innocent, often tacit, assumptions, whose statistical nature is hidden. For example, our "proofs" that enstrophy moves toward higher wavenumbers rest on the essentially statistical assumptions that a spectral peak will spread out (rather than sharpen), and that material lines get longer (rather than shorter). We cannot escape such assumptions, but we can hope to find the simplest and most compelling ones possible. Turbulence theory largely consists of linking *plausible statistical hypotheses* to interesting, even unexpected, consequences.

9. *More two-dimensional turbulence*

Now suppose that $\nu \neq 0$ and consider the statistically steady two-dimensional turbulence that arises from a stirring force acting at wavenumber k_1 (Figure 4.8). The energy and enstrophy put in by the stirring force spread to other wavenumbers by the nonlinear terms in the equations of the motion. At some high wavenumber, k_D, viscosity becomes effective, and energy and enstrophy are removed. If the container has size L, then the lowest wavenumber, k_0, has size $1/L$. If the flow is unbounded, then $k_0=0$.

First, consider the inertial range between k_1 and k_D. If k_D/k_1 is large, then there are many cascade steps between the stirring at k_1 and the dissipation near k_D. Within this inertial range the energy spectrum $E(k)$ then plausibly depends only on the wavenumber k; on ε, the rate of energy transfer past k to higher wavenumbers; and on η, the rate of enstrophy transfer past k. If all of the energy and enstrophy passing through the inertial range on $[k_1,k_D]$ is removed at wavenumbers *greater* than k_D, then

$$\eta > k_D^{\,2}\varepsilon \,. \tag{9.1}$$

Now let k_1 be fixed, and let $k_D \to \infty$. This corresponds to the limit $\nu \to 0$ of a very wide inertial range, with many cascade steps between k_1 and k_D. In this limit ε must vanish, or, by (9.1), η would blow up (which is impossible, because the stirring force supplies a *finite* enstrophy to the fluid, and the nonlinear interactions conserve enstrophy). We thus conclude that, in the inertial range on $[k_1,k_D]$, the rightward energy transfer is asymptotically *zero*, and the spectrum $E(k)$ therefore depends only on k and η. It then follows from dimensional analysis that

$$E(k) = C_1\,\eta^{2/3}k^{-3}, \qquad\qquad k_D \sim \left(\frac{\eta}{\nu^3}\right)^{1/6} \tag{9.2}$$

where C_1 is a universal constant.[14]

The spectrum at low wavenumbers is more problematic. Since the energy dissipated by the viscosity ν is asymptotically zero, the total energy of the flow must increase with time, and no statistically steady state is possible. If $k_0=0$, this energy moves toward ever-lower wavenumbers, perhaps following the similarity theory proposed by Batchelor (1969). In the more realistic case $k_0 \neq 0$ of bounded flow, the energy piles up near k_0. But suppose that *something* (another *type* of dissipation, an Ekman drag perhaps) *removes* this energy near k_0, so that an equilibrium state becomes possible. What then is the nature of the turbulence in the inertial range on $[k_0,k_1]$? By the same reasoning as above, we conclude that, in the asymptotic limit $k_0/k_1 \to 0$, the *enstrophy* transfer across $[k_0,k_1]$ vanishes, and the spectrum therefore depends only on k and ε, the rate of energy dissipation near k_0. Dimensional analysis then yields

$$E(k) = C_2 \varepsilon^{2/3} k^{-5/3}, \tag{9.3}$$

where C_2 is a universal constant. The spectrum (9.3) in the energy-cascading inertial range has the same form as in *three*-dimensional turbulence. Of course, in three-dimensional turbulence, the energy tranfer is from *large* to *small* scales of motion.

Meteorologists and oceanographers often use these results by imagining that atmosphere and ocean obey the equations for two-dimensional turbulence, and that the stirring force at wavenumber k_1 represents baroclinic instability injecting energy at scales of motion comparable to the deformation radius. (In Chapter 6 we pursue the much better strategy of generalizing the theory to equations that better apply to the atmosphere and ocean.) Then the above theory predicts a k^{-3} spectrum on wavenumbers between k_1 and k_D, the wavenumber at which the Rossby number Uk_D/f exceeds unity. At higher wavenumbers, rotation cannot keep the flow two-dimensional, and the enstrophy passes into smaller-scale three-dimensional turbulence.

Although observations support a k^{-3} spectrum in the ocean and atmosphere, there are at least three reasons to question this explanation:

(1) The dynamics (8.7) of pure two-dimensional turbulence omit too much of the physics. In particular, the beta-effect and density stratification are very important in the atmosphere and ocean.

(2) Even if we ignore objection (1), the hypotheses about two-dimensional turbulence required to establish (9.2) and (9.3) are not satisfied by the atmosphere and ocean. In neither fluid are the separations between k_0, k_1, and k_D large enough to justify the picture of a turbulent cascade. Furthermore, on scales larger than the deformation radius, the atmosphere and ocean show very large departures from statistical homogeneity and isotropy.

(3) Even if we ignore objections (1) and (2), the inertial range theory of two-dimensional turbulence is not strictly self-consistent.

In Chapter 6, we shall consider generalizations of (8.7) that partly answer objection (1), and we shall avoid the strong hypotheses criticized in objection (2). In the remainder of this section we look more closely at objection (3), arriving at a picture of enstrophy transfer to small spatial scales that is, in some respects, the antithesis of a cascade.

The argument leading to (9.2) supposes that the transfer of enstrophy past k in the k^{-3} inertial range on $[k_1, k_D]$ is *local* in wavenumber, that is, that interactions between very distant wavenumbers (*eddies* of very different sizes) do not strongly contribute. This justifies the picture of a turbulent *cascade* whose many cascade-steps erase the memory of the precise nature of the stirring force and lead to universal behavior. Now, in the picture of enstrophy transfer to smaller scales developed in Section 8, eddies of size k^{-1} are stretched out by the straining motion of the fluid, and the palinstrophy (8.24) increases as the result of the stretching. The mean-square strain-rate has the same spectrum,

$$Z(k) = k^2 E(k) \sim k^2\ k^{-3} = k^{-1}. \tag{9.4}$$

as the enstrophy, and all spatial scales *larger* than k^{-1} contribute to the velocity difference between one side of this eddy and the other. It follows that

$$\int_{k_1}^{k} Z(k)dk \sim \int_{k_1}^{k} k^{-1}\,dk = \ln\left(\frac{k}{k_1}\right). \tag{9.5}$$

is that part of the mean-square strain-rate that is effective in stretching out an eddy of size k^{-1}. According to (9.5), every wavenumber octave in the range $[k_1, k]$ contributes *equally* to the mean-square strain on the eddy of size k^{-1}. This violates (if only *just*) the localness-in-wavenumber hypothesis used to derive (9.2).

The inertial-range theory can be saved by an extension of the reasoning we used in the beta-model.[15] In the enstrophy-cascading inertial range, the enstrophy in the cascade step centered on k is

$$k\,Z(k), \tag{9.6}$$

(*cf*. (7.10)), and this amount of enstrophy is transferred to the next cascade step in a time *T(k),* now *nonlocally* determined as the inverse of the average strain rate acting on the eddy of size k^{-1}, namely

$$T(k) \sim \left[\int_{k_1}^{k} Z(k')dk'\right]^{-1/2}. \tag{9.7}$$

(We assume that the cascade is space-filling, that is, that $\beta = 1$.) Since the enstrophy transfer past every wavenumber is a constant at equilibrium, we must have[16]

$$kZ(k)\cdot\left[\int_{k_1}^{k} Z(k')dk'\right]^{1/2} \sim \eta \quad \text{(constant)}. \tag{9.8}$$

To solve (9.8) for $Z(k)$, let

$$f \equiv k\,Z \qquad \text{and} \qquad x \equiv \ln k. \tag{9.9}$$

Then (9.8) is

$$f(x)\left[\int_{x_1}^{x} f(x')dx'\right]^{1/2} \sim \eta, \tag{9.10}$$

with solution,

$$f(x) \sim \frac{\eta^{2/3}}{(x-x_1)^{1/3}}, \qquad x >> x_1. \tag{9.11}$$

Thus

$$Z(k) \sim \eta^{2/3} k^{-1}\left[\ln\left(\frac{k}{k_1}\right)\right]^{-1/3}. \tag{9.12}$$

That is,

$$E(k) \sim \eta^{2/3} k^{-3}\left[\ln\left(\frac{k}{k_1}\right)\right]^{-1/3}. \tag{9.13}$$

Eqn. (9.13) represents a correction to (9.2) that is almost undetectably small for inertial ranges of reasonable width. But more interesting than the precise form of this correction is the *picture* of the enstrophy range that it implies, in which the nonlinear transfer of enstrophy toward higher wavenumbers is very *nonlocal* in wavenumber. Eddies well inside the inertial range are stretched out by straining motions dominated by much larger and stronger eddies, on which the stretched eddies themselves have almost no effect. This suggests that the vorticity in inertial-range eddies behaves almost like a *passive scalar* in a velocity field with a *uniform* strain. The strain appears uniform because it is concentrated in spatial scales that are much larger than the eddies being strained.

Batchelor (1959) showed that the spectrum of a passive conserved scalar in a uniform straining field is exactly proportional to k^{-1}. We can obtain this result from the calculation in Section 14 of Chapter 1. There we considered a single sinusoidal component of the passive tracer $\theta(\mathbf{x},t)$ in a field of uniform *shear* $\partial u/\partial y=\alpha$. We found that, on scales at which θ-diffusion is not yet important, the amplitude of the sinusoid was conserved, but that the wavenumber magnitude at time t is $k\gamma(t)$, where k is the initial wavenumber, and $\gamma(t)=(1+\alpha^2t^2)^{1/2}$ in the special case considered in Chapter 1 (see eqn.(14.20) in Chapter 1). Batchelor considered the case of uniform *strain*, in which $\gamma(t)=e^{\sigma t}$ and σ is the strain rate. In either case, the θ-variance initially between k_1 and $k_2=k_1+dk$ (say) must equal the variance between $\gamma(t)k_1$ and $\gamma(t)k_2$ at later time t. Thus, if $\Theta(k)$ is the spectrum of θ, then

$$\Theta(k_1)dk = \Theta(\gamma k_1)d(\gamma k) \quad (9.14)$$

at equilibrium. But since (9.14) must hold for every *t* and γ, $\Theta(k) \sim k^{-1}$. (To see this, take the derivative of (9.14) with respect to γ, and set γ=1. Then use (9.14) with γ=1 as an "initial condition" on the resulting ordinary differential equation.) Therefore, if the small-scale vorticity behaves like a passive scalar, then $Z(k) \sim k^{-1}$ and hence $E(k) \sim k^{-3}$, in agreement with (9.2), but *without* the hypothesis of a local cascade.[17]

There have been numerous numerical studies of two-dimensional turbulence.[18] The numerical solutions usually show a spectral slope that is significantly steeper than k^{-3} and sometimes as steep as k^{-5}. The steeper slope is caused by the appearance of long-lived, isolated, axisymmetric vortices.[19] In the frequently studied case of unforced two-dimensional turbulence beginning from random initial conditions (often simply called *freely decaying turbulence*), a strong enstrophy cascade is initially present, but, as the cascade subsides, significant enstrophy remains trapped in the isolated vortices. These vortices interact conservatively (that is, without losing energy or enstrophy) except for infrequent close encounters that lead to the merger of like-signed vortices. In a typical merger, the two interacting vortices strip long filaments of vorticity from one another. The dissipation of these thin filaments represents a loss of enstrophy, but energy is approximately conserved. The final state consists of a few large vortices that have consumed all the others. Interestingly, the isolated vortices can often be traced all the way back to local vorticity extrema in the initial conditions.

In continually forced two-dimensional turbulence, the enstrophy cascade and the vortices coexist. In fact, it seems best to regard forced two-dimensional turbulence as *two* fluids — one fluid consisting of the isolated coherent vortices, and the other fluid consisting of the more randomly distributed vorticity field between the vortices. The overall spectrum (including the vortices) is much steeper than k^{-3}, but if a spectral analysis is performed only on the regions between the vortices, then the result is very close to k^{-3}. The regions of the coherent vortices contribute a k^{-6} component to the spectrum, and the total spectrum (which seems always to lie between these two extremes) depends upon the relative strengths of the two components, as determined by the details of the forcing.[20]

Figure 4.9 shows the vorticity (bottom) and streamfunction (top) in a numerical simulation of freely-decaying two-dimensional turbulence governed by (8.7).[21] The boundary condition is ψ=0 at the (rigid) boundaries of the box. The 256^2 gridpoints correspond to a maximum wavenumber of 128 in each horizontal direction. The initial conditions (Figure 4.9a) are random, with the energy peaked at wavenumber k=8. Let time be measured in units of the time required for a fluid particle to move a distance equal to the side of the box at the (initial) rms speed of the flow. By t=0.5 (Figure 4.9b), straining motions have produced elongated features in the vorticity field, corresponding to enstrophy transfer to smaller spatial scales. The energy-containing scales (as represented by the streamfunction field) have, on the other hand, increased. By t=1.0 (Figure 4.9c), isolated axisymmetric vortices become prominent. As time further increases, these vortices decrease in number and increase in strength, as the flow evolves toward an expected final state of two large vortices with opposite signs.

10. *Energy transfer in two and three dimensions*

In freely decaying (*i.e.* unforced) Navier-Stokes turbulence, the total energy evolves according to

$$\frac{d}{dt}\iiint \tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}\,d\mathbf{x} = \nu\iiint \mathbf{v}\cdot\nabla^2\mathbf{v}\,d\mathbf{x}\,. \tag{10.1}$$

But

$$\nabla^2\mathbf{v} = \nabla(\nabla\cdot\mathbf{v}) - \nabla\times(\nabla\times\mathbf{v}) = -\nabla\times\boldsymbol{\omega}\,, \tag{10.2}$$

and thus

$$\begin{aligned}\frac{d}{dt}\iiint \tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}\,d\mathbf{x} &= -\nu\iiint \mathbf{v}\cdot(\nabla\times\boldsymbol{\omega})\,d\mathbf{x} \\ &= -\nu\iiint [\boldsymbol{\omega}\cdot(\nabla\times\mathbf{v}) + \nabla\cdot(\boldsymbol{\omega}\times\mathbf{v})]\,d\mathbf{x} = -\nu\iiint \boldsymbol{\omega}\cdot\boldsymbol{\omega}\,d\mathbf{x}\end{aligned} \tag{10.3}$$

According to (10.3) the energy-dissipation rate

$$\varepsilon = \frac{\nu}{V}\iiint \boldsymbol{\omega}\cdot\boldsymbol{\omega}\,d\mathbf{x} \tag{10.4}$$

is proportional to the enstrophy

$$Z \equiv \iiint \boldsymbol{\omega}\cdot\boldsymbol{\omega}\,d\mathbf{x}\,. \tag{10.5}$$

Here, V is the volume of the fluid.

Eqns. (10.1-4) apply to flow in two *or* three dimensions. However, in two-dimensional flow, the enstrophy (10.5) can never exceed its initial value, because the nonlinear terms in the equations of motion conserve enstrophy, and dissipation always decreases the enstrophy. Thus, in two dimensions, the energy dissipation rate (10.4) vanishes with the viscosity coefficient ν. That is,

$$\lim_{\nu\to 0}\varepsilon = 0 \qquad \text{(in two dimensions)}\,. \tag{10.6}$$

Experiments suggest that *three*-dimensional turbulence behaves quite differently. If an impulsive stirring creates eddies with velocity scale U and lengthscale L, then the resulting three-dimensional turbulence is observed to decay on the *turn-over time-scale*, L/U, of the eddies. Thus

$$\varepsilon \sim \frac{U^3}{L} \qquad \text{(in three dimensions)}\,, \tag{10.7}$$

and is *independent* of ν.

The relation (10.7) underlies nearly every phenomenological theory of three-dimensional turbulence. But since (10.4) holds exactly, we must conclude that, if (10.7) is correct, then the enstrophy in three-dimensional turbulence becomes infinite as $\nu \to 0$. In other words, if (10.7) holds for arbitrarily small viscosity, then three-dimensional turbulence must be able to transfer energy to *arbitrarily small scales in a finite time*. The energy transfer in three-dimensional turbulence must be explosive.[22]

Is such a transfer consistent with Kolmogorov's theory? Consider the cascade step centered on wavenumber k, and let the next cascade step be centered on nk, where n is some fixed integer. (In the beta-model, we assumed that n=2, but now we shall be more general.) If the cascade is space-filling, then, according to K41, the time $T(k)$ required to transfer the energy from cascade step k to the next step is (cf. (9.7))

$$T(k) \sim \frac{1}{\sqrt{k^3 E(k)}} \propto k^{-2/3} \qquad (10.8)$$

for $E(k) \propto k^{-5/3}$. If the energy is initially at $k_1 = 1/L$, then the time required to reach infinite wavenumber is

$$T(k_1) + T(nk_1) + T(n^2 k_1) + \cdots \propto T(k_1) \sum_{r=0}^{\infty} n^{-2r/3} , \qquad (10.9)$$

which converges for all $n > 1$. Thus K41 is not obviously inconsistent with the requirement that the energy reach infinite wavenumber in a finite time.

In contrast, the inertial ranges of two-dimensional turbulence both require an infinite amount of time to transfer the energy or enstrophy across an infinite wavenumber interval. The time for energy in the two-dimensional energy-cascading inertial range to reach k=0 is given by (10.9) with n replaced by $1/n$. This obviously diverges — the terms in the series get bigger! In the k^{-3} enstrophy-cascading inertial range, the transfer time $T(k)$ between cascade steps depends only on η and k. Hence, by dimensional analysis,

$$T(k) \sim \eta^{-1/3} \qquad \text{(constant)}, \qquad (10.10)$$

and each step requires the same amount of time. Nonlocal corrections of the type considered in the previous section alter this result only logarithmically.

These results hint that the mechanism of transfer is very different in two- and three-dimensional turbulence. Now we offer a mechanistic picture of the energy transfer in two and three dimensions that seems to tie things together. This picture attempts to explain, in physical terms, why the energy transfer is oppositely directed in the two cases, and why the transfer of energy to high wavenumbers is so much more efficient in three dimensions.

First, recall that the average of the Navier-Stokes momentum equation is

$$\frac{\partial < v_i >}{\partial t} + < v_j > \frac{\partial < v_i >}{\partial x_j} - \frac{\partial < p >}{\partial x_i} - \nu \frac{\partial^2 < v_i >}{\partial x_j \partial x_j} = -\frac{\partial}{\partial x_j} < v_i' v_j' >, \tag{10.11}$$

where, as usual, the primes denote departures from the average, and the right-hand side of (10.11) is the divergence of the Reynolds stress. We associate the average flow with the large scales of the motion and the primed flow with the smaller scales. (This constitutes our *definition* of the averaging, if you like.) To form an equation for the energy in the large-scale motion, we multiply (10.11) by $<v_i>$, and integrate over the whole fluid. After integrations by parts,

$$\frac{d}{dt}\iiint \tfrac{1}{2}\langle v_i\rangle\langle v_i\rangle + \nu\iiint \frac{\partial\langle v_i\rangle}{\partial x_j}\frac{\partial\langle v_i\rangle}{\partial x_j} = -C, \tag{10.12}$$

where

$$C \equiv -\iiint \langle v_i v_j \rangle \frac{\partial\langle v_i\rangle}{\partial x_j}\, d\mathbf{x} \tag{10.13}$$

is the rate at which the nonlinear terms in the momentum equation convert large-scale energy to small-scale energy. (To show this beyond any doubt, we could form an equation for the rate of change of the energy $<v_i'v_i'>$ in small spatial scales. We would find that the term (10.13) occurs with the opposite sign.)

Now, from the previous lectures, we expect that C is *typically* positive in three-dimensional turbulence, and *typically* negative in two-dimensional turbulence. (The word *typically* is a reminder that all such statements are statements about statistical averages, and rest on assumptions about average behavior.) Consider the situation sketched in Figure 4.10, in which the large-scale velocity

$$\langle \mathbf{v}\rangle = (\bar{u}(y), 0, 0) \tag{10.14}$$

points everywhere in the x-direction, and varies only in y. Then C is given by

$$C = -\iiint \langle u' v' \rangle \frac{\partial \bar{u}}{\partial y}\, d\mathbf{x}. \tag{10.15}$$

In two dimensions (Figure 4.10, middle), the mean flow strains initially isotropic small-scale eddies (left) into the shape at the right. Thus, for $\partial<u>/\partial y$ positive as depicted, $<u'v'>$ becomes positive, and C is indeed negative. The Reynolds flux of x-momentum is directed toward positive y (that is, *up*-gradient), and there is a *negative* transfer of energy from the mean flow to the smaller scales of motion.

In three dimensions, vortex stretching is possible, and it becomes the primary mechanism for energy transfer between scales. We regard the small-scale motion as an initially isotropic collection of vortex tubes (Figure 4.10, bottom left). Tube A is stretched by the mean shear, and the magnitude of its vorticity therefore increases. On the other hand, vortex tube C is squashed, and its vorticity magnitude therefore decreases. Tube B is instantaneously unstretched. At a later time (Figure 4.10, bottom right) vortex tube A makes the dominant contribution to the Reynolds stress, and it contributes negatively to $\langle u'v' \rangle$. Thus, the Reynolds momentum flux is *down*-gradient, and C is positive.

Notes for Chapter 4.

1. Some writers call fluids in which $p=p(\rho)$ *barotropic,* but the term *homentropic* is preferable because in real fluids the pressure depends on density *and* entropy. Moreover, oceanographers use the term *barotropic* in a slightly different sense — to describe flows in which the horizontal velocity does not depend on depth.
2. In the language of tensor analysis, the contravariant vector **w** and the covariant vectors $\nabla\theta_i$ are *Lie-dragged* by the fluid motion. The Lie derivative of their product (a scalar) therefore vanishes; this is just (3.2).
3. See Moffatt (1969) and Moffatt and Tsinober (1992).
4. For a discussion of this point see Batchelor (1967), pp. 99-104.
5. However, one can show that the pressure term would, by itself, conserve the energy $u_i(\mathbf{k})u_i(\mathbf{k})^*$ in *each* wavenumber **k**. Thus pressure transfers energy between the different *directional* components of the velocity field, but not between the different wavenumbers.
6. Kolmogorov formulated his theory in **x**-space, but we follow most elementary treatments by explaining the theory in **k**-space. For an introduction to homogeneous turbulence, including Kolmogrov's theory, see Saffman (1968). For a retrospective on K41, including English translations of Kolmogorov's original papers and recent related work, see *Turbulence and Stochastic Processes: Kolmogorov's Ideas 50 Years On* (full reference in the bibliography under Kolmogorov (1941)) and the book by Frisch (1995).
7. See Landau and Lifshitz (1959, p.126). Kolmogorov (1962) revised the K41 theory, taking account of intermittency.
8. See Frisch et al. (1978) and Frisch (1995, pp.135-140).
9. The classic paper is Grant et al. (1962).
10. See Frisch (1995, pp. 127-133) and references therein.
11. The classic paper is Fjortoft (1953).
12. For a more detailed description of this process, see Weiss (1991).
13. See Dhar (1976).
14. See Kraichnan (1967). For reviews of two-dimensional turbulence, see Kraichnan and Montgomery (1980) and Vallis (1992).
15. See Kraichnan (1971b).
16. Of course (9.8) applies only to $k \gg k_1$. For wavenumbers near k_1, one must take account of the straining contributed by wavenumbers less than k_1, even though the spectrum at these low wavenumbers is flatter than in the enstrophy inertial range.
17. C. E. Leith (personal communication) has proposed still another explanation of the observed atmospheric k^{-3} range. Using an argument analogous to that proposed by O. M. Phillips for ocean wind waves, Leith suggests that the atmospheric energy spectrum represents the saturation spectrum for breaking Rossby waves.
18. See, for example, Lilly (1971), Herring et al. (1974), Herring and McWilliams (1985), Brachet et al. (1988), and Borue (1994).
19. See McWilliams (1984, 1990) and Carnevale et al. (1991).

20. See Benzi et al. (1986, 1987) and Farge et al. (1996).
21. Figure 4.9 shows a solution of (8.7) with vanishing viscosity (ν=0). However, I computed the vorticity-advection term in (8.7) using the third-order-upwind scheme proposed by Leonard (1984). This scheme has a *truncation error* corresponding to the presence of a term $-\nu_e \nabla^6 \psi$ on the right-hand side of (8.7), where ν_e is of the order of $U \Delta x^3$, U is the *local* fluid speed, and Δx is the grid-spacing. This *implicit* numerical viscosity is evidently sufficient to wipe out rapid oscillations on the scale of the grid. In contrast to the more conventional method of including an *explicit* eddy viscosity of the same form, the upwind scheme does not demand another boundary condition (besides ψ=0) at the solid walls.
22. This has led to a longstanding but as yet unproved conjecture that solutions of the three-dimensional Euler equations — the Navier-Stokes equations with ν=0 — develop singularities in a finite time. For a brief summary of the status of this problem, see Frisch (1995, pp.115-119).