Long-Wave Instability in Marginally Stable Shear Flows

N. J. Balmforth and W. R. Young
University of California at San Diego, La Jolla, California 92093-0230
(Received 15 July 1997)

A reduced description of disturbances on marginally stable shear flows is presented. The system is the Boussinesq equation coupled to a critical-layer, vorticity equation. [S0031-9007(97)04647-4]

PACS numbers: 47.20.Ft, 03.40.Gc

The reductive perturbation theories used by physicists typically fall into two classes: adiabatic elimination of strongly damped variables in dissipative systems, or averaging over fast oscillations in finite dimensional Hamiltonian systems. Our interest here is a third class which in a peculiar way combines aspects of both methods; this is the development of reduced descriptions for instabilities in nearly ideal fluids and plasmas. Near the ideal limit, the dissipative method is of no use. Moreover, because ideal fluids and plasmas are infinite dimensional Hamiltonian systems, there is a continuum of modal frequencies with no obvious time-scale separation that can be exploited to obtain simplifications. Yet simultaneously there is a peculiar form of (apparent) dissipation—Landau damping in plasmas and Orr-Kelvin decay in fluids. There are the essential physical ingredients which must be retained in any respectable approximation.

As a concrete example exemplifying the difficulties outlined above, we derive a reduced model of weakly nonlinear disturbances on marginally stable, high Reynolds number shear flows. Long-wave theory is suggested because shear flows, such as that shown in Fig. 1, first become marginally unstable to disturbances with infinite scale in the streamwise direction [1]. More specifically, flows that have negative vorticity everywhere and contain an inflection point in the velocity profile can become unstable through $k = 0$, where k is the streamwise (x) wave number.

Weakly nonlinear descriptions are subtle developments of the linear theory. But for shear flows the underlying linear description presents difficulties associated with the "critical level" singularity where the disturbance travels locally with the speed of the ambient flow [2]. The singularity introduces some difficult mathematics in the expansion; this is symptomatic of the modal continuum. Fortunately, a combination of long-wave theory with matched asymptotic analysis (using a development of the techniques in [3]) saves the day.

The expansion opens with a study of the linear stability of a basic state like that in Fig. 1; there is an inflection point at the center of the channel. This is the critical level of the marginally stable mode. In the bulk of the flow, the leading order solution is given by the marginal, linear eigenfunction. However, the modal amplitude, denoted $b(x,t)$ is undetermined. This is the first term of the "outer" solution of the matched asymptotics. Because this eigenfunction is regular, there is no evidence of the critical level singularity at this stage. But we must proceed to higher order to find an evolution equation for $b(x,t)$; at these higher orders in the outer solution, the unpalatable singularity of the critical level appears.

One way of dealing with the singularity is to assume that the vorticity is uniform at the critical level (that is, all of the derivatives of the vorticity vanish there). If we follow this route, we arrive at a standard long-wave equation, known as the Boussinesq equation [4]. However, this Boussinesq equation fails to capture the dynamics of the critical level, and it is difficult to reach any compelling conclusions about the development of disturbances on generic basic states.

An alternative way of dealing with the apparent singularity of the outer flow is to recognize that the critical level singularity reflects a failure of the outer expansion: there is a region, the "critical layer," surrounding this level in which the vorticity field is not slaved to the evolution of $b(x,t)$. This forces us to include a second field, $\zeta(x,y,t)$, the vorticity inside the critical layer, to complete the evolution equations. The final coupled system is then obtained on matching the inner and outer solutions in some intermediate region.

The key feature of the expansion, then, is that the outer fields appear singular as one approaches the critical layer. The strength of the singularity is determined by the curvature of the vorticity of the basic state at the critical level; that is, U_c''' in conventional notation and in

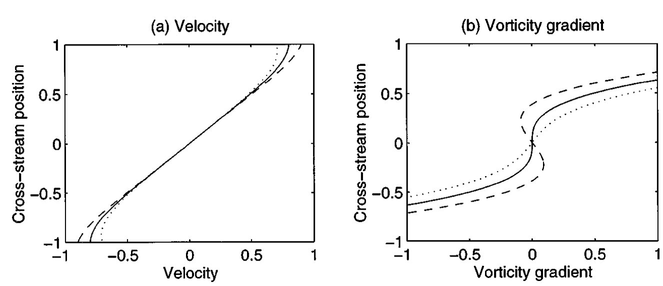

FIG. 1. Sample velocity profiles. The solid curves show a profile with a double inflection point ($U'' = U''' = 0$) at the critical level in the middle of the channel. The dashed ($\kappa < 0$) and dotted ($\kappa > 0$) curves are adjacent equilibria. The bad case is $\kappa < 0$ in which two new inflection points appear.

our notation $\kappa \propto -U_c'''$. To postpone the effects of this singularity until a manageable order in the expansion, we demand that U_c''' is small in a certain asymptotic sense. This constrains us to consider profiles that are adjacent to an equilibrium with a "double" inflection point as shown in Fig. 1. One important property of the profile is that, if κ is negative, the double inflection point splits into three. This leads to uncontrolled asymptotics and failure of our expansion (there are unstable modes associated with the new inflection points that break the ordering of the expansion). Therefore, we restrict attention to $\kappa > 0$ and a singly inflected flow.

The result of the reduction is the "Boussinesq critical layer equation." In dimensionless form, the system is

$$b_{tt} + \sigma b_{xx} + b_{xxxx} - \frac{1}{2}(b^2)_{xx} = \int_{-\infty}^{\infty} \zeta_{xx}\, dy\,, \quad (1)$$

$$\zeta_t + y\zeta_x - \lambda\zeta_{yy} + b_x\zeta_y = \kappa b_t\,, \quad (2)$$

where $-\infty < y < \infty$ is the coordinate within the critical layer.

There are three parameters in the equations, σ, λ, and κ. In addition, should we solve (1) and (2) on a finite, periodic domain in x, then there is also a dimensionless domain size, L. The parameter σ is equal to ± 1, depending on whether the flow is linearly unstable or stable. λ is the viscosity parameter (essentially the Haberman parameter of critical layer theory).

Without loss of generality, we require that the x average of b is zero; in the asymptotic expansion this requirement is used to distinguish the basic state from the disturbance, b. We also solve Eq. (2) subject to the boundary conditions, $\zeta \to 0$, as $|y| \to \pm\infty$. The integral on the right-hand side of (1) must then be evaluated as a principal value at infinity because it follows from (2) that $\zeta \sim y^{-1}$.

The Boussinesq critical layer system possesses a number of conservation laws. Define ϑ and ϖ by $b_t = \vartheta_x$ and $\vartheta_t = -\varpi_x$. Denote an integral over $0 < x < L$ by $\langle\,\rangle$ and define $Z_0 \int_{-\infty}^{\infty} \zeta\, dy$, $Z_1 \equiv \int_{-\infty}^{\infty}(y\zeta - \kappa\vartheta)\, dy$, and $Z_2 \equiv \int_{-\infty}^{\infty}(y^2\zeta - \kappa\vartheta y - \kappa\varpi)\, dy$. Then, $\langle b\rangle$, $\langle\vartheta\rangle$, $\langle xb\rangle_t$, $\langle Z_0\rangle$, and $\langle Z_1 + b\vartheta\rangle$ are constants of motion. Also,

$$\langle Z_2 + \sigma b^2 - \vartheta^2 - b_x^2 - b^3/3 - 2bZ_0\rangle_t = 2\lambda\langle Z_0\rangle\,; \quad (3)$$

we refer to (3) as the energy equation. This relation implies that the energy integral in (3) increases linearly with time, unless there is no mean vorticity in the disturbance. This is merely the secular growth of the usual Poincaré invariant in two-dimensional, viscous fluids. Here, it signifies that the x-averaged vorticity inside the critical diffuses viscously outwards [5].

One limit of the system (1) and (2) is the pure Boussinesq equation, which is a well-known model of marginally stable, long-wave disturbances [4]. To obtain this special case, we set $\kappa = 0$ and suppose that $\zeta = 0$ at $t = 0$. Then $\zeta = 0$ for all time, and the field $b(x,t)$ evolves in splendid isolation. The Boussinesq equation has an infinite number of conservation laws and an inverse scattering transform [6]. We do not know how much of this mathematical structure survives in the inviscid ($\lambda = 0$) version of the system (1) and (2) when $\zeta \neq 0$.

We can use (1) and (2) to obtain three basic results concerning viscous and inviscid shear flows.

Linear theory.—First we consider the linear initial-value problem. If we ignore the nonlinear terms in (1) and (2), and then look for solutions with dependence $\exp(ikx)$, we arrive at

$$b_{tt} + (k^2 - \sigma)k^2 b = -k^2 \int_{-\infty}^{\infty} \zeta\, dy\,, \quad (4)$$

$$\zeta_t + iky\zeta - \lambda\zeta_{yy} = \kappa b_t\,. \quad (5)$$

We solve these equations subject to initial conditions in which b and b_t are specified and $\zeta(x,y,0) = \zeta_0(y)e^{ikx}$. Equation (5) has the solution

$$\zeta = \kappa \int_0^t g(t-s,y)b_s(s)\, ds + e^{ikx}\int_{-\infty}^{\infty} \hat{\zeta}_0(q+kt)e^{\lambda[q^3-(q+kt)^3]/3k + iqy}\frac{dq}{2\pi}\,, \quad (6)$$

where $g(t,y) = \exp(-ikyt - \lambda k^2 t^3/3)$ and $\hat{\zeta}_0(q)$ is the Fourier transform ($\partial_y \to iq$) of the initial condition, $\zeta_0(y)$. On substituting Eq. (6) into (4), and making use of the identity $\int_{-\infty}^{\infty} g(t,y)\, dy = 2\pi|k|^{-1}\delta(t)$, we obtain

$$b_{tt} + \pi\kappa|k|b_t + (k^2 - \sigma)k^2 b = -k^2\hat{\zeta}_0(kt)e^{ikx - \lambda k^2 t^3/3}\,. \quad (7)$$

The homogeneous part of Eq. (7) has normal mode solutions $b = \exp[ik(x - ct)]$, where c satisfies

$$c^2 + i\pi\,\mathrm{sgn}(k)\kappa c + \sigma - k^2 = 0\,. \quad (8)$$

It is remarkable that this dispersion relation is independent of λ, provided only that $\lambda \neq 0$.

From (8), we see that there is instability when $\sigma > k^2$. The dispersion relation is shown in Fig. 2. The forcing term on the right-hand side of (7) arises from the vortical perturbations injected at $t = 0$. Thus (7) shows how the normal modes are excited by initial disturbances.

In (7) there is no difficulty in taking the inviscid limit; simply set $\lambda = 0$. Thus the viscous initial value problem limits identically to its inviscid counterpart. However,

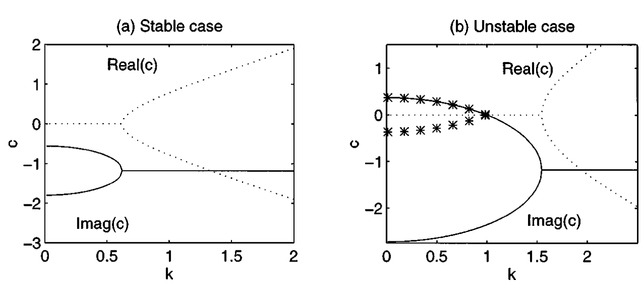

FIG. 2. Eigenvalues for (a) $\sigma = -1$ and (b) $\sigma = 1$. In (b), the inviscid eigenvalues are shown by $*$; there are no inviscid eigenvalues in case (a).

this is not true of the normal modes; Eq. (8) is *not* the inviscid dispersion relation. If we set $\lambda = 0$ in (5), the inviscid normal mode has an eigenfunction,

$$[\zeta, b] = e^{ik(x-ct)}[\kappa c/(c - y), 1], \tag{9}$$

and then, using $\int dy/(y - c) = i\pi\,\mathrm{sgn}(c_i)$, it follows that the inviscid dispersion relation is

$$c^2 + i\pi\,\mathrm{sgn}(c_i)\kappa c + \sigma - k^2 = 0 \qquad (\text{if } \lambda = 0). \tag{10}$$

The key difference between (8) and (10) is the $\mathrm{sgn}(c_i)$ in (10). This discontinuous function signifies that the inviscid dispersion relation is not analytic in c; in fact, it has a branch cut along the c_r axis; this locates the continuous spectrum.

Equation (10) implies that there are discrete inviscid modes only in the unstable band ($\sigma > k^2$). The inviscid eigenvalues are also shown in Fig. 2. From this figure it is evident that the unstable inviscid mode is equivalent to the unstable viscous mode. However, there is a stable inviscid mode that has no viscous counterpart [7] and *vice versa*.

The damped viscous modes obtained from (8) have a curious correspondence with inviscid eigenvalues obtained from (10): if one ignores the $\mathrm{sgn}(c_i)$, then the damped viscous modes appear as "inconsistent" solutions of the inviscid dispersion relation. We refer to these inconsistent solutions of (10) as quasimodes of the inviscid problem. The quasimodes appear in the initial value problem whether λ is finite or zero. In ideal plasma theory the decay associated with such quasimodes is Landau damping; here, we have the fluid analog. The distinction between the inviscid quasimodes and bona fide vicous modes can only be appreciated by examining the vorticity in (6) which, when $\lambda = 0$, contains a nonseparable and undamped term of the form $\exp(ikyt)$. Thus, the inviscid vorticity does not have normal mode form, and this is why the quasimodes escape the net cast by the normal mode ansatz. These subtleties evaporate with the addition of viscosity: the quasimodes disappear but are replaced by damped viscous normal modes that have intensely oscillatory vertical structure when $\lambda \to 0$ [8].

The viscous critical layer.—We next consider strongly viscous critical layers and take $\lambda \gg 1$. We then adiabatically eliminate ζ through the relation, $\lambda\zeta_{yy} - y\zeta_x = \kappa b_t$. This leads to

$$\begin{aligned}\int_{-\infty}^{\infty} \zeta_x\, dy &= \kappa \int_{-\infty}^{\infty} \frac{b_t(x',t) - b_t(x,t)}{x' - x}\, dx' \\ &= \pi\kappa\mathcal{H}[b_t],\end{aligned} \tag{11}$$

where $\mathcal{H}$ denotes a Hilbert transform. Hence the Boussinesq part of the system is

$$b_{tt} - \pi\kappa\mathcal{H}[b_{xt}] + \sigma b_{xx} - b_{xxxx} - (b^2/2)_{xx} = 0. \tag{12}$$

The nonlocal dissipative Hilbert transform term in (12) is anticipated by the factor $\mathrm{sgn}(k)$ in (8).

Equation (12) again has conservation laws. One of the more notable of these is the analog of the energy equation (3),

$$\langle \vartheta^2 + b_x^2 - \sigma b^2 + b^3/3\rangle_t = \pi\kappa\langle\vartheta\mathcal{H}[\vartheta_x]\rangle \le 0 \tag{13}$$

($\mathcal{H}[\vartheta_x]$ is a negative definite, pseudodifferential operator). Thus the energy functional must decrease in time. In fact, it is only stationary provided $\vartheta = 0$, that is, for steady solutions. These steady solutions are the steady solutions of the Boussinesq equation, and the energy integral can be identified as part of the Hamiltonian functional of that equation. Therefore, much as a particle in a potential well will sink to rest at the minimum of the potential, the Boussinesq critical layer system relaxes to a steady state that minimizes the Hamiltonian.

Inviscid equilibria.—Third, we establish some inviscid equilibrium solution to the coupled, Boussinesq-critical-layer equations using the techniques in [9].

If we neglect the viscous term, and look for traveling wave solutions with forms $b = b(x - ct) = b(\xi)$ and $\zeta = \zeta(x - ct, y) = \zeta(\xi, y)$, then Eqs. (1) and (2) reduce to

$$b_{\xi\xi} + (\sigma + c^2)b - \frac{1}{2}b^2 + P = \int_{-\infty}^{\infty} \zeta(\xi, y)\, dy, \tag{14}$$

$$(y - c)\zeta_\xi + b_\xi\zeta_y + c\kappa b_\xi = 0, \tag{15}$$

where P is a constant of integration, chosen to ensure that the x-average of b vanishes.

Equation (15) is solved by $\zeta + c\kappa y = \mathcal{Z}(\Psi)$, where $\Psi \equiv B(\xi) - (y - c)^2/2$ and $B(\xi) \equiv b(\xi) - b_{\min}$. We confine attention to the case in which b is a periodic function of ξ so that the (ξ, y) plane is divided into three regions as shown in Fig. 3. There are two "open streamline" regions, $\mathcal{O}^+$ and $\mathcal{O}^-$; the open streamlines are separated by a recirculating eddy of closed streamlines (i.e., the cat's eye). We denote the region of closed streamlines by $\mathcal{C}$. The separatrix is $\Psi = 0$ and, inside $\mathcal{C}, \Psi > 0$.

The function $\mathcal{Z}(\Psi)$ can have different branches in each of $\mathcal{C}$ and $\mathcal{O}^\pm$. We restrict attention to a simple class of solutions by taking $\zeta + c\kappa y = \pm\sqrt{-2\Psi}$, in $\mathcal{O}^\pm$, respectively; this ensures that $\zeta \to 0$ as $|y| \to \infty$. Within $\mathcal{C}$ we write $\zeta + c\kappa y = \mathcal{Z}_C(\Psi)$. Because of the antisymmetry of the vorticity in $\mathcal{O}^+$ and $\mathcal{O}^-$, we evaluate $\int_{-\infty}^{\infty} \zeta\, dy$ by integrating only over $\mathcal{C}$. Then, Eq. (14) can be written as

$$b_{\xi\xi} + (\sigma + c^2)b - \frac{1}{2}b^2 + P = 2\int_0^B \frac{\mathcal{Z}_C(\Psi)\, d\Psi}{\sqrt{B - \Psi}}. \tag{16}$$

The right-hand side of (16) is the Abel integral.

The integral equation (16) can be simplified in one of two ways. The first option is to specify the function $\mathcal{Z}_C$ and then evaluate the integral on the right of (16). This leads to a nonlinear oscillator equation which determines the stream function amplitude, $b(\xi)$. In essence, we fix

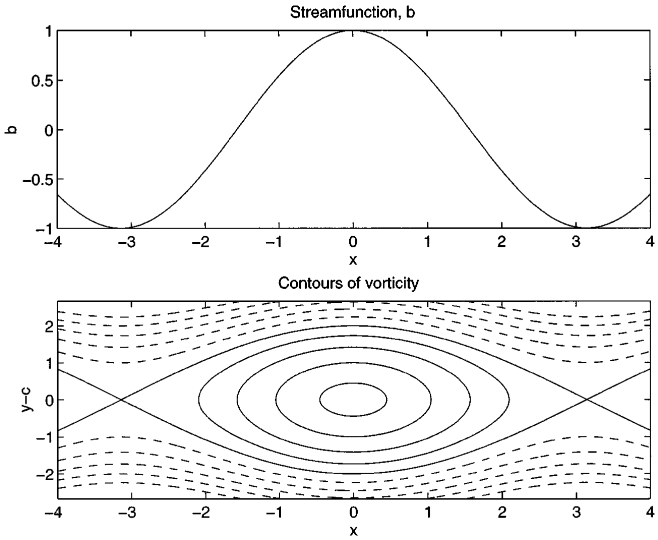

FIG. 3. Geometry of the inviscid equilibrium. In the cat's eye, the vorticity is given by $\zeta = \mathcal{Z}_C(\Psi) - c\kappa y$.

the vorticity distribution in the cat's eye, then find the associated stream function.

Alternatively, we may proceed by specifying $b(\xi)$, and then express the left-hand side of (16) as a known function of b. Then we solve the resulting Abel equation for $\mathcal{Z}_C$. This procedure determines the cat's eye vorticity distribution that corresponds to a given stream function.

It is clear that with any of these approaches we obtain an infinitude of inviscid equilibria. Two special cases of the construction are the following. With the selection $\mathcal{Z}_C(\Psi) = 0$, we recover all the equilibrium solutions of the Boussinesq equation, including the solitary wave that exists for $\sigma = -1$. In the limit of small amplitude, we may expand the terms in (16) in a series in powers of b. The leading order, linear terms lead to a condition that is equivalent to the condition for the existence of a neutral, linear normal mode. Subsequent nonlinear corrections allow us to build finite-amplitude solutions that limit to this neutral mode.

The Boussinesq critical layer system simplifies many problems that arise in shear flow dynamics; some issues open to future work are the following. Stewartson [10] has expressed misgivings concerning the relevance of nonlinear solutions which are obtained by invoking the Prandtl-Batchelor theorem to justify the assumption that the vorticity is uniform within the closed streamline region C (e.g., [3]). On the other hand, if one ignores viscosity completely, then whole classes of solutions can be manufactured; from a physical perspective, this freedom is equally unsatisfactory. The way out of this impasse is a proper treatment of the slightly viscous initial value problem, a task that is lighter if one starts with a reduced description such as (1) and (2).

Another direction lies in understanding how the critical layer modifies the dynamics of the Boussinesq equation. For example, if $\sigma = +1$, then the Boussinesq equation has solutions that blow up in finite time [5]. Can the Boussinesq blow up be halted by the reaction of the critical layer?

We close with some more general remarks about the coupled system (1) and (2). In reductions of dissipative problems, or in averaging methods, one is able to simplify the system by reducing the number of dimensions. If such methods were to work in the problem at hand, we would derive a long wave equation in the single coordinate, x, such as the Ginzburg-Landau equation or the pure Boussinesq equation. However, because of the continuous spectrum, at each order of the asymptotic expansion, singularities of increasingly higher degree appear at the critical level [11]. The cure is an inner expansion of the critical layer that leaves us with a system which still involves both x and y (see also [12]). In fact, at first sight (1) and (2) might appear to be as complicated as the Navier-Stokes equation. The examples we have presented show that this is not so.

This collaboration began at the Woods Hole Oceanographic Institutions's Summer Study Program on Geophysical Fluid Dynamics. We thank J. B. Keller for pointing out to us that most people prefer equations to solutions. N. J. B. acknowledges partial support from the Green Foundation. This research was supported by the National Science Foundation Award No. OCE 9529824.

[1] M. N. Rosenbluth and A. Simon, Phys. Fluids **7**, 557 (1964).

[2] K. Stewartson, IMA J. Appl. Math. **27**, 133 (1981); S. A. Maslowe, Annu. Rev. Fluid Mech. **18**, 405 (1986).

[3] D. J. Benney and R. F. Bergeron, Stud. Appl. Math. **48**, 181 (1969); R. E. Davis, J. Fluid Mech. **36**, 337 (1969); K. Stewartson, Geophys. Astrophys. Fluid Dyn. **9**, 185 (1978); T. Warn and H. Warn, Stud. Appl. Math. **59**, 37 (1978).

[4] F. J. Hickernell, Stud. Appl. Math. **69**, 1 (1983); K. R. Helfrich and J. Pedlosky, J. Fluid Mech. **251**, 377 (1993).

[5] R. Haberman, Stud. Appl. Math. **51**, 139 (1972); S. Brown and K. Stewartson, Geophys. Astrophys. Fluid Dyn. **10**, 1 (1978).

[6] V. E. Zhakarov, Sov. Phys. JETP **38**, 108 (1974); V. E. Zhakarov and A. B. Shabat, Funct. Anal. Appl. **8**, 226 (1974).

[7] This mode appears because of time reversibility; see C. C. Lin, *The Theory of Hydrodynamic Stability* (Cambridge University Press, Cambridge, 1955).

[8] N. J. Balmforth, "Stability of Vorticity Defects in Viscous Shear" (to be published).

[9] I. B. Bernstein, J. M. Greene, and M. D. Kruskal, Phys. Rev. **108**, 546 (1958).

[10] K. Stewartson, Fluid Dyn. Trans. Polish Acad. Sci. **7**, 101 (1975).

[11] J. D. Crawford, Phys. Fluids **2**, 97 (1995).

[12] T. Warn and P. Gauthier, Tellus **41A**, 115 (1989); S. M. Churilov and G. Shukhman, J. Fluid Mech. **318**, 189 (1996).