Inelastic collapse and clumping in a one-dimensional granular medium

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The dynamics of a one-dimensional gas of inelastic point particles is investigated. To model inelastic collisions, it is supposed that the relative velocity of two colliding particles is reduced by a factor r, where $0 < r < 1$. The constant r is the coefficient of restitution. Because the collisions are inelastic, particles can collide infinitely often in finite time so that the relative separations and velocities of adjacent particles on the line become zero. The minimal example of this "inelastic collapse" requires $r < 7 - 4\sqrt{3} \approx 0.0718$. With this restriction, three particles condense into a single lump in a finite time: The particle in the middle is sandwiched between the monotonically converging outer particles. When r is greater than $7 - 4\sqrt{3}$, more than three particles are needed to trigger inelastic collapse and it is shown that r is close to 1 the minimum number scales as $-\ln(1-r)/(1-r)$. The simplest statistical problem is the "cooling law" of a uniformly excited gas confined between inelastic boundaries. A scaling argument suggests that the mean square velocity (the "granular temperature") of the particles decreases like t^{-2}. Numerical simulations show that this scaling is correct only if the total number of particles in the domain is less than the number required to trigger collapse (e.g., roughly 88 if $r = 0.95$). When the number of particles is much greater than this minimum, and before the first collapse, clusters form throughout the medium. Thus a state with uniform particle density is unstable to the formation of aggregates and inelastic collapse is the finite-amplitude expression of this instability.

I. INTRODUCTION

Consider an ensemble of inelastic point particles, all with the same mass, moving on a line. We suppose that the collisions conserve momentum but dissipate kinetic energy. Thus the velocities after a collision, u_1' and u_2', are related to the velocities before collision, u_1 and u_2, by

$$u_1' = \tfrac{1}{2}(1-r)u_1 + \tfrac{1}{2}(1+r)u_2,$$
$$u_2' = \tfrac{1}{2}(1+r)u_1 + \tfrac{1}{2}(1-r)u_2. \qquad (1)$$

Here, $0 \leqslant r \leqslant 1$ is the coefficient of restitution, i.e., $u_1' - u_2' = -r(u_1 - u_2)$. If $r = 1$, the collisions are perfectly elastic and the system is the classical, one-dimensional perfect gas. If $r = 0$, the collisions are perfectly inelastic and we recover the system studied in Ref. 1. In between these two extremes, the system is a simple model of a one-dimensional "granular medium."

Reference 2 is a clear discussion of the physical basis of continuum approximations for granular media and Ref. 3 is a recent review. The study of granular media is motivated by a variety of astrophysical,[4] geophysical,[5,6] and industrial[7,8] problems. Elegant experiments with vertically vibrated granular layers provide unusual examples of pattern formation.[9]

The one-dimensional idealization is a nontrivial adjunct to these more realistic studies and has recently been investigated in Ref. 10. One surprising result of this work, arrived at independently by us, is that, with the inelastic dynamics in Eq. (1), it is possible for an infinite number of collisions to occur in a finite time.[10,11] Thus the relative velocity (and the separation) between adjacent inelastic particles can go to zero in finite time. We refer to this as "inelastic collapse," and in Sec. II, we study the phenomenon in more detail, paying particular attention to the elastic limit, $r \to 1$. We find that, as r becomes close to 1, inelastic collapse requires a number of particles that increases indefinitely. Thus the perfectly elastic case is a singular limit.

Section III discusses the simplest problem in the kinetic theory of granular media, viz., the cooling of a uniformly excited gas confined between inelastic walls.[2] This problem was used in Ref. 2 as a pedagogical example and a simple scaling argument, which we recapitulate, suggests that the mean square velocity decreases ultimately as t^{-2}. Our numerical simulations show that the scaling argument needs some qualifications in one dimension. The t^{-2} law applies unequivocally only if the number of particles in the gas is not large enough to trigger inelastic collapse (e.g., less than 16 when $r = 0.8$ and less than about 88 if $r = 0.95$).

Finally, in our conclusion, we speculate that inelastic collapse, and the associated velocity correlations, occur in higher dimensions. It is possible that these are related to the development of "inelastic microstructure" observed in two-dimensional simulations of granular flow.[12,13] And the singular nature of the limit $r \to 1$ points to difficulties with kinetic theories that draw an analogy between inelastic particles and molecules.

II. INELASTIC COLLAPSE

In this section, we document a phenomenon that is central to the statistical mechanics of a one-dimensional inelastic gas. We show that it is possible for inelastic particles to collide infinitely often in finite time. Because relative velocities approach zero exponentially with the collision number,

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and the number itself becomes infinite, a group of adjacent inelastic particles acquire exactly the same velocity in finite time. In addition, the interparticle separations also vanish completely. Of course, this is what happens when two perfectly inelastic particles[1] ($r = 0$) collide. Thus, even when r is nonzero, inelastic particles have a collective dynamics that is, in a sense, perfectly inelastic.

A. Inelastic collapse with small numbers of particles

The simplest example of inelastic collapse requires $r < 7 - 4\sqrt{3} \approx 0.0718$ and involves just three particles, as shown in Fig. 1(a). The two outer particles move monotonically toward each other and the one in the middle bounces to and fro in between. Because there are an infinite number of collision in finite time, the three particles condense into a single lump.

One can easily show that, after the two collisions in Fig. 1(a) the relation between the final and initial velocities is $\mathbf{u}' = \mathscr{M}\mathbf{u}$, where $\mathbf{u} \equiv (u_1, u_2, u_3)^T$ and $\mathscr{M}$ is a 3×3 matrix whose entries are quadratic polynomials in r. For collapse to occur, this matrix must have a least one real eigenvalue in the interval (0,1), so that the cycle in Fig. 1(a) endlessly repeats, but with geometrically smaller space and time scales in each successive cycle. This is the case when $r < r_c \equiv 7 - 4\sqrt{3}$. When r is slightly greater than r_c, the relevant eigenvalue becomes complex with an imaginary part proportional to $(r - r_c)^{1/2}$. This means that $\mathbf{u}$ rotates through a small angle after each cycle and eventually passes out of the octant defined by $u_1 > 0$, $u_2 > 0$, and $u_3 < 0$ and so the group disperses. But the number of collisions required to rotate $\mathbf{u}$ through an order one angle scales as $(r - r_c)^{-1/2} \gg 1$, and the three particles separate very slowly when r slightly exceeds r_c. These analytic results were confirmed by numerical simulation of the three-particle problem.

When r is greater than $7 - 4\sqrt{3}$, inelastic collapse requires the collective participation of more than three particles. Figures 1(b)–1(d) show two particles colliding with an inelastic wall (because of symmetry, this is equivalent to an interaction between four inelastic particles). When $r > 0.346\,015$, there is "quasispecular" reflection [Fig. 1(b)] in which the inner particle bounces off the wall twice. When $r = 0.346\,015$, the inner particle is stationary after its second collision with the outer particle [Fig. 1(c)]. If r is less than 0.346 015, then the inner particle bounces three (or more, as r decreases) times before it escapes from the wall. Finally, in Fig. 1(d), when $r < r_c \equiv 3 - 2\sqrt{2} \approx 0.171\,57$, the two particles collapse on the wall in finite time, i.e., there are an infinite number of bounces in finite time. And again, when r slightly exceeds r_c, the number of collisions required to disperse the four particles scales as $(r - r_c)^{-1/2} \gg 1$.

The critical coefficient of restitution for four-particle collapse, $r_c = 3 - 2\sqrt{2}$, was also given in Ref. 10. Both our calculation and that of Ref. 10 used the matrix method outlined at the beginning of this section.

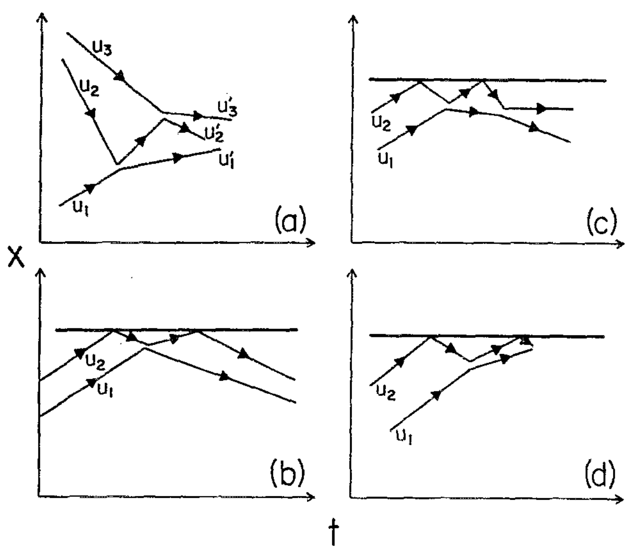


FIG. 1. Schematic examples of particle world lines (position, x, as a function of time, t). (a) Three particles collapse. Provided that $r < 7 - 4\sqrt{3}$, the two outer particles will keep moving together while the inner particle bounces back and forth in the middle. (b) Two particles bouncing off an inelastic wall when $r > 0.346\,015$. In this case, the inner particle collides with the wall twice. There is another collision between the two particles (not shown in this figure) at larger times. (c) At the critical value $r = 0.346\,015$ shown in this figure, the inner ball is stationary after its second collision with the outer ball. (d) When $r < 3 - 2\sqrt{2}$, the two particles collapse on the inelastic wall, i.e., there are an infinite number of collisions in finite time.

B. Inelastic collapse with large numbers of particles

As the coefficient of restitution r increases toward 1 so that the gas becomes more elastic, the number of particles required for collapse increases. For instance, with $r = 0.8$, we find that $N = 16$ particles bouncing off an inelastic wall collapse in finite time, but $N \leqslant 15$ particles do not.

Figure 2 shows the position-time trajectories of $N = 14$ and $N = 15$ particles as they collide with an inelastic wall located at $x = 0$. We use a "domino" initial condition in which the $N - 1$ particles closest to the wall are almost stationary and then the outer particle crashes into the end of the chain with $u = -1$.

An important distinction between $N = 14$ in Fig. 2(a) and $N = 15$ in Fig. 2(b) is the direction of the particle furthest from the wall at $t = 4$. In Fig. 2(a) the outer particle is moving away from the wall, while in Fig. 2(b) it is moving toward the wall. In the former case, the whole group is dispersing slowly. In the latter case, there is another medley of collisions at around $t = 15.3$, when the outer particle crashes into the bunch of 14 particles near the wall for the second time. However, this second burst of collisions does not produce inelastic collapse, and at $t = 16$, after 12 500 collisions, all 15 particles are moving away from the wall with very small velocities. This information is summarized in Fig. 3, which shows the number of collisions as a function of time for calculations with $N = 14$, 15, and 16 particles.

We believe that $N = 16$ is the minimum number of particles required for collapse when $r = 0.8$. Thus, in Fig. 3, the dotted curve, labeled 16, rises forever at about $t = 9.0$. In practical terms, we stopped the calculation after 411 500 collisions when the particle velocities were of order 10^{-64}.

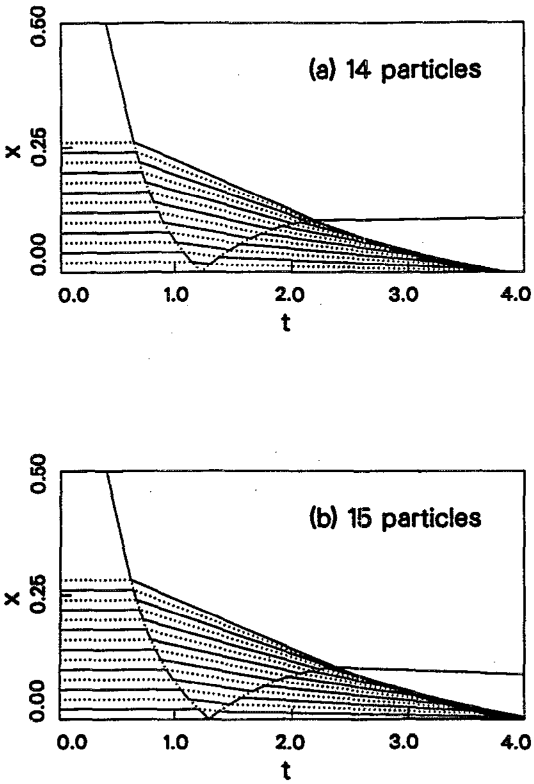


FIG. 2. Particle world lines for two calculations with $r = 0.8$. To emphasize the particle identities, every second world line is dashed. (a) A cushion of 13 stationary particles is struck by a rapidly moving particle at $t \approx 0.6$. This triggers a collision wave that passes through the cushion and reflects off the inelastic wall at $x = 0$. The collision wave passes back through the cushion and reaches the outer particle at about $t = 2.2$. The outer particle is left moving slowly away from the wall after the wave hits it. (b) This calculation differs from that of part (a) because the cushion now has 14 particles. At $t = 4$ the outer particle is still moving toward the wall and so it collides with the cushion again at about $t = 15.3$ (not shown).

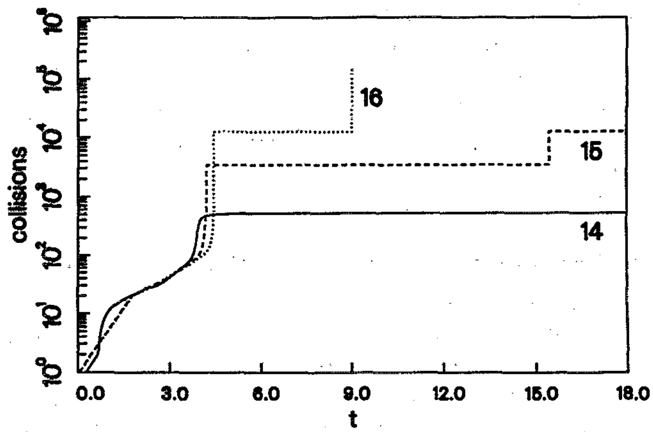


FIG. 3. Number of collisions as a function of time for three calculations. The curves labeled "14" and "15" refer to the simulations in parts (a) and (b) of Fig. 2. In both of these cases, the particles eventually disperse and the total number of collisions is finite. By contrast, the curve labeled "16" summarizes a calculation in which a cushion of 15 stationary particles is struck by a rapidly moving particle. In this case, inelastic collapse occurs at $t \approx 9.0$.

At this time, both the outermost particle and the center of mass are still moving toward the wall.

An analytic estimate of the minimum number of particles required for collapse when $r \approx 1$ is of interest. In Ref. 10, this minimum number of particles, $N_{\min}(r)$, has been estimated using an "independent collision wave" (ICW) approximation. We use the notation

$$p \equiv (1+r)/2, \quad q \equiv (1-r)/2 \tag{2}$$

in terms of which the ICW estimate[10] is

$$N_{\min}(r) \approx \pi/2q \quad \text{as } q \to 0. \tag{3}$$

In Sec. II D we present an alternative estimate of $N_{\min}(r)$ (the "cushion model"), which leads to

$$N_{\min}(r) \approx \ln(2/q)/2q \quad \text{as } q \to 0. \tag{4}$$

Our numerical simulations support Eq. (4) when $r \to 1$ and we conclude that the ICW approximation is inaccurate in this elastic limit. However, in the complementary limit $r \to 0$, the ICW becomes exact—it reproduces the earlier expressions for r_c with $N = 3$ and $N = 4$. From a number of simulations, it seems that the ICW approximation is accurate when $0 < r < 0.8$, but by $r = 0.9$ it begins to fail.

Notice that, when $r = 0.8$ ($q = 0.1$), Eq. (3) gives $N_{\min}(0.8) \approx 15.71$ and Eq. (4) gives $N_{\min}(0.8) \approx 14.98$. We indicated above that the numerical simulation gives $N_{\min}(0.8) = 16$. Thus one must go to smaller values of q, and larger numbers of particles, to distinguish between Eq. (3) and Eq. (4) and to reach the asymptotic regime.

The results from a series of calculations are summarized in Table I. In these simulations the N particles were projected at the inelastic wall as a "parallel beam." Thus, in the initial condition, all N particles are equally spaced and have the same velocity $u_i = -1$. (Actually, to avoid the coding problems, the velocities were perturbed from -1 by adding increments of 10^{-6}.)

Table I shows that, when $r = 0.9$ and $N = 36$, the group disperses after 6.14×10^6 collisions and the final mean square velocity is 4.58×10^{-10}. Notice that the ICW estimate in (3) predicts that collapse should have already occured at $N = 32$ or 33. In fact, we find that it is when $N = 37$ that the collision number increases indefinitely in finite time, i.e., the threshold for inelastic collapse has finally been exceeded. We stopped the simulation after 9.3×10^7 collisions and at this time the outermost particle (and many others) were still moving toward the wall. The very small kinetic energy (compared to that left in the dispersing cluster with $N = 36$) suggests that the addition of an extra particle has resulted in a qualitative change in the dynamics. The final column of the table shows that $N_{\min}(0.9)$ is accurately estimated by Eq. (4).

C. The effect of initial conditions

We were troubled by the possible effect of initial conditions on $N_{\min}(r)$. To address this point we experimented with three different initial conditions:

TABLE I. A summary of the numerical calculations. In each block of entries, we show the number of collisions required for a beam of N particles impinging on an inelastic wall to eventually disperse at that value of r. In the last line of each block, the number of collisions required for dispersion is certainly greater than the number indicated. This is our numerical estimate of $N_{\min}(r)$. The sudden drop in U^2 (relative to the penultimate entry) shows that there has been a qualitative change in the dynamics. An exception is the final block ($r=0.95$) where the collision numbers were too large to allow a conclusive check of Eq. (4).

r	N	Collisions	$U^2=\frac{1}{N}\Sigma u_i^2$	$\ln(2/q)/2q$
	14	2.68×10^3	1.35×10^{-11}	
0.8	15	9.93×10^3	3.26×10^{-18}	14.98
	16	$>4.0\times10^5$	$<10^{-115}$	
	20	1.65×10^4	2.38×10^{-9}	
0.85	21	3.45×10^4	1.14×10^{-8}	21.89
	22	$>1.68\times10^6$	$<10^{-201}$	
	33	6.72×10^5	2.41×10^{-6}	
	34	1.43×10^6	9.53×10^{-7}	36.89
0.9	35	3.01×10^6	2.13×10^{-7}	
	36	6.14×10^6	4.58×10^{-10}	
	37	$>9.3\times10^7$	$<10^{-72}$	
	44	1.5×10^7	1.49×10^{-6}	
	46	6.58×10^7	2.51×10^{-7}	48.90
0.92	47	1.35×10^8	6.43×10^{-8}	
	48	3.08×10^8	7.11×10^{-11}	
	49	$>2.0\times10^9$	$<10^{-37}$	
	60	6.85×10^5	9.51×10^{-9}	
0.95	65	1.31×10^7	7.50×10^{-6}	87.64
	70	3.07×10^8	4.15×10^{-6}	

(a) the "domino" configuration shown in Fig. 2;

(b) the "parallel beam" summarized in Table I and described above;

(c) an ensemble of 100 "random beam" simulations.

We have already indicated in our discussion above that both (a) and (b) led to the conclusion that $N_{\min}(0.8)=16$. In case (c), we took 100 different initial conditions, with positions uniformly distributed between 0 and 1, and velocities uniformly distributed between -0.1 and -1.1 (to ensure all particles were moving toward the same wall). All simulations ran until either $t=100$ or the collision count exceeded 4×10^5. With $r=0.8$ and $N=15$, all 100 realizations stopped at $t=100$ and at this time the mean square velocity defined in the fourth column of Table I, U^2, ranged from 10^{-30} to 10^{-15}. The number of collisions was between 8×10^3 and 1.6×10^4. But with $r=0.8$ and $N=16$, all 100 runs were halted because the collision count reached 4×10^5. At this point all 100 realizations had $U^2<10^{-107}$. Thus the threshold for inelastic collapse is not sensitive to initial conditions.

D. The cushion model

To motivate our approach to the estimate of $N_{\min}$ in Eq. (4) we refer to Fig. 2 and consider a "domino" initial condition in which $N-1$ balls are initially stationary and the Nth ball collides with the end of the chain and triggers the collision wave, which then passes through the assembly and reflects from the wall. The collision wave passes back through the chain and when it reaches the end it leaves the last ball either moving away from the wall [Fig. 2(a) with $N=14$] or toward the wall [Fig. 2(b) with $N=15$].

Our assumption is that the $N_{\min}(r)$ can be estimated by requiring that this last ball is moving toward the wall after the arrival of the reflected collision wave [Fig. 2(b) with $N=15$]. Thus the inner $N-1$ balls act as a "cushion" for the outermost ball. Actually $N=16$ balls are needed for collapse so that the cushion model does not lead to a precise value of $N_{\min}(0.8)$. But the results in the final column of Table I indicate that the estimate is accurate as $r\to1$.

We now calculate the critical number of balls required for collapse using the cushion model. We suppose that there are N balls and that positive velocities are directed toward the wall. Before any collisions occur

$$v_1=1,\quad v_2=v_3=\cdots=v_N=0, \tag{5}$$

and then after the first collision $v_1=q$, $v_2=p$, and all of the other velocities are still zero, i.e., the collision wave is now at ball 2. It is now straightforward to follow the collision wave through $N-1$ collisions so that

$$v_n=p^{n-1}q\quad(n\leqslant N-1),\quad v_N=p^{N-1}. \tag{6}$$

Note that, at this stage, the collision wave is at ball N so that the next interaction is with the inelastic wall.

When ball N reflects off the wall, its new speed is $v_N=-rp^{N-1}$. Now we can follow the collision wave as it makes its way back out through the assembly of $N-1$ slowly moving balls, i.e., ball N collides with ball $N-1$ so that $v_{N-1}=-rp^N+p^{N-2}q^2$. (It is not necessary to calculate the velocities of the slowly moving balls, such as v_N, after they have interacted with the collision wave twice—they all have small velocities toward the wall.) When the wave finally emerges at the end of the chain, one finds

$$v_1=-rp^{2(N-1)}+q^2[(1-p^{2(N-1)})/(1-p^2)]. \tag{7}$$

The critical condition is now obtained by requiring that v_1 in Eq. (7) is zero. Using $q\ll1$ and $N\gg1$ to simplify the resulting equation gives our estimate of $N_{\min}$ in Eq. (4).

III. COOLING OF A UNIFORMLY EXCITED GRANULAR MEDIUM

The simplest statistical problem for the one-dimensional inelastic gas is to suppose that $N\gg1$ inelastic particles are confined within the interval $0<x<l$ by inelastic walls. By analogy with Refs. 2 and 3 we speak of a "granular temperature," which is just the mean square velocity

$$U^2(t)\equiv\frac{1}{N}\sum_{i=1}^{i=N}u_i^2(t). \tag{8}$$

Here, $u_i(t)$ is the velocity of the ith particle at time t. We suppose that the walls have the same coefficient of restitution as the particles and are not moving. The gas is uniformly excited by picking $u_i(0)$ from some probability density function with zero mean and finite variance.

A. A scaling argument

We begin by recapitulating a simple scaling argument[2] that suggests $U\sim t^{-1}$ as the gas "cools." Numerical simulation shows that the argument is correct, provided that the number of particles in the gas is below the threshold for inelastic collapse.

The mean square velocity satisfies a rate equation

$$\frac{dU^2}{dt} = -c\tau^{-1}U^2, \tag{9}$$

where τ^{-1} is a collision frequency and c is a dimensionless constant. The idea is that each collision dissipates some fraction of the kinetic energy of the two participants. One also has the straightforward estimate $\tau^{-1} = U/a$, where $a \equiv l/N$ is the average distance between particles. Putting this into Eq. (9) and solving the rate equation gives $U \to 2a/ct$ as $t \to \infty$. We show below that this argument is incomplete because it makes no allowance for the strong velocity and position correlations that develop as precursors of inelastic collapse.

This scaling argument hints at a continuum explanation for the initial stages of inelastic collapse. If the particle density (proportional to $1/a$) is increased in some neighbourhood, then in that region the medium will cool more rapidly, because the collision rate is locally elevated. Thus the pressure falls more quickly where the density is high, and so particles in adjacent regions will be pushed by the resulting pressure gradient into those places with high density. Consequently, the density perturbations, which are responsible for nonuniform cooling, are reinforced by macroscopic motion.

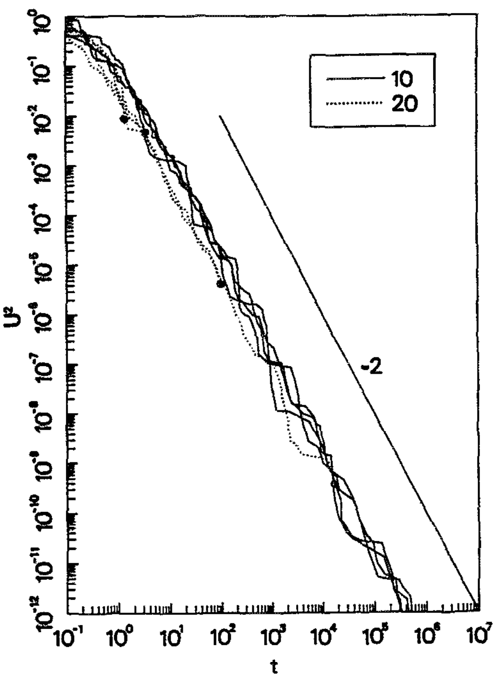


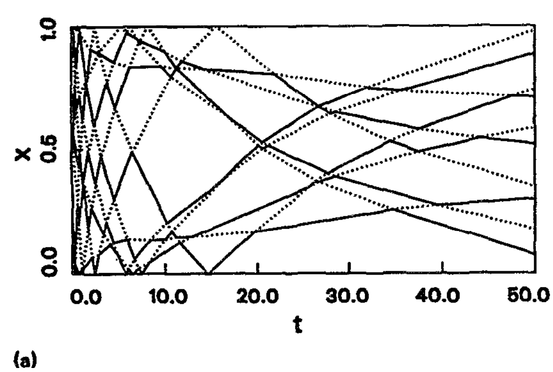
FIG. 4. The mean square velocity as a function of time for a cooling inelastic gas with $r = 0.8$. We show four realizations with $N = 10$ and 20. The t^{-2} regime is clear for realizations with $N = 10$. But in the other cases, the t^{-2} regime is transient and is halted by inelastic collapse (indicated by the heavy dots).

B. Numerical simulations of the cooling inelastic gas with $N = 10$ and 20

We simulated the cooling gas using $r = 0.8$ and different numbers of particles ($N = 10$, 20, 40, and 80) randomly placed in the interval $0 < x < 1$. The initial velocities are uniformly distributed in the interval $-\sqrt{3} < u_i < \sqrt{3}$, so that $U^2(0) \approx 1$ (with fluctuations of order $N^{-1/2}$). The walls at $x = 0$ and $x = 1$ are also inelastic. The results are summarized in Figs. 4 and 5.

Figure 4 shows $U^2(t)$ for eight realizations. The four realizations with $N = 10$ fully confirm the t^{-2} law. However, if N exceeds the threshold for inelastic collapse [which is $N_{\min}(0.8) = 16$], some of the particles come to rest in finite time. This is the case for the four realizations with $N = 20$. These all stop at the time of the first collapse when plots such as Fig. 3 show a vertical rise. (Of course, our collision-based simulation cannot evolve past this time.) But even when $N = 20$ there is a t^{-2} transient regime before the finite-time singularity stops the simulation. Simulations with $N = 60$ and $N = 80$ showed a similar behavior, and it is not surprising that the finite time singularity occurred earlier with larger numbers of particles.

Figure 5 shows two panoramic views of the cooling gas. In Fig. 5(a), with $N = 10$ particles, there is no collapse and the gas cools as damped collision waves pass back and forth through it. The $N = 10$ particles do not bunch up. In Fig. 5(b), with $N = 20$ particles, collapse occurs near the wall at $x = 1$. The particles near $x = 0$ do not participate in this event and so graphs of U^2 vs t, such as Fig. 4, show there is still some kinetic energy left in the gas after collapse.



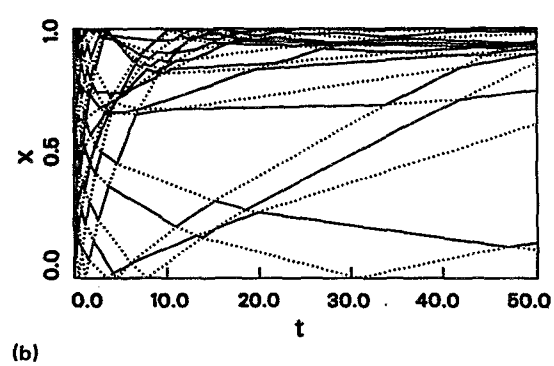


FIG. 5. Two examples of cooling simulations. Every second world line has been dashed to emphasize the particle identities. (a) In the upper panel, where $N = 10$, collapse does not occur. The collision waves passing back and forth through the gas are apparent and the particles do not bunch up. (b) In the lower panel, where $N = 20$, the gas collapses against the inelastic wall at $x = 1$.

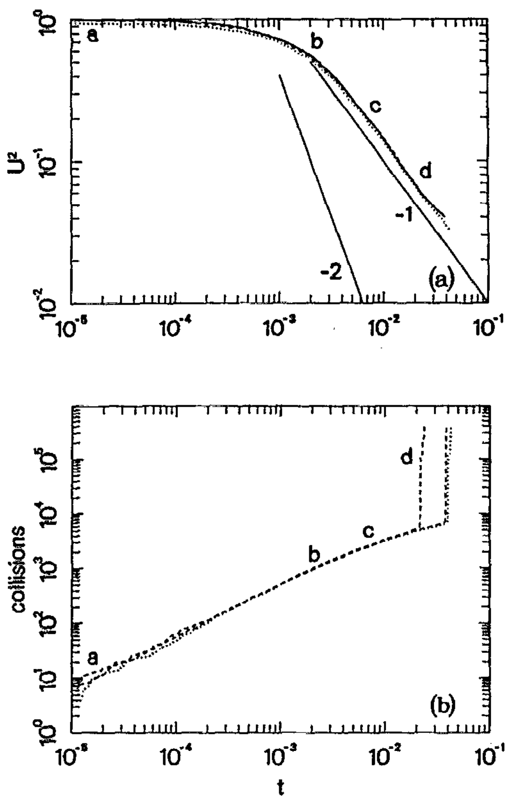


FIG. 6. Three realizations of a cooling simulation with $r = 0.8$ and $N = 1000$. (a) The granular temperature as a function of time. There is a rough t^{-1} fit but, because collapse brings all of the simulations to a halt, this tentative regime is not convincingly demonstrated. (b) Collision number as a function of time. The vertical rise indicates that collapse has occurred.

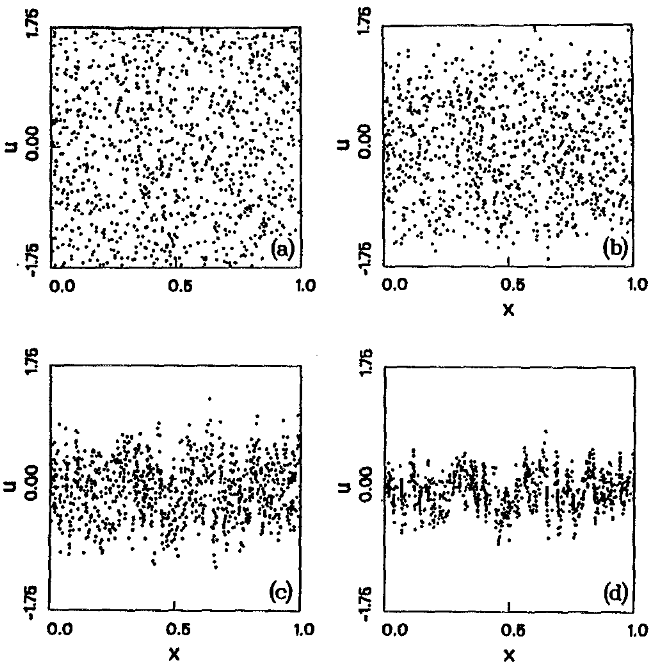


FIG. 7. The postion-velocity phase space of an $N = 1000$ realizations from Fig. 6. The four panels, (a)–(d), are indicated by the corresponding labels in Fig. 6. The final panel, (d), is on the vertically rising part of Fig. 6(b) and shows the velocity and position correlations accompanying collapse.

C. Numerical simulations of the cooling inelastic gas with $N = 1000$

In most applications, there are very large numbers of particles and the threshold for collapse is exceeded by many orders of magnitude. As an illustration of this regime we show in Fig. 6 the results of three simulations, all with $N = 1000$ and $r = 0.8$. Again there are inelastic walls at $x = 0$ and $x = 1$ and in each of the three realizations the initial conditions have uniformly distributed positions with $0 < x < 1$ and uniformly distributed velocities with $-\sqrt{3} < u < \sqrt{3}$. The simulations eventually stop evolving because of collapse. It is interesting that, with $N = 1000$ particles, the t^{-2} cooling law no longer applies, even as a transient scaling regime. Instead, as we have indicated in Fig. 6(a), there seems to be a rough t^{-1} fit. We return to this point below.

The "phase space" plots in Fig. 7 are a useful way of visualizing the cooling of this many-particle gas. The position and velocity of each particle is represented by a dot and the four parts of this figure, (a)–(d), correspond to the points indicated by (a)–(d) in both parts of Fig. 6. Thus Fig. 7(a) is essentially the initial condition in which the points are uniformly strewn in the rectangle $0 < x < 1$ and $-\sqrt{3} < u < \sqrt{3}$. The subsequent evolution to Fig. 7(d) shows the formation of strong velocity and position correlations in the inelastic gas. The scaling argument in Sec. III A fails because of these correlations.

Figure 8(a) shows a histogram of the particle positions for the state in Fig. 7(d). The interval $0 < x < 1$ was divided into 100 bins and the number of particles in each bin was counted. It is obvious that there is a significant clustering of particles: although the average number of particles in each bin is ten, the standard deviation is much greater than the Poisson value, $\sqrt{10}$. In particular, there are 38 particles in bin 45 and it is this largest clump that is responsible for the rising collision count in Fig. 6(b). This peak, and the second and third largest, are unchanged if the number of bins is increased to 200. The clumps in Fig. 8(a) are roughly of the size we would expect on the basis of the arguments from Sec. II. That is, with $r = 0.8$ it takes at least 16 particles bouncing against an inelastic wall, or 32 in the middle of the gas, to trigger collapse.

Figure 8(b) shows a histogram of the particle velocities from the state in Fig. 7(d). The kurtosis of this distribution,

$$\mathrm{Ku} \equiv U^{-4}\left(\frac{1}{N}\sum_{i=1}^{i=N} u_i^4\right), \tag{10}$$

is 2.97, i.e., very close to the Gaussian value of 3. In Fig. 9 we show $\mathrm{Ku}(t)$ for six $N = 1000$, $r = 0.8$ simulations. The three simulations that begin with $\mathrm{Ku} \approx 1.8$ are the three simulations in Fig. 6. The other three have initial velocity distributions chosen to give $\mathrm{Ku} \approx 6$. The six realizations have all reached the Gaussian value by the time of collapse. Although the kurtosis of the data in Fig. 8(b) is close to the

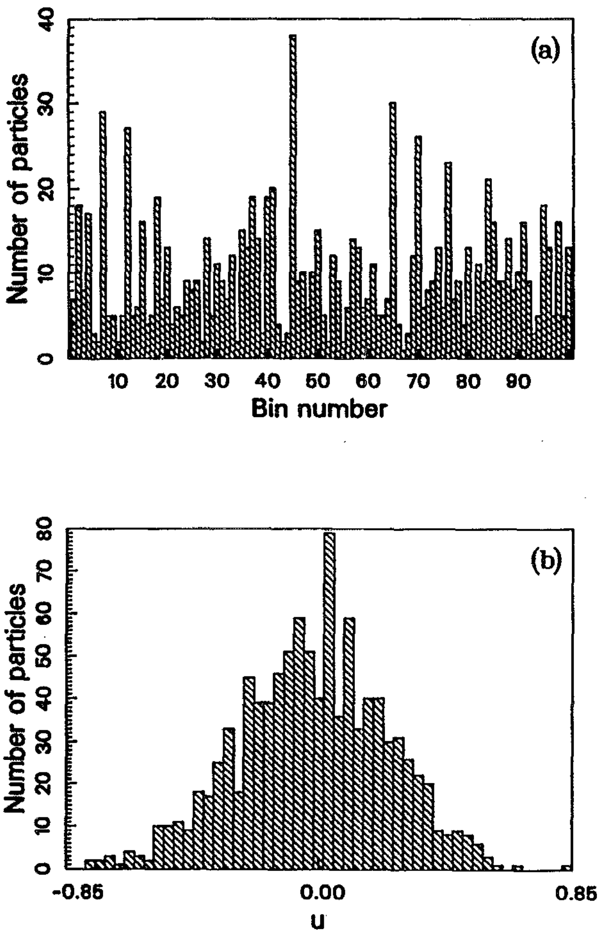


FIG. 8. (a) This figure shows a more detailed view of the particle density in Fig. 7(d). The position axis, $0<x<1$, is divided into 100 bins and the number of particles in each bin is counted. There is obvious clumping and, indeed, the largest agglomerations are unchanged if the bin size is halved. (b) The velocity distribution of the particles from Fig. 7(d).

Gaussian value, it fails the Kolmogorov–Smirnov test[14] for a Gaussian distribution. We believe this happens because the velocities of the particles are not independent random variables; they are grouped into clumps where all particles have almost the same velocity. Thus, although there are 1000 particles, there are not 1000 independent samples from the same velocity distribution. Furthermore, it is not surprising that the velocities of the clumps would have a Gaussian distribution. Since collisions conserve momentum, the velocity of a clump will just be the average velocity of all the particles that fall into the clump as it forms. By the central limit theorem, we expect this average to have a Gaussian distribution.

Finally, in Fig. 10 we examine the effects of changing the coefficient of restitution. The six simulations all have $N=1000$ particles and the same initial condition. In all six cases inelastic collapse brings the simulation to a halt and it is not surprising that this happens earlier when r is smaller. We mentioned before in our discussion of Fig. 6 that there is a possible t^{-1} cooling regime in the $r=0.8$ simulations. Figure 9 shows that, if a simple cooling law exists before collapse occurs, then the exponent seems to be a function of

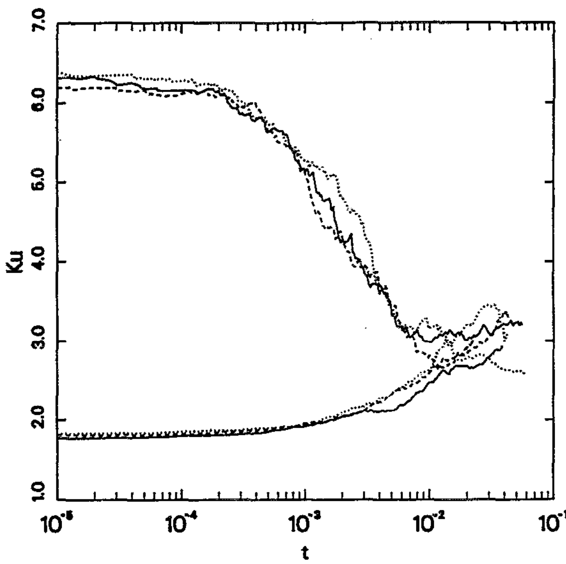


FIG. 9. The kurtosis, defined in Eq. (10), as a function of time for six realizations of the $r=0.8$ and $N=1000$ inelastic gas. The three simulations that begin with $\mathrm{Ku}\approx 1.8$ are the three simulations in Fig. 6. The other three have a piecewise constant initial velocity distribution, constructed to give $\mathrm{Ku}\approx 6$ and $U^2\approx 1$. By collapse time, all six realizations are close to the Gaussian value $\mathrm{Ku}=3$.

r. In particular, the more elastic simulations have steeper slopes. For instance, the case with $r=0.9$ is slightly steeper than the straight line of slope -1. And if N is fixed and r is increased toward 1, then we eventually expect to recover the t^{-2} regime demonstated in Fig. 4. Indeed, a simulation at $N=800$, $r=0.995$ (this is below the threshold for collapse) clearly showed t^{-2} cooling.

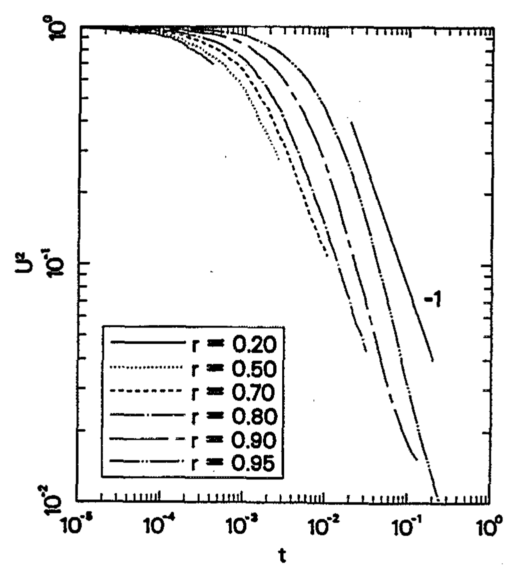


FIG. 10. Cooling laws for $N=1000$ particle gases with various coefficients of restitution r. In all six cases, the simulations are halted by inelastic collapse. There is some evidence of a transient scaling regime with a slope that steepens as r approaches 1.

IV. CONCLUSION AND DISCUSSION

We emphasize that, because of collapse, the model defined by the binary collision rule in Eq. (1) is not well formulated past a finite time. To advance simulations past this point we need to either append additional rules, encompassing simultaneous multiparticle collisions, or alter the dynamics. For instance, one might suppose that the coefficient of restitution is a function of the relative velocity of the impact so that very gentle collisions are either perfectly inelastic ($r = 0$) or perfectly elastic ($r = 1$). It seems likely that both of these choices will eliminate the singularity. Other alternatives include "particle-overlap" strategies, such as those described in Ref. 12. A thorough exploration of these possibilities is beyond the scope of this work. However, it seems that any reasonable scheme that finesses the barrier of an infinite collision count must retain the strong position and velocity correlations that develop as precursors of this phenomenon. These correlations are an inevitable consequence of inelastic dynamics in one dimension.

The development of position and velocity correlations is obvious when the number of particles in the gas, N, is much greater than the minimum number required for collapse, $N_{\min}$. Thus, in Fig. 7, we see the formation of clusters of particles of size $N_{\min}$. [Here, we refer to conditions in the middle of the gas, so that $N_{\min}$ is twice the value of Eq. (4).] The implication of this is that a state of uniform density is unstable to the formation of aggregates and that the wavelength of the instability is roughly $aN_{\min}$, where a is the average interparticle separation. This length scale is evident in Figs. 7(d) and 8(a) as the separation between the clusters.

A referee has suggested that "an infinite number of collisions in a finite time" is just another way of saying "in contact." It is true that, as a consequence of an infinite collision count, particles do come into contact, which is a well-known phenomenon in granular flows. However, we believe that inelastic collapse is a distinct process that differs from what is usually intended by saying that the particles come into contact. Inelastic collapse is a process that can occur in the heavily idealized model described in the first paragraph of this article: smooth particles with no static friction between them. At the very least, it serves as a counterexample to the view that sustained contact can result only from finite-sized, rough particles sliding relative to one another.

We emphasize the important restriction of our results to one dimension. We do not know if inelastic collapse occurs in two and three dimensions. But it is an interesting empirical fact[8,9] that a vertically vibrated granular bed does not bounce when it collides with the base of the containing vessel. In Ref. 9, where the bed consists of glass spheres with $r > 0.8$, the collision with the base is completely inelastic. These experimental results are consistent with the notion that an ensemble of particles, each with r rather close to 1, can have a collective behavior that is perfectly inelastic. However, a completely convincing experimental demonstration of inelastic collapse would require one to show that the bed does bounce when its thickness is below a well-defined threshold, and moreover, that the threshold thickness increases as $r \to 1$. We have not found any published experiments bearing directly on this.

Computer simulations of two dimensional granular media, such as those in Refs. 12 and 13, are another approach to this question. The "hard-disk" simulations in Ref. 13 do show the development of a spatial correlations between particle positions, but there is no indication that these are accompanied by an unbounded collision count. It may be that the total number of particles in these simulations (about 40) is below the threshold for inelastic collapse in two dimensions.

Insofar as they resemble the peaks in Fig. 8(a), the "microstructural agglomerations of discs" described in Ref. 12 are suggestive of inelastic collapse. Also, these agglomerations are more obvious when the coefficient of restitution is small and this is consistent with our one-dimensional intuition. But the algorithm used in Ref. 12 steps in discrete time intervals, so that, while it might accurately represent the initial development two-dimensional inelastic collapse, it cannot follow this phenomenon to its singular finale. Further, in both Refs. 12 and 13, the granular medium is subjected to external shear and presumably the associated stresses tend to disrupt the formation of lumps. The cooling simulations described in Sec. III are probably the most favorable conditions for inelastic collapse.

One certain conclusion of the present work is that the continuum theories of Refs. 2 and 3 do not apply to the strictly one-dimensional gas, except perhaps in a double limit in which $r \to 1$ and $N \to \infty$ so that one is always below the threshold for collapse.

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an infinite number of bounces at $T = 2u/g(1-r)$. The bouncing frequency is $(-1/\ln r)(T-t)^{-1}$, which is the "sound of the singularity" as the ball stops.

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