Asymptotics and perturbation theory

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March13th 2024

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Lecture 1

Making approximations

The long-term goal of this class (SIO203C/MAE294C) is to teach you how to obtain approximate solutions to applied-mathematical problems that can't be solved "exactly". A less ambitious goal of this introductory lecture is convince you that even problems with "exact solutions" are often best understood by ignoring the exact solution and looking at approximations. I'll talk in general terms about approximations and use some historical examples as illustrations. This involves some revision (I hope it's revision) of dimensional analysis and scaling.

1.1 Distance to the horizon

Start with a typical example: suppose you're standing on the deck of a ship and looking out into the distance. How far is the horizon? To estimate the distance to the horizon we approximate the Earth as perfect sphere with radius $R = 6\,371$km and suppose that the height of your eyes above sea level is $h = 10$m. The unknown distance to the horizon is d. The answer to this question has the form

$$d = f_1(R, h)\,, \tag{1.1}$$

where f_1 is an unknown function with two arguments; f_1 has dimensions of length.

It is the third quarter of your graduate career and I'm sure that by now you've seen several discussion of dimensional analysis. So you will not be surprised by the claim that the answer is simpler than (1.1) – it must have the form

$$\frac{d}{R} = f_2(\epsilon)\,, \tag{1.2}$$

where

$$\epsilon \stackrel{\text{def}}{=} \frac{h}{R} \tag{1.3}$$

is a dimensionless parameter. In (1.3) f_2 is an unknown dimensionless function with one argument. It is also clear that $f_2(0) = 0$. The function f_2 is much simpler than f_1 and so we have already made substantial progress.

Exercise: Dr. Kluge protests that

$$\frac{d}{h} = f_3\left(\epsilon\right) \tag{1.4}$$

is perfectly acceptable on dimensional grounds. Explain why Kluge's objection is captious.

With R and h above

$$\epsilon \approx 1.57 \times 10^{-6} \ll 1\,. \tag{1.5}$$

In these lectures ϵ will almost invariably denote a small dimensionless parameter and $\approx$ means approximate equality. (Sometimes ϵ is negative, or even complex; it is $|\epsilon|$ that is small.)

Some Pythagorean geometry shows that

$$d = \underbrace{\sqrt{2Rh + h^2}}_{f_1(h,R)} . \tag{1.6}$$

This is an exact solution for d. But I prefer the approximation obtained by neglecting h^2 relative to $2hR$ so that

$$d \approx \sqrt{2Rh} . \tag{1.7}$$

Plugging the numbers into (1.7) one finds that $d \approx 11.29$km.

Let's return to the step between (1.6) and (1.7). We can re-write (1.6) in dimensionless variables as

$$\frac{d}{R} = \underbrace{\sqrt{2\epsilon + \epsilon^2}}_{f_2(\epsilon)} . \tag{1.8}$$

To obtain (1.7), neglect ϵ^2 relative to ϵ so that

$$\frac{d}{R} \approx \sqrt{2\epsilon} . \tag{1.9}$$

We'll dwell further on the relative magnitudes of ϵ and ϵ^2 later in this lecture. The main point is that it is easier to see and understand the approximation in the non-dimensional formulation (1.8) than in the equivalent dimensional formulation (1.6).

Discussion: Why might we prefer the approximation (1.7) to the exact result (1.6)? Suppose you launch a drone so that you see the horizon from a height of 100m. It is obvious from (1.7) that the horizon is now about $\sqrt{10} \times 11.3$km away. In other words, the approximation is easier to use than the exact result. The Earth is not a perfect sphere: according to Wiki, distances from points on the surface of the Earth to the center vary between the polar radius 6 357km and the maximum equatorial radius 6 378km – perhaps this complication is more important than the difference between (1.6) and (1.7)? And of course the ocean is not flat: the actual horizon will be perturbed by surface gravity waves and geostrophic currents. And because of atmospheric refraction, light does not travel in straight lines. The order-ϵ^2 term we have neglected is probably far less important than these other complications. What is clear is that if want to improve on our first estimate in (1.7) we must consider a better model than a smooth sphere and contend with additional physics such as atmospheric refraction. These complications introduce additional small parameters of their own. This is a typical perturbation problem.

1.2 Regular perturbation series

Physical problems often devolve to analysis of limits ($\epsilon \to 0$ in the horizon example). Simplification occurs in the limit. Simplification can occur in three or four different ways. The example above is an easy case in which we simply neglect ϵ^2 relative to ϵ. Problems like this lead to a regular perturbation series (RPS).

An improved approximation

To systematically improve on the *leading-order* approximation in (1.9) we write

$$\frac{d}{R} = \sqrt{2\epsilon}\left(1 + \tfrac{\epsilon}{2}\right)^{1/2} \tag{1.10}$$

and recall the Taylor series

$$(1+x)^n = 1 + nx + \tfrac{1}{2}n(n-1)x^2 + O\left(x^3\right) . \tag{1.11}$$

Putting $n = 1/2$ and $x = \epsilon/2$ in this result we improve (1.9) to

$$\frac{d}{R} = \sqrt{2\epsilon}\left(1 + \tfrac{\epsilon}{4} - \tfrac{\epsilon^2}{32} + O(\epsilon^3)\right) . \tag{1.12}$$

Exercise: What is the radius of convergence of the series in (1.12)?

Notation O(): You should take $O(x^3)$ in (1.11) as meaning: "There are terms involving x^3, and also even smaller stuff varying like x^4, x^5 and so on. We know these terms are there but we can't be bothered calculating them." This is not the official definition of $O()$ – we'll get to that eventually.

As an introductory example of a *regular perturbation series* (RPS) let's suppose we don't know the Taylor series (1.11). We proceed from scratch. Let

$$x = \left(1 + \tfrac{\epsilon}{2}\right)^{1/2} \qquad \text{so that} \qquad x^2 = 1 + \tfrac{\epsilon}{2} . \tag{1.13}$$

With $\epsilon = 0$ we know that $x = \pm 1$. We expect that if $\epsilon \ll 1$ then x is close to either $+1$ or -1. We're inspired to look for a solution of the quasi-obvious form

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots \tag{1.14}$$

where $x_0 = \pm 1$. With some algebra

$$x^2 = x_0^2 + \epsilon 2x_0x_1 + \epsilon^2(2x_0x_2 + x_1^2) + \epsilon^3(2x_3x_2 + 2x_1x_2) + O(\epsilon^4) . \tag{1.15}$$

Exercise: Do you see the pattern? What is the order ϵ^4 term in (1.15)?

Substituting (1.15) into (1.13) and equating terms at the same order in ϵ we have

$$\epsilon^0 : \quad x_0^2 = 1 , \qquad \Rightarrow \qquad x_0 = \pm 1 , \tag{1.16}$$

$$\epsilon^1 : \quad 2x_0x_1 = \tfrac{1}{2} , \qquad \Rightarrow \qquad x_1 = \tfrac{1}{4x_0} = \pm\tfrac{1}{4} , \tag{1.17}$$

$$\epsilon^3 : \quad 2x_0x_2 + x_1^2 = 0 , \qquad \Rightarrow \qquad x_2 = -\tfrac{x_1^2}{2x_0} = \mp\tfrac{1}{32} . \tag{1.18}$$

This essentially the method of undetermined coefficients. It works provided that we start with the correct form for the answer in (1.14).

A more challenging example

You're probably not very impressed by the previous example. Let's consider a more challenging example: find the $\epsilon \ll 1$ solutions of

$$x\mathrm{e}^{-x} = \epsilon . \tag{1.19}$$

See figure 1.1 for a graphical visualization of the problem. There is a small root that approaches zero as $\epsilon \to 0$. there is also a large root that goes to infinity as $\epsilon \to 0$. If we put $\epsilon = 0$ in (1.19) then we have the exact solution $x = 0$. The large root has disappeared.

We can determine the small root by substituting an RPS,

$$x = \epsilon x_1 + \epsilon^2 x_2 + O\left(\epsilon^3\right) , \tag{1.20}$$

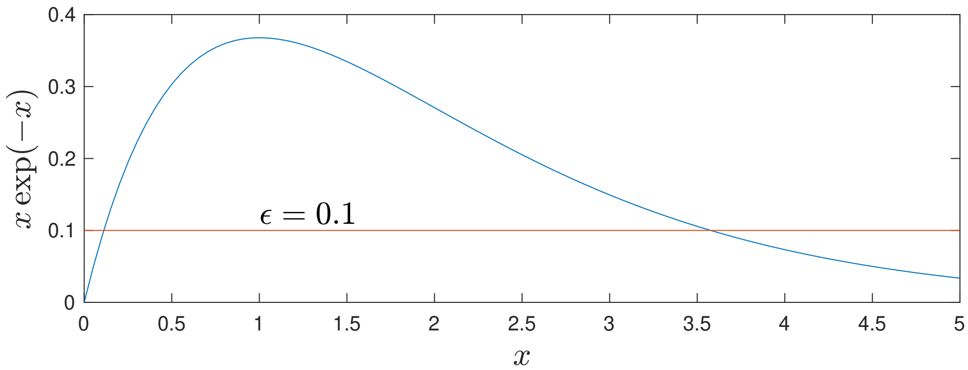

Figure 1.1: Graphical determination of the $\epsilon \ll 1$ solutions of (1.19).

into (1.19). Let's agree to discard terms of order ϵ^3. Then

$$\exp(-\epsilon x_1 - \epsilon^2 x_2 + O\left(\epsilon^3\right)) = 1 - \left[\epsilon x_1 + \epsilon^2 x_2 + O\left(\epsilon^3\right)\right] + \tfrac{1}{2}\left[\epsilon x_1 + O\left(\epsilon^2\right)\right]^2 + O\left(\epsilon^3\right) , \tag{1.21}$$

$$= 1 - \epsilon x_1 + \epsilon^2\left(\tfrac{1}{2}x_1^2 - x_2\right) + O\left(\epsilon^3\right) , \tag{1.22}$$

and

$$x\mathrm{e}^{-x} = \left[\epsilon x_1 + \epsilon^2 x_2 + O(\epsilon^3)\right] \times \left[1 - \epsilon x_1 + \epsilon^2\left(\tfrac{1}{2}x_1^2 - x_2\right) + O(\epsilon^3)\right] , \tag{1.23}$$

$$= \epsilon x_1 + \epsilon^2(x_2 - x_1^2) + O(\epsilon^3) . \tag{1.24}$$

It is irritating that we did unnecessary work back in (1.22) – we could have discarded the order-ϵ^2 term and maintained ϵ^2-accuracy in (1.24).

Substituting (1.24) into (1.19) and matching powers of ϵ we find

$$x_1 = 1 , \qquad \text{and} \qquad x_2 = x_1^2 = 1 . \tag{1.25}$$

Hence the small root is determined by the RPS

$$x = \epsilon + \epsilon^2 + O\left(\epsilon^3\right) . \tag{1.26}$$

Turning to the large root in figure 1.1 we have a much more difficult problem. The dependence of ϵ of this root is not obvious. This is an example of a *singular perturbation* problem: setting $\epsilon = 0$ makes a structural change in (1.19) – there is a single root $x = 0$. But if we take the limit $\epsilon \to 0$ then there is always a big root. Perhaps this big root goes off to infinity like ϵ^{-1}, or $\epsilon^{?}$. We'll return to this question in the next lecture.

1.3 Small parameters and really small parameters: ϵ versus ϵ^2

Figure 1.2 shows a three right-angled triangle with sides

$$1 , \qquad \epsilon \qquad \text{and} \qquad \sqrt{1+\epsilon^2} .$$

When ϵ is small one has trouble visually distinguishing this right-triangle from an isosceles triangle because the hypotenuse is very nearly equal to the long side. Using the binomial theorem and assuming that $\epsilon \ll 1$, the length of the hypotenuse is

$$\sqrt{1+\epsilon^2} = 1 + \frac{\epsilon^2}{2} + O\left(\epsilon^4\right) . \tag{1.27}$$

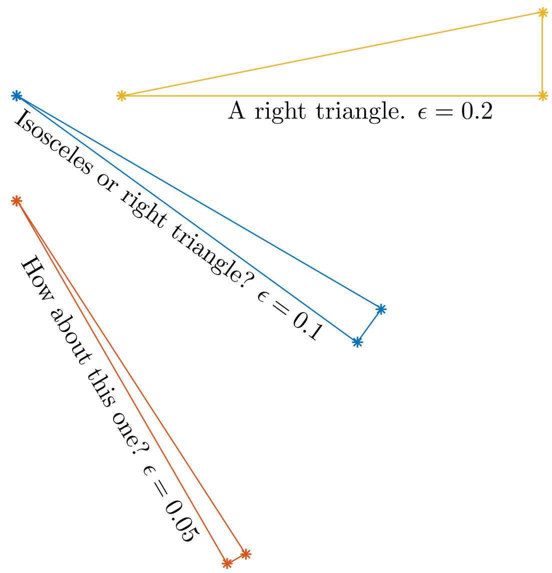

Figure 1.2: Three right triangles: the short side has length ϵ, the base has length 1 and the hypotenuse length $\sqrt{1+\epsilon^2}$.

So the difference between the long side and the hypotenuse is "order ϵ^2". With $\epsilon = 0.1$ this small difference is hard to see, and even more difficult if $\epsilon = 0.05$. On the other hand, the small angle in figure 1.2 is

$$\theta = \arctan(\epsilon) \approx \epsilon \,. \tag{1.28}$$

You have no difficulty seeing the order ϵ small angle and the small side of the triangle: to mistake the triangle for a line segment we'd have to make ϵ a lot less than 0.05.

As another example of the difference between ϵ and ϵ^2 consider the ellipse in figure 1.3. The eccentricity of this ellipse is $e = 0.2$ which is close to the eccentricity of the orbit of Mercury. I picked Mercury because it has the most eccentric orbit of the eight planets in the solar system. As you can see in figure 1.3, it is easy to mistake this ellipse for a circle. Kepler, analyzing data collected by Tycho Brahe, made that mistake: he thought that the orbit of Mars ($e = 0.09$) was a circle with the Sun off-center. Later Kepler realized that the orbit of Mars is actually a small-eccentricity ellipse with the Sun at a focal point. This confusion arises because the distance of the foci from the center of an ellipse is of order e, while the difference between the major and minor axes of an ellipse is of order e^2. Specifically, the curve in figure 1.3 is

$$x^2 + \frac{y^2}{1-e^2} = 1 \,, \tag{1.29}$$

and the focus $*$ is at $(x, y) = (e, 0)$, with $e = 0.2$.

1.4 Example: Rectification of the ellipse

Let's calculate the perimeter, ℓ, of the ellipse in figure 1.3. On dimensional grounds the perimeter is $2\pi a \times f(e)$ where the eccentricity e is the only dimensionless number in this problem and $2\pi a$ is the perimeter of a "comparison circle" (see the box "Anatomy of the Ellipse").

If a curve is specified as the graph of a function via

$$y = f(x) \,, \tag{1.30}$$

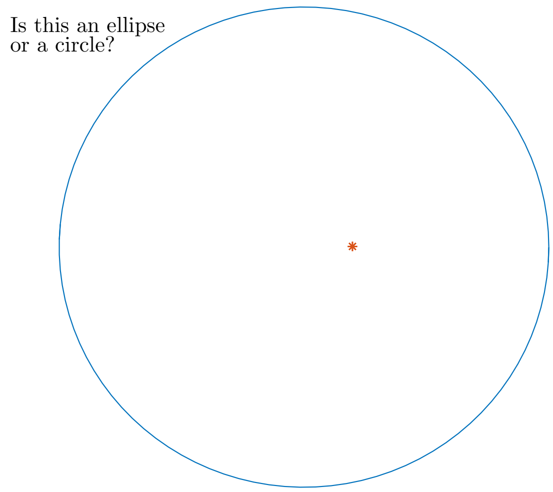

Figure 1.3: An ellipse with eccentricity $e = 0.2$. It looks like a circle doesn't it? The off-center point $*$ is at a focus.

then, using Pythagoras's theorem, $\mathrm{d}\ell = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}f)^2}$. Thus the length of the curve between x_1 and x_2 is

$$\ell = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2}\, \mathrm{d}x\,. \tag{1.31}$$

Suppose an ellipse is specified as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,. \tag{1.32}$$

For the portion of the ellipse above the x-axis (i.e. $y > 0$) we have

$$y = \underbrace{b\sqrt{1 - \frac{x^2}{a^2}}}_{f(x)}\,, \qquad \text{and} \qquad \frac{\mathrm{d}f}{\mathrm{d}x} = -\frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}}\,. \tag{1.33}$$

The semi-minor axis is $b = \sqrt{1 - e^2}\, a$, where e is the eccentricity. Combining these results, the perimeter integral in (1.31) is

$$\ell = 4\int_0^a \sqrt{\frac{a^2 - e^2x^2}{a^2 - x^2}}\, \mathrm{d}x\,, \tag{1.34}$$

$$= 4a\int_0^1 \sqrt{\frac{1 - e^2v^2}{1 - v^2}}\, \mathrm{d}v\,. \tag{1.35}$$

In going from (1.34) to (1.35) we've used the change of variable $x = av$ to tidy the integral so that it becomes non-dimensional and contains only the eccentricity e. We can try to evaluate the integral analytically by making a further substitution

$$v = \sin\theta\,, \qquad \text{and therefore} \qquad \frac{\mathrm{d}v}{\sqrt{1 - v^2}} = \mathrm{d}\theta\,. \tag{1.36}$$

The integral becomes

$$\ell = 4a\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\, \mathrm{d}\theta\,. \tag{1.37}$$

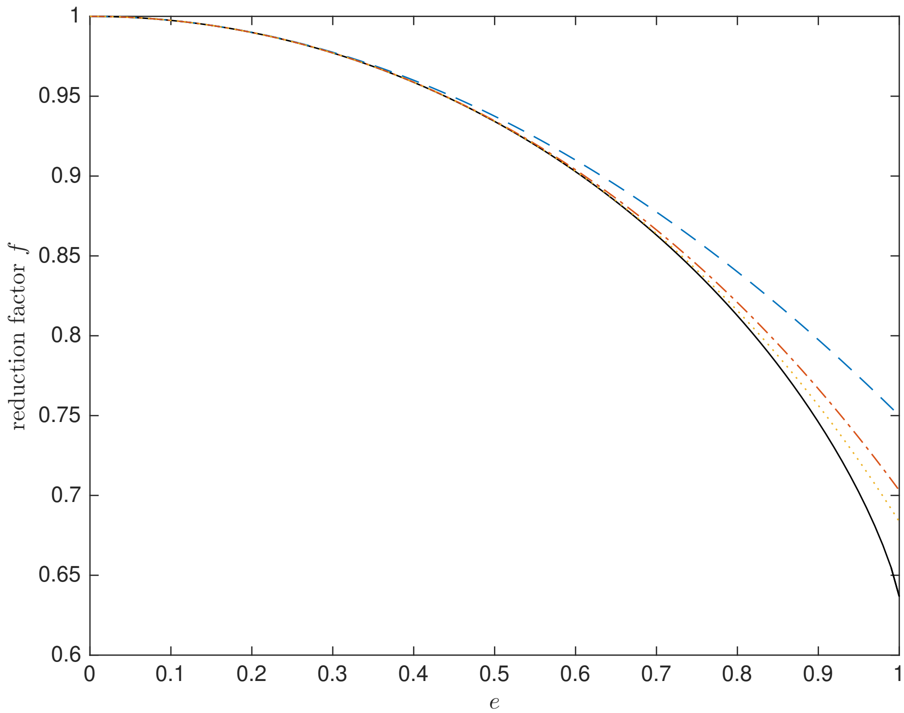

Figure 1.4: The "reduction factor" $f(e)$ in (1.38) is the solid black curve. The two-term approximation in (1.40) is the blue dashed curve. The three- and four-term approximations (the dash-dot and dotted curves) from (1.42) lie even closer to the black curve.

As a sanity check, notice that if $e = 0$ the perimeter in (1.37) is $2\pi a$.

Exercise: Make another sanity check by considering $e = 1$.

Let's write (1.37) as

$$\ell = 2\pi a \;\times\; \underbrace{\frac{2}{\pi}\int_0^{\pi/2}\sqrt{1 - e^2\sin^2\theta}\,\mathrm{d}\theta}_{f(e)}\,, \tag{1.38}$$

where $f(e)$ is a dimensionless "reduction factor" relative to a circle with radius a.

Small eccentricity

Students of applied mathematics used to learn to recognize elliptic integrals and many other special functions. These days students might use MATHEMATICA or something similar to discover that the integral in (1.37) and (1.38) is a "complete elliptic integral of the second kind" (see the box). Using elliptic integrals, the reduction factor can be written as an "exact analytic solution". It is difficult to deny that this exact answer is useful because both MATHEMATICA and MATLAB have elliptic integrals hardwired. If we are interested, however, in quickly and accurately estimating the perimeter of the near-circle in figure 1.3 then we can approximately evaluate (1.37) like this[1]

$$\ell \approx 4a \int_0^{\pi/2} 1 - \tfrac{1}{2} e^2 \sin^2\theta \, \mathrm{d}\theta \,, \tag{1.39}$$

$$= 2\pi a \underbrace{\left(1 - \tfrac{1}{4} e^2\right)}_{\approx f(e)} . \tag{1.40}$$

Figure 1.4 compares the approximation $f \approx 1 - e^2/4$ to the elliptic-function answer. With $e = 0.2$ the simple approximation is probably good enough for most purposes. Of course, to use an approximation with some confidence we must have some estimate of the size of the error.

Applied mathematics is concerned with making precise approximations in which the error is both understood and controllable. We should also strive to make the error smaller by some systematic method. Here we can do this by using more terms in the binomial expansion of $\sqrt{1 - e^2 \sin^2\theta}$. Let's use four terms and indicate the form of the first neglected term:

$$\sqrt{1 - e^2 \sin^2\theta} = 1 - \tfrac{1}{2} e^2 \sin^2\theta - \tfrac{1}{8} e^4 \sin^4\theta - \tfrac{1}{16} e^6 \sin^6\theta + O\left(e^8\right) . \tag{1.41}$$

Integrating over θ, our new improved approximation to the reduction factor $f(e)$ is

$$f = 1 - \tfrac{1}{4} e^2 - \tfrac{3}{64} e^4 - \tfrac{5}{256} e^6 + O\left(e^8\right) . \tag{1.42}$$

In figure 1.4 there is a systematic improvement as we use more terms in the series. (I used MATHEMATICA to compute the θ integrals above.)

A very eccentric ellipse

How about the other limit $e \to 1$? The ellipse degnerates into a line segment. It is obvious, both analytically and geometrically, that the perimeter is $\ell = 4a$. Suppose e is slightly less than one. How do we find the difference between ℓ and $4a$? This is a typical "asymptotic question": we have a simple result at the extreme parameter value $e = 1$. ($e = 1$ is extreme because the ellipse degenerates to a line segment.) We want to understand what happens close to, but not exactly at, this interesting value $e = 1$.

The equations look pretty if we define a small parameter ϵ by

$$\epsilon = \sqrt{1 - e^2} \,. \tag{1.43}$$

Then the perimeter of the very eccentric ellipse is

$$\ell(e) = 4a \int_0^{\pi/2} \sqrt{\cos^2\theta + \epsilon^2 \sin^2\theta} \, \mathrm{d}\theta \,. \tag{1.44}$$

We use the trusty approximation discussed previously:

$$\sqrt{\cos^2\theta + \epsilon^2 \sin^2\theta} = \cos\theta \left(1 + \epsilon^2 \tan^2\theta\right)^{1/2} \approx \cos\theta + \epsilon^2 \frac{\sin^2\theta}{2\cos\theta} \,. \tag{1.45}$$

[1] For the integrals

$$\frac{\pi}{2} = \int_0^{\pi/2} \cos^2\theta + \sin^2\theta \, \mathrm{d}\theta \,, \qquad \text{and by quarter-wavelength symmetry} \qquad \int_0^{\pi/2} \cos^2\theta \, \mathrm{d}\theta = \int_0^{\pi/2} \sin^2\theta \, \mathrm{d}\theta = \frac{\pi}{4} \,.$$

Substituting this approximation into (1.44) the result is a disaster

$$\ell(e) \approx 4a \int_0^{\pi/2} \cos\theta + \epsilon^2 \frac{\sin^2\theta}{2\,\cos\theta}\, \mathrm{d}\theta\,, \tag{1.46}$$

$$\overset{??}{\approx} 4a\left[1 + \epsilon^2 \infty\right]\,. \tag{1.47}$$

The second integrand in (1.46), namely

$$\frac{\sin^2\theta}{2\,\cos\theta}\,, \tag{1.48}$$

has a non-integrable singularity at $\theta = \pi/2$. We'll return to this example later in these lectures.

Discussion: Why has this reasonable approach to a simple geometric problem failed?

1.5 Example: Period of a pendulum

Following Galileo, suppose you observe a mass m swinging at the end of massless rigid rod, length ℓ, in a gravitational field with acceleration g. At the top of the swing the rod makes an angle θ_m (subscript m for "maximum") with the vertical. What can one say about the period, p, of this pendulum? From ℓ and g there is a time scale $\sqrt{\ell/g}$. The angle θ_m is non-dimensional. Hence by dimensional analysis

$$p = \sqrt{\frac{\ell}{g}}\, f(\theta_m)\,, \tag{1.49}$$

where f is a dimensionless function. It is a profound[2] fact that the mass m is "irrelevant" to p. The story is that Galileo realized that if $\theta_m \ll 1$ (small swings) then the unknown function above approaches a non-zero constant:

$$\lim_{\theta_m \to 0} f(\theta_m) = c_0\,. \tag{1.50}$$

Consideration of symmetry indicate that f should be an even function of its argument θ_m e.g. the minimum (meaning most negative) angle to the vertical is $-\theta_m$. Thus we expect that f has the expansion

$$f = c_0 + c_2\theta_m^2 + c_4\theta_m^4 + \cdots \tag{1.51}$$

In the discussion above have neglected damping e.g. air resistance as the bob swishes back and forth. As a problem you can determine the additional non-dimensional parameters required by this extra physics.

Discussion: Can we assume that $|\theta_m|$ does not appear in (1.51)?

To say more about f, we start with the equation of motion

$$m\ell\ddot{\theta} + mg\sin\theta = 0. \tag{1.52}$$

the first term is mass × acceleration in the θ-direction and the second term is the component of gravitational force along the θ-direction. We cancel m and write the pendulum equation as

$$\ddot{\theta} + \omega^2 \sin\theta = 0\,, \tag{1.53}$$

[2] Inertial and gravitational masses are the same. If the two masses were not equal there would be another non-dimensional parameter m_I/m_G.

where

$$\omega \stackrel{\text{def}}{=} \sqrt{\frac{g}{\ell}}\,. \tag{1.54}$$

Multiply (1.53) by $\dot\theta$ and integrate to obtain energy conservation in the form

$$\tfrac{1}{2}\dot\theta^2 + \omega^2\cos\theta_m - \omega^2\cos\theta = 0\,. \tag{1.55}$$

We've determined the constant of integration in (1.55) so that $\dot\theta = 0$ when $\theta = \pm\theta_m$

(I assume that you know and love the linearized version of (1.53), and that you also know that the small-angle approximation to the period is $p = 2\pi/\omega$. Our goal is to improve on the small-angle formula by finding the "first correction". In (1.51) we have used symmetry to anticipate that the first correction is proportional to θ_m^2. It is also plausible that $c_2 > 0$ i.e. bigger swings take longer.)

Separate variables in (1.55)

$$\omega\,\mathrm{d}t = \pm\frac{\mathrm{d}\theta}{\sqrt{2(\cos\theta - \cos\theta_m)}}\,. \tag{1.56}$$

We draw trajectories in the phase plane (figure 1.5) and argue that the period is given by:

$$p = \frac{2\sqrt{2}}{\omega}\int_0^{\theta_m}\frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_m}}\,. \tag{1.57}$$

With ingenuity the integral above can be converted into a complete elliptic integral of the first kind:

$$p = \frac{1}{\omega}\frac{2}{\pi}K\left(\sin\frac{\theta_m}{2}\right)\,. \tag{1.58}$$

Bah humbug.

Back up the truck to (1.57) and proceed with a brutal[3] small-angle expansion of the cosines

$$\cos\theta - \cos\theta_m = \tfrac{1}{2}(\theta_m^2 - \theta^2) - \tfrac{1}{24}(\theta_m^4 - \theta^4) + O\left(\theta_m^6\right)\,, \tag{1.59}$$

$$\approx \tfrac{1}{2}(\theta_m^2 - \theta^2)\left[1 - \tfrac{1}{12}(\theta_m^2 + \theta^2)\right]\,. \tag{1.60}$$

Therefore

$$\frac{1}{\sqrt{\cos\theta - \cos\theta_m}} \approx \frac{\sqrt{2}}{\sqrt{\theta_m^2 - \theta^2}}\left[1 - \tfrac{1}{12}(\theta_m^2 + \theta^2)\right]^{-1/2} \tag{1.61}$$

$$\approx \frac{\sqrt{2}}{\sqrt{\theta_m^2 - \theta^2}}\left[1 + \tfrac{1}{24}(\theta_m^2 + \theta^2)\right]\,. \tag{1.62}$$

Hence

$$p \approx \frac{4}{\omega}\int_0^{\theta_m}\frac{1 + \frac{1}{24}(\theta_m^2 + \theta^2)}{\sqrt{\theta_m^2 - \theta^2}}\mathrm{d}\theta\,. \tag{1.63}$$

With the change of variables $x = \theta/\theta_m$

$$p \approx \frac{4}{\omega}\int_0^1\frac{1 + \frac{\theta_m^2}{24}(1 + x^2)}{\sqrt{1 - x^2}}\mathrm{d}x \tag{1.64}$$

[3] See the problem 1.11 for elegance.

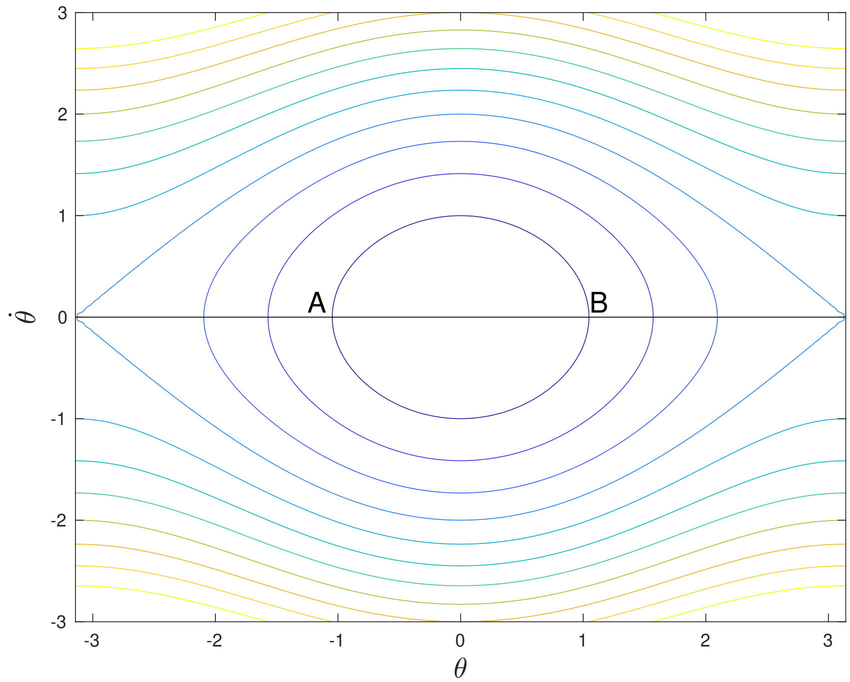

Figure 1.5: Phase plane of the pendulum (scale time so that $\omega \mapsto 1$). You should draw arrows on the curves above to show the direction of evolution. The expression for the period p in (1.57) is obtained by noting that it takes $p/2$ to go from A to B in the upper half plane. Because we're in the upper half plane take the $+$ in (1.56).

The integrals above can be evaluated with a further change of variables to $x = \sin v$. Instead, using MATHEMATICA, I get

$$p \approx \frac{2\pi}{\omega}\left(1 + \frac{\theta_m^2}{16}\right) . \tag{1.65}$$

The familiar small-angle approximation to p is an underestimate of the true period. At $\theta_m = \pi/6$ (a pretty big swing) the correction, $\theta_m^2/16$, is 1.7%. That seems like a small correction. But see the grandfather clock problem at the end of this lecture.

1.6 Some references

Two books that have shaped my view of perturbation methods and asymptotics are:

BO C.M. Bender & S.A. Orszag (1978), *Advanced Mathematical Methods for Scientists and Engineers*;

H E.J. Hinch (1991), *Perturbation Methods.*

For a more recent textbook see:

Ho Mark H. Holmes (2013), *Introduction to Perturbation Methods* (second edition).

More advanced, books are:

M P.D. Miller (2006), *Applied Asymptotic Analysis*;

N J.C. Neu (2015), *Singular Perturbation in the Physical Sciences*;

KC J. Kevorkian & J.D. Cole (1996) *Multiple Scale and Singular Perturbation Methods*

While I don't recommend it as a systematic reference, I enjoyed reading *Mathematical Understanding of Nature* by V.I. Arnold (some of the material in this lecture is based on Arnold).

Basic anatomy of the ellipse

An ellipse is a plane curve enclosing two focal points such that the sum of the distances to the two foci is constant for every point on the ellipse. In figure (1.6) the foci are on the x-axis at $x = \pm ea$ and ellipse is defined by

$$\underbrace{\sqrt{(x-ea)^2+y^2}}_{\stackrel{\text{def}}{=} r_+} + \underbrace{\sqrt{(x+ea)^2+y^2}}_{\stackrel{\text{def}}{=} r_-} = 2a\,. \tag{1}$$

If $e = 0$ then the ellipse becomes a circle with radius a. With some algebra you can show that (1) is equivalent to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,, \tag{2}$$

where

$$b = \sqrt{1-e^2}\,a\,.$$

The lengths a and b are the semi-major and semi-minor axes respectively. If $e \ll 1$ then the difference between a and b is order e^2.

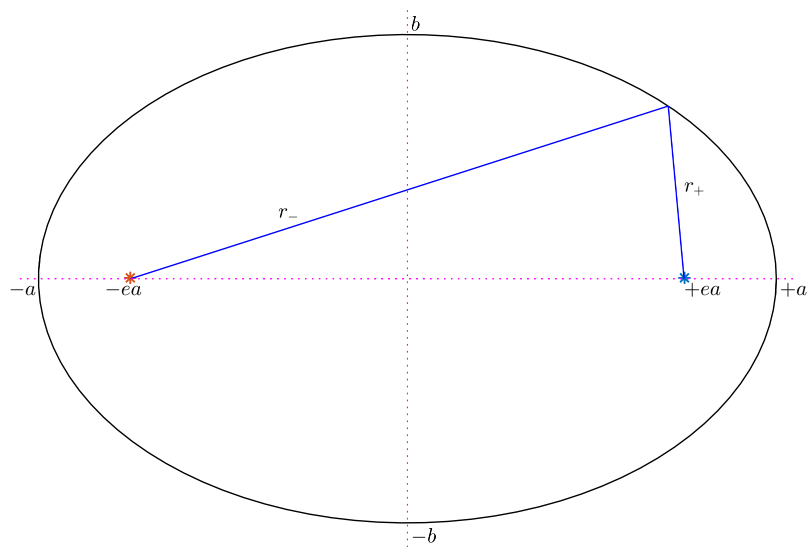

Figure 1.6: An ellipse with $e = 0.75$.

Lines from a focus point to a point P on the ellipse make equal angles with the tangent at P. Hence if the ellipse is a mirror and there is a light source at one of the foci then light rays reflecting specularly from the mirror all pass through the other focal point. *Focus* is latin for fireplace.

Complete Elliptic Integrals

In traditional notation the complete elliptic integral of the first kind is

$$\begin{aligned} K(k) &\stackrel{\text{def}}{=} \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}\,, \\ &= \frac{\pi}{2}\left[1+\frac{1^2}{2^2}k^2+\frac{1^2\,3^2}{2^2\,4^2}k^4+\frac{1^2\,3^2\,5^2}{2^2\,4^2\,6^2}k^4+\cdots\right]. \end{aligned}$$

The complete elliptic integral of the second kind is

$$\begin{aligned} E(k) &\stackrel{\text{def}}{=} \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\,\mathrm{d}\theta\,, \\ &= \frac{\pi}{2}\left[1-\frac{1}{2^2}k^2-\frac{1^2\,3}{2^2\,4^2}k^4-\frac{1^2\,3^2\,5}{2^2\,4^2\,6^2}k^4+\cdots\right]. \end{aligned}$$

The series above converge if $k^2 < 1$.

Other series include

$$\begin{aligned} K(k) &= \ln\frac{4}{k'}+\frac{1}{2}\left(\ln\frac{4}{k'}-1\right)k'^2+O\left(k'^4\ln k'\right)\,, \\ E(k) &= 1+\frac{1}{2}\left(\ln\frac{4}{k'}-\frac{1}{2}\right)k'^2+O\left(k'^4\ln k'\right)\,, \end{aligned}$$

where $k' \stackrel{\text{def}}{=} \sqrt{1-k^2}$.

The two integrals are related by

$$\frac{\mathrm{d}E}{\mathrm{d}k} = \frac{E-K}{k}\,, \qquad \text{and} \qquad \frac{\mathrm{d}K}{\mathrm{d}k} = \frac{1}{k}\left(\frac{E}{k'^2}-K\right).$$

Be aware there are slightly different notations out there e.g. MATLAB does not use the notation above. Read the documentation.

There are many, many more identities involving elliptic integrals. Online resources include Wikipedia, MathWorld and the Digital Library of Mathematical Functions (google DLMF).

1.7 Problems

Problem 1.1. Suppose you're on a ship and your eyes are $h = 30\text{m}$ above sea level. You see an island on the horizon. How far do you have to go, following the surface of the Earth (i.e. on a great-circle route), to reach the island? Assume that the Earth is a perfect sphere with radius $R = 6\,371\text{km}$. Give an exact answer as a formula involving inverse trigonometric functions et cetera, and a useful $h/R \ll 1$ approximation to the exact answer. Compare this great-circle distance with the distance to the horizon.

Problem 1.2. *(i)* A triangle in the plane can be specified uniquely by giving the length of the longest side – call it c – and the acute angles θ and ϕ that the two shorter sides make with the longest side. Use dimensional analysis to say what you can about the area of the triangle in terms of c, θ and ϕ. (Pretend you don't know trigonometry: leave an undetermined dimensionless function in the answer.) *(ii)* Consider the special case of a right-angled triangle with sides a, b and c. Divide the triangle into two sub-triangles by dropping a perpendicular onto the long side with length c. The total area is the sum of the areas of two right-angled subtriangles. Use this observation to prove Pythagoras's theorem. *(iii)* Spherical triangles don't satisfy Pythagoras's theorem. How far can you proceed with the spherical version this problem?

Problem 1.3. You can use the high school formula to exactly solve the quadratic equation

$$x^2 - \pi x + 2 = 0\,. \tag{1.66}$$

Notice that if we replace π by the approximation 3 then you can solve the equation by inspection. Define ϵ by $\pi = 3 + \epsilon$ and use an RPS to solve (1.66) neglecting terms of order ϵ^3 and smaller. Assess the accuracy of this solution.

Problem 1.4. Because 10 is close to 9 we suspect that $\sqrt{10}$ is close to $\sqrt{9} = 3$. *(i)* Define $x(\epsilon)$ by

$$x(\epsilon)^2 = 9 + \epsilon\,. \tag{1.67}$$

Assume that $x(\epsilon)$ has an RPS as in (1.14). Calculate the first four terms, x_0 through x_3. *(ii)* Take $\epsilon = 1$ and compare your estimate of $\sqrt{10}$ with a six decimal place computation. *(iii)* Solve (1.67) with the binomial expansion and verify that the resulting series is the same as the RPS from part *(ii)* What is the radius of convergence of the series?

Problem 1.5. Assume the Earth is a perfect sphere with radius $R = 6\,371\text{km}$ and it wrapped around the equator by a rope with length $2\pi R + \ell$, where $\ell = 1\,\text{meter}$. *(i)* As an easy warm-up calculate h if the rope is pulled to a uniform height h above the surface of the Earth. *(ii)* Suppose the rope is grabbed at a point and that point is hoisted vertically to a height H till the rope is taut – see figure 1.7. Estimate H by: *(i)* guessing an order of magnitude and *(ii)* perturbation theory based on $\ell \ll R$.

Problem 1.6. Show that the expansion of $f(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)$ is

$$f(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) = f(x_0) + \epsilon x_2 f'(x_0) + \epsilon^2 \big(x_2 f'(x_0) + \tfrac{1}{2} x_1^2 f''(x_0)\big) + O(\epsilon^3)\,. \tag{1.68}$$

If your OCD is strong, calculate some more terms and spot the pattern.

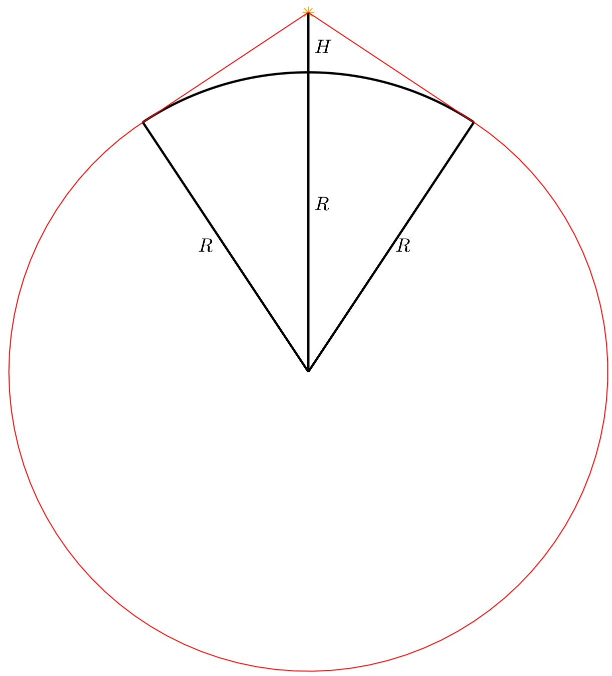

Figure 1.7: The rope is red with length $2\pi R + \ell$. Find H.

Problem 1.7. Consider an ellipse with semi-axes a and b ($a > b$) and perimeter ℓ. If the ellipse is a near-circle, $a \approx b$, then the perimeter ℓ might be estimated by

$$\ell_1 = \pi(a+b)\,, \qquad \text{or perhaps by} \qquad \ell_2 = 2\pi\sqrt{ab}\,. \tag{1.69}$$

Both formulas above give the right answer if $a = b$. (a) Which ℓ_n is best if $a \gg b$? (b) Which ℓ_n is best if $a \approx b$? (c) Determine α so that

$$\ell_3 = \alpha\ell_1 + (1-\alpha)\ell_2 \tag{1.70}$$

is the best possible approximation to ℓ in the case $a \approx b$.

Problem 1.8. Consider

$$B(n) = \int_0^\infty \frac{\mathrm{d}t}{(1+t^2)(1+t^n)} \tag{1.71}$$

Plot the integrand on the interval $0 < t < 2$ for $n = 2$, 4, 8 and 16. After studying this plot, devise a simple $n \gg 1$ approximation to $B(n)$ and test your approximation by comparison with a numerical evaluation of $B(n)$ with $0 \leq n \leq 40$. You'll know you've done this problem correctly if comparison of the numerical answer with your approximation is surprising. (So surprising that you'll see the need for analytic evaluation of $B(n)$.)

Problem 1.9. Figure 1.8 shows the path followed by a tipsy sailor from a bar at the origin of the (x, y)-plane to home at $(x, y) = (\ell, 0)$. The path is a sinusoid leaving the bar at an angle α and in figure 1.8 $\alpha = \pi/4$. How much longer is the sinusoidal path than the straight line? Answer this question by: *(i)* eyeballing the curve in figure 1.8 and guessing; *(ii)* constructing the integral that gives the arclength and evaluating it numerically; *(iii)* devising an approximation to the arc-length integral based on $\alpha \ll 1$, and then pressing your luck by using this approximation with $\alpha = \pi/4$.

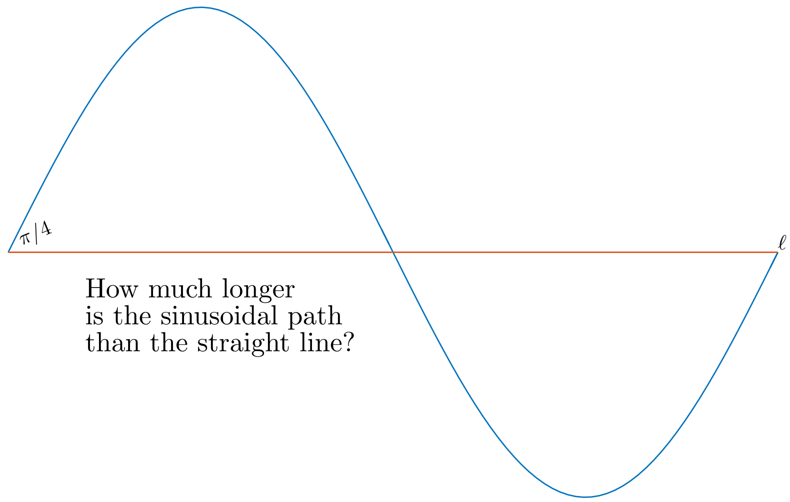

Figure 1.8: A tipsy walk.

Problem 1.10. Because of surface waves a snapshot of the sea-surface is

$$z = a\left[p\cos(kx+\alpha) + q\cos(ly+\beta)\right] , \tag{1.72}$$

where a (meters) is the amplitude of the waves field. The waves are small amplitude so that the sea-surface slope is small i.e. $\{ka, la\} \ll 1$. How much extra surface area (relative to the flat undisturbed surface) do these waves produce?

Problem 1.11. Recall that the period of a pendulum is

$$p = \frac{2\sqrt{2}}{\omega}\int_0^{\theta_m} \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_m}} . \tag{1.73}$$

Simplify the integral above by substituting $\sin(\theta/2) = \sin(\theta_m/2)\sin\psi$. Show that

$$p = 2\pi\sqrt{\frac{\ell}{g}}\left[1 + \tfrac{1}{4}\sin^2\frac{\theta_m}{2} + \tfrac{9}{64}\sin^4\frac{\theta_m}{2} + \cdots\right] . \tag{1.74}$$

Problem 1.12. A grandfather clock swings at a maximum angle $\theta_m = 5°$ to the vertical. How many seconds does the clock lose or gain each day if it is adjusted to keep perfect time when the swing is $\theta_m = 2°$? (Use results from the lecture.)

Problem 1.13. Suppose that the pendulum in section 1.5 is damped by air resistance. Assume that the turbulent drag law

$$\mathrm{Drag} = C_D \rho A u^2 \tag{1.75}$$

applies. In (1.75) ρ is the density of air, A is the cross-section area of the mass, and u is the velocity of the mass through the air. *(i)* Check that the drag above has the dimensions of force = mass × acceleration. *(ii)* How is (1.49) modified to account for this additional physics? *(iii)* Write the enhanced version of (1.53) accounting for drag.

Problem 1.14. As an elementary analog of the eccentric ellipse problem investigate the iintegral

$$A(\epsilon) \stackrel{\text{def}}{=} \int_0^1 \sqrt{(1-x)^2 + \epsilon^2 x^2}\, \mathrm{d}x . \tag{1.76}$$

Evaluate $A(0)$. Can you determine the $\epsilon \ll 1$ correction to $A(0)$?

Lecture 2

Dominant balance and iteration

After this lecture you'll be able to solve, or greatly simplify, almost any equation you encounter.

2.1 Consistent dominant balances

Rosencrantz: Consistency is all I ask!

Guildenstern: Give us this day our daily mask.

A typical equation is

$$A(x) + B(x) + C(x) + D(x) + E(x) + F(x) = 0\,. \tag{2.1}$$

where A through F are functions of the unknown x. We try to solve the equation above by finding a balance between two of the terms – a *two-term dominant balance* – and dropping the other terms in the equation. A key step in dominant balance is to very the consistency of the approximation. For example, suppose we retain terms A and D and solve the simplified equation

$$A(x) + D(x) \approx 0\,. \tag{2.2}$$

We now possess an approximation to x – call it y. Consistency requires that

$$\{A(y), D(y)\} \gg \{B(y), C(y), E(y), F(y)\}\,. \tag{2.3}$$

Note carefully that we do not require that the neglected terms be much less than one – consistency only requires that the neglected terms are much less than the retained terms.

Example: A quadratic equation

Consider the quadratic equation

$$\epsilon x^2 + x - 1 = 0\,, \tag{2.4}$$

with $\epsilon \ll 1$. Let's proceed using the method that worked in the first lecture: pick the low-hanging fruit by setting $\epsilon = 0$ and solving the simplified equation with two terms. Obviously if $\epsilon = 0$ then $x = 1$. With small but non-zero ϵ there is a consistent two-term dominant balance between the second and third terms in (2.4). This means that if $x \approx 1$ then the neglected term ϵx^2 is much less than the retained terms $\{x, 1\}$. Then we can grind out the answer with an RPS – details follow.

RPS around $x = 1$: How does the root near $x = 1$ changes with ϵ? We use an RPS:

$$x = \underbrace{1}_{x_0} + \epsilon x_1 + \epsilon^2 x_2 + \cdots \tag{2.5}$$

Substituting into the quadratic equation (2.4) we have

$$\epsilon \left(1 + 2\epsilon x_1 + \epsilon^2 2x_2 + \epsilon^2 x_1^2\right) + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + O\left(\epsilon^4\right) = 0\,. \tag{2.6}$$

Now match up powers of ϵ:

$$\epsilon^1: \qquad 1 + x_1 = 0\,, \qquad \Rightarrow \qquad x_1 = -1\,, \tag{2.7}$$

$$\epsilon^2: \qquad 2x_1 + x_2 = 0\,, \qquad \Rightarrow \qquad x_2 = 2\,, \tag{2.8}$$

$$\epsilon^3: \qquad 2x_2 + x_1^2 + x_3 = 0\,, \qquad \Rightarrow \qquad x_3 = -5\,. \tag{2.9}$$

To summarize

$$x = 1 - \epsilon + 2\epsilon^2 - 5\epsilon^3 + O\left(\epsilon^4\right)\,. \tag{2.10}$$

The procedure is never going to help us find the missing root of (2.4).

More than you need to know: Because (2.4) is a quadratic equation there are two roots and we've only located one. To find the missing root Let's peek at the exact answer to this problem. Using the formula for the solution of a quadratic equation, we find

$$x = \frac{-1 \pm \sqrt{1 + 4\epsilon}}{2\epsilon}\,. \tag{2.11}$$

But it is not entirely straightforward to get both leading-order solutions from (2.11).
Taking the plus sign in (2.11), we have the easy case

$$x = \frac{-1 + \sqrt{1 + 4\epsilon}}{2\epsilon} \approx 1\,, \qquad \text{as } \epsilon \to 0. \tag{2.12}$$

We obtained this root with a lot less fuss by just ditching the term ϵx^2 in (2.4). In fact if we just need the leading-order approximation, then the exact answer is more trouble than its worth.
We obtain the missing root by taking the minus sign in (2.11):

$$x = \frac{-1 - \sqrt{1 + 4\epsilon}}{2\epsilon} \approx -\frac{1}{\epsilon}\,, \qquad \text{as } \epsilon \to 0. \tag{2.13}$$

We see that the missing root is going to minus infinity as $\epsilon \to 0$. The term we blithely dropped, namely ϵx^2, is therefore $O(\epsilon^{-1})$. Dropping a big term is a big mistake.
Using MATHEMATICA I expanded the two answers (2.11) as

$$x = \begin{cases} 1 - \epsilon + 2\epsilon^2 - 5\epsilon^3 + \cdots \\ -\epsilon^{-1} - 1 + \epsilon - 2\epsilon^2 + 5\epsilon^3 + \cdots \end{cases} \tag{2.14}$$

Two-term dominant balance

We are missing a root. the missing root by looking for another two-term dominant balance in (2.4):

$$\underbrace{\epsilon x^2 + x}_{\text{dominant balance?}} \qquad -1 = 0\,. \tag{2.15}$$

The balance above implies that $x \approx -\epsilon^{-1}$. The balance is *consistent* because the neglected term in (2.15) (the -1) is smaller than the two retained terms as $\epsilon \to 0$.

Once we know that x is varying as ϵ^{-1} we can *rescale* by introducing

$$X \stackrel{\text{def}}{=} \epsilon x\,. \tag{2.16}$$

The variable X remains finite as $\epsilon \to 0$, and substituting (2.16) into (2.4) we find that X satisfies the rescaled equation

$$X^2 + X - \epsilon = 0\,. \tag{2.17}$$

Now we can find the big root via an RPS

$$X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + O(\epsilon^3). \tag{2.18}$$

Exercise: Verify that (2.18) reproduces the expansion of $x(\epsilon)$ that begins with $-\epsilon^{-1}$ in (2.14).

The introduction of X in (2.16) is "only" a change in notation, and (2.17) is completely equivalent to (2.4). But notation matters: in terms of x the problem is singular while in terms of X the problem is regular. The importance of rescaling, and notation, is a main message from this simple example.

Comment: The equation

$$\epsilon x^2 + x - 1 = 0 \tag{2.19}$$

has three terms and so there are three different two-term balances. In the discussion above we balanced the second and third terms (to get $x \approx 1$). We also balanced the first and second terms to get $x \approx -1/\epsilon$. We have not tried balancing the first and third terms. Let's do it and see what happens. Assume that

$$\epsilon x^2 + x - 1 = 0\,. \tag{2.20}$$

We're provisionally dropping the green term above. This balance implies that $x \propto \epsilon^{-1/2}$ and suggests the rescaling $X = \epsilon^{1/2} x$. But this is wrong – we dropped x relative ϵx^2 and 1. But if $x \propto \epsilon^{-1/2}$ then x is bigger than both ϵx^2 and 1. So this third dominant balance is inconsistent. This is a relief – we're solving a quadratic equation and we don't want to find a third solution.

2.2 Some quartic examples of two-term dominant balance

A quartic equation

Now consider

$$\epsilon x^4 + x - 1 = 0 \tag{2.21}$$

Quartic polynomials can be solved exactly, but the formula is complicated. Instead we use the method of dominant balance to find approximations to all four roots.

There is a consistent two-term balance between the second and third terms i.e. $x \approx 1$, and ϵx^4 is consistently less than the retained terms as $\epsilon \to 0$. We can obtain more terms with $x = 1 + \epsilon x_1 + \cdots$ Instead let's find the other three roots.

Try the two-term dominant balance

$$\epsilon x^4 - 1 \overset{?}{\approx} 0\,, \qquad \Rightarrow \qquad x \sim \epsilon^{-1/4}\,. \tag{2.22}$$

But the neglected term is $x \sim \epsilon^{1/4}$ and $\epsilon^{-1/4}$ is much larger than the retained terms $\{\epsilon x^4, 1\}$. This dominant balance is inconsistent.

The final two-term dominant balance is

$$\epsilon x^4 + x \overset{?}{\approx} 0\,, \qquad \Rightarrow \qquad x \sim \epsilon^{-1/3}\,. \tag{2.23}$$

The neglected term, 1, is much less than the retained terms $\epsilon^{-1/3} \gg 1$. This dominant balance is consistent. Thus the leading-order solutions of the quartic are

$$x = 1\,, \qquad \text{and} \qquad x = \epsilon^{-1/3}\{-1, \mathrm{e}^{\pm \mathrm{i}\pi/3}\}\,. \tag{2.24}$$

To systematically investigate the $\epsilon^{-1/3}$ solutions we rescale with $x = \epsilon^{-1/3} X$. The re-scaled equation is

$$X^4 + X - \epsilon^{1/3} = 0\,. \tag{2.25}$$

Now we can develop an RPS using powers of $\epsilon^{1/3}$:

$$X = X_0 + \epsilon^{1/3} X_1 + \epsilon^{2/3} X_2 + \cdots \tag{2.26}$$

where $X_0 = \{-1, \mathrm{e}^{\pm \mathrm{i}\pi/3}\}$.

Another quartic equation

Now consider

$$\epsilon^2 x^4 + 7x^3 + \epsilon x + 11 = 0\,. \tag{2.27}$$

The low hanging fruit is picked by setting $\epsilon = 0$. This results in the cubic equation

$$7x^3 + 11 \approx 0\,. \tag{2.28}$$

This is a consistent two-term dominant balance because the neglected terms are small relative to retained terms. We find three solutions,

$$x = (11/7)^{1/3}\{-1\,, \mathrm{e}^{\pm \mathrm{i}\pi/3}\}\,. \tag{2.29}$$

Further terms are obtained by an RPS $x = x_0 + \epsilon x_1 + \cdots$.

The fourth solution is obtained by the dominant balance

$$\epsilon^2 x^4 + 7x^3 \overset{?}{\approx} 0\,, \qquad \Rightarrow \qquad x \approx -\frac{7}{\epsilon^2}\,. \tag{2.30}$$

Neglected term ϵx is then of order $\epsilon^{-1} \gg 1$. Nonetheless this is a consistent dominant balance because the retained terms are of order ϵ^{-2} which is much larger than ϵ^{-1} as $\epsilon \to 0$. To determine the development of the large root we rescale with $X \overset{\text{def}}{=} \epsilon^2 x$ to obtain

$$X^4 + 7X^3 + \epsilon^5 X + 11\epsilon^6 = 0\,. \tag{2.31}$$

This example emphasizes that neglecting big terms is OK provided that the neglected terms are dwarfed by even bigger retained terms. We are always thinking of the limit $\epsilon \to 0$ and the disparity between neglected and retained terms becomes ever larger as $\epsilon \to 0$.

Yet another quartic equation

We discuss solution of the quartic polynomial

$$\underbrace{\epsilon x^4}_{T_1} + \underbrace{\epsilon^2 x^3}_{T_2} + \underbrace{\epsilon x^2}_{T_3} - \underbrace{x}_{T_4} + \underbrace{7}_{T_5} = 0 \tag{2.32}$$

as a final example of dominant balance in quartic polynomials.

In (2.32) there are five terms labelled T_1 through T_5. We're going to solve this problem by finding two-term dominant balances. There are ten pairs of terms. So a brutal approach is to examine all ten pairs and find the balances which are consistent approximations as $\epsilon \to 0$.

Balance T_4 and T_5: so that that $x \overset{?}{\approx} 7$. Neglected terms T_1, T_2 and T_3 are:

$$T_1 = \epsilon x^4 = O(\epsilon)\,, \qquad T_2 = \epsilon^2 x^3 = O(\epsilon^2)\,, \qquad T_3 = \epsilon x^2 = O(\epsilon^1)\,.$$

All terms above are smaller than the retained terms $(T_4, T_5) \propto \epsilon^0$ as $\epsilon \to 0$. This is a consistent dominant balance and we can plug in the RPS $x = 7 + \epsilon x_1 + \cdots$ into (2.32) with every expectation of success. You should do this.

Balance T_3 and T_4: this means that $\epsilon x^2 - x \overset{?}{\approx} 0$, implying that $x = 0$ or $x = O(\epsilon^{-1})$. Only $x = O(\epsilon^{-1})$ might be helpful. In this case the neglected terms are

$$T_1 = \epsilon x^4 = O(\epsilon^{-3}), \qquad T_2 = \epsilon^2 x^3 = O(\epsilon^{-1}), \qquad T_5 = 7 = O(\epsilon^0).$$

This dominant balance is inconsistent because the neglected term $T_1 = O(\epsilon^{-3})$ is much larger than the retained terms $(T_3, T_4) = O(\epsilon^{-1})$ as $\epsilon \to 0$. No joy here.

Balance T_3 and T_5: this means that $\epsilon x^2 + 7 \overset{?}{\approx} 0$, implying that $x = O(\epsilon^{-1/2})$. Neglected terms are

$$T_1 = \epsilon x^4 = O(\epsilon^{-1}), \qquad T_2 = \epsilon^2 x^3 = O(\epsilon^{1/2}), \qquad T_4 = x = O(\epsilon^{-1/2}).$$

As $\epsilon \to 0$, neglected term $T_1 \propto \epsilon^{-1}$ is much greater than retained terms $(T_3, T_5) \propto \epsilon^0$: this dominant balance is inconsistent. Nothing to see here – move on.

Balance T_1 and T_5: this means that $\epsilon x^4 + 7 \overset{?}{\approx} 0$, implying that $x = O(\epsilon^{-1/4})$. The neglected terms are

$$T_2 = \epsilon^2 x^3 = O(\epsilon^{5/4}), \qquad T_3 = \epsilon x^2 = O(\epsilon^{1/2}), \qquad T_4 = x = O(\epsilon^{-1/4}).$$

The neglected term $T_4 = O(\epsilon^{-1/4})$ is much bigger than the retained terms $(T_1, T_5) = O(\epsilon^0)$: this is another inconsistent dominant balance.

Balance T_1 and T_4: this means that $\epsilon x^4 - x \overset{?}{\approx} 0$, implying that $x = O(\epsilon^{-1/3})$. The neglected terms are

$$T_2 = \epsilon^2 x^3 = O(\epsilon^1), \qquad T_3 = \epsilon x^2 = O(\epsilon^{1/3}), \qquad T_5 = 7 = O(\epsilon^0).$$

Hallelujah – this works. The three neglected terms are all smaller than the retained terms $(T_1, T_4) \propto \epsilon^{-1/3}$ as $\epsilon \to 0$.

Exercise: It is sporting to examine the other five two-term balances and show that all of these are inconsistent.

The consistent dominant balance between T_1 and T_4 says that

$$\epsilon x^4 - x \approx 0 \tag{2.33}$$

is a good $\epsilon \to 0$ leading-order approximation to (2.32). Equation (2.33) has three useful solutions – the three cube roots of one. The fourth root, $x = 0$ can be thought of as reproducing the other consistent dominant balance, $x \approx 7 \ll \epsilon^{-1/3}$. We now possess all four $\epsilon \to 0$ roots of (2.32).

To nail down the three singular roots, introduce

$$\delta \overset{\text{def}}{=} \epsilon^{1/3}, \qquad \text{and} \qquad X \overset{\text{def}}{=} \delta x. \tag{2.34}$$

The rescaled polynomial is

$$X^4 + \delta^4 X^3 + \delta^2 X^2 - X + 7\delta = 0. \tag{2.35}$$

The good dominant balance is immediately revealed by setting $\delta = 0$ and the RPS is

$$X = X_0 + \delta X_1 + O(\delta^2). \tag{2.36}$$

Noting that $X^4 = X_0^4 + 4\delta X_0^3 X_1 + O(\delta^2)$ we see that the first two orders are

$$\delta^0: \qquad X_0(X_0^3 - 1) = 0\,, \tag{2.37}$$

$$\delta^1: \qquad (4X_0^3 - 1)X_1 + 7 = 0\,. \tag{2.38}$$

The three leading order solutions are $X_0 = 1$, and $X_0 = \mathrm{e}^{\pm 2\pi \mathrm{i}/3}$. In all three cases $X_1 = -3/7$.

Summary: In our original notation the leading-order behaviour of the four roots of (2.32) is

$$x \sim \left\{7\,, \epsilon^{-1/3}\,, \epsilon^{-1/3}\mathrm{e}^{\pm 2\pi \mathrm{i}/3}\right\}\,, \qquad \text{as } \epsilon \to 0. \tag{2.39}$$

Exercise: Find leading-order expressions for all six roots of

$$\epsilon^2 x^6 - \epsilon x^4 - x^3 + 8 = 0\,, \qquad \text{as } \epsilon \to 0. \tag{2.40}$$

The answer is in section 7.2 of **BO**.

Exercise: Construct a quartic polynomial in x, with coefficients $\epsilon^{\text{some power}}$, which presents a three-term dominant balance as $\epsilon \to 0$.

2.3 Iteration, also known as recursion

Now let's consider the method of iteration. Iteration requires a bit of initial ingenuity. But in cases where the form of the expansion is not obvious, iteration is essential. (One of the strengths of **H** is that it emphasizes the utility of iteration.)

Introductory example of iteration

Consider the quadratic equation

$$(x-1)(x-2) = \epsilon x\,. \tag{2.41}$$

If we interested in the effect of ϵ on the root $x = 1$ then we rearrange this further as

$$x = 1 + \epsilon \underbrace{\frac{x}{x-2}}_{\stackrel{\text{def}}{=} f(x)}\,. \tag{2.42}$$

We iterate by first dropping the ϵ-term on the right — this provides the initial guess $x^{(0)} = 1$. At the next iteration we keep the ϵ-term with f evaluated at $x^{(0)}$:

$$x^{(1)} = 1 + \epsilon f\left(x^{(0)}\right) = 1 - \epsilon\,. \tag{2.43}$$

We continue to improve the approximation with more and more iterates:

$$x^{(n+1)} = 1 + f\left(x^{(n)}\right)\,. \tag{2.44}$$

A few more times through the loop with MATHEMATICA produces:

$$x^{(2)} = \frac{1+\epsilon^2}{1+\epsilon}\,, \tag{2.45}$$

$$x^{(3)} = \frac{1+\epsilon-\epsilon^2-\epsilon^3}{1+2\epsilon-\epsilon^2}\,, \tag{2.46}$$

$$x^{(4)} = \frac{1+2\epsilon-2\epsilon^2+2\epsilon^3+\epsilon^4}{1+3\epsilon-\epsilon^2+\epsilon^3}\,, \tag{2.47}$$

$$x^{(5)} = \frac{1+3\epsilon-2\epsilon^2+2\epsilon^3-3\epsilon^4-\epsilon^5}{1+4\epsilon-\epsilon^4}\,. \tag{2.48}$$

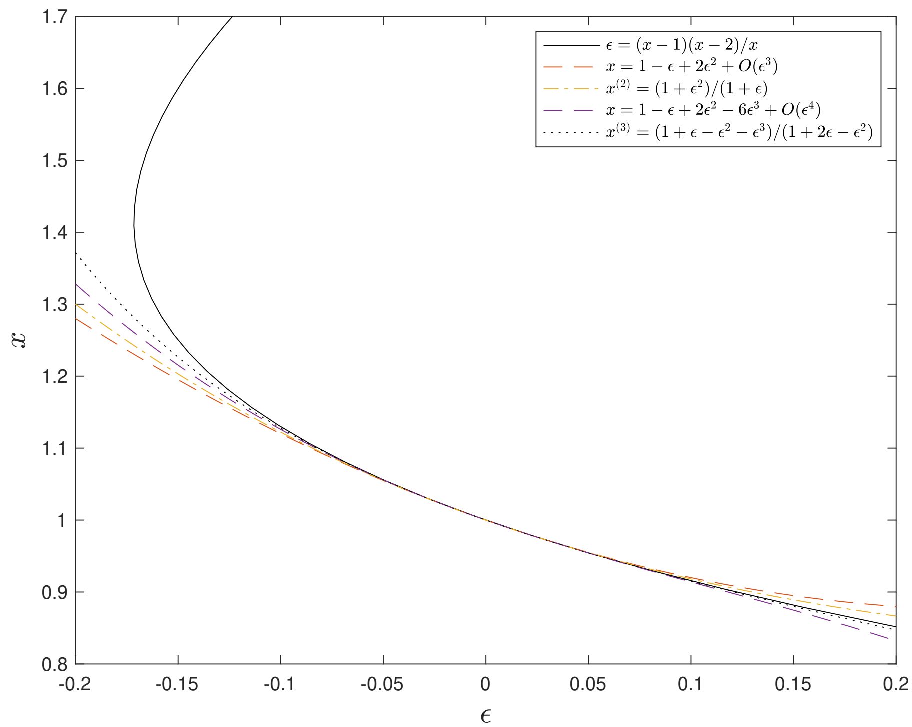

Figure 2.1: Comparison of the iterates $x^{(2)}$ and $x^{(3)}$ with power series having the same formal accuracy. The iterates are a better approximation to the answer than truncated power series.

The exact answer is

$$x = [3 + \epsilon - \sqrt{1 + 6\epsilon + \epsilon^2}]/2\,, \tag{2.49}$$

$$= 1 - \epsilon + 2\epsilon^2 - 6\epsilon^3 + 22\epsilon^4 - 90\epsilon^5 + O(\epsilon^6) \tag{2.50}$$

The expansion of the successive iterates is

$$x^{(1)} = 1 - \epsilon\,, \tag{2.51}$$

$$x^{(2)} = 1 - \epsilon + 2\epsilon^2 - 2\epsilon^3 + 2\epsilon^4 - 2\epsilon^5 + O(\epsilon^6)\,, \tag{2.52}$$

$$x^{(3)} = 1 - \epsilon + 2\epsilon^2 - 6\epsilon^3 + 14\epsilon^4 - 34\epsilon^5 + O(\epsilon^6)\,, \tag{2.53}$$

$$x^{(4)} = 1 - \epsilon + 2\epsilon^2 - 6\epsilon^3 + 22\epsilon^4 - 74\epsilon^5 + O(\epsilon^6)\,, \tag{2.54}$$

$$x^{(5)} = 1 - \epsilon + 2\epsilon^2 - 6\epsilon^3 + 22\epsilon^4 - 90\epsilon^5 + O(\epsilon^6)\,. \tag{2.55}$$

Wrong terms are red. Every pass through the iteration loop provides another correct term in the RPS.

But why do we insist on expanding the iterates in a power series in ϵ? Perhaps the unexpanded iterates in (2.45) through (2.48) are superior to the RPS? In fact they are: see the comparison in figure 2.1. Iteration is producing a *Padé approximation* to the solution. Padé approximations have superior convergence properties because they are not limited by singularities in the complex ϵ-plane[1]. This example shows that an RPS is not always the best approximation. Also it may be faster to bash out the first one or two iterates than an equivalent RPS.

[1] See **BO** for a discussion of Padé approximation.

Exercise: Use iteration to locate the root near $x = 2$.

Example: Considering the equation

$$4 - x^2 = \epsilon \ln x \,, \tag{2.56}$$

with $0 < \epsilon \ll 1$, we see that there is a positive real solution close to $x = 2$. To improve on $x \approx 2$ we rewrite the equation as

$$x = 2 - \frac{\epsilon \ln x}{2 + x} \,. \tag{2.57}$$

If we drop the ϵ-term we get a first approximation $x^{(1)} = 2$, and the next iterate is

$$x^{(2)} = 2 - \frac{\epsilon \ln 2}{4} \,, \tag{2.58}$$

and again

$$x^{(3)} = 2 - \frac{\epsilon \ln\left(2 - \frac{\epsilon \ln 2}{4}\right)}{4 - \frac{\epsilon \ln 2}{4}} \,. \tag{2.59}$$

We can develop an RPS by simplifying $x^{(3)}$ as

$$x^{(3)} = 2 - \frac{\epsilon}{4}\left(1 + \epsilon\frac{\ln 2}{16}\right)\Big[\ln 2 + \underbrace{\ln\left(1 - \epsilon\frac{\ln 2}{8}\right)}_{= -\epsilon\frac{\ln 2}{8} + O(\epsilon^2)}\Big] + O(\epsilon^3) \,, \tag{2.60}$$

$$= 2 - \frac{\ln 2}{4}\epsilon + \frac{\ln 2}{64}(2 - \ln 2)\,\epsilon^2 + O(\epsilon^3) \,. \tag{2.61}$$

Another example of iteration

As another example of iteration, consider the equation

$$x = \epsilon \mathrm{e}^{-x^2} \,. \tag{2.62}$$

There is a root near $x = 0$. But if x is close to zero, then $\mathrm{e}^{-x^2} \approx 1$, and in this case $x \approx \epsilon$. These considerations motivate the iterative scheme

$$x^{(0)} = \epsilon \,, \qquad \text{and} \qquad x^{(n+1)} = \epsilon \exp\left(-x^{(n)2}\right) \,. \tag{2.63}$$

Going twice through the loop

$$x^{(1)} = \epsilon \mathrm{e}^{-\epsilon^2} \,, \qquad \text{and} \qquad x^{(2)} = \epsilon \mathrm{e}^{-\epsilon^2 \mathrm{e}^{-2\epsilon^2}} \,. \tag{2.64}$$

The expansion of the first iterate is $x^{(1)} = \epsilon - \epsilon^3 + O(\epsilon^5)$. Iteration is a more expeditious route to this approximation than RPS machinery.

Convergence of iteration

Usually we can't prove that an RPS converges. The only way of proving convergence is to have a simple expression for the form of the n'th term. In realistic problems this is not available. One just has to be satisfied with consistency and hope for the best.

But with iteration there is a simple result. Suppose that $x = x_*$ is the solution of

$$x = f(x) \,. \tag{2.65}$$

Start with a guess $x = x_0$ and proceed to iterate with $x_{n+1} = f(x_n)$. If an iterate x_n is close to the solution x_* then we have

$$x = x_* + \eta_n \,, \qquad \text{with } \eta_n \ll 1. \tag{2.66}$$

The next iterate is:

$$x_* + \eta_{n+1} = f(x_* + \eta_n)\,, \tag{2.67}$$

$$= x_* + \eta_n f'(x_*) + O(\eta_n^2)\,, \tag{2.68}$$

and therefore

$$\eta_{n+1} = f'(x_*)\eta_n\,. \tag{2.69}$$

The sequence η_n will decrease exponentially if

$$|f'(x_*)| < 1\,. \tag{2.70}$$

If the condition above is satisfied, and the first guess is good enough, then the iteration converges onto x_*. This is a loose version of the *contraction mapping theorem*

2.4 Double roots

Consider

$$\underbrace{x^2 - 2x + 1}_{(x-1)^2} - \epsilon f(x) = 0\,, \tag{2.71}$$

where $f(x)$ is some function of x. Section 1.3 of **H** discusses the case $f(x) = x^2$ – with a surfeit of testosterone we attack the general case.

We try the RPS:

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots \tag{2.72}$$

We must expand $f(x)$ with a Taylor series:

$$f\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\right) = f(x_0) + \epsilon x_1 f'(x_0) + \epsilon^2\left[x_2 f'(x_0) + \tfrac{1}{2}x_1^2 f''(x_0)\right] + O\left(\epsilon^3\right)\,. \tag{2.73}$$

This is not as bad as it looks — we'll only need the first term, $f(x_0)$, though that may not be obvious at the outset.

The leading term in (2.71) is

$$x_0^2 - 2x_0 + 1 = 0\,, \qquad \Rightarrow \qquad x_0 = 1\,,\ \text{(twice)}. \tag{2.74}$$

There is a double root. At next order there is a problem:

$$\epsilon^1 : \qquad \underbrace{2x_1 - 2x_1}_{=0} - f(1) = 0\,. \tag{2.75}$$

Unless $f(1)$ happens to vanish, we're stuck. The problem is that we assumed that the solution is delivered by the RPS in (2.72), and it turns out that this assumption is wrong. The perturbation method kindly tells us this by producing the contradiction in (2.75).

Iteration to the rescue

To find the correct form of the expansion we use iteration: rewrite (2.71) as

$$x = 1 \pm \sqrt{\epsilon f(x)}\,. \tag{2.76}$$

Starting with $x^{(0)} = 1$, iterate with

$$x^{(n+1)} = 1 \pm \sqrt{\epsilon f\left(x^{(n-1)}\right)}\,. \tag{2.77}$$

At the first iteration

$$x^{(1)} = 1 \pm \sqrt{\epsilon f\,(1)}\,. \tag{2.78}$$

There is a $\sqrt{\epsilon}$ which was not anticipated by the RPS back in (2.72).

Exercise: Go through another iteration cycle to find $x^{(2)}$.

Iteration has shown us the way forward: we proceed assuming that the correct RPS is probably

$$x = x_0 + \epsilon^{1/2}x_1 + \epsilon x_2 + \epsilon^{3/2}x_3 + \cdots \tag{2.79}$$

At leading order we find $x_0 = 1$, and at next order

$$\epsilon^{1/2}: \qquad 2x_1 - 2x_1 = 0\,. \tag{2.80}$$

This is surprising, but it is not a contradiction: x_1 is not determined at this order. We have to endure some suspense – we go to next order and find

$$\epsilon^1 \;: \qquad \underbrace{2(x_0-1)x_2}_{=0} + x_1^2 - f(x_0) = 0\,, \qquad \Rightarrow \qquad x_1 = \pm\sqrt{f(1)}\,. \tag{2.81}$$

The RPS has now managed to reproduce the first iterate $x^{(1)}$. Going to order $\epsilon^{3/2}$, we find that x_3 is undetermined and

$$x_2 = \tfrac{1}{2}f'(1)\,. \tag{2.82}$$

The solution we constructed is

$$x = 1 \pm \sqrt{\epsilon f(1)} + \frac{\epsilon}{2}f'(1) + O\left(\epsilon^{3/2}\right)\,. \tag{2.83}$$

This example teaches us that a perturbation "splits" double roots. The splitting is rather large: adding the order ϵ perturbation in (2.71) moves the roots apart by order $\sqrt{\epsilon} \gg \epsilon$. This sensitivity to small perturbations is obvious geometrically — draw a parabola P touching the x-axis at some point, and move P downwards by small distance. The small movement produces two roots separated by a distance that is clearly much greater than the small vertical displacement of P. If P moves upwards (corresponding to $f(1) < 0$ in the example above) then the roots split off along the imaginary axis.

Non-uniformity

Consider the cubic equation

$$(x-a)(x-b)(x-c) = \epsilon f(x)\,. \tag{2.84}$$

Suppose that a, b, c and $f(x)$ are known and that $\epsilon \ll 1$. If $\epsilon = 0$ then we can easily solve the equation: $x = a$, $x = b$ and $x = c$ are all solutions. How are these solutions perturbed if ϵ is non-zero but small?

We use an RPS

$$x = x_0 + \epsilon x_1 + O(\epsilon^2). \tag{2.85}$$

Consider the case with $x_0 = a$. At order ϵ^1 we find

$$x_1(a-b)(a-c) = f(a)\,, \tag{2.86}$$

with solution

$$x_1 = \frac{f(a)}{(a-b)(a-c)}\,. \tag{2.87}$$

Thus the two-term RPS is

$$x = a + \frac{\epsilon f(a)}{(a-b)(a-c)} + O\left(\epsilon^2\right)\,. \tag{2.88}$$

Exercise: Obtain (2.88) in one line with iteration.

This is great unless the denominator $(a-b)(a-c)$ in (2.88) is very small e.g. if $a-b$ is same size as ϵ then the ostensible small correction,

$$\frac{\epsilon f(a)}{(a-b)(a-c)}\,, \tag{2.89}$$

is not small and the RPS likely fails. This happens if either b or c is close to a. Of course if $b = a$ or if $c = a$ then it is a complete disaster – we're dividing by zero in (2.89). The message is that double roots, and near double roots, obstruct an RPS.

Exercise: Assuming that there is not a problem with double roots, find x_1 if we start with $x_0 = b$ and $x_0 = c$.

Suppose that a and b are close. Introduce

$$\xi \stackrel{\text{def}}{=} \frac{a+b}{2}\,, \qquad \text{and} \qquad \nu \stackrel{\text{def}}{=} \frac{a-b}{2}\,. \tag{2.90}$$

Close means that $\nu \ll \xi$. With this change of notation the problem is now

$$(x-\xi-\nu)(x-\xi+\nu)(x-c) = \epsilon f(x)\,, \tag{2.91}$$

where $\nu \ll 1$ and $\epsilon \ll 1$. We remain agnostic about the relative size of these two small parameters. The problem is that there is a near double root at $x = \xi$ – the parameter ν controls proximity to the double root, while ϵ controls the size of the perturbation.

(If c is not close to ξ then there is an isolated single root near $x = c$. We can proceed as before and develop an RPS $x = c + \epsilon x_1 + \cdots$. Nailing down this isolated root presents no challenges. Press on to the more difficult case.)

For the near double root, we divide by $x - c$ and write the equation as

$$(x-\xi-\nu)(x-\xi+\nu) = \epsilon \underbrace{\frac{f(x)}{x-c}}_{\stackrel{\text{def}}{=} g(x)}\,. \tag{2.92}$$

All the action is in the neighbourhood of $x = \xi$ and very plausibly (2.92) can be approximated as

$$(x-\xi-\nu)(x-\xi+\nu) \approx \epsilon g(\xi)\,. \tag{2.93}$$

We multiply out the left hand side and rearrange to obtain

$$(x-\xi)^2 \approx \nu^2 + \epsilon g(\xi)\,. \tag{2.94}$$

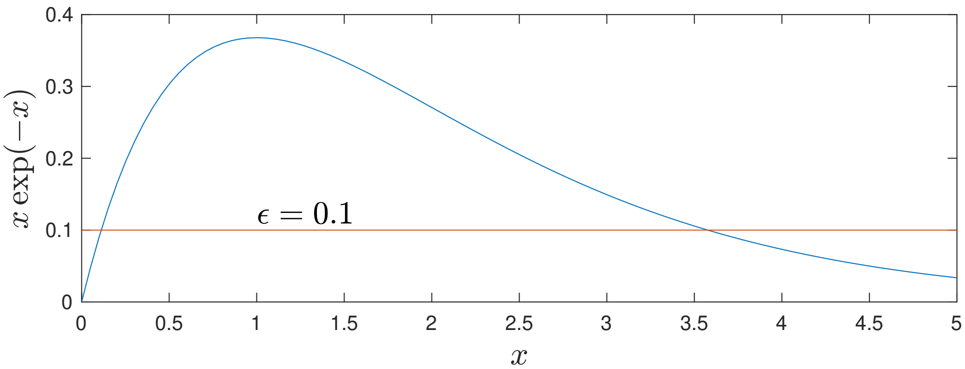

Figure 2.2: Graphical determination of the $\epsilon \ll 1$ solutions of (2.96).

Let's admit that taking a square root is not a big deal so now[2]

$$x \approx \xi \pm \sqrt{\nu^2 + \epsilon g(\xi)}\,. \tag{2.95}$$

The interesting case is if ν and ϵ are both small and ϵ is the same size as ν^2. This is a *distinguished limit*: take $\epsilon \to 0$ and $\nu \to 0$ with ϵ/ν^2 fixed. The formula above works in this limit, and also works if $1 \gg \nu^2 \gg \epsilon$, or if $\nu^2 \ll \epsilon \ll 1$.

2.5 An example with logarithms

I'll discuss the example from **H** section 1.4:

$$x\mathrm{e}^{-x} = \epsilon\,. \tag{2.96}$$

It is easy to see that if $0 < \epsilon \ll 1$ there is a small solution and a big solution – see figure 2.2. It is straightforward to find the small solution in terms of ϵ. Here we discuss the more difficult problem of finding the big solution.

Exercise: Show that the small solution is $x(\epsilon) = \epsilon + \epsilon^2 + \frac{3}{2}\epsilon^3 + O(\epsilon^4)$.

To get a handle on (2.96), we take the logarithm and write the result as

$$x = L_1 + \ln x\,, \tag{2.97}$$

where

$$L_1 \stackrel{\text{def}}{=} \ln \frac{1}{\epsilon}\,. \tag{2.98}$$

Note if $0 < \epsilon < 1$ then $\ln \epsilon < 0$. To avoid confusion over signs it is best to work with the large positive quantity L_1.

Now observe that if $x \to \infty$ then there is a consistent two-term dominant balance in (2.97): $x \approx L_1$. This is consistent because the neglected term, namely $\ln x$, is much less than x as $x \to \infty$. We can improve on this first approximation using the iterative scheme

$$x^{(n+1)} = L_1 + \ln x^{(n)} \qquad \text{with} \qquad x^{(0)} = L_1\,. \tag{2.99}$$

The first iteration gives

$$x^{(1)} = L_1 + L_2\,, \tag{2.100}$$

[2] I'm writing $\approx$ in (2.95) because I've been sloppy about estimating the size of the neglected terms.

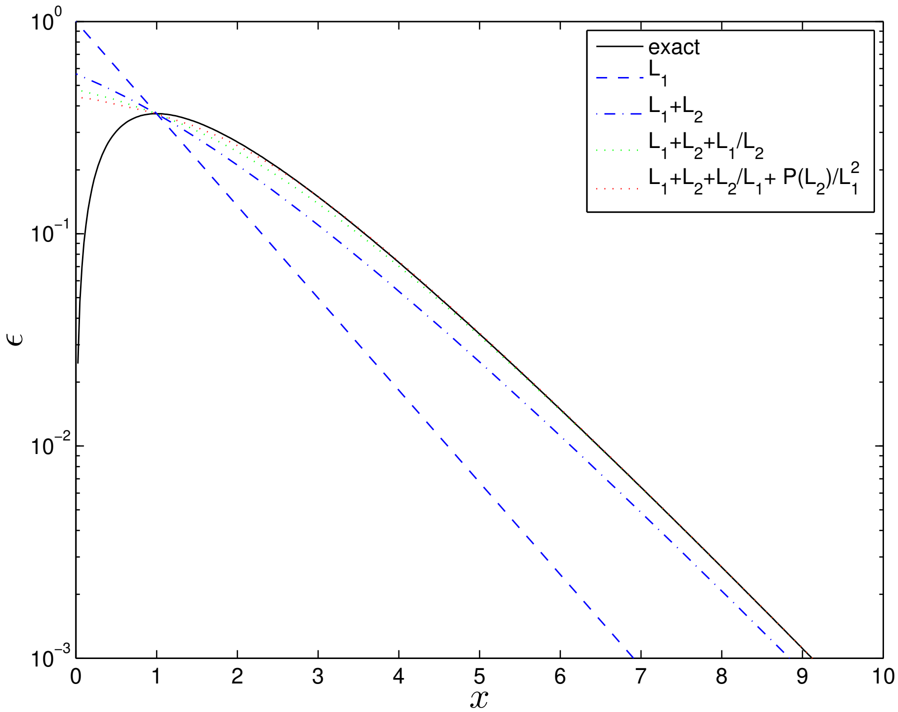

Figure 2.3: Comparison of $\epsilon = x\mathrm{e}^{-x}$ with increasingly accurate small-ϵ approximations to the inverse function $\epsilon(x)$.

where $L_2 \stackrel{\text{def}}{=} \ln L_1$ is the iterated logarithm.

The second iteration[3] is

$$x^{(2)} = L_1 + \ln(L_1 + L_2) \ , \tag{2.101}$$

$$= L_1 + L_2 + \ln\left(1 + \frac{L_2}{L_1}\right) \ , \tag{2.102}$$

$$= L_1 + L_2 + \frac{L_2}{L_1} - \tfrac{1}{2}\left(\frac{L_2}{L_1}\right)^2 + \cdots \tag{2.103}$$

We don't need L_3.

[3]We're using the Taylor series

$$\ln(1+\eta) = \eta - \tfrac{1}{2}\eta^2 + \tfrac{1}{3}\eta^3 + \tfrac{1}{4}\eta^4 + \cdots$$

At the third iteration a pattern starts to emerge

$$
\begin{aligned}
x^{(3)} &= L_1 + \ln\left(L_1 + L_2 + \frac{L_2}{L_1} - \tfrac{1}{2}\left(\frac{L_2}{L_1}\right)^2 + \cdots\right) , \\
&= L_1 + L_2 + \ln\left(1 + \frac{L_2}{L_1} + \frac{L_2}{L_1^2} - \tfrac{1}{2}\frac{L_2^2}{L_1^3} + \cdots\right) , \\
&= L_1 + L_2 + \left(\frac{L_2}{L_1} + \frac{L_2}{L_1^2} - \tfrac{1}{2}\frac{L_2^2}{L_1^3} + \cdots\right) - \tfrac{1}{2}\left(\frac{L_2}{L_1} + \frac{L_2}{L_1^2} + \cdots\right)^2 + \tfrac{1}{3}\left(\frac{L_2}{L_1} + \cdots\right)^3 \cdots \\
&= L_1 + L_2 + \frac{L_2}{L_1} + \frac{L_2 - \tfrac{1}{2}L_2^2}{L_1^2} + \frac{\tfrac{1}{3}L_2^3 - \tfrac{3}{2}L_2^2 + \cdots}{L_1^3} + \cdots
\end{aligned}
\tag{2.104}
$$

The final $\cdots$ above indicates a fraction with L_1^4 in the denominator.

The philosophy is that as one grinds out more terms the earlier terms in the developing expansion stop changing and a stable pattern emerges. In this example the expansion has the form

$$
x = L_1 + L_2 + \sum_{n=1}^{\infty} \frac{P_n(L_2)}{L_1^n} , \tag{2.105}
$$

where P_n is a polynomial of degree n. This was not guessable from (2.96).

The Lambert W function

The Lambert W-function is defined implicitly via

$$z = W\mathrm{e}^{W} \,. \tag{1}$$

In the complex plane (1) defines a multivalued function $W_k(z)$ where k is an integer $\{0, \pm 1, \pm 2 \cdots\}$. The figure shows two branches $W_0(x)$ and $W_{-1}(x)$. These are the real branches which exist only if $x > -1/e$.

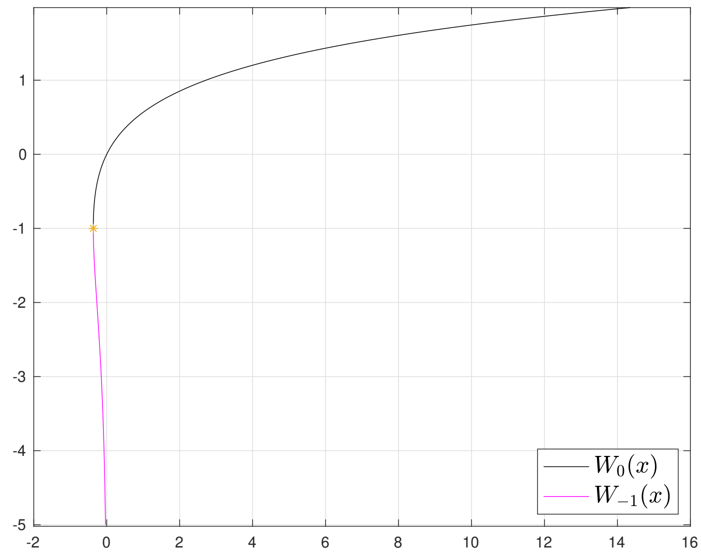

Figure 2.4: The two real branches of the Lambert W functions. The nose $*$ is at $(-\mathrm{e}^{-1}, -1)$.

The Lambert W-function, also known as the omega function and the product logarithm; try help `ProductLog` in MATHEMATICA and `lambertw` in MATLAB.

The implicit function theorem

There are many versions of the implicit function theorem. I'll state the *analytic* implicit function theorem. Suppose that x_0 is a complex number and

$$f(x_0, 0) = 0\,, \tag{1}$$

where $f(x, \epsilon)$ is analytic at the point $(x, \epsilon) = (x_0, 0)$. Analytic means that $f(x, \epsilon)$ has a convergent power series expansion in non-negative powers of $x - x_0$ and ϵ.

Provided that

$$\frac{\partial f}{\partial x}(x_0, 0) \neq 0\,, \tag{2}$$

then there are constants p and q such that for every ϵ in the disc $|\epsilon| < p$, equation (1) has a unique simple root $x = x(\epsilon)$ in the disc $|x - x_0| < q$. In addition $x(\epsilon)$ is an analytic function of ϵ and $x(0) = x_0$.

The condition in equation (2) is the same as saying that x_0 is a simple root of $f(x, 0) = 0$. The fuss about multiple roots results from failure, and near failure, of (2).

2.6 Problems

Problem 2.1. Show that $\epsilon \to 0$ expansion of the roots of $\epsilon x^3 + x - 1 = 0$ is

$$x = 1 - \epsilon + O(\epsilon^2), \qquad \text{and} \qquad x = \pm\frac{\mathrm{i}}{\sqrt{\epsilon}} - \frac{1}{2} \pm \sqrt{\epsilon}\,\frac{3\,\mathrm{i}}{8} + O\left(\epsilon^1\right). \tag{2.106}$$

Problem 2.2. $x(t)$ is defined via the initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \exp\left(\frac{x}{10}\right) - x, \qquad \text{with IC} \qquad x(0) = 0. \tag{2.107}$$

Find $\lim_{t\to\infty} x(t)$ to three significant figures. (From a mid-term exam.)

Problem 2.3. $x(t)$ is defined via the initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1.005 - x - \mathrm{e}^{-x}, \qquad \text{with IC} \qquad x(0) = 0. \tag{2.108}$$

Find $\lim_{t\to\infty} x(t)$ to two significant figures. (From a mid-term exam.)

Problem 2.4. Consider the quartic polynomial

$$\epsilon^2 x^4 + \epsilon x^3 + \epsilon x^2 - x + 7 = 0. \tag{2.109}$$

Find the leading order $\epsilon \to 0$ approximation to all roots.

Problem 2.5. Consider a quadratic equation, $ax^2 + bx + c = 0$, and suppose that $b^2 \gg ac$ (all coefficients are real). Use dominant balance (not the exact solution) to obtain a simple approximation to both roots. Test drive your approximation on $x^2 + 3x + 1/2 = 0$.

Problem 2.6. *(i)* Find a two-term approximation to all five roots of

$$x^5 - x + \epsilon = 0. \tag{2.110}$$

Take $\epsilon = 1/4$ and compare your approximation to a numerical solution (e.g. use the MATLAB command `roots`). *(ii)* Suppose that $\eta = \epsilon^{-1}$. Find a two-term approximation to the five roots in the limit $\eta \to 0$ (which is the same as the limit $\epsilon \to \infty$).

Problem 2.7. Consider the transcendental equation

$$x^2 - 1 = \epsilon \mathrm{e}^{x^2}. \tag{2.111}$$

If $\epsilon = 0$ there is a root $x = 1$. Find the first three terms in the $\epsilon \to 0$ regular perturbation expansion of this root.

Problem 2.8. Consider $(x+1)^7 = \epsilon x$ with $\epsilon \ll 1$. How rapidly do the 7 roots vary from $x = -1$ as ϵ increases from zero? Give the first three terms in the expansion.

Problem 2.9. Find two-term, $\epsilon \to 0$ approximations to all roots of

$$x^3 + 5x^2 + 4x + \epsilon = 0, \tag{2.112}$$

and

$$y^3 - y^2 + \epsilon = 0, \tag{2.113}$$

and

$$\epsilon^3 z^4 + \epsilon z^2 - z + 1 = 0. \tag{2.114}$$

Problem 2.10. Develop perturbation solutions to

$$x^3 - (6 + \epsilon + \epsilon^2)x^2 + (12 + 3\epsilon + 3\epsilon^2 + 2\epsilon^3)x - 8 - 2\epsilon - 3\epsilon^2 - 2\epsilon^3 - \epsilon^4 = 0 \tag{2.115}$$

finding the first three terms in the approximation for each root, $x = x_0 + \epsilon^a x_a + \epsilon^{2a} x_{2a}$, and determining a along the way.

Problem 2.11. Find rescalings for the roots of

$$\epsilon^2 x^3 - (1 - \epsilon + 3\epsilon^2)x^2 + (3 - 3\epsilon + 2\epsilon^2 - \epsilon^3)x - 2 + 3\epsilon - \epsilon^3 = 0 \tag{2.116}$$

and thence find two non-trivial terms in the approximation for each root using *(i)* iteration and *(ii)* series expansion.

Problem 2.12. Here is a medley of algebraic perturbation problems, mostly from **BO** and **H**. Use perturbation theory to find two-term approximations ($\epsilon \to 0$) to all roots of:

(a) $x^2 + x + 6\epsilon = 0\,,$
(b) $x^3 - \epsilon x - 1 = 0\,,$
(c) $x^3 + \epsilon x^2 - x - \epsilon = 0\,,$
(d) $\epsilon^2 x^3 + x^2 + 2x + \epsilon = 0\,,$
(e) $\epsilon x^3 + x^2 - 2x + 1 = 0\,,$
(f) $\epsilon x^3 + x^2 + (2 + \epsilon)x + 1 = 0\,,$
(g) $\epsilon x^3 + x^2 + (2 - \epsilon)x + 1 = 0\,,$
(h) $\epsilon x^4 - x^2 - x + 2 = 0\,,$
(i) $\epsilon x^8 - \epsilon^2 x^6 + x - 2 = 0\,,$
(j) $\epsilon x^8 - \epsilon x^6 + x - 2 = 0\,,$
(k) $\epsilon^2 x^8 - \epsilon x^6 + x - 2 = 0\,,$
(l) $x^3 - x^2 + \epsilon = 0\,.$

Problem 2.13. Consider $y(\epsilon, a)$ defined as the solution of

$$\epsilon y^a = \mathrm{e}^{-y}\,. \tag{2.117}$$

Note that $a = -1$ is the example (2.96). Use the method of iteration to find a few terms in the $\epsilon \to 0$ asymptotic solution of (2.117) – "few" means about as many as in (2.104). Consider the case $a = +1$; use MATLAB to compare the exact solution with increasingly accurate asymptotic approximations (e.g., as in Figure 2.3).

Problem 2.14. Let us continue problem 2.13 by considering numerical convergence of iteration in the special case $a = 1$. Figure 2.5 shows numerical iteration of

$$y_{n+1} = \ln\frac{1}{\epsilon} - \ln y_n\,. \tag{2.118}$$

With $\epsilon = 0.25$ everything is hunky-dory. At $\epsilon = 0.35$ the iteration is converging, but it is painfully slow. And at $\epsilon = 0.45$ it all goes horribly wrong. Explain this failure of iteration. To be convincing your explanation should include a calculation of the magic value of ϵ at which numerical iteration fails. That is, if $\epsilon > \epsilon_*$ then the iterates do not converge to the solution of $\epsilon y = \mathrm{e}^{-y}$. Find ϵ_*.

Problem 2.15. Find a three-term approximation to the real solutions of

$$\mathrm{e}^{x - x^2} = \epsilon x^2\,, \qquad \text{as } \epsilon \to 0\,. \tag{2.119}$$

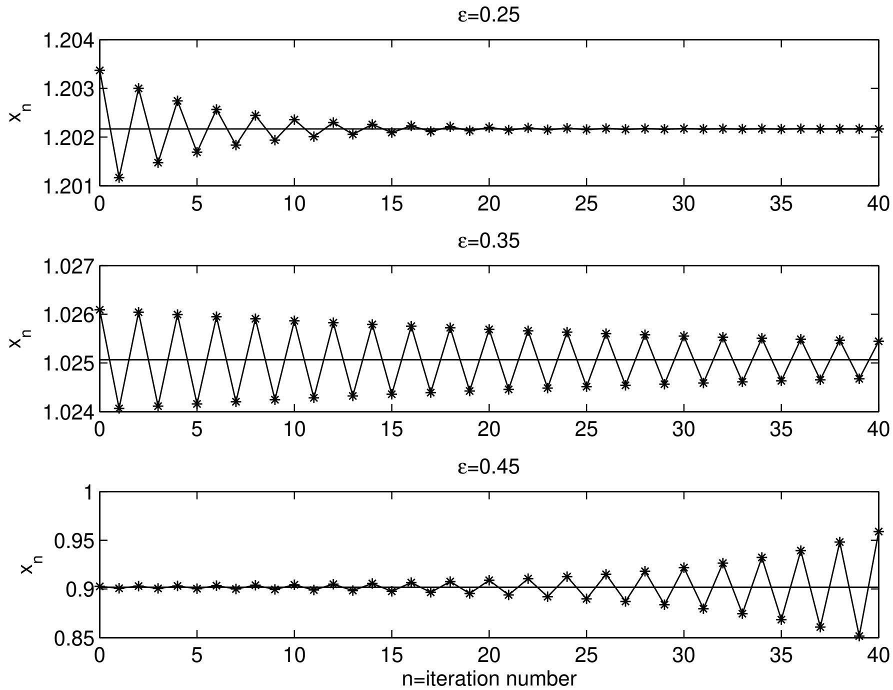

Figure 2.5: Figure for problem 2.14. Numerical iteration of $y_{n+1} = \ln \frac{1}{\epsilon} - \ln y_n$. At $\epsilon = 0.45$ the iteration diverges. In all three cases we start x_0 within 0.1% of the right answer.

Problem 2.16. Find two- or three- term approximations to all real solutions of

$$x^2 - 1 = \mathrm{e}^{\epsilon x}, \qquad \text{as } \epsilon \to 0. \tag{2.120}$$

Using figure 2.3 as an example, and considering the largest positive root, use MATLAB to compare your approximation with the exact relation.

Problem 2.17. Find a two-term approximation to all positive real roots of $x^2 - 4 = \epsilon \ln x$ as $\epsilon \to 0$.

Problem 2.18. Consider $z(\epsilon)$ defined as the solution to

$$z^{\frac{1}{\epsilon}} = \mathrm{e}^z. \tag{2.121}$$

(i) Use MATLAB to make a graphical analysis of this equation with $\epsilon = 1/5$ and $\epsilon = 1/10$. Convince yourself that as $\epsilon \to 0$ there is one root near $z = 1$, and second, large root that recedes to infinity as $\epsilon \to 0$. *(ii)* Use an iterative method to develop an $\epsilon \to 0$ approximation to the large solution. Calculate a few terms so that you understand the form of the expansion. *(iii)* Use MATLAB to compare the exact answer with approximations of various orders e.g., as in Figure 2.3. *(iv)* Find the dependance of the other root, near $z = 1$, on ϵ as $\epsilon \to 0$.

Problem 2.19. Find the $x \gg 1$ solution of

$$\mathrm{e}^{\mathrm{e}^x} = 10^{10} x^{10} \exp\left(10^{10} x^{10}\right)$$

with one significant figure of accuracy. (I think you can do this without a calculator if you use $\ln 2 \approx 0.69$ and $\ln 10 \approx 2.30$.)

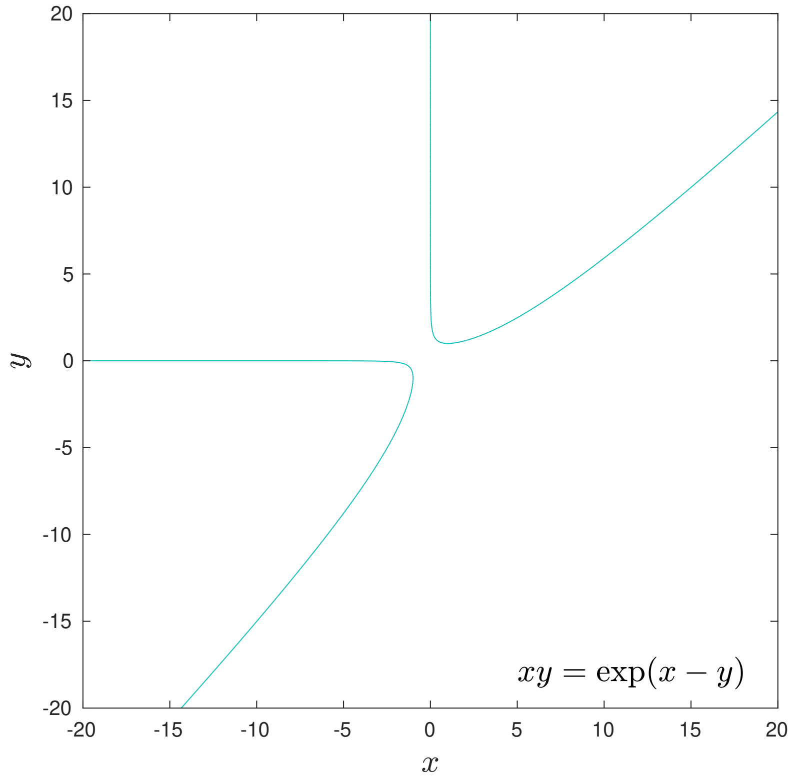

Figure 2.6: The function defined implicitly by 2.122.

Problem 2.20. The relation

$$xy = \mathrm{e}^{x-y} \tag{2.122}$$

implicitly defines y as a function of x, or vice versa. See figure 2.6. View y as a function x, and determine the $x \to \infty$ behavior of this function. Calculate enough terms to guess the form of the expansion. Then consider $x \to 0$ and do the same.

Lecture 3

Ordinary differential equations

3.1 Initial value problems: the projectile problem

If one projects a particle vertically upwards from the surface of the Earth at $z = 0$ with speed u then the projectile reaches a maximum height $h = u^2/2g_0$ and returns to the ground at $t = 2u/g_0$ (ignoring air resistance). The particle spends as much time going up as coming down. At least that's what happens if the gravitational acceleration g_0 is constant and if there is no air resistance. But a better model is that the gravitational acceleration is

$$g(z) = \frac{g_0}{(1 + z/R)^2}\,, \tag{3.1}$$

where $g_0 = 9.81\text{m s}^{-2}$, $R = 6\,371$kilometers and z is the altitude. The particle stays aloft longer than $2u/g_0$ because gravity is weaker up there.

Let's use perturbation theory to calculate the correction to the time aloft due to the small decrease in the force of gravity. But first, before the perturbation expansion, we begin with a complete formulation of the problem. We must solve the second-order autonomous differential equation

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = -\frac{g_0}{(1 + z/R)^2}\,, \tag{3.2}$$

with the initial condition

$$t = 0: \qquad z = 0 \qquad \text{and} \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = u\,. \tag{3.3}$$

We require the time τ at which $z(\tau) = 0$. If $R = \infty$ we recover the elementary problem with uniform gravity.

An important part of this problem is *non-dimensionalizing* and identifying the small parameter used to organize a perturbation expansion. We use the elementary problem ($R = \infty$) to motivate the following definition of non-dimensional variables

$$\bar{z} \stackrel{\text{def}}{=} \frac{g_0 z}{u^2}\,, \qquad \text{and} \qquad \bar{t} \stackrel{\text{def}}{=} \frac{g_0 t}{u}\,. \tag{3.4}$$

To recast the problem using non-dimensional variables

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{g_0}{u}\frac{\mathrm{d}}{\mathrm{d}\bar{t}}\,, \qquad \text{and therefore} \qquad \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \left(\frac{g_0}{u}\right)^2 \frac{\mathrm{d}^2}{\mathrm{d}\bar{t}^2}\frac{u^2}{g_0}\bar{z} = g_0\,\frac{\mathrm{d}^2 \bar{z}}{\mathrm{d}\bar{t}^2}\,. \tag{3.5}$$

Putting these expressions into (3.2) we obtain the non-dimensional problem

$$\frac{\mathrm{d}^2\bar{z}}{\mathrm{d}\bar{t}^2} + \frac{1}{(1+\epsilon\bar{z})^2} = 0\,, \tag{3.6}$$

where

$$\epsilon \stackrel{\text{def}}{=} \frac{u^2}{Rg_0}\,. \tag{3.7}$$

We must also non-dimensionalize the initial conditions in (3.3):

$$\bar{t} = 0: \qquad \bar{z} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{t}} = 1\,. \tag{3.8}$$

At this point we have done nothing more than change notation. The original problem was specified by three parameters, g_0, u and u. The non-dimensional problem is specified by a single parameter ϵ, which might be large, small, or in between. If we're interested in balls and bullets fired from the surface of the Earth then $\epsilon \ll 1$.

OK, so assuming that $\epsilon \ll 1$ we try a regular perturbation expansion on (3.6). We also drop all the bars that decorate the non-dimensional variables: we can restore the dimensions at the end of the calculation and it is just too onerous to keep writing all those little bars. The regular perturbation expansion is

$$z(t) = z_0(t) + \epsilon z_1(t) + \epsilon^2 z_2(t) + O\left(\epsilon^3\right)\,. \tag{3.9}$$

We use the binomial theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + O\left(x^3\right)\,, \tag{3.10}$$

with $n = -2$ to expand the nonlinear term:

$$(1+\epsilon z)^{-2} = 1 - 2\epsilon z + 3\epsilon^2 z^2 + O\left(\epsilon^3\right)\,. \tag{3.11}$$

Introducing (3.9) into the expansion above gives

$$(1+\epsilon z)^{-2} = 1 - 2\epsilon z_0 + \epsilon^2(3z_0^2 - 2z_1) + O\left(\epsilon^3\right)\,. \tag{3.12}$$

So matching up equal powers of ϵ in (3.6) (and denoting time derivatives by dots) we obtain the first three terms in perturbation hierarchy:

$$\begin{aligned}
\ddot{z}_0 &= -1\,, && \text{with } z_0(0) = 0\,, \quad \dot{z}_0(0) = 1\,,\\
\ddot{z}_1 &= 2z_0\,, && \text{with } z_1(0) = 0\,, \quad \dot{z}_1(0) = 0\,,\\
\ddot{z}_2 &= 2z_1 - 3z_0^2\,, && \text{with } z_2(0) = 0\,, \quad \dot{z}_2(0) = 0\,.
\end{aligned}$$

Above we have the first three terms in a hierarchy of *linear* equations of the form

$$Lz_{n+1} = R(z_0, \cdots z_n)\,, \tag{3.13}$$

where the linear operator is

$$L \stackrel{\text{def}}{=} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\,. \tag{3.14}$$

To solve each term in the hirerarchy we must invert this linear operator, being careful to use the correct initial equations that with $n \geq 1$, $z_{n+1}(0) = \dot{z}_{n+1}(0) = 0$.

The solution of the first two equations is

$$z_0(t) = t - \frac{t^2}{2}, \qquad \text{and} \qquad z_1(t) = \frac{t^3}{3} - \frac{t^4}{12}. \tag{3.15}$$

To obtain $z_2(t)$ we integrate

$$\ddot{z}_2 = -3t^2 + \frac{11t^3}{3} - \frac{11t^4}{12}, \tag{3.16}$$

to obtain

$$z_2(t) = -\frac{t^4}{4} + \frac{11t^5}{60} - \frac{11t^6}{360}. \tag{3.17}$$

Thus the expanded solution is

$$z(t) = t - \frac{t^2}{2} + \epsilon\left(\frac{t^3}{3} - \frac{t^4}{12}\right) + \epsilon^2\left(-\frac{t^4}{4} + \frac{11t^5}{60} - \frac{11t^6}{360}\right) + O\left(\epsilon^3\right). \tag{3.18}$$

We assume that the time aloft, $\tau(\epsilon)$, also has a perturbation expansion

$$\tau(\epsilon) = \tau_0 + \epsilon\tau_1 + \epsilon^2\tau_2 + O\left(\epsilon^3\right). \tag{3.19}$$

The terms in this expansion are determined by solving:

$$z_0\left(\tau_0 + \epsilon\tau_1 + \epsilon^2\tau_2\right) + \epsilon z_1\left(\tau_0 + \epsilon\tau_1\right) + \epsilon^2 z_2\left(\tau_0\right) = O\left(\epsilon^3\right). \tag{3.20}$$

We have ruthlessly ditched all terms of order ϵ^3 into the garbage heap on the right of (3.20). The left side is a polynomial of order τ^6 so there are six roots. One of these roots is $\tau = 0$ and another root is close to $\tau = 2$. The other four roots are artificial creatures of the perturbation expansion and should be ignored – if we want the time aloft then we focus on the root near $\tau = 2$ by taking $\tau_0 = 2$ in (3.19). Expanding the z_n's in a Taylor series about $\tau_0 = 2$, we have:

$$z_0(2) + (\epsilon\tau_1 + \epsilon^2\tau_2)\dot{z}_0(2) + \tfrac{1}{2}(\epsilon\tau_1)^2\ddot{z}_0(2) + \epsilon z_1(2) + \epsilon^2\tau_1\dot{z}_1(2) + \epsilon^2 z_2(2) = O\left(\epsilon^3\right). \tag{3.21}$$

Now we can match up powers of ϵ:

$$\begin{aligned} z_0(2) &= 0\,, \\ \tau_1\dot{z}_0(2) + z_1(2) &= 0\,, \\ \tau_2\dot{z}_0(2) + \tfrac{1}{2}\tau_1^2\ddot{z}_0(2) + \tau_1\dot{z}_1(2) + z_2(2) &= 0\,. \end{aligned}$$

Solving[1] these equations, one finds

$$\tau = 2 + \frac{4}{3}\epsilon + \frac{4}{5}\epsilon^2 + O(\epsilon^3)\,.$$

The Taylor series above is another procedure for generating the expansion of a regularly perturbed root of a polynomial.

[1]Some intermediate results $\dot{z}_0(2) = -1$, $z_1(2) = 4/3$, $\dot{z}_1(2) = 4/3$ and $z_2(2) = -4/45$.

Attempted solution of the projectile problem by iteration

We're considering the differential equation

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \frac{1}{(1+\epsilon z)^2} = 0\,, \tag{3.22}$$

again. Our first iterate is

$$z^{(0)} = t - \frac{t^2}{2}\,, \tag{3.23}$$

which is the same as the first term in the earlier RPS. To obtain the next iterate, $z^{(1)}(t)$, we try to solve

$$\frac{\mathrm{d}^2 z^{(1)}}{\mathrm{d}t^2} + \frac{1}{\left(1+\epsilon\left(t-\frac{t^2}{2}\right)\right)^2} = 0\,, \tag{3.24}$$

with the initial condition

$$z^{(1)} = 0\,, \qquad \dot{z}^{(1)}(0) = 1\,. \tag{3.25}$$

We could assault this problem with MATHEMATICA:

```
DSolve[{z''[t] + 1/(1 + (t - t^2/2))^2 == 0 , z[0] == 0 , z'[0] == 1}, z[t], t]
```

However the answer is not presentable in polite company. In this example, the RPS back in (3.18) is definitely superior to iteration.

3.2 Boundary value problems: belligerent drunks

Imagine a continuum of drunks random-walking along a stretch of sidewalk, the x-axis, that lies between bars at $x = 0$ and $x = \ell$. When a drunk collides with another drunk they have a certain probability of mutual destruction: a fight breaks out that may result in the death of one or both participants. We desire the density of drunks on the stretch of sidewalk between $x = 0$ and $x = \ell$. The mathematical description of this problem is based on the density (drunks per meter) $u(x,t)$, which is governed by the partial differential equation

$$u_t = \kappa u_{xx} - \mu u^2\,, \tag{3.26}$$

with boundary conditions at the bars

$$u(0,t) = u(\ell,t) = U\,. \tag{3.27}$$

We're modeling the bars using a Dirichlet boundary condition — there is constant density at each bar and drunks spill out onto the sidewalk. The parameter μ models the lethality of the interaction between pairs of drunks.

Exercise: How would the formulation change if the drunks are not belligerent? They peacefully ignore each other. But instead, drunks have a constant probability per unit time of dropping dead. How does the formulation of the continuum model change? (In this case you might prefer to think of $u(x,t)$ as the concentration of a radioactive element, rather than drunken walkers.)

If we integrate (3.26) from $x = 0$ to ℓ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\ell u\,\mathrm{d}x = \left[\kappa u_x\right]_0^\ell - \int_0^\ell \mu u^2\,\mathrm{d}x\,. \tag{3.28}$$

You should be able to interpret each term in this budget.

First order of business is to non-dimensionalize the problem. How many control parameters are there? With the definitions

$$\bar{t} \stackrel{\text{def}}{=} \frac{4\kappa t}{\ell^2}\,, \qquad \bar{x} \stackrel{\text{def}}{=} \frac{2x}{\ell} - 1\,, \qquad \text{and} \qquad \bar{u} \stackrel{\text{def}}{=} \frac{u}{U}\,, \tag{3.29}$$

we quickly find that the scaled boundary value problem is

$$u_t = u_{xx} - \alpha u^2\,, \qquad \text{with BCs} \qquad u(\pm 1) = 1\,. \tag{3.30}$$

There is a single control parameter

$$\alpha \stackrel{\text{def}}{=} \frac{\ell^2 \mu U}{4\kappa}\,. \tag{3.31}$$

We made an aesthetic decision to put the boundaries at $x = \pm 1$. This means we are using $\ell/2$ as the unit of length and $\ell^2/4\kappa$ as the unit of time.

Looking for a steady solution ($u_t = 0$) to the partial differential equation, we consider the nonlinear boundary value problem

$$u_{xx} = \alpha u^2\,, \qquad \text{with BCs} \qquad u(\pm 1) = 1. \tag{3.32}$$

The weakly interacting limit $\alpha \ll 1$

If $\alpha \ll 1$ – the weakly interacting limit – we can use an RPS

$$u = u_0(x) + \alpha u_1(x) + \cdots \tag{3.33}$$

and

$$u^2 = u_0^2 + \alpha 2 u_0 u_1 + \alpha^2\left(2u_0u_2 + u_1^2\right) + \cdots \tag{3.34}$$

The leading-order problem is

$$u_{0xx} = 0\,, \quad \text{with BCs} \quad u_0(\pm 1) = 1\,, \tag{3.35}$$

and the solution is simply

$$u_0(x) = 1\,. \tag{3.36}$$

At subsequent orders, the BCs are homogeneous. For example, the first-order problem is

$$u_{1xx} = \underbrace{u_0^2}_{=1}\,, \qquad \text{with BCs} \qquad u_1(\pm 1) = 0\,. \tag{3.37}$$

The solution is

$$u_1(x) = -\tfrac{1}{2}(1 - x^2)\,. \tag{3.38}$$

At second order

$$u_{2xx} = 2u_0u_1 = -\left(1 - x^2\right)\,, \qquad \text{with} \qquad u_2(\pm 1) = 0\,. \tag{3.39}$$

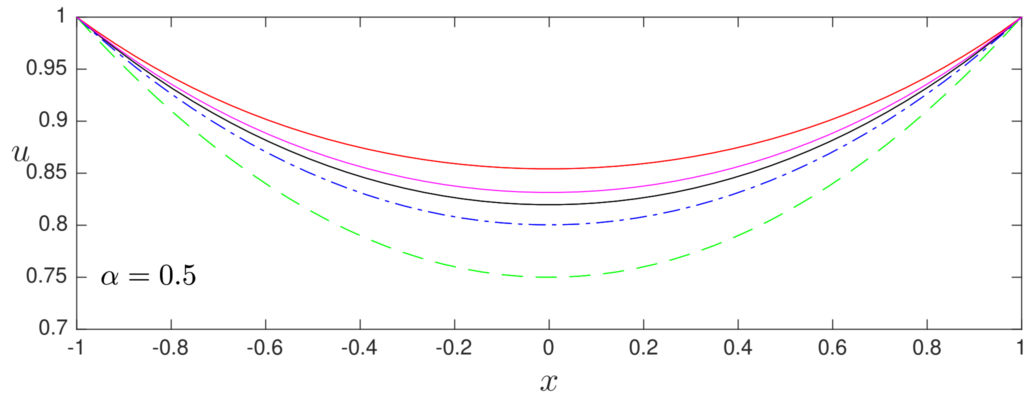

Figure 3.1: Comparison of the perturbation solution for $u(x, 0.5)$ with numerical solution obtained by the MATLAB routine `bvp4c`; the `bvp4c` solution is the black solid curve. The approximation $1 + \alpha u_1$ is green dashed, the three-term expansion is red solid, the four-term expansion is blue dash-dot and the five term expansion in (3.45) is cayan solid.

The solution is

$$u_2(x) = \frac{x^4}{12} - \frac{x^2}{2} + \frac{5}{12} = \tfrac{1}{12}\left(1 - x^2\right)\left(5 - x^2\right) . \tag{3.40}$$

For those with obsessive-compulsive tendencies it is always tempting to calculate more terms: the next term is

$$u_{3xx} = 2u_0 u_2 + u_1^2 , \qquad \text{with} \qquad u_3(\pm 1) = 0 , \tag{3.41}$$

with solution

$$u_3 = -\tfrac{1}{72}\left(1 - x^2\right)\left(31 - 8x^2 + x^4\right) . \tag{3.42}$$

And another

$$u_{4xx} = 2u_0 u_3 + 2u_1 u_2 , \qquad \text{with} \qquad u_4(\pm 1) = 0 , \tag{3.43}$$

with solution

$$u_4(x) = \tfrac{1}{504}\left(1 - x^2\right)\left(251 - 71x^2 + 13x^4 - x^6\right) . \tag{3.44}$$

I solved the boundary value problems in (3.41) and (3.43) with the MATHEMATICA routine `DSolve`. Figure 3.1 compares the perturbation solution at $\alpha = 0.5$ with a numerical solution obtained using the MATLAB routine `bvp4c`. Even with 5 terms the agreement is only so so — we need more terms. Using the five-term perturbation series above at $x = 0$, the concentration at the center of the domain is

$$u(0, \alpha) = 1 - \frac{\alpha}{2} + \frac{5\alpha^2}{12} - \frac{31\alpha^3}{72} + \frac{251\alpha^4}{504} + O\left(\alpha^5\right) . \tag{3.45}$$

We extend this series below in (3.58)

At every step of the perturbation hierarchy we are inverting the linear operator $\mathrm{d}^2/\mathrm{d}x^2$ with homogeneous Dirichlet boundary conditions. You should recognize that all the regular perturbation problems we've seen have this structure. There is a general result – the implicit function theorem – which assures us that if we know how to solve these reduced linear problems, with invertible linear operators, then the original problem has a solution for some sufficiently small value of the expansion parameter (α in the problem above).

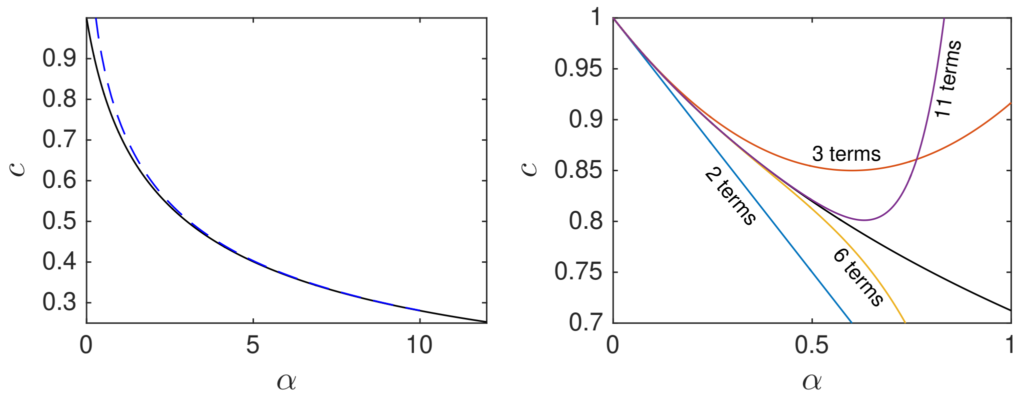

Figure 3.2: In both panels the solid black curve is $c(\alpha)$ defined implicitly by (3.50); the integral is evaluated with the MATLAB routine `integral`. In the left panel the implicit solution is compared with the $\alpha \to \infty$ approximation in (3.54). In the right panel the implicit solution is compared to the series in (3.58).

Example: Let's consider a different approach to solving the boundary value problem

$$u_{xx} = \alpha u^2 \,, \qquad \text{with BCs} \qquad u(\pm 1) = 1 \,. \tag{3.46}$$

Following our discussion of energy conservation for nonlinear oscillators, we multiplying the equation by u_x and integrate to obtain

$$\tfrac{1}{2} u_x^2 = \tfrac{\alpha}{3} u^3 + \text{constant} \,. \tag{3.47}$$

Let $c(\alpha) \stackrel{\text{def}}{=} u(0, \alpha)$ be the unknown concentration at the center of the domain, where $u_x(0, \alpha) = 0$. Evaluating the equation above at $x = 0$ we see that the constant of integration is $-\alpha c^3/3$. Next, take the square root, separate the variables and integrate from $x' = 0$ to $x > 0$ to obtain

$$\int_c^u \frac{\mathrm{d}u'}{\sqrt{u'^3 - c^3}} = \sqrt{\frac{2\alpha}{3}}\, x \,. \tag{3.48}$$

(We take the positive square root because if $x > 0$ then $u_x > 0$.) Evaluating (3.48) at $x = 1$ we obtain

$$\int_c^1 \frac{\mathrm{d}u'}{\sqrt{u'^3 - c^3}} = \sqrt{\frac{2\alpha}{3}} \,. \tag{3.49}$$

Tidy up by changing variables to $v = u'/c$:

$$\sqrt{\frac{2\alpha c}{3}} = \int_1^{c^{-1}} \frac{\mathrm{d}v}{\sqrt{v^3 - 1}} \,. \tag{3.50}$$

The expression in (3.50) is convenient for numerical work: we can graph the relation between c and α by specifying c in the range $0 < c < 1$ and evaluating α by numerical quadrature (see figure 3.2). Note it is not necessary to use `fzero`.

The form in (3.50) is useful in the limit $\alpha \to \infty$ and $c \to 0$: an asymptotic approximation to c is obtained by

$$\sqrt{\frac{2\alpha c}{3}} = \int_1^\infty \frac{\mathrm{d}v}{\sqrt{v^3 - 1}} - \int_{c^{-1}}^\infty \frac{\mathrm{d}v}{\sqrt{v^3 - 1}} \,, \tag{3.51}$$

$$\approx 2\sqrt{\pi}\frac{\Gamma(7/6)}{\Gamma(2/3)} - \int_{c^{-1}}^\infty \frac{\mathrm{d}v}{v^{3/2}} \,, \tag{3.52}$$

$$= \underbrace{2\sqrt{\pi}\frac{\Gamma(7/6)}{\Gamma(2/3)}}_{=2.42865} - 2\sqrt{c} \,. \tag{3.53}$$

Solving (3.53) for c, we obtain the approximation

$$c \sim \left(\frac{2.42865}{2 + \sqrt{2\alpha/3}} \right)^2 \,, \qquad \text{as } \alpha \to \infty. \tag{3.54}$$

The left panel of figure 3.2 compares this approximation with the result from numerical evaluation of the integral in (3.50).

Note I have written $\sim$ in (3.53) — this is unjustified because I haven't displayed the asymptotic sequence used to construct the approximation, nor indicated how one might obtain more terms. On the other hand, the large-α comparison with the "exact" solution is splendid – careful justification of the approximation (3.54) seems pointless. This is often the case when we compare asymptotic solutions with numerical solutions.

One can also use (3.50) to reproduce and extend the $\alpha \to 0$ approximation in (3.45). Changing variables to $w = v - 1$:

$$\sqrt{2\alpha c} = \int_0^f \frac{\mathrm{d}w}{w} \left(1 + w + \tfrac{1}{3}w^2\right)^{-1/2} , \tag{3.55}$$

where $f \stackrel{\text{def}}{=} c^{-1} - 1 \ll 1$. Using MATHEMATICA

$$\sqrt{2\alpha c} = \int_0^f \frac{\mathrm{d}w}{w} \left(1 - \frac{w}{2} + \frac{5w^2}{24} + \cdots\right) , \tag{3.56}$$

$$= 2f^{1/2} - \frac{f^{3/2}}{3} + \frac{f^{5/2}}{12} + \cdots \tag{3.57}$$

Squaring the expression above and using `InverseSeries` to express c in terms of α one eventually arrives at

$$c = 1 - \frac{\alpha}{2} + \frac{5\alpha^2}{12} - \frac{31\,\alpha^3}{72} + \frac{251\,\alpha^4}{504} - \frac{5599\,\alpha^5}{9072} + \frac{43615\,\alpha^6}{54432} - \frac{10657285\,\alpha^7}{9906624} + \frac{25157603\,\alpha^8}{16982784} - \frac{4452284365\,\alpha^9}{2139830784} + \frac{241448268505\,\alpha^{10}}{81313569792} + O\left(\alpha^{11}\right) . \tag{3.58}$$

The coefficients are growing slowly — this series may have a non-zero radius of convergence. The right panel of figure 3.2 compares partial sums of (3.58) to the exact solution. This example is frustrating: the function $c(\alpha)$ is only slightly bent in the range $0 \le \alpha \le 1$. Yet the series (3.58) is ineffective beyond about $\alpha = 0.5$. This may be a job for Padé summation.

3.3 Failure of RPS: singular perturbation problems

Let's close by giving a few examples of differential equation that do not obligingly yield to RPS, dominant balance and iteration.

Boundary layers

First, consider the boundary value problem (3.32) with $\alpha = \epsilon^{-1} \gg 1$. In terms of ϵ, the problem is

$$\epsilon u_{xx} = u^2 , \qquad \text{with BCs} \qquad u(\pm 1) = 1 . \tag{3.59}$$

We try the RPS

$$u = u_0(x) + \epsilon u_1(x) + \cdots \tag{3.60}$$

The leading order is

$$0 = u_0^2 , \qquad \text{with BCs} \qquad u_0(\pm 1) = 1 . \tag{3.61}$$

Immediately we see that there is no solution to the leading-order problem.

What's gone wrong? Let's consider a linear problem with the same issues:

$$\epsilon v_{xx} = v , \qquad \text{with BCs} \qquad v(\pm 1) = 1 . \tag{3.62}$$

Again the RPS fails because the leading-order problem,

$$0 = v_0 , \qquad \text{with BCs} \qquad v_0(\pm 1) = 1 , \tag{3.63}$$

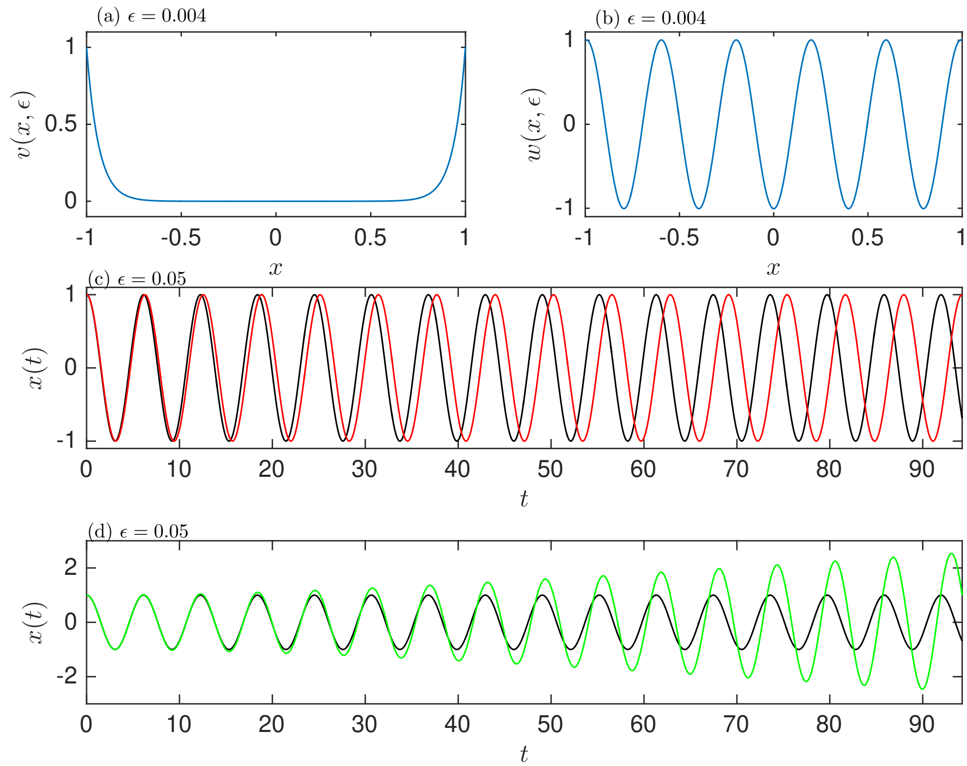

Figure 3.3: Solutions of some linear ODEs. **Exercise:** match the panels above with the ODEs in section 3.3

has no solution. The advantage of a linear example is that we can exhibit the exact solution:

$$v = \frac{\cosh(x/\sqrt{\epsilon})}{\cosh(1/\sqrt{\epsilon})}\,, \tag{3.64}$$

see figure 3.3(a). The exact solution has *boundary layers* near $x = -1$ and $x = +1$. In these regions v varies rapidly so that the term ϵv_{xx} in (3.62) is not small relative to v. Note that the leading order interior solution, $v_0 = 0$ is a good approximation to the correct solution *outside the boundary layers*. In this interior region the exact solution is exponentially small:

$$v(0,\epsilon) = \frac{1}{\cosh(1/\sqrt{\epsilon})} \sim 2\mathrm{e}^{-1/\sqrt{\epsilon}}\,, \qquad \text{as } \epsilon \to 0. \tag{3.65}$$

Our attempted RPS is using ϵ^n as gauge functions and as $\epsilon \to 0$

$$2\mathrm{e}^{-1/\sqrt{\epsilon}} = O\left(\epsilon^n\right)\,, \qquad \text{for all } n \geq 0. \tag{3.66}$$

As far as the ϵ^n gauge is concerned, $\mathrm{e}^{-1/\sqrt{\epsilon}}$ is indistinguishable from zero.

The problem in both examples above is that the small parameter ϵ multiplies the term with the most derivatives. Thus the leading-order problem in the RPS is of lower order than the exact problem. In fact, in the examples above, the leading-order problem is not even a differential equation.

Rapid oscillations

Another linear problem that defeats a regular perturbation expansion is

$$\epsilon w_{xx} = -w\,, \qquad \text{with BCs} \qquad w(\pm 1) = 1\,. \tag{3.67}$$

The exact solution, shown in figure 3.3(b), is

$$w = \frac{\cos(x/\sqrt{\epsilon})}{\cos(1\sqrt{\epsilon})} \,. \tag{3.68}$$

In this case the solution is rapidly varying throughout the domain. The term ϵw_{xx} is *never* smaller than w.

Secular errors

Let's consider a more subtle problem:

$$\ddot{x} + (1+\epsilon)x = 0 \,, \qquad \text{with ICs} \qquad x(0) = 1 \,, \qquad \text{and} \qquad \dot{x}(0) = 0 \,. \tag{3.69}$$

The exact solution of this oscillator problem is

$$x(t,\epsilon) = \cos\left(\sqrt{1+\epsilon}\, t\right) \,. \tag{3.70}$$

In this case it looks like the RPS

$$x(t,\epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots \tag{3.71}$$

might work. The leading-order problem is

$$\ddot{x}_0 + x_0 = 0 \,, \qquad \text{with ICs} \quad x_0(0) = 1 \,, \quad \text{and} \quad \dot{x}_0(0) = 0 \,. \tag{3.72}$$

The solution is

$$x_0 = \cos t \,. \tag{3.73}$$

In fact, this RPS does work for some time – see figure 3.3(c). But eventually the exact solution (3.70) and the leading-order approximation in (3.73) have different signs. That's a bad error if $x_0(t)$ is a clock.

Maybe we can improve the approximation by calculating the next term? The order ϵ^1 problem is

$$\ddot{x}_1 + x_1 = -\cos t \,, \tag{3.74}$$

with homogeneous initial conditions

$$x_1(0) = 0 \,, \qquad \text{and} \qquad \dot{x}_1(0) = 0 \,. \tag{3.75}$$

I hope you recognize a resonantly forced oscillator when you see it: the solution of (3.74) is

$$x_1 = -\tfrac{1}{2} t \sin t \,. \tag{3.76}$$

Thus the perturbation solution is now

$$x = \cos t - \epsilon \tfrac{1}{2} t \sin t + O\left(\epsilon^2\right) \,. \tag{3.77}$$

This first-order "correction" makes matters worse — see figure 3.3(d). The RPS in (3.77) is "disordered" once $\epsilon t = O(1)$: we don't expect an RPS to work if the higher order terms are larger than the earlier terms. Clearly there is a problem with direct perturbative solution of an elementary problem.

In this example the term ϵx is small relative to the other two terms in differential equation at all time. Yet the small error slowly accumulates over long times $\sim \epsilon^{-1}$. Astronomers call this a *secular* error[2]. We did not face secular errors in the projectile problem because we were solving the differential equation only for the time aloft, which was always much less than $1/\epsilon$.

[2]From Latin *saecula*, meaning a long period of time. Saecula saeculorum is translated literally as "in a century of centuries", or more poetically as "forever and ever", or "world without end".

3.4 Dominant balance and ODEs

Dominant balance is an important method for local analysis of ordinary differential equations.

Example: Dawson's function

Consider the first-order linear differential equation

$$D' + 2xD = 1\,, \tag{3.78}$$

with initial condition $D(0) = 0$. Using an integrating factor we obtain the solution

$$D(x) = \mathrm{e}^{-x^2} \int_0^x \mathrm{e}^{t^2}\,\mathrm{d}t\,. \tag{3.79}$$

The function $D(x)$ occasionally occurs in the solution of diffusion problems. It is important enough to have a name: Dawson's function. Dawson's function is hardwired in MATLAB and other computational environments. See the graph in figure 3.4.

Small x**:** How does D behave as $x \to 0$? We seek a two-term dominant balance in (3.78). There is only one choice consistent with the initial condition $D(0) = 0$:

$$D' \overset{?}{\approx} 1\,, \qquad \Rightarrow \qquad D \approx x\,, \qquad \text{as } x \to 0. \tag{3.80}$$

This is consistent because the neglected term is $2xD = 2x^2 \ll x$ as $x \to 0$.

If we need more terms in the series then we can use iteration. Rearrange (3.78) to

$$D' = 1 - \underbrace{2xD}_{\text{small}}\,, \tag{3.81}$$

and use the scheme

$$D^{(n+1)\prime} = 1 - 2xD^{(n)}\,. \tag{3.82}$$

With $D^{(0)} = x$ we find

$$D^{(1)} = x - \tfrac{2}{3}x^3\,, \qquad \text{and} \qquad D^{(2)} = x - \tfrac{2}{3}x^3 + \tfrac{4}{15}x^5\,. \tag{3.83}$$

Each time through the loop we get an extra term in Taylor series expansion of $D(x)$.

Once we know that the solution of this problem is a Taylor series it may be more efficient to substitute a series into the equation and determine the coefficients by matching powers of x. But dominant balance followed by a one or two passes through the iteration loop is the best way to get started.

Large x**:** It seems more difficult to understand the $x \gg 1$ behaviour of D. The integral representation in (3.79) is particularly opaque $\cdots$

Dominant balance to the rescue. Balancing the second and third terms in (3.78) we have

$$2xD \overset{?}{\approx} 1\,, \qquad \Rightarrow \qquad D \sim \frac{1}{2x}\,, \qquad \text{as } x \to \infty. \tag{3.84}$$

As $x \to \infty$ the neglected term D' is much less than the retained terms: this is a consistent $x \to \infty$ dominant balance and it compares well with the MATLAB result in figure 3.4.

We can now use iteration to improve on (3.84). Rewrite (3.78) as

$$D = \frac{1}{2x} - \frac{D'}{2x}\,. \tag{3.85}$$

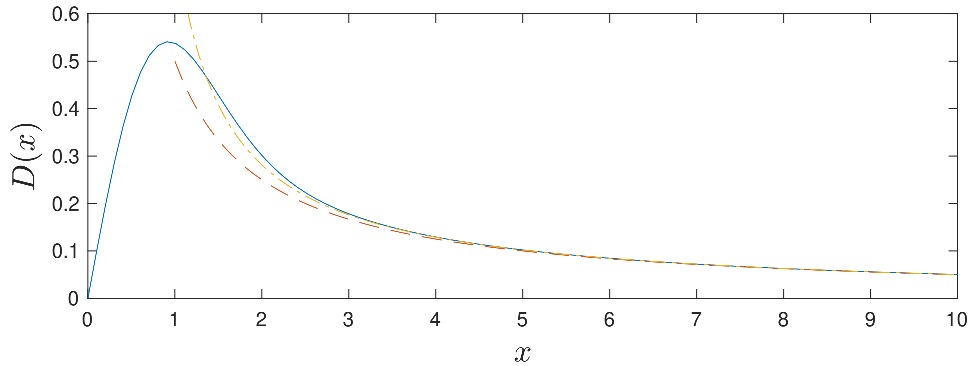

Figure 3.4: Dawson's function and some $x \gg 1$ approximations.

The iterative scheme is

$$D^{(n+1)} = \frac{1}{2x} - \frac{D^{(n)\prime}}{2x} \tag{3.86}$$

where $D^{(0)} = 1/2x$. Going twice through the loop I found

$$D^{(1)}(x) = \frac{1}{2x} + \frac{1}{4x^3}\,, \qquad \text{and} \qquad D^{(2)}(x) = \frac{1}{2x} + \frac{1}{4x^3} + \frac{3}{8x^4}\,. \tag{3.87}$$

Figure 3.4 compares $D^{(0)}(x)$ and $D^{(1)}(x)$ with MATLAB's Dawson function. At large x this is splendid.

Exercise: Consider $y' + 2xy = 1$ with $y(0) = y_0$. Using the integrating factor solution show that all solutions approach $1/2x$ as $x \to \infty$.

Example: A singularly forced oscillator

Consider

$$\ddot{y} + y = t^{-1/2}\,, \tag{3.88}$$

with initial conditions $x(0) = \dot{x}(0) = 0$.

Small t: There is a good dominant balance $\ddot{y} \approx t^{-1/2}$ leading to

$$y \approx \frac{4}{3}t^{3/2}\,, \qquad \text{as } t \to 0. \tag{3.89}$$

We can obtain more terms by writing

$$\ddot{y} = t^{-1/2} - \underbrace{y}_{\text{small}} \tag{3.90}$$

and iterating. The first iteration shows that $y = \frac{4}{3}t^{3/2} - \frac{16}{105}t^{7/2} + O(t^{11/2})$ as $t \to 0$. This small-time approximation compares well with numerical solution: see figure 3.5.

Large t: There is a consistent dominant balance $y \approx t^{-1/2}$. The neglected term is $\ddot{y} = \frac{3}{4}t^{-5/2} \ll t^{-1/2}$ as $t \to \infty$. Again use iteration

$$y = \frac{1}{t} - \ddot{x}\,, \qquad \Rightarrow \qquad y \approx \frac{1}{t} - \frac{3}{4t^{5/2}}\,. \tag{3.91}$$

There is a second consistent dominant balance

$$\ddot{y} + y = 0\,, \qquad \Rightarrow \qquad y \approx a\cos t + b\sin t\,. \tag{3.92}$$

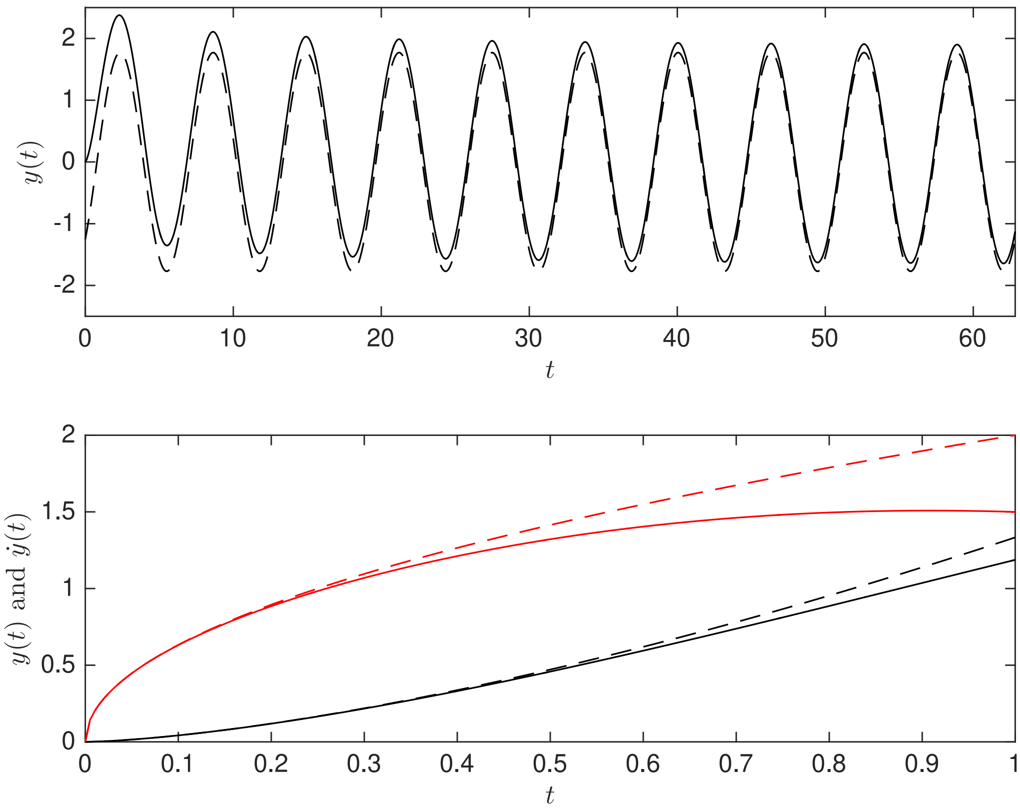

Figure 3.5: The solid curve in the upper panel shows the solution in (3.96); the dashed curve is the large-time approximation in (3.98). The solid curves in the lower panel are y and $\dot{y}$ from (3.96) and the dashed curves show the small-time approximations in $y \approx 4t^{3/2}/3$.

As $t \to \infty$ the neglected term $t^{-1/2}$ is much less than the $O(1)$ oscillation. Because this equation is linear the large t solution is

$$y \approx a\cos t + b\sin t + \frac{1}{t^{1/2}} - \frac{3}{4t^{5/2}} + O(t^{-9/2}). \tag{3.93}$$

We cannot determine a and b, except by solving the equation with reduction of order or the Green's function method.

Exercise: Show that the dominant balance, $\ddot{x} \approx t^{-1/2}$, is inconsistent as $t \to \infty$.

Discussion: Pitfalls in numerical solution of this IVP.

Details: The Green's function solution is

$$\begin{aligned} y &= \int_0^t \frac{\sin(t-t')}{\sqrt{t'}}\,\mathrm{d}t', \\ &= \sin t \int_0^t \frac{\cos t'}{\sqrt{t'}}\,\mathrm{d}t' - \cos t \int_0^t \frac{\sin t'}{\sqrt{t'}}\,\mathrm{d}t'. \end{aligned} \tag{3.94}$$

With the change of variables

$$v \stackrel{\text{def}}{=} \sqrt{\frac{2t'}{\pi}}, \qquad \text{and} \qquad \frac{\mathrm{d}t'}{\sqrt{t'}} = \sqrt{2\pi}\mathrm{d}v, \tag{3.95}$$

we express the integrals in (3.94) in terms of the *Fresnel integrals*:

$$y = \sqrt{2\pi}\,\sin t\,\underbrace{\int_0^{\sqrt{2t/\pi}} \cos\left(\frac{\pi v^2}{2}\right)\mathrm{d}v}_{C(\sqrt{2t/\pi})} - \sqrt{2\pi}\,\cos t\,\underbrace{\int_0^{\sqrt{2t/\pi}} \sin\left(\frac{\pi v^2}{2}\right)\mathrm{d}v}_{S(\sqrt{2t/\pi})} \tag{3.96}$$

The Fresnel integrals $C(z)$ and $S(z)$ are `fresnels` and `fresnelc` in MATLAB and their main properties are summarized in books on special functions and on the **DLMF**. So the result in (3.96) is not completely useless. For instance, the **DLMF** tells us that

$$\lim_{z\to\infty} C(z) = \lim_{z\to\infty} S(z) = \frac{1}{2}\,. \tag{3.97}$$

Therefore at large times the solution in (3.96) becomes

$$y \approx \sqrt{\frac{\pi}{2}}\,\sin t - \sqrt{\frac{\pi}{2}}\,\cos t\,. \tag{3.98}$$

The solution is summarized in figure 3.5. At large times the numerical solution does not agree very well with the approximation (3.98). Can you quantitatively explain the difference between th esolid and dashed curves in the upper panel of figure 3.5?

3.5 Problems

Problem 3.1. *(i)* Consider the projectile problem with linear drag:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \mu \frac{\mathrm{d}z}{\mathrm{d}t} = -g_0\,, \tag{3.99}$$

and the initial conditions $z(0) = 0$ and $\mathrm{d}z/\mathrm{d}t = u$. Find the solution with no drag, $\mu = 0$, and calculate the time aloft, τ. *(ii)* Suppose that the drag is small – make this precise by non-dimensionalizing the equation of motion and exhibiting the relevant small parameter ϵ. Hint: non-dimensionalize so that $(g_0, u) \mapsto (1, 1)$. *(iii)* Use a RPS to determine the first correction to τ associated with non-zero drag. *(iv)* Find the time to reach maximum altitude. Does the projectile take longer going up or coming down? *(v)* Integrate the non-dimensional differential equation exactly and obtain a transcendental equation for $\tau(\epsilon)$. Solve this transcendental equation approximately in the limit $\epsilon \to 0$. Make sure the $\epsilon \to 0$ solution agrees with the earlier RPS.

Problem 3.2. Consider the projectile problem with quadratic drag:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \nu \left|\frac{\mathrm{d}z}{\mathrm{d}t}\right| \frac{\mathrm{d}z}{\mathrm{d}t} = -g_0\,, \tag{3.100}$$

and the initial conditions $z(0) = 0$ and $\mathrm{d}z/\mathrm{d}t = u$. *(i)* Explain why the absolute value $|\dot{z}|$ in (3.100) is necessary if this term is to model air resistance. *(ii)* What are the dimensions of the coefficient ν? Nondimensionalize the problem so there is only one control parameter. *(iii)* Suppose that ν is small. Use a regular perturbation expansion to determine the first correction to the time aloft. *(iv)* Solve the nonlinear problem exactly and obtain a transcendental equation for the time aloft. (This is complicated.)

Problem 3.3. In this problem we use energy conservation to obtain a solution to the projectile problem which is superior to (3.18). *(i)* From the non-dimensional equation of motion (3.6), show that

$$\tfrac{1}{2}\dot{z}^2 - \frac{1}{\epsilon}\frac{1}{1+\epsilon z} = \tfrac{1}{2} - \tfrac{1}{\epsilon}\,. \tag{3.101}$$

(ii) Find the maximum height reached by the projectile, $z_{\max}$, in terms of ϵ. *(iii)* Show that the time aloft is given exactly by

$$\tau = 2z_{\max}\int_0^1 \sqrt{\frac{1+a\xi}{1-\xi}}\,\mathrm{d}\xi\,, \qquad \text{with} \qquad a(\epsilon) \overset{\text{def}}{=} \frac{\epsilon}{2-\epsilon}\,. \tag{3.102}$$

(iv) Evaluate the integral above exactly. (e) Use MATHEMATICA or some other tool to obtain the $\epsilon \ll 1$ expansions

$$\tau = \frac{4}{2-\epsilon}\left[1 + \frac{a}{3} - \frac{a^2}{15} + \frac{a^3}{35} - \frac{a^4}{63} + \frac{a^5}{99} - \frac{a^6}{143} + O\left(a^7\right)\right], \tag{3.103}$$

and

$$\tau = 2 + \frac{4\epsilon}{3} + \frac{4\epsilon^2}{5} + \frac{16\epsilon^3}{35} + \frac{16\epsilon^4}{63} + \frac{32\epsilon^5}{231} + \frac{32\epsilon^6}{429} + O\left(\epsilon^7\right). \tag{3.104}$$

Which series is superior?

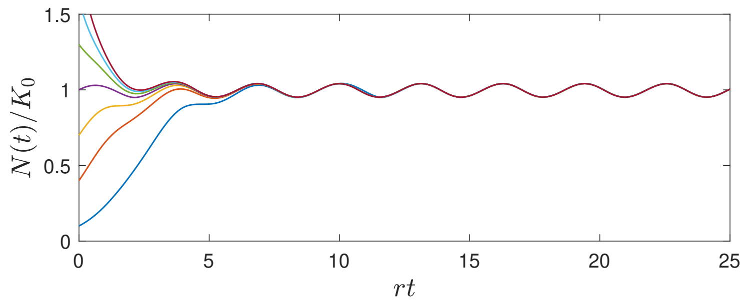

Figure 3.6: Numerical solution of (3.107) with various initial conditions. In this illustration $K_1/K_0 = 0.1$ and $\omega/r = 2$. At large time all initial conditions convergence to a periodic solution that lags the carrying capacity.

Problem 3.4. *(i)* Consider a ball that is dropped from a height h, with gravity g_0. Show that if the ball bounces elastically on an ideal hard surface then the period of the bounce is $\sqrt{8h/g_0}$. To model an upwards repulsive force that acts smoothly as the ball approaches $z = 0$ we use

$$\ddot{z} = -g_0 + \frac{\beta}{z^5}\,, \tag{3.105}$$

with $\beta > 0$. The initial conditions are

$$z(0) = h\,, \qquad \text{and} \qquad \dot{z}(0) = 0\,. \tag{3.106}$$

If h is very large then the repulsive force is initially negligible and the ball falls freely for some time. *(ii)* Nondimensionalize the problem and identify the non-dimensional parameter that quantifies "h is very large". *(iii)* Is the time to return to $z = h$ greater or less than $\sqrt{8h/g}$? *(iv)* Find the first correction to the bounce period resulting from this model of an elastic bounce. *(v)* Repeat *(iv)*, replacing z^5 by z^n in (3.105).

Problem 3.5. Consider the logistic equation with a periodically varying carrying capacity:

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)\,, \qquad \text{with} \qquad K = K_0 + K_1 \cos\omega t\,. \tag{3.107}$$

The initial condition is $N(0) = N_0$. *(i)* Based on the $K_1 = 0$ solution, non-dimensionalize this problem. Show that there are three control parameters. *(ii)* Suppose that K_1 is a perturbation i.e., $K_1/K_0 \ll 1$. The numerical solution in Figure 3.6 shows that eventually the initial condition is "forgotten" and all solutions converge to a periodic oscillation about the mean carrying capacity K_0. Use perturbation theory to determine the amplitude and phase of the long-term oscillation.

Problem 3.6. Consider a partial differential equation analog to the boundary value problem in (3.32). The domain is the disc $r = \sqrt{x^2 + y^2} < a$ in the (x, y)-plane and the problem is

$$u_{xx} + u_{yy} = \alpha u^2\,, \qquad \text{with BC:} \qquad u(a, \theta) = U\,. \tag{3.108}$$

Following the discussion in section 3.2, compute three terms in the RPS.

Problem 3.7. Let's make a small change to the formulation of the belligerent-drunks example in (3.26) and (3.27). Suppose that we model the bars using a Neumann boundary condition. This means that the flux of drunks, rather than the concentration, is prescribed at $x = 0$ and ℓ: the boundary condition in (3.27) is changed to

$$\kappa u_x(0,t) = -F\,, \qquad \text{and} \qquad \kappa u_x(\ell,t) = F\,, \tag{3.109}$$

where F, with dimensions drunks per second, is the flux entering the domain from the bars. Try to repeat *all calculations* in section 3.2, including the analog of the $\beta \ll 1$ perturbation expansion. You'll find that it is not straightforward and that a certain amount of ingenuity is required to understand the weakly interacting limit with fixed-flux boundary conditions.

Problem 3.8. First read the section entitled **Boundary layers**. Inspired by the example in that section, find an approximate solution of the boundary value problem:

$$10^{-12} v_{xx} = \mathrm{e}^{x^4}\, v\,, \qquad \text{with BCs} \qquad v(\pm 1) = 1\,. \tag{3.110}$$

If you can do this, you'll be on on your way to understanding boundary layer theory.

Problem 3.9. Consider the non-dimensional oscillator problem

$$\ddot{x} + \beta \dot{x} + x = 0\,, \tag{3.111}$$

with the initial conditions

$$x(0) = 0\,, \qquad \text{and} \qquad \dot{x}(0) = 1\,. \tag{3.112}$$

(i) Supposing that $\beta > 2$, solve the problem exactly. *(ii)* Show that if $\beta \gg 1$ then the long-time behaviour of your exact solution is

$$x \propto \mathrm{e}^{-t/\beta}\,, \tag{3.113}$$

i.e., the displacement very slowly decays to zero. *(iii)* Motivated by this exact solution, "rescale" the problem (and the initial condition) by defining the slow time

$$\tau \overset{\text{def}}{=} \frac{t}{\beta}\,, \tag{3.114}$$

and $X(\tau) = ?x(t)$. Show that with a suitable choice of ?, the rescaled problem is

$$\epsilon X_{\tau\tau} + X_\tau + X = 0\,, \qquad \text{with the IC:} \quad X(0) = 0\,, \quad X_\tau(0) = 1\,. \tag{3.115}$$

Make sure you give the definition of $X(\tau)$ and $\epsilon \ll 1$ in terms of the parameter $\beta \gg 1$ and the original variable $x(t)$. *(iv)* Try to solve the rescaled problem (3.115) using an RPS

$$X(\tau, \epsilon) = X_0(\tau) + \epsilon X_1(\tau) + \cdots \tag{3.116}$$

Discuss the miserable failure of this approach by analyzing the dependence of the exact solution from part *(i)* on β. That is, simplify the exact solution to deduce a useful $\beta \to \infty$ approximation, and explain why the RPS (3.116) cannot provide this useful approximation.

Problem 3.10. Consider a medium $-\ell < x < \ell$ in which the temperature $\theta(x,t)$ is determined by

$$\theta_t - \kappa \theta_{xx} = \alpha \mathrm{e}^{\beta\theta}\,, \tag{3.117}$$

with boundary conditions $\theta(\pm\ell, t) = 0$. The right hand side is a heat source due to an exothermic chemical reaction. The simple form in (3.117) is obtained by linearizing the Arrhenius law. The medium is cooled by the cold walls at $x = \pm\ell$. *(i)* Put the problem into the non-dimensional form

$$\Theta_T - \Theta_{XX} = \epsilon \mathrm{e}^{\Theta} \qquad \text{with BCs} \qquad \Theta(\pm 1, \epsilon) = 0\,. \tag{3.118}$$

Your answer should include a definition of the dimensionless control parameter ϵ in terms of κ, α, β and ℓ. *(ii)* Assuming that $\epsilon \ll 1$, calculate the *steady* solution $\Theta(X, \epsilon)$ using a regular perturbation expansion. Obtain two or three non-zero terms and check your answer by showing that the "central temperature" is

$$C(\epsilon) \stackrel{\text{def}}{=} \Theta(0, \epsilon)\,, \tag{3.119}$$

$$= \frac{\epsilon}{2} + \frac{5\epsilon^2}{24} + \frac{47\epsilon^3}{360} + O\left(\epsilon^4\right)\,. \tag{3.120}$$

(iii) Develop an approximate solution with iteration. *(iv)* Integrate the steady version of (3.118) exactly and deduce that:

$$\underbrace{\mathrm{e}^{-C/2} \tanh^{-1} \sqrt{1 - \mathrm{e}^{-C}}}_{\stackrel{\text{def}}{=} F(C)} = \sqrt{\frac{\epsilon}{2}}\,. \tag{3.121}$$

(Use MATHEMATICA to do the integral.) Plot the function $F(C)$ and show that there is no steady solution if $\epsilon > 0.878$. *(v)* Based on the graph of $F(C)$, if $\epsilon < 0.878$ then there are *two* solutions. There is the "cold solution", calculated perturbatively in (3.120), and there is a second "hot solution" with a large central temperature. Find an asymptotic expression for the hot central temperature as $\epsilon \to 0$.

Problem 3.11. Consider the perturbed first-order autonomous differential equation

$$\dot{x} = f(x) + \epsilon\, p(x, t)\,, \qquad \text{with IC } x(0) = 0. \tag{3.122}$$

If we use an RPS, $x_0(t) + \epsilon x_1(t) + \cdots$, then the leading-order term is defined by

$$\dot{x}_0 = f(x_0)\,, \qquad \text{with IC } x_0(0) = 0. \tag{3.123}$$

(i) Show that

$$x(t) = x_0(t) + \epsilon f_0\left(x_0(t)\right) \int_0^t \frac{p\left(x_0(t'), t'\right)}{f\left(x_0(t')\right)}\, \mathrm{d}t' + O\left(\epsilon^2\right)\,. \tag{3.124}$$

(ii) Check the formula above by considering the special perturbations $p(x, t) = f(x)$ and $p(x, t) = s(t) f(x)$ where s is some function of t alone.

Lecture 4

What is asymptotic?

4.1 An example: the erf function

We consider the error function

$$\mathrm{erf}(z) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-t^2}\,\mathrm{d}t\,. \tag{4.1}$$

The upper panel of Figure 4.1 shows erf, and the complementary error function

$$\mathrm{erfc}(z) \stackrel{\text{def}}{=} 1 - \mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t\,, \tag{4.2}$$

on the real line.

The series on the right of

$$\mathrm{e}^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} \tag{4.3}$$

has infinite radius of convergence i.e., e^{-t^2} is an *entire* function in the complex t-plane. Thus we can simply integrate term-by-term in (4.1) to obtain a series for $\mathrm{erf}(z)$ that converges in the entire complex plane:

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-)^n z^{2n+1}}{(2n+1)n!}\,, \tag{4.4}$$

$$= \underbrace{\frac{2}{\sqrt{\pi}} \left(z - \tfrac{1}{3}z^3 + \tfrac{1}{10}z^5 - \tfrac{1}{42}z^7 + \tfrac{1}{216}z^9 - \tfrac{1}{1320}z^{11}\right)}_{=\mathrm{erf}_6(z)} + R_6\,, \tag{4.5}$$

where $\mathrm{erf}_6(z)$ is the sum of the first six terms and $R_6(z)$ is the remainder after 6 terms.

The lower panel of Figure 4.1 shows that erf_n (the sum of the first n nonzero terms) provides an excellent approximation to erf if $|x| < 1$. With matlab we find that

$$\frac{\mathrm{erf}(1) - \mathrm{erf}_{10}(1)}{\mathrm{erf}(1)} = 1.6217 \times 10^{-8}\,, \qquad \text{and} \qquad \frac{\mathrm{erf}(2) - \mathrm{erf}_{10}(2)}{\mathrm{erf}(2)} = 0.0233\,. \tag{4.6}$$

The Taylor series is useful if $|z| < 1$, but as $|z|$ increases past 1 convergence is slow. Moreover some of the intermediate terms are very large and there is a lot of destructive cancellation

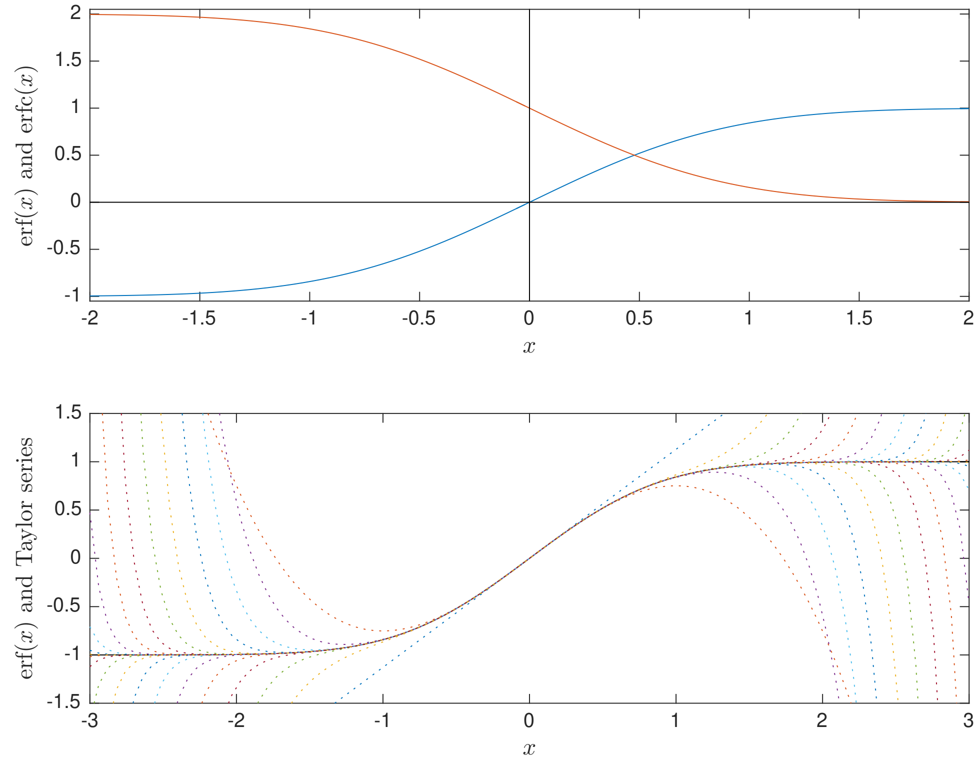

Figure 4.1: Upper panel: the blue curve is erf(x) and the red curve is erfc(x). Lower panel shows erf(x) and truncated Taylor series $\mathrm{erf}_n(x)$, with $n = 1, 2 \cdots$, 20.

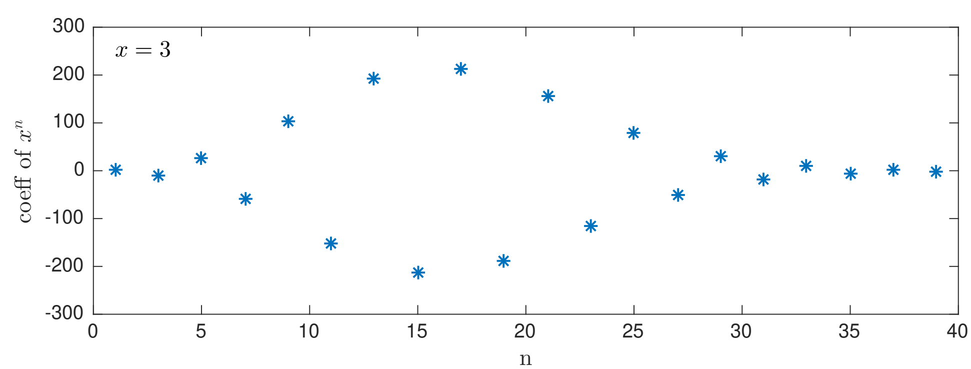

Figure 4.2: The terms in the Taylor series (4.5) with $x = 3$. The sum of the series — that is erf(3) — is very close to 1. But there are cancellations between terms of order ±200 before convergence takes hold. The problem quickly gets worse: at $x = 4$ the largest terms in the series exceed 10^5.

between terms of different signs. Figure 4.2 shows that this cancellation is bad at $z = 3$, and it gets a lot worse as $|z|$ increases. Thus, because of round-off error, a computer with limited precision cannot accurately sum the convergent Taylor series if $|z|$ is too large. Convergence is not as useful as one might think.

Now let's consider an approximation to $\mathrm{erf}(x)$ that's good for large[1] x. We work with the complementary error functions in (4.2) and use integration by parts

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \left(-\frac{1}{2t}\right) \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{-t^2}\, \mathrm{d}t\,, \tag{4.7}$$

$$= \frac{2}{\sqrt{\pi}} \frac{\mathrm{e}^{-x^2}}{2x} - \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{\mathrm{e}^{-t^2}}{2t^2}\, \mathrm{d}t\,. \tag{4.8}$$

If we discard the final term in (4.8) we get a useful approximation[2]

$$\mathrm{erfc}(x) \sim \frac{\mathrm{e}^{-x^2}}{\sqrt{\pi} x}\,, \qquad \text{as } x \to \infty. \tag{4.9}$$

The upper panel of Figure 4.3 shows that this *leading-order asymptotic approximation* is reliable once x is greater than about 2 e.g., at $x = 2$ the error is 10.5%, and at $x = 4$ the error is less than 3%.

Exercise: If we try integration by parts on erf (as opposed to erfc) something bad happens: try it and see.

Why does the approximation in (4.9) work? Notice that the final term in (4.8) can be bounded like this

$$\frac{2}{\sqrt{\pi}} \int_x^\infty \frac{\mathrm{e}^{-t^2}}{2t^2}\, \mathrm{d}t = \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{1}{4t^3} \times 2t\mathrm{e}^{-t^2}\, \mathrm{d}t\,, \tag{4.10}$$

$$\leq \frac{2}{\sqrt{\pi}} \frac{1}{4x^3} \int_x^\infty 2t\mathrm{e}^{-t^2} \mathrm{d}t\,, \tag{4.11}$$

$$= \frac{2}{\sqrt{\pi}} \frac{\mathrm{e}^{-x^2}}{4x^3}\,. \tag{4.12}$$

The little trick we've used above in going from (4.10) to (4.11) is that

$$t \geq x\,, \qquad \Rightarrow \qquad \frac{1}{4t^3} \leq \frac{1}{4x^3}\,. \tag{4.13}$$

Pulling the $(4x)^{-3}$ outside, we're left with an elementary integral. Variants of this maneuverer appear frequently in the asymptotics of integrals (try the exercise below).

Using the bound in (4.23) in (4.8) we have

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \frac{\mathrm{e}^{-x^2}}{2x} + \left[\text{something which is much less than } \frac{2}{\sqrt{\pi}} \frac{\mathrm{e}^{-x^2}}{2x} \text{ as } x \to \infty.\right] \tag{4.14}$$

Thus as $x \to \infty$ there is a dominant balance in (4.14) between the left hand side and the first term on the right. The final term is smaller than the other two terms by a factor of at least x^{-2}.

[1]We restrict attention to the real line: $z = x + \mathrm{i}y$. The situation in the complex plane is tricky — we'll return to this later. We also defer the definition *asymptotic approximation*.

[2]The $\sim$ in (4.9) denotes "asymptotic equivalence" and is defined in section 4.2. In (4.9) it means that

$$\lim_{x\to\infty} \sqrt{\pi} x \mathrm{e}^{x^2} \mathrm{erfc}(x) = 1\,.$$

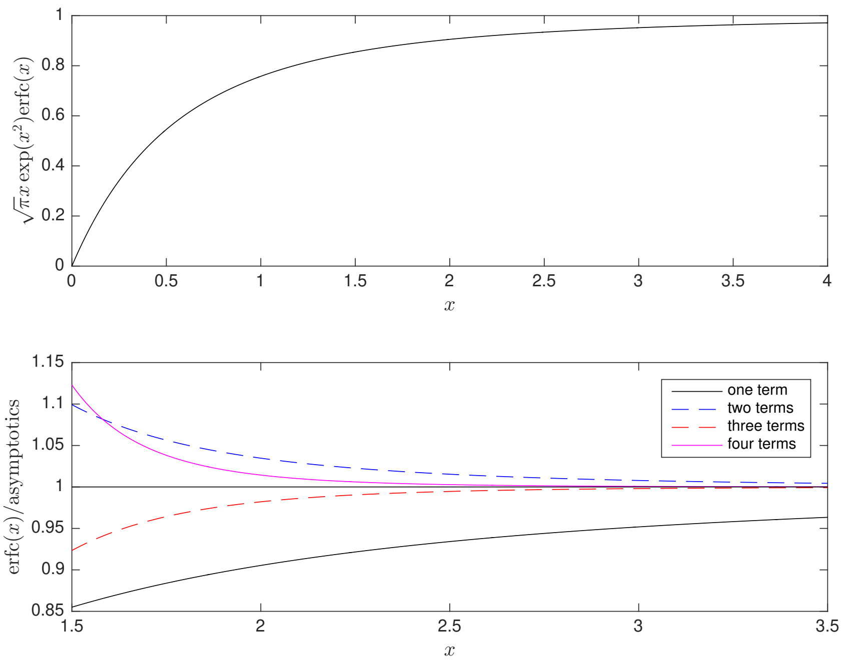

Figure 4.3: Upper panel shows $\mathrm{erfc}(x)$ divided by the leading order asymptotic approximation on the right of (4.9); as $x \to \infty$ the ratio approaches 1. The lower panel shows $\mathrm{erfc}(x)$ divided by an n-term truncation of (4.28) with $n = 1$, 2, 3 and 4.

Exercise: Prove that

$$\int_x^\infty \frac{\mathrm{e}^{-t}}{t^N}\,\mathrm{d}t < \frac{\mathrm{e}^{-x}}{x^N}\,. \tag{4.15}$$

One more term

We can develop an *asymptotic series* if we integrate by parts successively starting with (4.8):

$$\mathrm{erfc}(z) = \frac{\mathrm{e}^{-x^2}}{\sqrt{\pi}x} - \frac{1}{\sqrt{\pi}}\int_x^\infty \frac{1}{t^2}\left(-\frac{1}{2t}\right)\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-t^2}\,\mathrm{d}t\,, \tag{4.16}$$

$$= \frac{\mathrm{e}^{-x^2}}{\sqrt{\pi}x}\left(1 - \frac{1}{2x^2}\right) + \underbrace{\frac{3}{2\sqrt{\pi}}\int_x^\infty \frac{\mathrm{e}^{-t^2}}{t^4}\,\mathrm{d}t\,.}_{R_2} \tag{4.17}$$

We use the same trick to bound the remainder:

$$R_2 = -\frac{3}{4\sqrt{\pi}}\int_x^\infty \frac{-2t\mathrm{e}^{-t^2}}{t^5}\,\mathrm{d}t < \frac{3}{4\sqrt{\pi}x^5}\int_x^\infty \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-t^2}\,\mathrm{d}t = \frac{3}{4\sqrt{\pi}x^5}\mathrm{e}^{-x^2}\,. \tag{4.18}$$

As $x \to \infty$ the remainder $R_2(x)$ is much less than the second term in the series, so we can suppress some information and write

$$\mathrm{erfc}(x) = \frac{\mathrm{e}^{-x^2}}{\sqrt{\pi}x}\left[1 - \frac{1}{2x^2} + O\left(\frac{1}{x^4}\right)\right]\,. \tag{4.19}$$

The big O notation used above is explained in section 4.2 — it means that x^4 times the term $O(x^{-4})$ is bounded by some constant as $x \to \infty$. You can see that the constant identified by the inequality (4.18) is in fact $3/4$.

Yet more terms: the asymptotic series

Exercise: show that
$$\underbrace{\int_z^\infty t^{-q}\mathrm{e}^{-t^2}\,\mathrm{d}t}_{J_q} = \tfrac{1}{2}z^{-(q+1)}\mathrm{e}^{-z^2} - \tfrac{1}{2}(q+1)J_{q+2}\,. \tag{4.20}$$

Using the result in the exercise above we integrate by parts N times and obtain an exact expression for $\mathrm{erfc}(x)$:

$$\mathrm{erfc}(x) = \underbrace{\frac{\mathrm{e}^{-x^2}}{\sqrt{\pi}x}\sum_{n=0}^{N-1}(2n-1)!!\left(-\frac{1}{2x^2}\right)^n}_{N\text{ terms}} + \underbrace{(-1)^N(2N-1)!!\frac{2}{\sqrt{\pi}}\int_x^\infty \frac{\mathrm{e}^{-t^2}}{(2t^2)^N}\,\mathrm{d}t}_{R_N}\,. \tag{4.21}$$

Above, $R_N(x)$ is the remainder after N terms and the "double factorial" is $7!! = 7\cdot 5\cdot 3\cdot 1$ etc. To bound the remainder we use our trick again:

$$|R_N| = \frac{2\,(2N-1)!!}{\sqrt{\pi}}\int_x^\infty \frac{(\mathrm{e}^{-t^2})_t}{2t\times(2t^2)^N}\,\mathrm{d}t\,, \tag{4.22}$$
$$\leq \frac{(2N-1)!!}{\sqrt{\pi}2^N x^{2N+1}}\,\mathrm{e}^{-x^2}\,. \tag{4.23}$$

We have shown that
$$\frac{|R_N|}{N\text{th term of the series}} \leq \frac{2N-1}{(2x)^2}\,, \tag{4.24}$$

or equivalently
$$|R_N| \leq \text{term } N+1 \text{ in the asymptotic series}\,. \tag{4.25}$$

Thus the first term we neglect in the expansion is an upper bound on the error as $x\to\infty$. And if we fix N and increase x then the approximation to $\mathrm{erfc}(x)$ obtained by dropping the remainder gets better and better. But the limits

$$x\to\infty \qquad \text{and} \qquad N\to\infty \tag{4.26}$$

don't "commute". In other words, if we fix x at some large value, such as $x=3$, and increase N then the approximation gets better for a while, but then goes horribly wrong. This behaviour is illustrated in figure 4.4 which shows how

$$\text{relative error} \overset{\text{def}}{=} \frac{N\text{-term approximation to erfc}(x)}{\mathrm{erfc}(x)} - 1 \tag{4.27}$$

depends on both N and x in our erf example.

Numerical use of asymptotic series — the optimal stopping rule

Suppose an unreasonable person insists on ignoring the simple limit $x\to\infty$ and instead demands the best answer at a fixed value of x, such as $x=2$. How many terms in the series

$$\mathrm{erfc}(x) \sim \frac{\mathrm{e}^{-x^2}}{x\sqrt{\pi}}\left(1 - \frac{1}{2x^2} + \frac{1\times 3}{(2x^2)^2} - \frac{1\times 3\times 5}{(2x^2)^3} + \frac{1\times 3\times 5\times 7}{(2x^2)^4} + O\left(x^{-10}\right)\right) \tag{4.28}$$

should one use to appease this tyrant? The numerators above are growing very quickly so at a fixed value of x this series for $\mathrm{erfc}(x)$ diverges as we add more terms. But Figure 4.4 shows

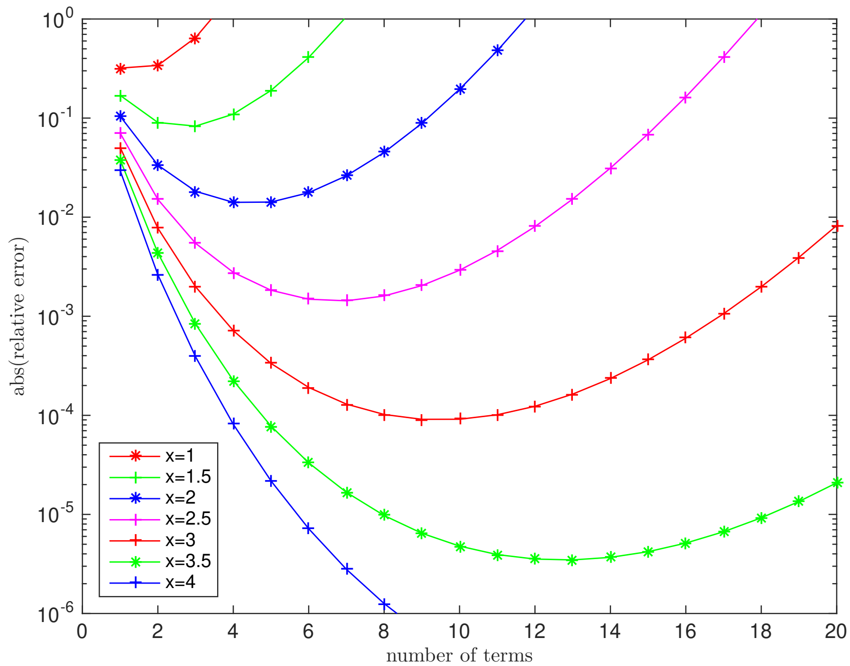

Figure 4.4: The absolute value of the relative error as a function of the number of terms used in the asymptotic series (4.28).

that at fixed x there is an optimal value of N at which the relative error is smallest. How do we find this best asymptotic estimate?

We showed above in (4.24) and (4.25) that as $x \to \infty$ the remainder $R_N(x)$ is less than the $(N+1)$st term in the series. Thus a good place to stop summing is just before the smallest term in the series: we know the remainder is less than this smallest term. In practice we get good accuracy if we use the *optimal stopping rule*: locate the smallest term in the series and add all the previous terms. Do not include the smallest term in this sum.

The optimal stopping rule is a rule of thumb not a precise result — the remainder R_N is less than the $(N+1)$st term only when x is sufficiently large i.e., in the limit $x \to \infty$. We have no assurance that this inequality applies at a particular value of x.

We illustrate the optimal stopping rule by estimating erfc(2). With $x = 2$ the sum (4.28) is

$$\underbrace{\mathrm{erfc}(2)}_{4.67773\times10^{-3}} \sim \underbrace{\frac{\mathrm{e}^{-4}}{2\sqrt{\pi}}}_{5.16675\times10^{-3}} \Big(1 - \underbrace{\frac{1}{8}}_{0.0125} + \underbrace{\frac{3}{64}}_{0.046875} - \underbrace{\frac{15}{512}}_{0.0292969} + \underbrace{\frac{105}{4096}}_{0.0256348} - \underbrace{\frac{945}{32768}}_{0.0288391} + \underbrace{\frac{10395}{262144}}_{0.0396538} + \cdots\Big)\,. \tag{4.29}$$

The smallest term is 105/4096. The optimal approximation is obtained by stopping before the smallest terms:

$$0.0051667\Big(1 - \frac{1}{8} + \frac{3}{64} - \frac{15}{512}\Big) = 0.00461172\,. \tag{4.30}$$

The relative error is 0.0141116, or about 1.4%.

We get a much better answer by including *half of the smallest term* in the asymptotic series:

$$0.0051667\Big(1 - \frac{1}{8} + \frac{3}{64} - \frac{15}{512} + \frac{1}{2}\,\frac{105}{4096}\Big) = 0.00467795\,. \tag{4.31}$$

With this mysterious improvement the relative error is now -0.000046. We should explain why adding half of the smallest term works so well. (Bender & Orszag and Hinch don't mention this.....)

Exercise: $\mathrm{erfc}(1) = 0.157299$ and the leading-order approximation is $\mathrm{e}^{-1}/\sqrt{\pi} = 0.207554$. The relative error is therefore 0.31948 which seems unfortunately large. Show that according to the optimal stopping rule the leading-order approximation is optimal. Does adding half of the smallest term significantly reduce the error?

4.2 Landau symbols

Let's explain the frequently used "Landau symbols". In asymptotic calculations the Landau notation is used to suppress information while still maintaining some precision.

Gauge functions

First we need to explain gauge functions. These are simple functions that we use to compare a complicated $f(\epsilon)$ with. The gauge functions we've used most frequently are

$$\phi_0(\epsilon) = \epsilon^0\,, \quad \phi_1(\epsilon) = \epsilon^1\,, \quad \phi_1(\epsilon) = \epsilon^2\,, \quad \text{and so on with } \phi_n(\epsilon) = \epsilon^n. \tag{4.32}$$

More generally, a sequence of gauge functions $\{\phi_0\,, \phi_1\,, \cdots\}$ is asymptotically ordered if

$$\frac{\phi_{n+1}(\epsilon)}{\phi_n(\epsilon)} \to 0\,, \qquad \text{as } \epsilon \to 0. \tag{4.33}$$

In practice the ϕ's are combinations of powers and logarithms:

$$\epsilon^n\,, \qquad \ln\epsilon\,, \qquad \epsilon^m(\ln\epsilon)^p\,, \qquad \ln\ln\epsilon \text{ etc.} \tag{4.34}$$

Exercise Suppose $\epsilon \to 0$. Arrange the following gauge functions in order, from the largest to the smallest:

$$\epsilon\,, \qquad \ln\left(\ln\frac{1}{\epsilon}\right), \qquad \mathrm{e}^{-\ln^2\epsilon}\,, \qquad \mathrm{e}^{1/\sqrt{\epsilon}}\,, \qquad \epsilon^0\,, \qquad \ln\frac{1}{\epsilon} \tag{4.35}$$

$$\mathrm{e}^{-1/\epsilon}\,, \qquad \epsilon^{1/3}\,, \qquad \epsilon^{1/\pi}\,, \qquad \epsilon\ln^2\frac{1}{\epsilon}\,, \qquad \frac{1}{\ln\frac{1}{\epsilon}}\,, \qquad \epsilon^{\ln\epsilon}\,. \tag{4.36}$$

Big Oh

We frequently use "big Oh" — in fact I've done this without defining O! One says $f(\epsilon) = O(\phi(\epsilon))$ as $\epsilon \to 0$ if we can find an ϵ_0 and a number A such that

$$|f(\epsilon)| < A|\phi(\epsilon)|\,, \qquad \text{whenever } \epsilon < \epsilon_0.$$

Both ϵ_0 and A have to be independent of ϵ. Application of the big Oh notation much easier than this definition suggests. Here are some $\epsilon \to 0$ examples

$$\sin 32\epsilon = O(\epsilon)\,, \qquad \sin 32\epsilon = O(\epsilon^{1/2})\,, \qquad \epsilon^5 = O(\epsilon^2)\,,$$
$$\cos\epsilon - 1 = O(\epsilon^{1/2})\,, \qquad \epsilon + \epsilon^2\sin\frac{1}{\epsilon} = O(\epsilon)\,,$$
$$\sin\frac{1}{\epsilon} = O(1)\,, \qquad \mathrm{e}^{-1/\epsilon} = O(\epsilon^n) \text{ for all } n.$$

The expression

$$\cos\epsilon = 1 - \frac{\epsilon^2}{2} + O(\epsilon^3) \qquad \text{means} \quad \cos\epsilon - 1 + \frac{\epsilon^2}{2} = O(\epsilon^3)\,. \tag{4.37}$$

In some of the cases above

$$\lim_{\epsilon\to 0} \frac{f(\epsilon)}{\phi(\epsilon)} \tag{4.38}$$

is zero, and that's good enough for O. Also, according to our definition of O, the limit in (4.38) may not exist — all that's required is that ratio $f(\epsilon)/\phi(\epsilon)$ is *bounded* by a constant independent of ϵ as $\epsilon \to 0$. One of the examples above illustrates this case.

The big Oh notation can be applied to other limits in obvious ways. For example, as $x \to \infty$

$$\sin x = O(1)\,, \qquad \sqrt{1+x^2} = O(x^2)\,, \qquad \ln\cosh x = O(x)\,. \tag{4.39}$$

As $x \to 1$

$$\ln\left(1 + x + x^2\right) - x = O(x^2)\,. \tag{4.40}$$

Little Oh

Very occasionally — almost never — we need "little Oh". We say $f(\epsilon) = o(\phi(\epsilon))$ if for every positive δ there is an ϵ_0 such that

$$|f(\epsilon)| < \delta|\phi(\epsilon)|\,, \qquad \text{whenever } \epsilon < \epsilon_0.$$

Another way of saying this is that

$$f(\epsilon) = o(\phi(\epsilon)) \qquad \Leftrightarrow \qquad \lim_{\epsilon\to 0} \frac{f(\epsilon)}{\phi(\epsilon)} = 0\,. \tag{4.41}$$

Obviously $f(\epsilon) = o(\phi(\epsilon))$ implies $f(\epsilon) = O(\phi(\epsilon))$, but not the reverse. Here are some examples

$$\ln(1+\epsilon) = o(\epsilon^{1/2})\,, \qquad \cos\epsilon - 1 + \frac{\epsilon^2}{2} = o(\epsilon^3)\,, \qquad \mathrm{e}^{o(\epsilon)} = 1 + o(\epsilon)\,. \tag{4.42}$$

The trouble with little Oh is that it hides too much information: if something tends to zero we usually want to know how it tends to zero. For example

$$\ln(1 + 2\mathrm{e}^{-x} + 3\mathrm{e}^{-2x}) = o\left(\mathrm{e}^{-x/2}\right)\,, \quad \text{as } x \to \infty, \tag{4.43}$$

is not as informative as

$$\ln(1 + 2\mathrm{e}^{-x} + 3\mathrm{e}^{-2x}) = O\left(\mathrm{e}^{-x}\right)\,, \quad \text{as } x \to \infty. \tag{4.44}$$

Asymptotic equivalence

Finally "asymptotic equivalence" $\sim$ is useful. We say $f(\epsilon) \sim \phi(\epsilon)$ as $\epsilon \to 0$ if

$$\lim_{\epsilon\to 0} \frac{f(\epsilon)}{\phi(\epsilon)} = 1\,. \tag{4.45}$$

Notice that

$$f(\epsilon) \sim \phi(\epsilon)\,, \qquad \Leftrightarrow \qquad f(\epsilon) = \phi(\epsilon)\left[1 + o(\epsilon)\right]\,. \tag{4.46}$$

Some $\epsilon \to 0$ examples are

$$\epsilon + \frac{\sin\epsilon}{\ln(1/\epsilon)} \sim \epsilon\,, \qquad \text{and} \qquad \sqrt{1+\epsilon} - 1 \sim \frac{\epsilon}{2}\,. \tag{4.47}$$

Some $x \to \infty$ examples are

$$\sinh x \sim \frac{\mathrm{e}^x}{2}\,, \qquad \text{and} \qquad \frac{x^3}{1+x^2} + \sin x \sim x\,, \qquad \text{and} \qquad x + \ln\left(1 + e^{2x}\right) \sim 3x\,. \tag{4.48}$$

Exercise: Show by counterexample that $f(x) \approx g(x)$ as $x \to \infty$ does not imply that $\frac{\mathrm{d}f}{\mathrm{d}x} \approx \frac{\mathrm{d}g}{\mathrm{d}x}$, and that $f(x) \approx g(x)$ as $x \to \infty$ does not imply that $\mathrm{e}^f \approx \mathrm{e}^g$.

4.3 The definition of asymptoticity

Asymptotic power series

Consider a sum based on the simplest gauge functions ϵ^n:

$$\sum_{n=0}^{\infty} a_n \epsilon^n\,. \tag{4.49}$$

This sum is an $\epsilon \to 0$ asymptotic approximation to a function $f(\epsilon)$ if

$$\lim_{\epsilon\to 0} \frac{f(\epsilon) - \sum_{n=0}^{N} a_n \epsilon^n}{\epsilon^N} = 0\,. \tag{4.50}$$

The numerator in the fraction above is the remainder after summing $N+1$ terms, also known as $R_{N+1}(\epsilon)$. So the series in (4.49) is asymptotic to the function $f(\epsilon)$ if the remainder $R_{N+1}(\epsilon)$ goes to zero faster than the last retained gauge function ϵ^N. We use the notation $\sim$ to denote an asymptotic approximation:

$$f(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n\,, \text{ as } \epsilon \to 0. \tag{4.51}$$

The right hand side of (4.51) is called an *asymptotic power series* or a Poincaré series, or an asymptotic representation of $f(\epsilon)$.

Our erf-example satisfies this definition with $\epsilon = x^{-1}$. If we retain only one term in the series (4.28) then the remainder is

$$R_1 = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \frac{\mathrm{e}^{-t^2}}{2t^2}\,\mathrm{d}t\,. \tag{4.52}$$

In (4.11) we showed that

$$\frac{R_1}{\mathrm{e}^{-x^2}/\sqrt{\pi}x} \le \frac{1}{4x^2}\,. \tag{4.53}$$

Thus as $x \to \infty$ the remainder is much less than the last retained term. According to the definition above, this is the first step in justifying the asyptoticness of the series.

Exercise: Show from the definition of asymptoticity that

$$\mathrm{e}^{-1/\epsilon} \sim 0 + 0\,\epsilon + 0\,\epsilon^2 + 0\,\epsilon^3 + \cdots \qquad \text{as } \epsilon \downarrow 0. \tag{4.54}$$

A problem with applying the definition is that one has to be able to say something about the remainder in order to determine if a series is asymptotic. This is not the case with convergence. For example, one can establish the convergence of

$$\sum_{n=0}^{\infty} \ln(n+2)\, x^n\,, \tag{4.55}$$

without knowing the function to which this mysterious series converges. Convergence is an intrinsic property of the coefficients $\ln(n+2)$. The ratio test shows that the series in (4.55) converges if $|x| < 1$ and we don't have to know what (4.55) is converging to. On the other hand, asympoticity depends on *both* the function *and* the terms in the asymptotic series.

Example The famous *Stieltjes* series

$$S(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} (-)^n n! x^n \tag{4.56}$$

does not converge unless $x = 0$. In fact, as it stands, $S(x)$ does not define a function of x. $S(x)$ is a *formal power series*. And we can't say that $S(x)$ is an asymptotic series because we have to ask asymptotic to what? But now observe that

$$n! = \int_0^\infty t^n \mathrm{e}^{-t}\, \mathrm{d}t\,, \tag{4.57}$$

and substitute this integral representation of $n!$ into the sum (4.56). There is a moment of pleasure when we realize that if we exchange the order of integration and summation then we can evaluate the sum to obtain

$$F(x) \stackrel{\text{def}}{=} \int_0^\infty \frac{\mathrm{e}^{-t}}{1+xt}\, \mathrm{d}t\,. \tag{4.58}$$

Because of the dubious steps between (4.56) and (4.58), I've simply defined $F(x)$ by the integral above. But now that we have a well defined function $F(x)$, we're entitled to ask is the sum $S(x)$ asymptotic to $F(x)$ as $x \to 0$? The answer is yes.

The proof is integration by parts, which yields the identity

$$F(x) = 1 - x + 2!x^2 - 3!x^3 + \cdots (-1)^{(N-1)}(N-1)!\, x^{N-1} + \underbrace{(-1)^N N! x^N \int_0^\infty \frac{\mathrm{e}^{-t}}{(1+xt)^{N+1}}\, \mathrm{d}t}_{=R_N}\,. \tag{4.59}$$

It is easy show that

$$|R_N(x)| \le N! x^N\,, \tag{4.60}$$

and therefore

$$\lim_{x\to 0} \frac{R_N(x)}{(N-1)!\, x^{N-1}} = 0\,. \tag{4.61}$$

Above we're comparing the remainder to the last retained term in the truncated series. Because the ratio goes to zero in the limit the series is asymptotic.

Exercise: Find another function with the same $x \to 0$ asymptotic expansion as $F(x)$ in (4.58).

Example: *Dawson's integral* is

$$D(x) \stackrel{\text{def}}{=} \mathrm{e}^{-x^2} \int_0^x \mathrm{e}^{t^2}\, \mathrm{d}t\,. \tag{4.62}$$

The integrand is strongly peaked near $t = x$, where the integrand is equal to e^{x^2}. The width of this peak is order x^{-1}. Thus we expect that the answer is something like

$$D(x) \sim \mathrm{e}^{-x^2} \frac{?}{x} \mathrm{e}^{x^2} = \frac{?}{x}\,, \tag{4.63}$$

where ? is an unidentified number.

To more precisely estimate $D(x)$ for $x \gg 1$ we try IP:

$$\int_0^x \mathrm{e}^{t^2}\, \mathrm{d}t = \int_0^x \frac{1}{2t} \frac{\mathrm{d}\mathrm{e}^{t^2}}{\mathrm{d}t}\, \mathrm{d}t\,, \tag{4.64}$$

$$= \left[\frac{\mathrm{e}^{t^2}}{2t}\right]_0^x + \int_0^x \frac{\mathrm{e}^{t^2}}{2t^2}\, \mathrm{d}t\,. \tag{4.65}$$

The expression above is meaningless — we've taken a perfectly sensible integral and written it as the difference of two infinities.

A correct approach is to split the integral like this

$$\int_0^x e^{t^2}\,dt = \int_0^1 e^{t^2}\,dt + \int_1^x \frac{1}{2t}\frac{de^{t^2}}{dt}\,dt\,, \tag{4.66}$$

$$= \int_0^1 e^{t^2}\,dt + \left[\frac{e^{t^2}}{2t}\right]_1^x + \int_1^x \frac{e^{t^2}}{2t^2}\,dt\,, \tag{4.67}$$

$$= \underbrace{\int_0^1 e^{t^2}\,dt - \tfrac{1}{2}e}_{\text{a number}} + \frac{e^{x^2}}{2x} + \underbrace{\int_1^x \frac{e^{t^2}}{2t^2}\,dt}_{R}\,, \tag{4.68}$$

$$\sim \frac{e^{x^2}}{2x}\,, \qquad \text{as } x \to \infty. \tag{4.69}$$

Thus

$$D(x) \sim \frac{1}{2x}\,, \qquad \text{as } x \to \infty. \tag{4.70}$$

Back in (4.66) we split the range at $t = 1$ — this was an arbitrary choice. We could split at another arbitrary value such as $t = 32.2345465$. The point is that as $x \to \infty$ all the terms on the right of (4.68) are much less than the single dominant term $e^{x^2}/2x$. If we want the next term in (4.70), then that comes from performing another IP on the next biggest term on the right of (4.68), namely

$$R(x) = \int_1^x \frac{e^{t^2}}{2t^2}\,dt\,. \tag{4.71}$$

To show that (4.69) is a valid asymptotic approximation according to the definition of Poincaré — with $\epsilon = x^{-1}$ and $N = 1$ in definition (4.50) — we should show that $R(x)$ in (4.71) is very much less than the leading term, or in other words that

$$\lim_{x\to\infty} \frac{\int_1^x e^{t^2}/2t^2\,dt}{e^{x^2}/2x} = 0\,. \tag{4.72}$$

Exercise: Use l'Hôpital's rule to verify the result above.

A generalized definition of aymptoticity

Many – but not all – of the expansion expansions you'll encounter have the form of an asymptotic power series as in (4.49) and (4.50). But in the previous lectures we saw examples with fractional powers of ϵ and $\ln \epsilon$ and $\ln[\ln(1/\epsilon)]$. These expansions have the form

$$f(\epsilon) \sim \sum_{n=0}^{\infty} a_n \phi_n(\epsilon)\,, \tag{4.73}$$

where $\{\phi_n\}$ is an asymptotically ordered set of gauge functions. The definition of asymptoticity is generalized to say that the sum on the right of (4.73) is an asymptotic approximation to $f(\epsilon)$ as $\epsilon \to 0$ if

$$\lim_{\epsilon\to 0} \frac{f(\epsilon) - \sum_{n=0}^{N} a_n\phi_n(\epsilon)}{\phi_N(\epsilon)} = 0\,. \tag{4.74}$$

In other words, once ϵ is sufficiently small the remainder is less than the last term.

Example: Using the $x \to \infty$ gauge functions $\{x^{n/12}\}$, where n is an integer, we have the generalized asymptotic expansion

$$\frac{x^{1/2} + x^{1/3}}{x^{1/12} + 1} = x^{5/12} - x^{1/3} + 2x^{1/4} - 2x^{1/6} + 2x^{1/12} - 2 + O\left(x^{-1/12}\right)\,. \tag{4.75}$$

Example: Another example produced by

```
Series[x^(x - x^2), {x, 0, 3}]
```

in MATHEMATICA is

$$x^{x-x^2} \sim 1 + x\ln x + x^2(\tfrac{1}{2}\ln^2 x - \ln x) + x^3(\tfrac{1}{6}\ln^3 x - \ln^2 x) + O(x^4)\,. \tag{4.76}$$

Evidently in this example the $x \to 0$ gauge functions are $x^p \ln^q x$ where p and q are non-negative integers.

Uniqueness

If a function has an asymptotic expansion in terms of a particular set of gauge function then that expansion is unique. For example, using the $\theta \to 0$ gauge functions θ^n, the function $\sin 2\theta$ can be expanded as

$$\sin 2\theta = 2\theta - \frac{4\theta^3}{3} + O\left(\theta^5\right)\,, \tag{4.77}$$

and that's the only asymptotic expansion of $\sin\theta$ using θ^n. In this sense asymptotic expansions are unique.

The converse is not true: two different functions might share an asymptotic expansion because they differ by a quantity that is asymptotically smaller than every gauge function. For example, as $\theta \downarrow 0$

$$\sin 2\theta + \mathrm{e}^{-1/\theta} \sim \sum_{n=0}^{\infty} (-1)^n \frac{(2\theta)^{2n+1}}{(2n+1)!}\,. \tag{4.78}$$

The right of (4.78) is also the asymptotic expansion of $\sin 2\theta$ in terms of the gauge functions θ^n.

A given function can also have multiple asymptotic expansions in terms of different gauge functions. For example, consider the $\theta \to 0$ gauge functions $\sin^n\theta$, for which

$$\sin 2\theta = 2\sin^2\theta - \sin^3\theta + O\left(\sin^5\theta\right)\,. \tag{4.79}$$

Or gauge functions $\tan^n\theta$, for which

$$\sin 2\theta = 2\tan\theta - 2\tan^3\theta + O\left(\tan^5\theta\right)\,. \tag{4.80}$$

Manipulation of asymptotic expansions

If we have two $\epsilon \to 0$ asymptotic power series

$$f \sim \sum_{n=0}^{\infty} a_n\epsilon^n\,, \qquad \text{and} \qquad g \sim \sum_{n=0}^{\infty} a_n\epsilon^n\,. \tag{4.81}$$

then we can do what comes naturally as far as adding, multiplying and dividing these expansions.

If f and g are represented by the generalized asymptotic series in (4.73) then we have a minor problem with multiplication: $\phi_m\phi_n$ may not be a member of our set of gauge functions. In this case we can simply enlarge the set of gauge functions — provided that the expanded set can be ordered as $\epsilon \to 0$. (I can't think of an example in which this is not possible.)

Exercise: Noting that

$$\frac{1}{\epsilon(1+\epsilon)} \sim \frac{1}{\epsilon} \quad \text{as } \epsilon \to 0, \tag{4.82}$$

is

$$\exp\left(\frac{1}{\epsilon(1+\epsilon)}\right) \sim \mathrm{e}^{1/\epsilon}\,? \tag{4.83}$$

Asymptotic series can be integrated: if

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n \,, \qquad \text{as } x \to x_0, \tag{4.84}$$

then

$$\int_{x_0}^{x} f(t)\, \mathrm{d}t \sim \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1} \,, \qquad \text{as } x \to x_0. \tag{4.85}$$

Asymptotic series cannot in general be differentiated. Thus

$$x + \sin x \sim x \,, \qquad \text{as } x \to \infty, \tag{4.86}$$

but the derivative $1 + \cos x$ is not asymptotic to 1. Note however that **BO** section 3.8 discusses some useful special cases in which differentiation is permitted.

4.4 The Taylor series, with remainder

We can use integration by parts to prove that a function $f(x)$ with n derivatives can be represented *exactly* by n terms of a Taylor series, plus a remainder. The fundamental theorem of calculus is

$$f(x) = f(a) + \underbrace{\int_a^x f'(\xi)\, \mathrm{d}\xi}_{R_1} \,. \tag{4.87}$$

If we drop the final term, $R_1(x)$, we have a one-term Taylor series for $f(x)$ centered on $x = a$. To generate one more terms we integrate by parts like this

$$f(x) = f(a) + \int_a^x f'(\xi) \frac{\mathrm{d}}{\mathrm{d}\xi}(\xi - x)\, \mathrm{d}\xi \,, \tag{4.88}$$

$$= f(a) + (x-a) f'(a) - \int_a^x f''(\xi)(\xi - x)\, \mathrm{d}\xi \,. \tag{4.89}$$

And again

$$f(x) = f(a) + (x-a) f'(a) - \int_a^x f''(\xi) \frac{\mathrm{d}}{\mathrm{d}\xi} \frac{1}{2} (\xi - x)^2\, \mathrm{d}\xi \,, \tag{4.90}$$

$$= f(a) + (x-a) f'(a) + \frac{1}{2} f''(a)(x-a)^2 + \underbrace{\frac{1}{2} \int_a^x f'''(\xi)(\xi - x)^2\, \mathrm{d}\xi}_{R_3} \,. \tag{4.91}$$

If $f(x)$ has n-derivatives we can keep going till we get

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}}_{=n \text{ terms, let's call this } f_n(x)} + R_n(x) \,, \tag{4.92}$$

where the remainder after n-terms is

$$R_n(x) = \frac{1}{(n-1)!} \int_a^x f^{(n)}(\xi)(x - \xi)^{n-1}\, \mathrm{d}\xi \,. \tag{4.93}$$

Using the first mean value theorem, the remainder can be represented as

$$R_n(x) = \frac{f^{(n)}(\bar{x})}{n!}(x-a)^n\,, \tag{4.94}$$

where $\bar{x}$ is some unknown point in the interval $[a, x]$. This is the form given in section 4.6 of **RHB**.

Some remarks about the result in (4.92) through (4.94) are:

(1) $f(x)$ need not have derivatives of all order at the point x: the representation in (4.92) and (4.94) makes reference only to derivatives of order n, and that is all that is required.

(2) Using (4.94), we see that the ratio of $R_n(x)$ to the last retained term in the series is proportional to $x-a$ and therefore vanishes as $x \to a$. Thus, according to our definition in (4.50), $f_n(x)$ is an asymptotic expansion of $f(x)$.

(3) The convergence of the truncated series $f_n(x)$ as $n \to \infty$ is not assumed: (4.92) is exact. The remainder $R_n(x)$ may decrease up to a certain point and then start increasing again.

(4) Even if $f_n(x)$ diverges with increasing n, we may obtain a close approximation to $f(x)$ — with a small remainder R_n — if we stop summing at a judicious value of n.

(5) The difference between the convergent case and the divergent case is that in the former instance the remainder can be made arbitrarily small by increasing n, while in the latter case the remainder cannot be reduced below a certain minimum.

Above we are recapitulating many remarks we made previously regarding the asymptotic expansion of erf.

Example: Taylor series, even when they diverge, are still asymptotic series. Let's investigate this with the elementary problem:

$$x(\epsilon)^2 = 9 + \epsilon\,. \tag{4.95}$$

Before taking this class you could have solved this problem by arguing that

$$x(\epsilon) = 3\left(1 + \tfrac{\epsilon}{9}\right)^{1/2}\,, \tag{4.96}$$

and then recollecting the standard Taylor series

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!}z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3 + \cdots \tag{4.97}$$

The perturbation method is laboriously reproducing the special case $\alpha = 1/2$ and $z = \epsilon/9$.

You should recall from your undergraduate education that the radius of convergence of (4.97) is limited by the nearest singularity to the origin in the complex z-plane. With $\alpha = 1/2$ the nearest singularity is the branch point at $z = -1$. So the series in problem **1.1** *converges* provided that $\epsilon < 9$. Let us ignore this red flag and use the Taylor series with $\epsilon = 16$ to estimate $x(16) = \sqrt{25} = 5$. We calculate a lot of terms with the mathematica command:

```
Series[Sqrt[9 + u], {u, 0, 8}].
```

This produces the series

$$x(\epsilon) = 3 + \frac{\epsilon}{6} - \frac{\epsilon^2}{216} + \frac{\epsilon^3}{3\,888} - \frac{5\epsilon^4}{279\,936} + \frac{7\epsilon^5}{5\,038\,848} - \frac{7\epsilon^6}{60\,466\,176} + \frac{11\epsilon^7}{1\,088\,391\,168} - \frac{143\epsilon^8}{156\,728\,328\,192} + O\left(\epsilon^9\right)\,.$$

Thus

$$x(16) \sim 3 + \frac{8}{3} - \underbrace{\frac{32}{27}}_{1.18519} + \underbrace{\frac{256}{243}}_{1.0535} - \underbrace{\frac{2560}{2187}}_{1.17055} + \underbrace{\frac{28672}{19683}}_{1.45669} - \underbrace{\frac{114688}{59049}}_{1.94225} + \cdots \tag{4.98}$$

The fourth term is the smallest term. Stopping short of the smallest term, the sum of the first three terms is

$$x(16) \approx \frac{121}{27} = 4.48148\,, \tag{4.99}$$

which is a relative error of about 10%. If we include half of the smallest term then

$$x(16) \approx \frac{1217}{243} = 5.00823\,, \tag{4.100}$$

with relative error 0.00165. This is a good result when working with the "small" parameter 16/9.

4.5 Large-s behaviour of Laplace transforms

The $s \to \infty$ behaviour of the Laplace transform

$$\bar{f}(s) \stackrel{\text{def}}{=} \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t \tag{4.101}$$

provides a typical and important example of IP. But before turning to IP, we argue that as $\Re s \to \infty$, the maximum of the integrand in (4.101) is determined by the rapidly decaying e^{-st} and is therefore at $t = 0$. In fact, e^{-st} is appreciably different from zero only in a peak at $t = 0$, and the width of this peak is $s^{-1} \ll 1$. Within this peak $t = O(s^{-1})$ the function $f(t)$ is almost equal to $f(0)$ (assuming that $f(0)$ is non-zero) and thus

$$\bar{f}(s) \approx f(0) \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}t = \frac{f(0)}{s}\,. \tag{4.102}$$

This argument suggests that the large s-behaviour of the Laplace transform of any function $f(t)$ with a Taylor series around $t = 0$ is given by

$$\int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-st} \left[f(0) + t f'(0) + \frac{t^2}{2!} f''(0) + \cdots \right] \mathrm{e}^{-st}\,\mathrm{d}t\,, \tag{4.103}$$

$$\sim \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{f''(0)}{s^3} + \cdots \tag{4.104}$$

This heuristic[3] answer is in fact a valid asymptotic series.

We obtain an improved version of (4.104) using successive integration by parts starting with (4.101):

$$\bar{f}(s) = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{f''(0)}{s^2} + \cdots + \frac{f^{(n-1)}(0)}{s^n} + \underbrace{\frac{1}{s^n} \int_0^\infty \mathrm{e}^{-st} f^{(n)}(t)\mathrm{d}t}_{R_n}\,. \tag{4.105}$$

The improvement over (4.104) is that on the right of (4.105), IP has provided an explicit expression for the remainder $R_n(s)$.

Example: A Laplace transform. Find the large-s behaviour of the Laplace transform

$$\mathcal{L}\left[\frac{1}{\sqrt{1+t^2}}\right] = \int_0^\infty \frac{\mathrm{e}^{-st}}{\sqrt{1+t^2}}\,\mathrm{d}t\,. \tag{4.106}$$

When s is large the function e^{-st} is non-zero only in a peak located at $t = 0$. The width of this peak is $s^{-1} \ll 1$. In this region the function $(1+t^2)^{-1/2}$ is almost equal to one. Hence heuristically

$$\int_0^\infty \frac{\mathrm{e}^{-st}}{\sqrt{1+t^2}}\,\mathrm{d}t \approx \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}t = \frac{1}{s}\,. \tag{4.107}$$

[3] See the next section, *Watson's lemma*, for justification.

This is the correct leading-order behaviour.

To make a more careful estimate we can use integration by parts:

$$\mathcal{L}\left[\frac{1}{\sqrt{1+t^2}}\right] = -\frac{1}{s}\int_0^\infty \frac{1}{\sqrt{1+t^2}}\frac{\mathrm{d}e^{-st}}{\mathrm{d}t}\,\mathrm{d}t\,, \tag{4.108}$$

$$= -\frac{1}{s}\left[\frac{e^{-st}}{\sqrt{1+t^2}}\right]_0^\infty - \frac{1}{s}\int_0^\infty \frac{te^{-st}}{(1+t^2)^{3/2}}\mathrm{d}t\,, \tag{4.109}$$

$$= \frac{1}{s} - R_1(s)\,. \tag{4.110}$$

As $s\to\infty$ the remainder $R_1(s)$ is negligible with respect to s^{-1} and the heuristic (4.107) is confirmed. Why is $R_1(s)$ much smaller than s^{-1} in the limit? Notice that in the integrand of R_1

$$\frac{te^{-st}}{(1+t^2)^{3/2}} \le te^{-st}\,, \qquad \text{and therefore} \qquad R(s) < \frac{1}{s}\int_0^\infty te^{-st}\mathrm{d}t = \frac{1}{s^2}\,. \tag{4.111}$$

The estimates between (4.108) and (4.111) are a recap of arguments we've been making in the previous lectures. The proof of Watson's lemma below is just a slightly more general version of these same estimates.

To get more terms in the asymptotic expansion we invoke Watson's lemma, so as $s\to\infty$:

$$\mathcal{L}\left[\frac{1}{\sqrt{1+t^2}}\right] = \int_0^\infty e^{-st}\left[1 - \frac{t^2}{2} + \frac{3t^4}{8} - \frac{5t^6}{16} + O(t^8)\right]\mathrm{d}t\,, \tag{4.112}$$

$$\sim \frac{1}{s} - \frac{1}{s^3} + \frac{9}{s^5} - \frac{225}{s^7} + O\left(s^{-9}\right)\,. \tag{4.113}$$

Because of the rapid growth of the numerators this is clearly an asymptotic series. The Taylor series of $(1+t^2)^{-1/2}$ does not converge beyond $t=1$. The limited radius of convergence doesn't matter: Watson's lemma assures us that we get the right asymptotic expansion even if we integrate into the region where the Taylor series diverges. In fact, the expansion of the integral is asymptotic, rather than convergent, *because* we've integrated a Taylor series beyond its radius of convergence.

We obtain the entire asymptotic series by noting that

$$\frac{1}{\sqrt{1-4x}} = 1 + 2x + 6x^2 + 20x^3 + 70x^4 + \cdots \tag{4.114}$$

where the coefficient of x^n above is the "central binomial coefficient" $(2n)!/(n!)^2$. Thus, with $x=-t^2/4$, we have

$$\mathcal{L}\left[\frac{1}{\sqrt{1+t^2}}\right] \sim \sum_{n=1}^\infty \frac{(2n)!}{(n!)^2}(-1)^n\left(\tfrac{1}{2}\right)^{2n}\int_0^\infty t^{2n}e^{-st}\,\mathrm{d}t\,, \tag{4.115}$$

$$= \frac{1}{s}\sum_{n=0}^\infty (-1)^n\left(\frac{(2n)!}{n!}\right)^2\frac{1}{(2s)^{2n}}\,. \tag{4.116}$$

Example: Another Laplace transform. Consider

$$\mathcal{L}\left[\frac{H(t)}{\sqrt{1-t^2}}\right] = \int_0^1 \frac{e^{-st}}{\sqrt{1-t^2}}\,\mathrm{d}t\,, \tag{4.117}$$

$$\sim \frac{1}{s} + \frac{1}{s^3} + \frac{9}{s^5} + \frac{225}{s^7} + O\left(s^{-9}\right)\,. \tag{4.118}$$

This is the same as (4.113), except that all the signs are positive. The integrable singularity at $t=1$ makes only an exponentially small contribution as $s\to\infty$.

Example: Yet another Laplace transform. Find the large-s behaviour of the Laplace transform

$$\mathcal{L}\left[\sqrt{1+e^t}\right] = \int_0^\infty e^{-st}\underbrace{\sqrt{1+e^t}}_{f(t)}\,\mathrm{d}t\,. \tag{4.119}$$

In this case $f(0)=\sqrt{2}$ and we expect that the leading order is

$$\bar{f} \sim \frac{\sqrt{2}}{s}\,. \tag{4.120}$$

Let's confirm this using IP:

$$\bar{f}(s) = \frac{\sqrt{2}}{s} - \frac{1}{s}\int_0^\infty \mathrm{e}^{-st} \underbrace{\frac{\mathrm{e}^t}{2\sqrt{1+\mathrm{e}^t}}}_{f'(t)} \mathrm{d}t\,. \tag{4.121}$$

Notice that in this example $f'(t) \sim \mathrm{e}^{t/2}$ as $t \to \infty$, and thus we cannot bound the remainder using $\max_{t>0} f'(t)$. Instead, we bound the reminder like this

$$R_1 = \frac{1}{s}\int_0^\infty \mathrm{e}^{-(s-\frac{1}{2})t} \underbrace{\frac{1}{2\sqrt{1+\mathrm{e}^{-t/2}}}}_{\le\frac{1}{2}} \mathrm{d}t < \frac{1}{s}\frac{1}{2s-1}\,. \tag{4.122}$$

This maneuver works in examples with $f(t) \sim \mathrm{e}^{\alpha t}$ as $t \to \infty$.

Example: A thinly disguised Laplace transform. Consider

$$S(x) \stackrel{\text{def}}{=} \int_0^1 \mathrm{e}^{xt^7}\,\mathrm{d}t \tag{4.123}$$

as $x \to \infty$. The integrand is strongly peaked near $t = 1$. The height of the peak is e^x and the width of the peak is $1/x$. We expect $S(x) \sim ?\mathrm{e}^x/x$ as $x \to \infty$.
Changing variable to $v = 1 - t^7$ we obtain the Laplace transform

$$S(x) = \frac{\mathrm{e}^x}{7}\int_0^1 \frac{\mathrm{e}^{-xv}\,\mathrm{d}v}{(1-v)^{6/7}}\,, \tag{4.124}$$

$$\sim \frac{\mathrm{e}^x}{7}\int_0^\infty \mathrm{e}^{-xv}(1 - \tfrac{6}{7}v + \cdots)\,\mathrm{d}v\,, \tag{4.125}$$

$$= \frac{\mathrm{e}^x}{7}\left(\frac{1}{x} - \frac{6}{7} + \cdots\right)\,. \tag{4.126}$$

Example: The other limit Consider

$$R(x) \stackrel{\text{def}}{=} \int_0^1 \mathrm{e}^{-xt^7}\,\mathrm{d}t \tag{4.127}$$

as $x \to \infty$.

Example: An integral with a parameter. Consider

$$I(x,\nu) \stackrel{\text{def}}{=} \int_0^\infty t^\nu \mathrm{e}^{-x\sinh t}\mathrm{d}t\,. \tag{4.128}$$

The minimum of $\phi(t) = \sinh t$ is at $t = 0$, so

$$I(x) \sim \int_0^\infty t^\nu \mathrm{e}^{-xt}\mathrm{d}t \sim \frac{\Gamma(\nu+1)}{x^{\nu+1}}\,, \qquad \text{as } x \to \infty. \tag{4.129}$$

To get the next term in the asymptotic series, keep one more term in the expansion of $\sinh t$:

$$\mathrm{e}^{-x\sinh t} \approx \mathrm{e}^{-xt}\mathrm{e}^{xt^3/6 - xt^5/120+\cdots} \approx \mathrm{e}^{-xt}\left(1 - \frac{xt^3}{6} + O(xt^5)\right)\,. \tag{4.130}$$

Thus

$$I(x) \sim \int_0^\infty t^\nu \mathrm{e}^{-xt}\left(1 - \frac{xt^3}{6} + O(xt^5)\right)\mathrm{d}t\,, \tag{4.131}$$

$$\sim \frac{\Gamma(\nu+1)}{x^{\nu+1}} - \frac{\Gamma(\nu+4)}{6x^{\nu+3}} + O\left(x^{-\nu-5}\right)\,. \tag{4.132}$$

Notice we have to keep the dominant term xt up in the exponential.
If we desire more terms, and are obliged to justify the heuristic above, we should change variables with $u = \sinh t$ in (4.128), and use Watson's lemma. The transformed integral is a formidable Laplace transform:

$$I(x,\nu) \stackrel{\text{def}}{=} \int_0^\infty \mathrm{e}^{-xu}\ln^\nu\left(\sqrt{1+u^2}+u\right)\frac{\mathrm{d}u}{\sqrt{1+u^2}}\,. \tag{4.133}$$

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$$\frac{\ln^\nu\left(\sqrt{1+u^2}+u\right)}{\sqrt{1+u^2}} = u^\nu\left[1-\frac{3+\nu}{6}u^2+\frac{135+52\nu+5\nu^2}{360}u^4+O(u^6)\right]. \qquad (4.134)$$

The coefficient of u^{2n} in this expansion is a polynomial — let's call it $(-)^n P_n(\nu)$ — of order n. Substituting (4.134) into (4.133) and integrating term-by-term

$$I(x,\nu) \sim \frac{1}{x^{\nu+1}}\left[\Gamma(\nu+1)-\frac{P_2(\nu)}{x^2}\Gamma(\nu+3)+\frac{P_4(\nu)}{x^4}\Gamma(\nu+5)+O(x^{-6})\right]. \qquad (4.135)$$

4.6 Watson's Lemma

All the examples in the previous section are a special cases of Watson's lemma. So let's prove the lemma by considering a Laplace transform

$$\bar{f}(s) = \int_0^\infty \mathrm{e}^{-st}\, t^\xi g(t)\,\mathrm{d}t\,, \qquad (4.136)$$

where the factor t^ξ includes whatever singularity exists at $t = 0$; the singularity must be integrable i.e., $\xi > -1$. We assume that the function $g(t)$ has a Taylor series with remainder

$$g(t) = \underbrace{g_0 + g_1 t + \cdots g_n t^n}_{n+1 \text{ terms}} + R_{n+1}(t)\,. \qquad (4.137)$$

This is a $t \to 0$ asymptotic expansion in the sense that there is some constant K such that

$$|R_{n+1}| < K t^{n+1}\,. \qquad (4.138)$$

Notice we are not assuming that the Taylor series converges.

Of course, we do assume convergence of the Laplace transform (4.136) as $t \to \infty$, which most simply requires that $f(t) = t^\xi g(t)$ eventually grows no faster than $\mathrm{e}^{\gamma t}$ for some γ. Notice that the possibility of a finite upper limit in (4.136) is encompassed if $f(t)$ is zero once $t > T$.

With these modest constraints on $t^\xi g(t)$:

$$\bar{f}(s) = \underbrace{\int_0^\infty \mathrm{e}^{-st}\, t^\xi \left(g_0 + g_1 t + \cdots g_n t^n\right)\mathrm{d}t}_{I_1} + \underbrace{\int_0^\infty \mathrm{e}^{-st} t^\xi R_{n+1}(t)\,\mathrm{d}t}_{I_2}\,. \qquad (4.139)$$

The second integral in (4.139) is

$$I_2 < K\int_0^\infty \mathrm{e}^{-st} t^{n+1+\xi}\,\mathrm{d}t = O\left(\frac{1}{s^{\xi+n+2}}\right). \qquad (4.140)$$

Using

$$\int_0^\infty \mathrm{e}^{-st} t^{\xi+n}\,\mathrm{d}t = \frac{\Gamma(n+\xi+1)}{s^{n+\xi+1}}\,, \qquad (4.141)$$

we integrate I_1 term-by-term and obtain Watson's lemma:

$$\bar{f}(s) \sim g_0\frac{\Gamma(\xi+1)}{s^{\xi+1}} + g_1\frac{\Gamma(\xi+2)}{s^{\xi+2}} + \cdots + g_n\frac{\Gamma(\xi+n+1)}{s^{\xi+n+1}} + O\left(\frac{1}{s^{\xi+n+2}}\right). \qquad (4.142)$$

Watson's lemma justifies doing what comes naturally.

4.7 Problems

Problem 4.1. *(i)* Find a leading-order $x \to \infty$ asymptotic approximation to

$$A(x;p,q) \stackrel{\text{def}}{=} \int_x^\infty \mathrm{e}^{-pt^q}\,\mathrm{d}t\,. \tag{4.143}$$

Show that the remainder is asymptotically negligible as $x \to \infty$. Above, p and q are both positive real numbers.

Problem 4.2. Find two terms in the $x \to \infty$ behaviour of

$$F(x) = \int_0^x \frac{\mathrm{e}^{-v}}{v^{1/3}}\,\mathrm{d}v\,. \tag{4.144}$$

Problem 4.3. *(i)* Use integration by parts to find the leading-order term in the $x \to \infty$ asymptotic expansion of the *exponential integral:*

$$\mathrm{E}_1(x) \stackrel{\text{def}}{=} \int_x^\infty \frac{\mathrm{e}^{-v}}{v}\,\mathrm{d}v\,. \tag{4.145}$$

Show that this approximation is asymptotic i.e., prove that the remainder is asymptotically less than the leading term as $x \to \infty$. *(ii)* With further integration by parts, find an expression for the n'th term, and the remainder after n terms. *(iii)* Show that the remainder after N terms is asymptotically less than the N'th terms as $x \to \infty$.

Problem 4.4. Consider the first-order differential equation:

$$y' - y = -\frac{1}{x}\,, \qquad \text{with the condition} \qquad \lim_{x\to\infty} y(x) = 0\,. \tag{4.146}$$

(i) Find a valid two-term dominant balance in the differential equation and thus deduce the leading-order asymptotic approximation to $y(x)$ for large positive x. *(ii)* Use an iterative procedure to deduce the full asymptotic expansion of $y(x)$. *(iii)* Is the expansion convergent? *(iv)* Use the integrating function method to solve the differential equation exactly in terms of the exponential integral in (4.145). Use MATLAB (`help expint`) to compare the exact solution of (4.146) with asymptotic expansions of different order. Summarize your study as in Figure 4.5.

Problem 4.5. The exponential integral of order n is

$$\mathrm{E}_n(x) \stackrel{\text{def}}{=} \int_x^\infty \frac{\mathrm{e}^{-t}}{t^n}\,\mathrm{d}t\,. \tag{4.147}$$

Show that

$$\mathrm{E}_{n+1}(x) = \frac{\mathrm{e}^{-x}}{nx^n} - \frac{1}{n}\mathrm{E}_n(x)\,. \tag{4.148}$$

Find the leading-order asymptotic approximation to $\mathrm{E}_n(x)$ as $x \to \infty$.

Problem 4.6. *(i)* Solve the differential equation

$$y' - xy = -1\,, \qquad \text{with} \qquad \lim_{x\to\infty} y(x) = 0\,, \tag{4.149}$$

in terms of erf and use the results from this lecture to find the full asymptotic expansion of the solution as $x \to \infty$. *(ii)* Find this expansion without explicit solution of the ODE in (4.149): identify a two-term $x \to \infty$ balance in the ODE, and then proceed to higher order via iteration or some other scheme.

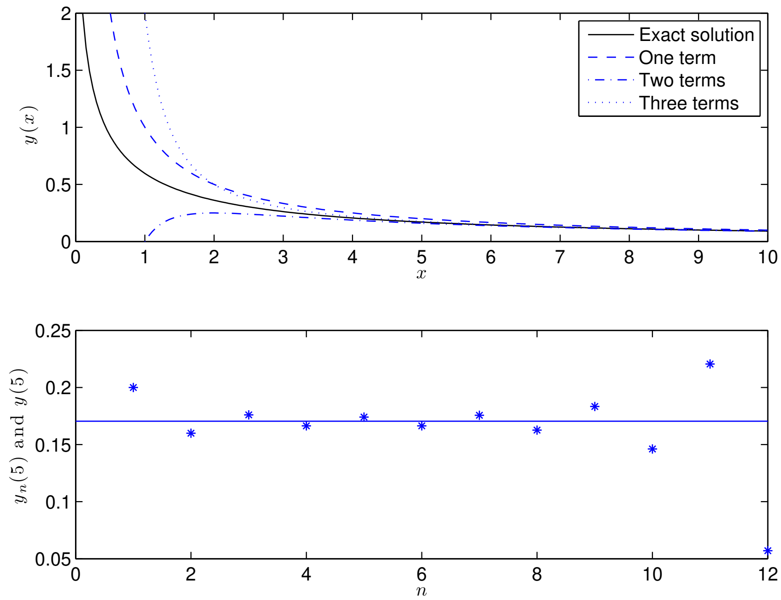

Figure 4.5: Solution of problem 4.4. Upper panel compares the exact solution with truncated asymptotic series. Lower panel shows the asymptotic approximation at $x = 5$ as a function of the truncation order n i.e., $n = 1$ is the one-term approximation. The solid line is the exact answer.

Problem 4.7. Find an example of a infinitely differentiable function satisfying the inequalities

$$\max_{0<x<1} |f(x)| < 10^{-10}\,, \qquad \text{and} \qquad \max_{0<x<1} \left|\frac{\mathrm{d}f}{\mathrm{d}x}\right| > 10^{10}\,. \tag{4.150}$$

This is why the differential operator $\mathrm{d}/\mathrm{d}x$ is "unbounded": $\mathrm{d}/\mathrm{d}x$ can take a small function and turn it into a big function.

Problem 4.8. Prove that

$$\int_0^\infty \frac{\mathrm{e}^{-t}}{1+xt^2}\,\mathrm{d}t \sim \sum_{n=0}^{\infty} (-1)^n (2n)! x^n\,, \qquad x \to 0\,. \tag{4.151}$$

Problem 4.9. True or false as $x \to \infty$

$$(i)\ x + \frac{1}{x} \overset{?}{\sim} x\,, \qquad (ii)\ x + \sqrt{x} \overset{?}{\sim} x\,, \qquad (iii)\ \exp\left(x + \frac{1}{x}\right) \overset{?}{\sim} \exp(x)\,, \tag{4.152}$$

$$(iv)\ \exp\left(x + \sqrt{x}\right) \overset{?}{\sim} \exp(x)\,, \qquad (v)\ \cos\left(x + \frac{1}{x}\right) \overset{?}{\sim} \cos x\,, \qquad (v)\ \frac{1}{x} \overset{?}{\sim} 0\,? \tag{4.153}$$

Problem 4.10. Let's investigate the Stieltjes series $S(x)$ in (4.56) and the function $F(x)$ in (4.58) *(i)* Compute the integral $F(0.1)$ numerically. *(ii)* With $x = 0.1$, compute partial sums of the divergent series (4.56) with $N = 2, 3, 4, \cdots 20$. Which N gives the best approximation to $F(0.1)$? *(iii)* I think the best answer is obtained by truncating the series $S(0.1)$ just before the smallest term. Is that correct?

Problem 4.11. *(i)* Obtain the leading-order asymptotic approximation for the integral

$$\int_{-1}^{1} \mathrm{e}^{xt^3}\,\mathrm{d}t\,, \qquad \text{as } x \to \infty. \tag{4.154}$$

(ii) Justify the asympoticness of the expansion. *(iii)* Find the leading-order asymptotic approximation for $x \to -\infty$.

Problem 4.12. In our evaluation of $\mathrm{Ai}(0)$ we encountered a special case, namely $n = 3$, of the integral

$$Z(n, x) \stackrel{\text{def}}{=} \int_0^{\pi/(2n)} \mathrm{e}^{-x \sin n\theta} \, \mathrm{d}\theta \,. \tag{4.155}$$

Convert $Z(n, x)$ to a Laplace transform and use Watson's lemma to obtain the first few terms of the $x \to \infty$ asymptotic expansion.

Problem 4.13. In lecture 3 we obtained the full asymptotic series for $\mathrm{erfc}(z)$ via IP:

$$\mathrm{erfc}(x) \sim \frac{\mathrm{e}^{-x^2}}{\sqrt{\pi} x} \sum_{n=0}^{\infty} (2n-1)!! \left(-\frac{1}{2x^2}\right)^n \,. \tag{4.156}$$

Obtain this result by making a change of variables that converts $\mathrm{erfc}(z)$ into a Laplace transform, and then use Watson's lemma.

Problem 4.14. Use integration by parts to find $x \to \infty$ asymptotic approximations of the integrals

$$A(x) = \int_0^x \mathrm{e}^{-t^4} \, \mathrm{d}t \,, \qquad B(x) = \int_0^x \mathrm{e}^{+t^4} \, \mathrm{d}t \,, \qquad C(x) = \int_0^\infty \mathrm{e}^{-xt} \ln(1+t^2) \, \mathrm{d}t \,, \tag{4.157}$$

$$D(x) = \int_0^\infty \frac{\mathrm{e}^{-xt}}{t^a (1+t)} \, \mathrm{d}t \,, \quad \text{with } a < 1; \qquad E(x) = \int_1^\infty \mathrm{e}^{-xt^p} \, \mathrm{d}t \,, \quad \text{with } p > 0. \tag{4.158}$$

In each case obtain a two-term asymptotic approximation and exhibit the remainder as an integral. Explain why the remainder is smaller than the second term as $x \to \infty$.

Problem 4.15. Using repeated IP, find the full $x \to \infty$ asymptotic expansion of Dawson's integral (4.62). Is this series convergent?

Problem 4.16. Consider $f(x) = (1+x)^{5/2}$, and the corresponding Taylor series $f_n(x)$ centered on $x = 0$. *(i)* Show that for $n \geq 3$ and $x > 0$:

$$R_n < \frac{f^{(n)}(0)}{n!} x^n \,,$$

i.e. the remainder is smaller than the first neglected term for all positive x. *(ii)* The Taylor series converges only up to $x = 1$. But suppose we desire $f(2) = 3^{5/2}$. How many terms of the series should be summed for best accuracy? Sum this optimally truncated series and compare with the exact answer. *(iii)* Argue from the remainder in (4.94) that the error can be reduced by adding *half* the first neglected term. Compare this corrected series with the exact answer.

Lecture 5

Geometric perturbation of PDEs

Let's consider some perturbation problems presented by partial differential equations — a main novelty is perturbation of geometry.

5.1 Thermal diffusion in solids

We'll use the diffusion equation as a main example. The conservation law for energy in a solid with non-uniform temperature is

$$Q_t + \boldsymbol{\nabla}\cdot\boldsymbol{F} = 0\,, \tag{5.1}$$

where the heat content and energy flux are

$$Q = \rho c T\,, \qquad \text{and} \qquad \boldsymbol{F} = -\varkappa\boldsymbol{\nabla}T\,. \tag{5.2}$$

Above T is the temperature (K), c is the heat capacity (J/K kg), ρ is the density (kg/m^3) and and $\varkappa$ is the conductivity. The flux $\boldsymbol{F}$ has dimensions Watts per square meter.

Assuming that c, ρ and $\varkappa$ are all constant, (5.1) is rewritten as the diffusion equation

$$T_t = \kappa\nabla^2 T\,, \tag{5.3}$$

where $\kappa \stackrel{\text{def}}{=} \varkappa/\rho c$ is the thermal diffusivity and ∇^2 is the Laplacian operator.

Diffusion a through a slab

The simplest solution of (5.3) is T =constant. The second simplest is constant flux. For example, consider a slab of thickness h with temperature $T = 0$ at $z = 0$ and $T = \tau$ at $z = h$. We look for a steady solution that depends only on z, $T(z)$. So the problem is

$$\kappa T_{zz} = 0\,, \qquad \text{with BCs} \qquad T(0) = 0\,, \qquad T(h) = \tau\,. \tag{5.4}$$

The solution is

$$T = \frac{z\tau}{h}\,. \tag{5.5}$$

The heat flux through the slab is

$$\boldsymbol{F} = -\underbrace{\rho c\kappa}_{\varkappa}\,\frac{\tau}{h}\hat{\boldsymbol{z}}\,. \tag{5.6}$$

($\hat{\boldsymbol{z}}$ is a unit vector pointing along the z-axis.) If $\tau > 0$ then the top of the slab is hot and the bottom is cold. The heat flux $\boldsymbol{F}$ points downwards, from the top to the bottom. Is that sign intuitive?

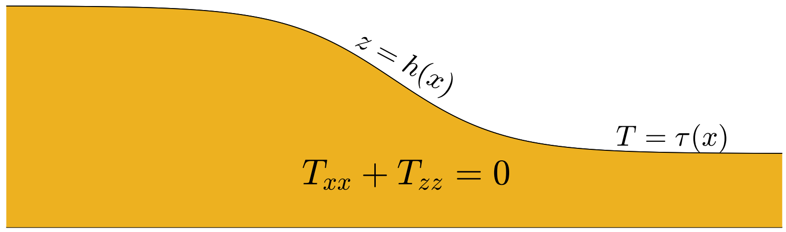

Figure 5.1: Heat conduction through a slab with non-uniform thickness.

5.2 Slow variations: diffusion through a slab with slowly changing thickness

Suppose the thickness of the slab is not constant e.g. the slab occupies the region

$$-\infty < x < \infty\,, \qquad \text{and} \qquad 0 < z < h(x)\,. \tag{5.7}$$

For extra fun we also suppose that the top temperature is non-uniform, $T(x,h) = \tau(x)$. The bottom temperature is still $T(x,0) = 0$. (We can have even more fun by making the bottom temperature non-uniform in x and the bottom surface vary in x. These complications are only algebraic.)

The slab thickness $h(x)$ has order of magnitude H and h changes on a length scale L. For example, we consider a model such as

$$h = H\,\frac{3 + \mathrm{e}^{x/L}}{1 + \mathrm{e}^{x/L}}\,. \tag{5.8}$$

In this example $h(x)$ varies from $3H$ at $x = -\infty$ to H at $x = +\infty$: see figure 5.1. A factor of three is a big change in thickness. Suppose that $\tau(x)$ provides a temperature scale T_*. Then we non-dimensionalize with

$$\bar{x} \stackrel{\text{def}}{=} \frac{x}{L}\,, \qquad \bar{z} \stackrel{\text{def}}{=} \frac{z}{H}\,, \qquad \bar{T} \stackrel{\text{def}}{=} \frac{T}{T_*}\,. \tag{5.9}$$

The non-dimensional problem is

$$\nu^2 \bar{T}_{\bar{x}\bar{x}} + \bar{T}_{\bar{z}\bar{z}} = 0\,, \qquad \text{where} \qquad \nu \stackrel{\text{def}}{=} \frac{H}{L}\,. \tag{5.10}$$

Boundary conditions are

$$\bar{T}(\bar{x},0) = 0\,, \qquad \text{and} \qquad \bar{T}(\bar{x},\bar{h}) = \bar{\tau}\,. \tag{5.11}$$

Geometry is challenging: we are confronted by $T_{xx} + T_{zz} = 0$ within a complicated 2D domain. (Complicated because there is no way to separate variables.) But if $\nu \ll 1$ there is a remarkable simplification.

An aspect-ratio expansion: a one-term dominant balance

Proceed dropping all the bars used to distinguish non-dimensional variables. Or even better, use the dimensional equations inserting an ν^2 in front of T_{xx}. Use the order parameter ν to keep

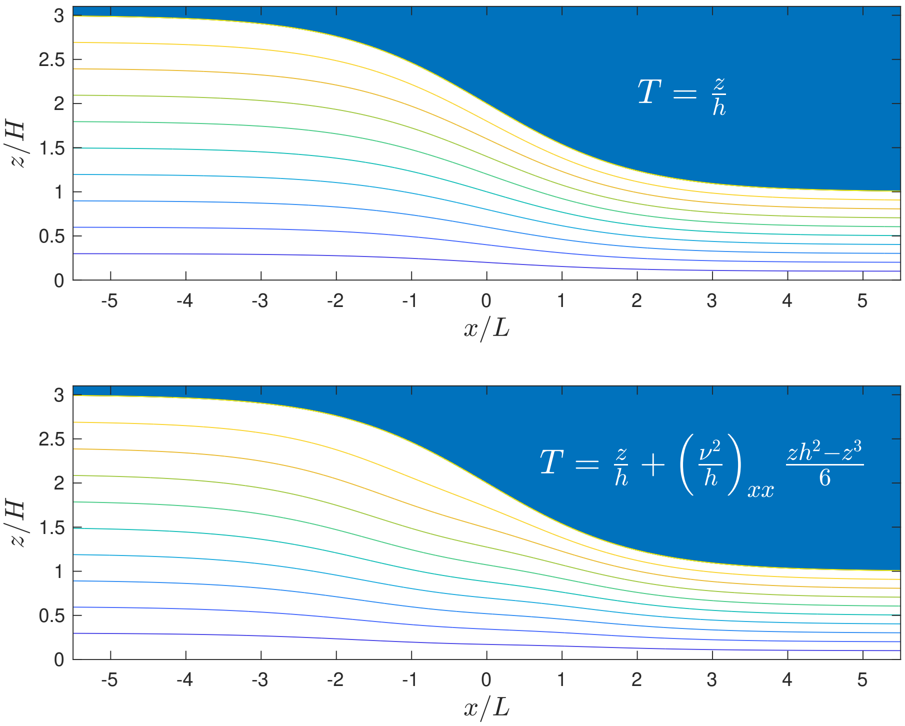

Figure 5.2: Illustration of thermal diffusion through a slab with slowly varying thickness and uniform top temperature $\tau = 1$. The top panel shows the leading-order solution and the bottom panel the solution in (5.15). In a futile attempt to see the effect of the order ϵ^2 correction I use $\nu = \sqrt{2}$.

track of different terms in the expansion (as below) and set $\nu = 1$ at the end of the calculation. (We use ν, rather than ϵ, to denote the small parameter – later in this lecture we use ϵ for another small parameter.)

If $\nu \ll 1$ there is a one term dominant balance in (5.10). The subsequent expansion is

$$T = \frac{\tau z}{h} + \nu^2 T_2(x, z) + \nu^4 T_4(x, z) + \cdots \tag{5.12}$$

The T_2 problem is

$$T_{2zz} = -\left(\frac{\tau}{h}\right)'' z\,, \qquad \text{with BCs} \qquad T_2(x, 0) = T_2(x, h) = 0\,. \tag{5.13}$$

Above $'$ denotes an x-derivative. The solution of (5.13) is straightforward. The reconstituted temperature is

$$T = \frac{\tau z}{h} + \nu^2 \left(\frac{\tau}{h}\right)'' \tfrac{1}{6}\left(zh^2 - z^3\right) + O(\nu^4)\,. \tag{5.14}$$

Example: To illustrate this solution let's take $\tau = 1$ and use the non-dimensional slab thickness

$$h = \frac{3 + \mathrm{e}^x}{1 + \mathrm{e}^x}\,, \qquad \text{with} \qquad \left(\frac{1}{h}\right)'' = \frac{2\mathrm{e}^x(\mathrm{e}^x - 3)}{(3 + \mathrm{e}^x)^3}\,. \tag{5.15}$$

In figure 5.2 I compare the leading-order term, $T_0 = z/h$, with the corrected temperature in (5.14). In this illustration I'm using an unreasonably large value $\nu = \sqrt{2}$. With smaller values, such as $\nu = 1$, there was no perceptible difference between the two panels. This might be because

$$\max_{0<\xi<1} \tfrac{1}{6}\left(\xi - \xi^3\right) = \frac{1}{9\sqrt{3}} = 0.0642 \ll 1\,. \tag{5.16}$$

This suggests that the approximation is excellent. It would be interesting to calculate the next term which will involve a fifth-order polynomial in z/h.

Heat flux

The heat flux through the bottom, $\boldsymbol{F} \cdot \hat{\boldsymbol{z}}$, is proportional to

$$T_z(x,0) = \frac{\tau}{h} + \frac{h^2}{6}\left(\frac{\tau}{h}\right)'' . \tag{5.17}$$

Is this the same heat flux that comes out the top of the slab?

```
%% contour plot of the corrugated-slab temperature
% tau=1, h = [n+1+exp(x)]/[1+exp(x)], sig = 1/[n+1+exp(x)]
% T = (z/h) + (1/h)'' (z h^2 - z^3)/6
% (1/h)'' = n exp(x)[exp(x)-1]/[1+exp(x)]^3
clc
clf
epsn = sqrt(2);
n=2;
x = linspace(-5.5,5.5,200);
h = (n+1+exp(x))./(1 + exp(x));
z = linspace(0,n+1);

[xx,zz] = meshgrid(x,z);
hh = (n+1+exp(xx))./(1 + exp(xx));
TT0 = zz./hh;
figure(1)
movegui('west')
subplot(2,1,1)
area(x,h,n+1+0.1)
hold on
V=[0:0.1:1];
contour(xx,zz,TT0,V)
axis([min(x), max(x), 0, n+1+0.1])
text(2,2.1,'$T = \frac{z}{h}$',...
    'Interpreter','latex','fontsize',18,'Color',[1 1 1])
xlabel('$x/L$','Interpreter','latex','fontsize',14)
ylabel('$z/H$','Interpreter','latex','fontsize',14)

%% the correction
secDeriv = ( n*exp(xx).*(1+n-exp(xx)) )./(1+n+exp(xx)).^3;
TT1 = secDeriv.*(zz.*hh.^2- zz.^3)/6;
TTT = TT0 + epsn^2*TT1;
figure(1)
subplot(2,1,2)
area(x,h,n+1+0.1)
hold on
V=[0:0.1:1];
contour(xx,zz,TTT,V)
axis([min(x), max(x), 0, n+1+0.1])
text(0.7,2.2,'$T =\frac{z}{h}+\left(\frac{\nu^2}{h}\right)_{xx}\frac{z h^2-z^3}{6}$',...
   'Interpreter','latex','fontsize',18,'color',[1 1 1])
xlabel('$x/L$','Interpreter','latex','fontsize',14)
ylabel('$z/H$','Interpreter','latex','fontsize',14)
hold off
```

Matlab code slabTemp2023 that produced figure 5.2. The command “area” is handy.

5.3 Slow variations: diffusion along a slab with slowly changing thickness

We consider diffusion of heat within a slab with boundaries at $z = \pm h(x)$. The outward unit normal vectors to the surface of the slab are therefore

$$@z = h(x): \qquad \boldsymbol{n} = \frac{\hat{\boldsymbol{z}} - h'\hat{\boldsymbol{x}}}{\sqrt{1+h'^2}}, \qquad \text{and} \qquad @z = -h(x): \qquad \boldsymbol{n} = \frac{-\hat{\boldsymbol{z}} - h'\hat{\boldsymbol{x}}}{\sqrt{1+h'^2}}. \tag{5.18}$$

The unsteady diffusion equation is

$$T_t = \kappa \left(T_{xx} + T_{zz}\right), \tag{5.19}$$

and we suppose that there is no flux of heat through the boundaries. This means that $\boldsymbol{n} \cdot \boldsymbol{\nabla} T = 0$, or

$$@z = h(x): \qquad \frac{T_z - h'T_x}{\sqrt{1+h'^2}} = 0, \qquad \text{and} \qquad @z = -h(x): \qquad -\frac{T_z + h'T_x}{\sqrt{1+h'^2}} = 0. \tag{5.20}$$

(We could simplify by cancelling the factor $\sqrt{1+h'^2}$ from the boundary conditions above. But if you needed to impose a flux of heat through the boundary then there'd be a non-zero right hand side and this factor will not cancel.)

The problem is completed by specifying an initial condition

$$T(x, z, 0) = I(x, z). \tag{5.21}$$

The initial condition is going to provoke some discussion later.

Now let's simplify this problem by considering the slowly varying limit in which

$$\frac{H}{L} \to 0. \tag{5.22}$$

We continue to use the notation $\nu \overset{\text{def}}{=} H/L$.

If we non-dimensionalize using the anisotropic scaling $\bar{x} = x/L$ and $\bar{z} = z/H$ then every x-derivative in the system above appears with a ν in front of it. We scale time as $\bar{t} = t/\tau$ where τ is a TBD time scale. Thus the diffusion equation is

$$\frac{H^2}{\kappa\tau} T_{\bar{t}} = \nu^2 T_{\bar{x}\bar{x}} + T_{\bar{z}\bar{z}}. \tag{5.23}$$

It is necessary to pick $\tau = L^2/\kappa$ i.e. τ is the time scale for the diffusion of heat *along* the slab. With this definition of τ the diffusion equation is

$$\nu^2 \left(T_{\bar{t}} - T_{\bar{x}\bar{x}}\right) = T_{\bar{z}\bar{z}}. \tag{5.24}$$

5.4 Slight variations: a slab with small corrugations

Now consider a different type of geometric perturbation: suppose that the slab is slightly deformed so that the thickness is no longer uniform. Let's suppose that the solid slab is now the region

$$-\infty < x < \infty, \qquad h_{\max} \cos kx < y < H, \tag{5.25}$$

The lower boundary is corrugated. We assume that there is a gap between the crests and the flat lid i.e. $h_{\max} < H$. The problem is to solve Laplace's equation

$$T_{xx} + T_{zz} = 0 \tag{5.26}$$

with boundary conditions

$$T(x, h(x)) = 0, \qquad T(x, H) = T_*. \tag{5.27}$$

If $h_{\max} = 0$ then we recover the simple solution $T = T_* z/H$.

The method of slow variations used in the previous section will work if $kH \ll 1$. But now we want an answer with no restriction on the size kH. Instead we use a different method based on the assumption that $kh_{\max} \ll 1$ i.e. the corrugations are small.

Scale using the non-dimensional variables

$$\bar{T} \stackrel{\text{def}}{=} \frac{T}{T_*}, \qquad \text{and} \qquad (\bar{x}, \bar{z}) = k(x, z). \tag{5.28}$$

The non-dimensional parameters are

$$\epsilon \stackrel{\text{def}}{=} kh_{\max}, \qquad \text{and} \qquad \beta \stackrel{\text{def}}{=} kH. \tag{5.29}$$

In non-dimensional variables the bottom is $\bar{z} = \epsilon \cos x$ and the top is $\bar{z} = \beta$. $h_{\max}$ is less than H and therefore $\epsilon < \beta$. The heat flux through the bottom of the corrugated slab is

$$F = \rho c \kappa k T_* \left\langle \bar{T}_{\bar{z}}(\bar{x}, 0; \beta, \epsilon) \right\rangle, \tag{5.30}$$

where $\langle\rangle$ is an x-average and $\bar{T}(\bar{x}, \bar{z}; \beta, \epsilon)$ is the solution to the non-dimensional problem.

Dropping the decoration, the non-dimensional problem is

$$T_{xx} + T_{zz} = 0, \tag{5.31}$$

with boundary conditions

$$T(x, \epsilon \cos x) = 0, \qquad \text{and} \qquad T(x, \beta) = 1. \tag{5.32}$$

Small amplitude ripples, $\epsilon \ll 1$

Let's make $h_{\max}$ very small, holding all other parameters fixed. In non-dimensional variables this small-ripple limit is $\epsilon \to 0$ with β fixed and order unity. We assume that the solution $T(x, z)$ of Laplace's equation (5.31) with the BC in (5.32) is an RPS

$$T = T_0(x, z) + \epsilon T_1(x, z) + \epsilon^2 T_2(x, z) + \cdots \tag{5.33}$$

The complication is that the lower boundary condition in (5.32) becomes

$$T_0(x, \epsilon \cos x) + \epsilon T_1(x, \epsilon \cos x) + \epsilon^2 T_2(x, \epsilon \cos x) + \cdots = 0. \tag{5.34}$$

In (5.34) ϵ is appearing in two different places. The standard trick (probably originating with Stokes in 1847) is to use a Taylor series about $z = 0$ to transfer the boundary condition in (5.34) to $z = 0$. Thus each term in (5.34) is expanded like this

$$T_n(x, \epsilon \cos x) = T_n(x, 0) + \epsilon \cos x \, T_{nz}(x, 0) + \tfrac{1}{2}\epsilon^2 \cos^2 x \, T_{nzz}(x, 0) + \cdots \tag{5.35}$$

Substituting (5.35) into (5.34) and matching up powers of ϵ we obtain the $z = 0$ boundary conditions

$$T_0 = 0, \tag{5.36}$$

$$T_1 + \cos x \, T_{0z} = 0, \tag{5.37}$$

$$T_2 + \cos x \, T_{1z} + \tfrac{1}{2} \cos^2 x \, T_{0zz} = 0, \tag{5.38}$$

$$T_3 + \cos x \, T_{2z} + \tfrac{1}{2} \cos^2 x \, T_{1zz} + \tfrac{1}{6} \cos^3 x \, T_{0zzz} = 0, \tag{5.39}$$

$$T_4 + \cos x \, T_{3z} + \tfrac{1}{2} \cos^2 x \, T_{2zz} + \tfrac{1}{6} \cos^3 x \, T_{1zzz} + \tfrac{1}{24} \cos^4 x \, T_{0zzzz} = 0 \tag{5.40}$$

and so on. The boundary condition at the top is easy

$$T_0(x, \beta) = 1, \qquad \text{and} \qquad T_{n\geq 1}(x, \beta) = 0. \tag{5.41}$$

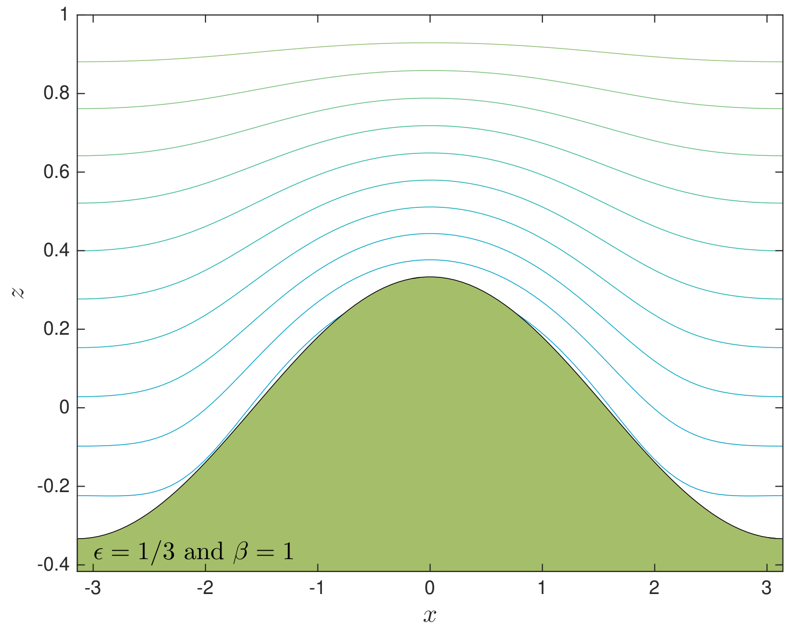

Figure 5.3: Contour plot of T in (**??**).

To complete the formulation of the perturbation problem, each $T_n(x,y)$ satisfies Laplace's equation

$$\left(\partial_x^2 + \partial_z^2\right) T_n = 0\,. \tag{5.42}$$

The solution is:

$$\begin{aligned} T &= \tfrac{z}{\beta} - \epsilon \tfrac{S_1}{\beta} - \epsilon^2 \tfrac{\coth\beta}{2\beta^2}\left[\beta - z + \beta S_2\right] \\ &\quad - \epsilon^3 \left[\left(\tfrac{\coth\beta\coth 2\beta}{2\beta} - \tfrac{1}{8\beta}\right) S_3 + \left(\tfrac{\coth\beta\coth 2\beta}{2\beta} + \tfrac{\coth\beta}{2\beta^2} - \tfrac{3}{8\beta}\right) S_1\right] + O(\epsilon^4)\,, \end{aligned} \tag{5.43}$$

where

$$S_n(x,z) \stackrel{\text{def}}{=} \frac{\sinh n(\beta - z)}{\sinh n\beta}\cos nx\,. \tag{5.44}$$

This solution is illustrated in figure 5.3.

Enhanced heat flux through the corrugated slab

Let's calculate the heat flux flowing through the corrugated slab. Heat is diffusing in through the hot flat top at $z = \beta$, and out through the cold corrugated bottom $z = \epsilon\cos x$. From (**??**), the flux in through the flat top at $z = \beta$ is

$$T_z(x,\beta) = \frac{1}{\beta} + \epsilon\frac{\cos x}{\beta\sinh\beta} + \tfrac{1}{2}\epsilon^2\frac{\coth\beta}{\beta^2}\left[1 + \frac{2\beta\cos 2x}{\sinh 2\beta}\right] + O\left(\epsilon^3\right)\,. \tag{5.45}$$

From (5.30) the averaged flux through the slab is

$$F = F_*\beta\left\langle T_{\bar z}(\bar x,\beta)\right\rangle\,, \tag{5.46}$$

$$= F_* \underbrace{\left(1 + \epsilon^2\frac{\coth\beta}{2\beta} + O(\epsilon^4)\right)}_{\stackrel{\text{def}}{=}\chi(\beta,\epsilon)}\,. \tag{5.47}$$

In (5.47) $\chi(\beta,\epsilon)$ is a flux enhancement factor resulting from the corrugations.

Is it physically intuitive that these corrugations increase the flux of heat through the slab? One can argue that the places where the slab is thin (above the hills) are short circuits. This suspicion is confirmed in figure 5.3 which shows that the temperature gradient is increased over the hills and decreased over the valleys. But this modulation is the order ϵ term in (5.45), which is proportional to $\cos x$ and therefore integrates to zero. The flux enhancement χ in (**??**) is order ϵ^2 — the big gradients over the hills more than compensate for the small gradients over the valleys. This effect is subtle and I don't have a totally satisfactory physical explanation.

We should also calculate the heat content of the slab

$$Q(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{\epsilon \cos x}^{\beta} T(x, z) \, \mathrm{d}z \mathrm{d}x \, . \tag{5.48}$$

```
%% contour plot of the corrugated-slab temperature
clc
epsn = 1/3; beta = 1;
xx= linspace(-pi,pi);zz = linspace(-epsn,beta);
[X,Z] = meshgrid(xx,zz);
S1 = sinh(beta - Z)/(beta*sinh(beta));
S2 = sinh(2*(beta - Z))/(2*beta*sinh(2*beta));
T = Z/beta - epsn*cos(X).*S1 - 0.5*epsn^2*(coth(beta)/beta)...
          *((beta-Z)/beta + 2*beta*cos(2*X).*S2);
figure(1)
contour(X,Z,T,20)
xlabel('$x$','interpreter','latex','fontsize',16)
ylabel('$z$','interpreter','latex','fontsize',16)
hold on
% use "area", rather than "fill"
height = epsn*cos(xx);
area(xx,height,-1.25*epsn)
text(-3,-0.37,'$\epsilon = 1/3$ and $\beta=1$',...
                    'interpreter','latex','fontsize',16)
```

Matlab code that produced figure 5.3 is above. The command “area” is very handy.

Remark: We can take the limit $\beta \to \infty$ in (**??**), to obtain

$$\beta T \to z - \epsilon\, \mathrm{e}^{-z} \cos x - \tfrac{1}{2}\epsilon^2 \left[1 + \mathrm{e}^{-2z} \cos 2x\right] . \tag{5.49}$$

A less algebra-intensive problem, corresponding to removing the lid at $z = H$ to ∞, is to consider the region $z > \epsilon \cos x$ and require that at great distances above the corrugated boundary the temperature is $T \to z + c(\epsilon)$, where $c(\epsilon)$ is to be determined. The lower boundary condition is $T(x, \epsilon \cos) = 0$. Taking $\beta \to \infty$ in (**??**) we have $c(\epsilon) = -\epsilon^2/2 + O(\epsilon^4)$. Although this special case is easier, and illustrates the method of boundary perturbation, it is less physically interesting because there is no "flux enhancement".

Remark: The other limit is small β: if $\beta = O(\epsilon)$ then the series in (**??**) through (**??**) becomes disordered. For instance, simplifying (5.45) with the assumption $\beta \ll 1$ we have

$$T_z(x, \beta) = \frac{1}{\beta} + \frac{\epsilon}{\beta^2} \cos x + O(\epsilon^2/\beta^3) . \tag{5.50}$$

If β is as small as ϵ then the terms in this perturbation series are no longer decreasing — this non-uniformity indicates failure of the method. The problem is that our approximate solution assumed that $\partial_z \sim k$ and this cannot be true if the vertical thickness of the slab, H, is significantly less than the exponential decay scale k^{-1}. See the next section....

5.5 Slow variations

The previous section discussed a problem in which the variations in the boundary geometry are small i.e. the height of the corrugations is much less the average thickness of the slab. But if we lower the lid so there is only a small gap above the hill tops then the corrugations have the same magnitude as the thickness of the slab. We can no longer assume small corrugations. Instead there is a different approximation: the thickness of the slab is changing on the horizontal length scale k^{-1}, which is much greater than both the slab thickness H and the height of the corrugations $h_{\max}$. In this sense the thickness of the slab is slowly varying.

So lets consider the small β case. We proceed by writing

$$\beta = \epsilon \alpha , \tag{5.51}$$

where $\alpha > 1$ and consider the limit $\epsilon \to 0$ with α fixed. The thickness-in-z of the slab is now of order ϵ everywhere so it makes sense to "rescale" the vertical coordinate

$$z \stackrel{\text{def}}{=} \frac{z}{\epsilon} , \qquad \Rightarrow \qquad \partial_z = \epsilon^{-1} \partial_\zeta . \tag{5.52}$$

The re-scaled problem is

$$\epsilon^2 T_{xx} + T_{\zeta\zeta} = 0 , \tag{5.53}$$

with boundary conditions

$$T(x, \cos x) = 0 , \qquad \text{and} \qquad T(x, \alpha) = 1 . \tag{5.54}$$

Now look for a regular perturbation solution

$$T = T_0(x, \zeta) + \epsilon^2 T_2(x, \zeta) + O(\epsilon^4) . \tag{5.55}$$

The leading-order problem is

$$T_{0\zeta\zeta} = 0 , \tag{5.56}$$

with boundary conditions

$$T_0(x, \cos x) = 0 , \qquad \text{and} \qquad T_0(x, \alpha) = 1 . \tag{5.57}$$

In (5.56) there is a "one-term dominant balance". One-term should be simpler than the two-term dominant balances that have figured so prominently in our earlier discussion. But the one-term dominant balance is subtle: the main point is that (5.56) has a nontrivial general soluton

$$T_0(x, \zeta) = A(x) + B(x)\zeta . \tag{5.58}$$

The boundary conditions (5.57) imply that

$$0 = A + B\cos x\,, \qquad \text{and} \qquad 1 = A + B\,. \tag{5.59}$$

Hence the leading-order solution is

$$T_0(x,\zeta) = \frac{\zeta - \cos x}{\alpha - \cos x}\,. \tag{5.60}$$

We can now calculate the heat flux through the slab

$$\langle T_{0\zeta}(x,1)\rangle = \frac{1}{\pi}\int_0^\pi \frac{\mathrm{d}x}{\alpha - \cos x}\,, \tag{5.61}$$

$$= \frac{1}{\sqrt{\alpha^2 - 1}}\,. \tag{5.62}$$

Example: Poiseuille flow through a tube with slowly changing radius.

5.6 A slowly rotating self-gravitating mass

A mass M of incompressible, self-gravitating and non-rotating fluid, with uniform density ρ, will be in hydrostatic equilibrium as a sphere of radius

$$\bar{r} \stackrel{\text{def}}{=} \left(\frac{3M}{4\pi\rho}\right)^{1/3}\,. \tag{5.63}$$

The gravitational potential is determined by solving

$$\nabla^2\phi = 4\pi G\rho\chi\,, \tag{5.64}$$

where

$$\chi = \begin{cases} 1\,, & \text{inside the mass;} \\ 0\,, & \text{outside the mass.} \end{cases} \tag{5.65}$$

Solving the Poisson equation (5.64), we obtain the well known spherically symmetric solution

$$\phi = -\frac{GM}{\bar{r}}\begin{cases} \frac{3\bar{r}^2 - r^2}{2\bar{r}^2}\,, & \text{inside the sphere;} \\ \frac{\bar{r}}{r}\,, & \text{outside the sphere}\,. \end{cases} \tag{5.66}$$

ϕ and ϕ_r are continuous at $r = a$; the second derivative is discontinuous at the surface of the sphere

$$\phi_{rr}(\bar{r}^+) - \phi_{rr}(\bar{r}^-) = -\frac{3g}{\bar{r}}\,, \tag{5.67}$$

where the gravitational acceleration at the surface is

$$g \stackrel{\text{def}}{=} \frac{GM}{\bar{r}^2}\,. \tag{5.68}$$

The liquid mass is in equilibrium because $\boldsymbol{\nabla}\phi$ is normal to the surface of the sphere. Equivalently, the surface of the sphere is an equipotential surface.

Solid body rotation, the geopotential and the equilibrium condition

Now suppose that the mass is in slow solid-body rotation about the z-axis with angular velocity $\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{z}}$. The mass is no longer perfectly spherical – the equator is slightly bulged out and the poles are flattened. If we knew the shape of the mass we could determined the potential ϕ by solving (5.64). Assuming that the rotation is weak, and that the deformation from a perfect sphere is small, we proceed to perturbatively solve (5.64) in combination with the momentum equation.

In solid body rotation

$$\boldsymbol{u} = \Omega\hat{\boldsymbol{z}} \times \boldsymbol{x} \tag{5.69}$$

where $\boldsymbol{x} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$ is position relative to the center of mass and $\boldsymbol{u}$ the fluid velocity. Thus the acceleration is

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \Omega\hat{\boldsymbol{z}} \times \boldsymbol{u}\,, \tag{5.70}$$

$$= \Omega^2\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{z}} \times \boldsymbol{x})\,, \tag{5.71}$$

$$= -\Omega^2\left(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}\right)\,, \tag{5.72}$$

$$= -\boldsymbol{\nabla}\tfrac{1}{2}\Omega^2(x^2+y^2)\,. \tag{5.73}$$

The momentum equation,

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}\left(\frac{p}{\rho} + \phi\right)\,, \tag{5.74}$$

is therefore equivalent to

$$\frac{p}{\rho} + \phi - \Omega^2\tfrac{1}{2}(x^2+y^2) = \text{constant}. \tag{5.75}$$

In (5.75) the gravitational potential ϕ is obtained from the solution of the Newtonian potential equation (5.64). The effective potential

$$\phi - \frac{\Omega^2}{2}(x^2+y^2) \tag{5.76}$$

in (5.75) is known as the *geopotential*.

The equilibrium condition is that at the unknown surface of the mass the pressure is zero and (5.75) becomes

$$@r = s(\theta): \qquad \phi - \tfrac{1}{2}\Omega^2 r^2\sin^2\theta = \text{constant}\,. \tag{5.77}$$

Above, $r = s(\theta)$ is the location of the surface and θ is the usual polar angle in spherical coordinates.

Later we need a expression for the unit normal $\boldsymbol{n}$ to the surface. The vector $\boldsymbol{n}$ is proportional to $\boldsymbol{\nabla}[r - s(\theta)]$, or

$$\boldsymbol{n} = \frac{s\hat{\boldsymbol{r}} - s_\theta\hat{\boldsymbol{\theta}}}{\sqrt{s^2+s_\theta^2}}\,. \tag{5.78}$$

Thus the normal derivative of the potential, evaluated on the surface of the mass, is

$$\underbrace{\boldsymbol{n}\cdot\boldsymbol{\nabla}\phi|_{r=s}}_{\stackrel{\mathrm{def}}{=}\phi_n} = \frac{s^2\phi_r - s_\theta\phi_\theta}{s\sqrt{s^2+s_\theta^2}}\,. \tag{5.79}$$

The normal derivative above is continuous at the surface.

Perturbative solution

Scale analysis identifies a single dimensionless parameter

$$\epsilon = \frac{\Omega^2}{\rho G} = \frac{4\pi}{3}\frac{\Omega^2\bar{r}}{g}\,. \tag{5.80}$$

"Slow rotation" means that $\epsilon \ll 1$. Instead of non-dimensionalizing the problem and expanding in ϵ, we live dangerously by expanding in terms of the dimensional parameter Ω^2:

$$s = \bar{r} + \Omega^2 s_1(\theta) + O\left(\Omega^4\right)\,, \tag{5.81}$$

and

$$\phi = \underbrace{-\frac{GM}{r}}_{\phi_0} + \Omega^2\phi_1 + O\left(\Omega^4\right)\,. \tag{5.82}$$

The boundary condition (5.77) is first transferred to $r = \bar{r}$ via the expansion

$$\phi(s(\theta), \theta) = \phi(\bar{r}) + \underbrace{(s - \bar{r})}_{=\Omega^2 s_1} \phi_r(\bar{r}, \theta) + O\left(\Omega^4\right) . \tag{5.83}$$

Thus at the convenient boundary

$$@r = \bar{r} : \qquad \phi + \Omega^2 \left(s_1\, \phi_r - \tfrac{1}{2}\bar{r}^2 \sin^2\theta\right) + O\left(\Omega^4\right) = \text{constant}\,. \tag{5.84}$$

The normal derivative in (5.79) is

$$@r = \bar{r} : \qquad \phi_n = \phi_r(\bar{r}, s) + \Omega^2 \left(s_1 \phi_{rr} - \frac{s_{1\theta}\phi_\theta}{\bar{r}^2}\right) + O\left(\Omega^4\right) . \tag{5.85}$$

The problem for ϕ_1 is therefore

$$\nabla^2 \phi_1 = 0\,, \tag{5.86}$$

with

$$@a : \qquad \phi_1 = -g s_1 + \tfrac{1}{2}\bar{r}^2 \sin^2\theta + \text{constant.} \tag{5.87}$$

In addition, continuity of the normal derivative of ϕ implies that

$$\phi_{1r}(\bar{r}^+) - \phi_{1r}(\bar{r}^-) = -s_1 \underbrace{\left(\phi_{0rr}(\bar{r}^+) - \phi_{0rr}(\bar{r}^-)\right)}_{=-3g/\bar{r}} \tag{5.88}$$

This ϕ_1-problem is forced by

$$\tfrac{1}{2}\sin^2\theta = \tfrac{1}{3} - \tfrac{1}{3}P_2(\theta) \tag{5.89}$$

where $P_2(\mu) = (3\mu^2 - 1)/2$ is the second Legendre polynomial. Thus we try to solve the ϕ_1-problem with

$$s_1 = \alpha \bar{r}\, P_2(\cos\theta)\,, \tag{5.90}$$

and

$$\phi_1 = \beta\, P_2(\cos\theta) \begin{cases} (r/\bar{r})^2\,, & \text{inside the mass;} \\ (r/\bar{r})^{-3}\,, & \text{outside the mass.} \end{cases} \tag{5.91}$$

The construction in (5.91) satisfies Laplace's equation (5.86) and also continuity of ϕ_1 at the surface.

Substituting (5.90) and (5.91) into (5.87) we obtain

$$\beta = -\bar{r} g \alpha - \tfrac{1}{3}\bar{r}^2\,. \tag{5.92}$$

Noting that $\phi_{1r}(\bar{r}^+) - \phi_{1r}(\bar{r}^-) = -5\beta P_2/\bar{r}$, the normal derivative condition in (5.88) gives

$$5\beta = -3g\bar{r}\alpha\,. \tag{5.93}$$

Solving for α and β one obtains

$$\alpha = -\frac{5}{6}\frac{\bar{r}}{g} \qquad \text{and} \qquad \beta = \frac{\bar{r}^2}{2}\,. \tag{5.94}$$

To summarize, the surface of the mass is $r = s(\theta)$ where

$$s = \bar{r} - \frac{5}{6}\frac{\bar{r}^2\Omega^2}{g} P_2(\cos\theta)\,. \tag{5.95}$$

The polar radius ($\theta = 0$) is

$$r_{\text{pol}} = \bar{r} - \frac{5}{6}\frac{\bar{r}^2\Omega^2}{g}\,, \tag{5.96}$$

and the equatorial radius ($\theta = \pi/2$) is

$$r_{\rm eq} = \bar{r} + \frac{5}{12}\frac{\bar{r}^2\Omega^2}{g}\,. \tag{5.97}$$

Thus the "flattening" is

$$\frac{r_{\rm eq} - r_{\rm pol}}{\bar{r}} = \frac{5}{4}\frac{\bar{r}\Omega^2}{g}\,. \tag{5.98}$$

The external potential is

$$\phi = -\frac{\bar{r}^2 g}{r} + P_2(\cos\theta)\frac{\Omega^2\bar{r}^5}{2r^3}\,. \tag{5.99}$$

5.7 Problems

Problem 5.1. Consider the partial differential equation

$$\kappa \left(C_{xx} + C_{zz} \right) - \mu C = 0 \tag{5.100}$$

in the region above $z = h(x)$, with $h(x) = h_{\max} \cos kx$. The boundary conditions are $C(x, h_{\max} \cos kx) = C_*$ and $C(x, z) \to 0$ as $z \to \infty$. *(i)* Describe a physical situation governed by this boundary value problem. *(ii)* Solve the problem with $h_{\max} = 0$. *(iii)* Based on your exact solution, non-dimensionalize the problem with non-zero $h_{\max}$ and determine the non-dimensional control parameters. *(iv)* Use perturbation theory to find the first effects of small non-zero $h_{\max}$ on the "inventory"

$$A \stackrel{\text{def}}{=} \frac{k}{2\pi} \int_{h(x)}^{\infty} \int_0^{2\pi/k} C(x,z) \, \mathrm{d}x \, \mathrm{d}z \, . \tag{5.101}$$

(I think you'll have to go to second order in $h_{\max}$.)

Problem 5.2. Consider the diffusion problem

$$\psi_{xx} + \psi_{yy} = -\mathrm{e}^{-y} \tag{5.102}$$

in the "corrugated half-plane" defined by

$$-\infty < x < \infty \, , \qquad \text{and} \qquad \epsilon \cos kx < y \, . \tag{5.103}$$

At the wavy boundary:

$$\psi(x, \epsilon \cos kx) = 0 \, . \tag{5.104}$$

The condition at infinity is

$$\lim_{y \to \infty} \psi(x, y) = A(\epsilon, k) \, , \tag{5.105}$$

where $A(\epsilon, k)$ is an unknown function. *(i)* Solve the problem with $\epsilon = 0$ and show that $A(0, k) = 1$. *(ii)* Use a perturbation expansion ($\epsilon \ll 1$) to determine the first non-zero correction to $A = 1$.

Problem 5.3. Consider 2D potential flow (no vorticity) around an cylindrical object whose cross section in the (x, y)-plane is a slightly distorted circle

$$r = a \left(1 - \epsilon \sin^2 \theta \right) \, . \tag{5.106}$$

Using a stream function $\psi(x, y)$, with $u = -\psi_y$ and $v = \psi_x$, the mathematical problem is

$$\nabla^2 \psi = 0 \, , \tag{5.107}$$

where

$$\nabla^2 = \partial_x^2 + \partial_y^2 = \partial_r^2 + r^{-1} \partial_r + r^{-2} \partial_\theta^2 \tag{5.108}$$

is the Laplacian operator. Boundary conditions are $\psi = 0$ on the surface of the body and $\psi \to -Uy$ at great distances from the body. *(i)* Review the standard solution for potential flow around a circular cylinder i.e. the case $\epsilon = 0$. This solution is in all fluid mechanics textbooks. Above I'm using cylindrical coordinates r and θ that feature prominently in those textbooks. *(ii)* Non-dimensionalize the problem and identify all non-dimensional control parameters. *(iii)* Use the boundary perturbation method to find the first effects of small distortion, $\epsilon \ll 1$. Visualize the solution with MATLAB or some other computational tool.

Problem 5.4. Consider Laplace's equation,

$$\phi_{xx} + \phi_{yy} = 0 \, , \tag{5.109}$$

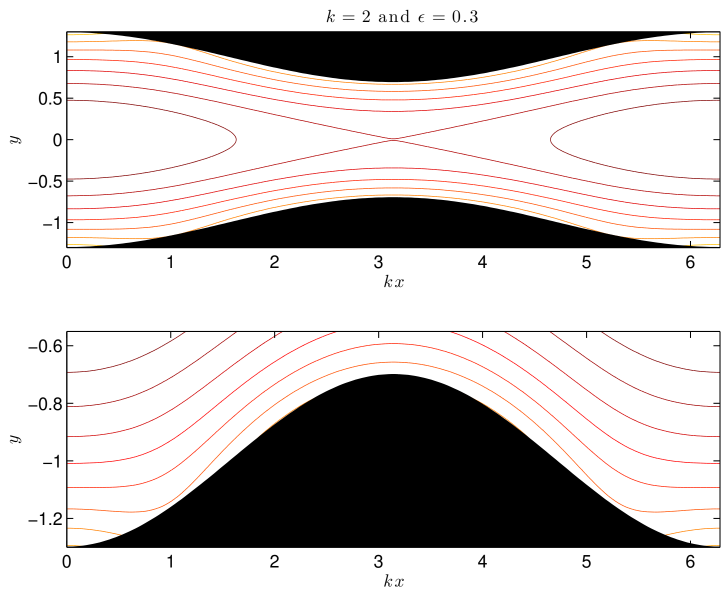

Figure 5.4: My solution to problem 5.5 with $k = 2$ and $\epsilon = 0.3$. The lower panel is an expanded view of the lower boundary. Some errors in the boundary condition $T = 0$ are evident near $kx = 0$ and 2π. At this largish value of ϵ another term wouldn't hurt.

in a domain which is a periodic-in-x channel with walls at $y = \pm(1 + \epsilon \cos kx)$. The boundary condition on the walls is

$$(\boldsymbol{\nabla}\phi + \hat{\boldsymbol{\imath}}) \cdot \hat{\boldsymbol{n}} = 0\,, \tag{5.110}$$

where $\hat{\boldsymbol{n}}$ is the outward normal and $\hat{\boldsymbol{\imath}}$ is the unit vector in the x-direction. Obtain two terms in the expansion of

$$J(\epsilon) \stackrel{\text{def}}{=} \iint \phi_x \,\mathrm{d}x\mathrm{d}y\,. \tag{5.111}$$

Problem 5.5. Consider a uniformly heated 2D metal ribbon of width $2h(x)$. The ribbon is cooled by fixing $T(x, h(x)) = 0$ at the two boundaries. Thus the steady state temperature is determined by

$$T_{xx} + T_{yy} = -1\,, \qquad \text{for} - h(x) < y < +h(x)\,, \tag{5.112}$$

where $h \stackrel{\text{def}}{=} 1 + \epsilon \cos kx$. The boundary conditions are

$$T(x, \pm h) = 0\,. \tag{5.113}$$

We particularly desire the heat content of the ribbon

$$J(\epsilon, k) \stackrel{\text{def}}{=} \frac{k}{2\pi} \int_0^{2\pi/k}\!\!\int_{-h}^{h} T \,\mathrm{d}x\mathrm{d}y\,. \tag{5.114}$$

Use the boundary perturbation method to show that

$$J = \frac{2}{3} + \epsilon^2 \left(1 - k \tanh k\right) + O\left(\epsilon^4\right)\,. \tag{5.115}$$

Lecture 6

Boundary Layers

6.1 Stommel's dirt pile

Consider a pile of dirt formed by a rain of sediment falling onto a conveyor belt. The belt stretches between $x = 0$ and $x = \ell$ and moves to the left with speed $-c$: see the figure. If $h(x,t)$ denotes the height of a sandpile, then a very simple model is

$$h_t - c h_x = s + \kappa h_{xx}\,, \tag{6.1}$$

with boundary conditions

$$h(0,t) = 0\,, \qquad \text{and} \qquad h(\ell,t) = 0\,. \tag{6.2}$$

The term $s(x)$ on the right of (6.1) is the rate (meters per second) at which sand is falling from the sky onto the belt.

We can make a sanity check by integrating (6.1) from $x = 0$ to $x = \ell$:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\ell h(x,t)\,\mathrm{d}x}_{\text{rate of accumulation}} = \underbrace{\int_0^\ell s(x,t)\,\mathrm{d}x}_{\text{sedimentation from above}} + \underbrace{\kappa h_x(\ell,t) - \kappa h_x(0,t)}_{\text{loss of dirt by falling over the edges}}\,. \tag{6.3}$$

The advective term, $c h_x$, does not contribute to the budget above — advection is moving dirt but because $h = 0$ at the boundaries advection is not directly contributing to the fall of dirt over the edges.

Exercise: Find the steady solution of (6.1) and (6.2) if the conveyor belt is switched off i.e. $c = 0$.

The steady solution with a uniform source

If the sedimentation rate, $s(x,t)$, is a constant then we can easily obtain the steady state ($t \to \infty$) solution:

$$h(x,\infty) = \frac{s\ell}{c}\frac{1-\mathrm{e}^{-cx/\kappa}}{1-\mathrm{e}^{-c\ell/\kappa}} - \frac{sx}{c}\,. \tag{6.4}$$

If the diffusion is very weak, meaning that

$$\epsilon \stackrel{\text{def}}{=} \frac{\kappa}{c\ell} \ll 1\,, \tag{6.5}$$

then there is a region of rapid variation, the *boundary layer*, at $x = 0$. This is where all the sand accumulated on the conveyor belt is pushed over the edge. Obviously if we reverse the direction of the belt, then the boundary layer will move to $x = \ell$. We assume that $c > 0$ so that $\epsilon > 0$.

If we introduce non-dimensional variables

$$\bar{x} \stackrel{\text{def}}{=} \frac{x}{\ell}\,, \qquad \text{and} \qquad \bar{h} = \frac{ch}{s\ell}\,, \tag{6.6}$$

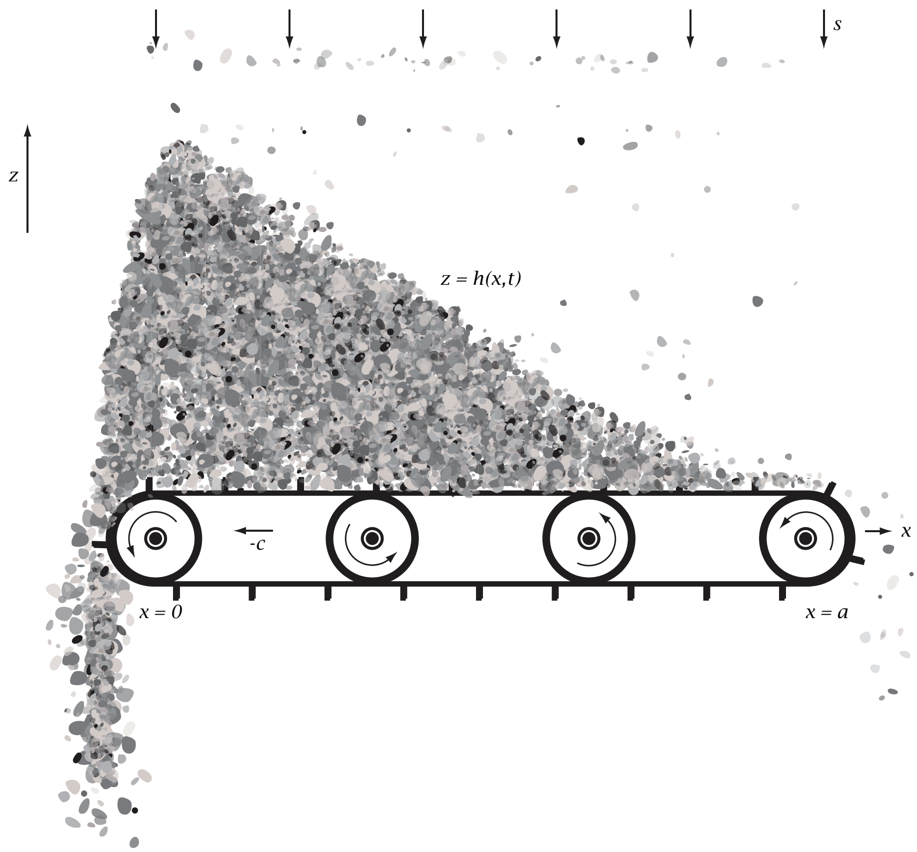

Figure 6.1: Stommel's boundary-layer problem.

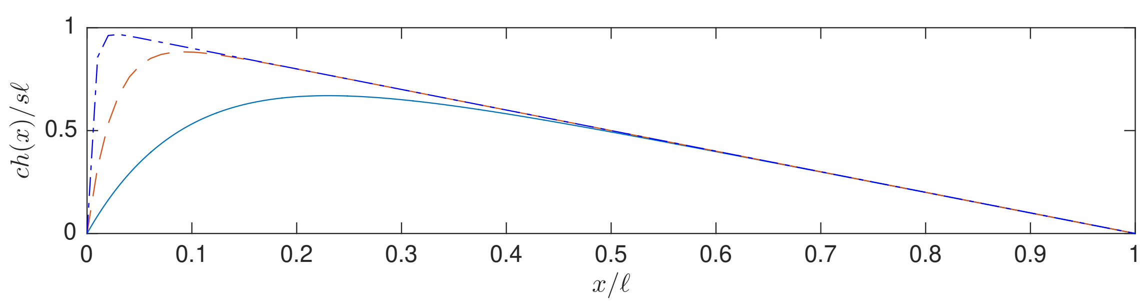

Figure 6.2: The solution in (6.7). The solid curve is $\epsilon = 0.1$, the dashed curve is $\epsilon = 0.025$ and the dash-dot curve is $\epsilon = 0.005$.

then the solution in (6.4) is

$$h(x,\epsilon) = \frac{1-\mathrm{e}^{-x/\epsilon}}{1-\mathrm{e}^{-1/\epsilon}} - x\,. \tag{6.7}$$

This solution is shown in figure 6.2 with various values of ϵ. We can consider two different limiting processes in (6.7):

1. *The outer limit:* $\epsilon \to 0$, with x fixed. Under this limit, the exact solution in (6.7) is $h \to 1 - x$. The outer limit produces a good approximation to the exact $h(x,\epsilon)$, except close to $x = 0$ where the boundary condition is not satisfied.

2. *The inner limit:* $\epsilon \to 0$ with $X \stackrel{\text{def}}{=} x/\epsilon$ fixed. Under this limit the exact solution in (6.7) is $h \to 1 - \mathrm{e}^{-X}$. The inner limit produces a good approximation to the solution within the *boundary layer*. This is a small region in which x is order ϵ. It is vital to understand that the term ϵh_{xx} is leading order within the boundary layer, and enables the solution to satisfy the boundary condition at $x = 0$.

Thus the function in (6.7) has two different asymptotic expansions. Each expansion is limited by non-uniformity as $\epsilon \to 0$.

6.2 Leading-order solution of the dirt-pile model

We want to take the inner and outer limits directly in the differential equation, *before* we have a solution. Understanding how to do this in the Stommel problem is one goal this lecture.

To make the problem a little more interesting, suppose that the sedimentation rate is some function of x:

$$s = s_{\max}\bar{s}\left(\frac{x}{\ell}\right). \tag{6.8}$$

We use $s_{\max}$ to define the non-dimensional $\bar{h}$ back in (6.6). Dropping the bars, the non-dimensional problem is

$$\epsilon h_{xx} + h_x = -s\,, \tag{6.9}$$

with boundary conditions

$$h(0) = h(1) = 0\,. \tag{6.10}$$

We're going to use boundary layer theory to obtain a quick and dirty leading-order solution of this problem. We'll return later to a more systematic discussion.

The outer expansion

Start the attack on (6.9) with a regular perturbation expansion

$$h(x,\epsilon) = h_0(x) + \epsilon h_1(x) + \epsilon^2 h_2(x) + \cdots \tag{6.11}$$

We're assuming that as $\epsilon \to 0$ with fixed x — the **outer limit** — that the solution has the structure in (6.19). Note that in the outer limit the h_n's in (6.11) are independent of ϵ.

Exercise: Consider the special case $s = 1$, with the exact solution in (6.7). Does the outer limit of that exact solution agree with the assumption in (6.11)?

The leading order is

$$h_{0x} = -s\,, \tag{6.12}$$

and we can solve this problem as

$$\underbrace{h_0(x) = \int_x^1 s(x')\,\mathrm{d}x'\,,}_{\text{correct}} \qquad \text{or perhaps as} \qquad \underbrace{h_0(x) = -\int_0^x s(x')\,\mathrm{d}x'}_{\text{incorrect}}\,. \tag{6.13}$$

Looking at the exact solution in (6.7) we know that the correct choice satisfies the BC at $x = 1$. If you think about conveyor belts and falling dirt then this choice of boundary condition is also "physically obvious". Whether it is obvious or not, we proceed satisfying the BC at $x = 1$:

$$h_0(x) = \int_x^1 s(x')\,\mathrm{d}x' \qquad \text{(the correct outer solution).} \tag{6.14}$$

We return later to show that the alternative ends in tears.

The inner expansion, and a quick-and-dirty matching argument

The inner region is the region near $x = 0$ in which (6.14) doesn't work. We define

$$X \stackrel{\text{def}}{=} \frac{x}{\delta}\,, \qquad \text{so that} \qquad \frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{\delta}\frac{\mathrm{d}}{\mathrm{d}X}\,. \tag{6.15}$$

δ is the boundary layer thickness — we're pretending that δ is unknown. Using the inner variable X, the problem (6.9) becomes is

$$\underbrace{\epsilon\delta^{-2}h_{XX} + \delta^{-1}h_X}_{\text{two term balance}} = -s(\delta X)\,. \tag{6.16}$$

We get a nice two-term balance if

$$\delta = \epsilon\,. \tag{6.17}$$

With this definition of δ we have the rescaled problem

$$h_{XX} + h_X = -\epsilon s(\epsilon X)\,. \tag{6.18}$$

Now attack (6.18) with a a regular perturbation expansion

$$h(x,\epsilon) = H_0(X) + \epsilon H_1(X) + \epsilon^2 H_2(X) + \cdots \tag{6.19}$$

In (6.19) we're assuming that the H_n's are independent of ϵ.

At leading order

$$H_{0XX} + H_{0X} = 0\,, \qquad \text{with solution} \qquad H_0 = A_0\left(1 - \mathrm{e}^{-X}\right)\,. \tag{6.20}$$

We've satisfied the BC at $X = 0$. But we still have an unknown constant A_0.

To determine A_0 we insist that "the inner limit of the outer solution is equal to the outer limit of the inner limit solution". This means that there is a region of overlap in which

$$\underbrace{A_0\left(1 - \mathrm{e}^{-X}\right)}_{\text{Inner solution}} \approx \underbrace{\int_x^1 s(x')\,\mathrm{d}x'}_{\text{Outer solution}}\,. \tag{6.21}$$

For instance, if $x = O(\epsilon^{1/2}) \ll 1$ then $X = O(\epsilon^{-1/2}) \gg 1$, and (6.21) tells us that

$$A_0 = \int_0^1 s(x')\,\mathrm{d}x'\,. \tag{6.22}$$

Construction of a uniformly valid solution

With A_0 determined by (6.22) we have completed the leading-order solution. We can combine our two asymptotic expansions into a single *uniformly valid* solutions using the recipe

$$\text{uniformly valid} = \text{outer} + \text{inner} - \text{match}\,, \tag{6.23}$$

$$= \int_x^1 s(x')\,\mathrm{d}x' + \int_0^1 s(x')\,\mathrm{d}x'\,\left(1 - \mathrm{e}^{-X}\right) - \int_0^1 s(x')\,\mathrm{d}x'\,, \tag{6.24}$$

$$= \int_x^1 s(x')\,\mathrm{d}x' - \int_0^1 s(x')\,\mathrm{d}x'\,\mathrm{e}^{-x/\epsilon}\,. \tag{6.25}$$

This is also known as the *composite expansion.*

Why can't we have a boundary layer at $x = 1$?

Now we return to (6.13) and discuss what happens if we make the incorrect choice

$$h_0(x) \overset{?}{=} -\int_0^x s(x')\,\mathrm{d}x' \,. \tag{6.26}$$

This outer solution satisfies the BC at $x = 0$.

On physical grounds we are deeply suspicious of (6.26): the height of the dirt pile is negative. Unless the dirt is adhering to the bottom of the belt this makes no sense at all. We proceed by abandoning physical arguments and discussing the failure of (6.26) as a mathematical issue.

We try to put a boundary layer at $x = 1$. Again we introduce a boundary-layer coordinate:

$$X \overset{\text{def}}{=} \frac{x-1}{\delta}\,, \qquad \text{so that } \frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{\delta}\frac{\mathrm{d}}{\mathrm{d}X}\,. \tag{6.27}$$

The dominant balance argument convinces us that $\delta = \epsilon$, and using (6.19) we find exactly the same leading-order solution as before:

$$H_0 = A_0\left(1 - \mathrm{e}^{-X}\right)\,, \qquad \text{except that now} \qquad X = \frac{x-1}{\delta}\,. \tag{6.28}$$

$H_0(X)$ above satisfies the BC at $X = 0$, which is the same as $x = 1$. But now when we attempt to match the outer solution in (6.26) it all goes horribly wrong: we take the limit $X \to -\infty$ and the exponential explodes. It is impossible to match the outer solution (6.26) with the inner solution in (6.28).

6.3 Stommel's problem at infinite order

To expose the complete structure of higher-order boundary-layer problems let us discuss the form of the high-order terms in Stommel's problem. Recall our model for the steady state sandpile is

$$\epsilon h_{xx} + h_x = -s\,. \tag{6.29}$$

We assume that the source $s(x)$ has the Taylor series expansion around $x = 0$:

$$s(x) = s_0 + s_0' x + \tfrac{1}{2}x^2 s_0'' + \cdots\,, \tag{6.30}$$

and around $x = 1$:

$$s(x) = s_1 + (x-1)s_1' + \tfrac{1}{2}(x-1)^2 s_1'' + \cdots \tag{6.31}$$

The outer solution

The leading-order outer problem is

$$h_{0x} = -s\,, \qquad \Rightarrow \qquad h_0 = \int_x^1 s(x')\,\mathrm{d}x'\,, \tag{6.32}$$

and the following orders are

$$h_{1x} = -h_{0xx} = +s_x\,, \qquad \Rightarrow \qquad h_1 = s(x) - s_1\,, \tag{6.33}$$

$$h_{2x} = -h_{1xx} = -s_{xx}\,, \qquad \Rightarrow \qquad h_2 = s_1' - s_x(x)\,, \tag{6.34}$$

$$h_{3x} = -h_{2xx} = +s_{xxx}\,, \qquad \Rightarrow \qquad h_3 = s_{xx}(x) - s_1''\,. \tag{6.35}$$

At every order $h_n(1) = 0$. It is clear how this series continues to higher order. We can assemble the first three terms of the outer solution as

$$h = \int_0^1 s(x')\,\mathrm{d}x' - \int_0^x s(x')\,\mathrm{d}x' + \epsilon\left[s(x) - s_1\right] + \epsilon^2\left[s_1' - s_x(x)\right] + O(\epsilon^3)\,. \tag{6.36}$$

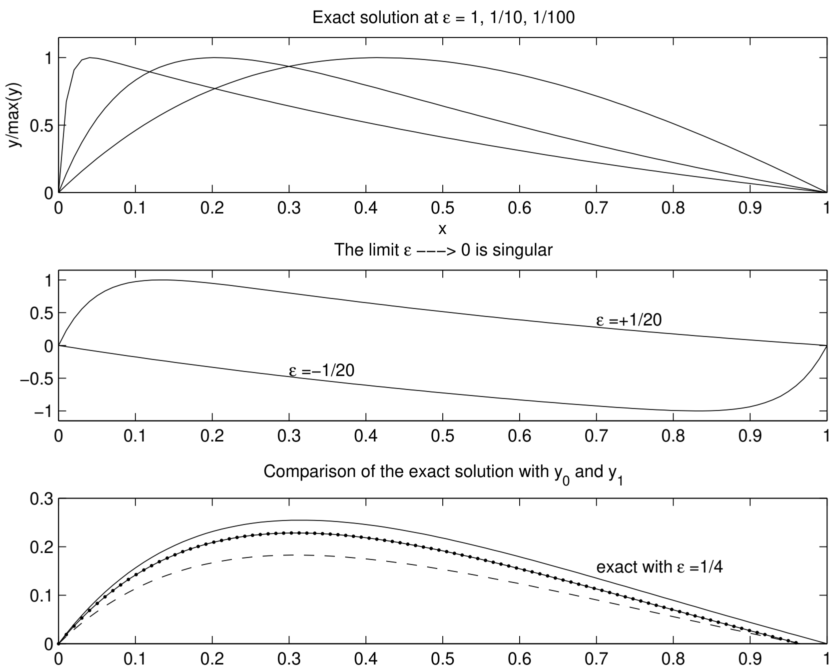

Figure 6.3: Solution with $a = -1$.

The boundary-layer solution

In the boundary layer, we *must* expand the source in a Taylor series

$$s(\epsilon X) = s_0 + \epsilon X s_0' + \tfrac{1}{2}\epsilon^2 X^2 s_0'' + \cdots \tag{6.37}$$

If we don't expand the source then there is no way to collect powers of ϵ and maintain our assumption that the H_n's in

$$h(x,\epsilon) = H_0(X) + \epsilon H_1(X) + \epsilon^2 H_2(X) + \cdots \tag{6.38}$$

are independent of ϵ. The RPS above leads to

$$H_{0XX} + H_{0X} = 0\,, \qquad \Rightarrow \qquad H_0 = A_0\left(1 - \mathrm{e}^{-X}\right)\,, \tag{6.39}$$

$$H_{1XX} + H_{1X} = -s_0\,, \qquad \Rightarrow \qquad H_1 = A_1\left(1 - \mathrm{e}^{-X}\right) - s_0\left(X - 1 + \mathrm{e}^{-X}\right)\,, \tag{6.40}$$

$$H_{2XX} + H_{2X} = -s_0' X\,, \qquad \Rightarrow \qquad H_2 = A_2\left(1 - \mathrm{e}^{-X}\right) - s_0'\left(\tfrac{1}{2}X^2 - X + 1 - \mathrm{e}^{-X}\right)\,. \tag{6.41}$$

At every order we've satisfied the boundary condition $H_n(0) = 0$. Matching determines the constants A_n.

Matching

In the matching region $X \gg 1$ and we simplify the boundary layer solution by neglecting all the exponentially small terms involving e^{-X}. This gives

$$h \sim \underbrace{A_0}_{H_0} + \underbrace{\epsilon A_1 - \epsilon s_0\,(X-1)}_{\epsilon H_1} + \underbrace{\epsilon^2 A_2 - \epsilon^2 s_0'\left(\tfrac{1}{2}X^2 - X + 1\right)}_{\epsilon^2 H_2} + O(\epsilon^3)\,. \tag{6.42}$$

We rewrite the outer solution in (6.33) through (6.35) in terms of $X = x/\epsilon$ and take the inner limit, keeping terms of order ϵ^2:

$$h \sim \underbrace{\int_0^1 s(x')\,\mathrm{d}x' - \epsilon s_0 X - \tfrac{1}{2}\epsilon^2 s_0' X^2}_{h_0} + \epsilon\underbrace{\left[s_0 + \epsilon X s_0' - s_1\right]}_{h_2} + \epsilon^2\underbrace{\left[s_1' - s_0'\right]}_{h_2} + O(\epsilon^3)\,. \tag{6.43}$$

The inner limit of $h_0(x)$ produces terms of all order in ϵ — above we've explicitly written only terms up to $O(\epsilon^2)$.

A shot-gun marriage between these different expansions (6.42) and (6.43) of the *same function* $h(x,\epsilon)$ implies that

$$A_0 = \int_0^1 s(x')\,\mathrm{d}x' , \qquad A_1 = s_0 - s_1 , \qquad A_2 = s_1' - s_0' . \tag{6.44}$$

All the other terms in (6.42) and (6.43) match. Terms from h_0 match terms from H_1 and H_2, and from H_3 if we continue to higher order. It is interesting that the boundary layer constants A_1 and A_2 involve properties s_1 and s_1' of the source at $x = 1$.

The special case $s(x) = 1$

This special case is very simple: the infinite-order uniform solution is

$$H = \underbrace{1 - \mathrm{e}^{-X}}_{H_0} - \epsilon \underbrace{X}_{H_1} . \tag{6.45}$$

And the infinite-order outer solution is simply

$$h = \underbrace{1 - x}_{h_0} . \tag{6.46}$$

All the higher-order terms are zero. With the recipe

$$\text{uniform} = \text{outer} + \text{inner} - \text{match} , \tag{6.47}$$

we assemble an infinite-order uniform approximation:

$$h_{\text{uni}}(x) = 1 - x - \mathrm{e}^{-x/\epsilon} . \tag{6.48}$$

The exact solution is

$$h(x) = \frac{1 - \mathrm{e}^{-x/\epsilon}}{1 - \mathrm{e}^{-1/\epsilon}} - x ; \tag{6.49}$$

this differs from the infinite-order approximation by the exponentially small $\mathrm{e}^{-1/\epsilon}$.

Example: To save chalk, in the lecture we might use the particular source function $s = \mathrm{e}^{a(x-1)}$ and assign the general $s(x)$ as reading.

The problem is

$$\epsilon h_{xx} + h_x = -\mathrm{e}^{a(x-1)} , \qquad \text{with BCs} \qquad h(0) = h(1) = 0 . \tag{6.50}$$

The interior solution, to infinite order, is

$$h = \underbrace{\left(1 - \epsilon a + \epsilon^2 a^2 - \epsilon^3 a^3 + \cdots\right)}_{\frac{1}{1+\epsilon a}} \frac{1}{a} \left[1 - \mathrm{e}^{a(x-1)}\right] . \tag{6.51}$$

No matter how many terms we calculate, we will never satisfy the $x = 0$ boundary condition.

After re-scaling, the boundary layer problem is

$$H_{XX} + H_X = -\epsilon \mathrm{e}^{-a} \underbrace{\mathrm{e}^{a\epsilon X}}_{1 + a\epsilon X + \cdots} \tag{6.52}$$

The expansion

$$H = H_0(X) + \epsilon H_1(X) + \cdots \tag{6.53}$$

then leads to

$$H_{0XX} + H_{0X} = 0 , \tag{6.54}$$

$$H_{1XX} + H_{1X} = -\mathrm{e}^{-a} , \tag{6.55}$$

$$H_{2XX} + H_{2X} = -a\mathrm{e}^{-a} X . \tag{6.56}$$

Before matching the solution of the boundary-layer problem is

$$H = A_0\big(1-\mathrm{e}^{-X}\big) - \epsilon \mathrm{e}^{-a}X + \epsilon A_1\big(1-\mathrm{e}^{-X}\big) + \epsilon^2\mathrm{e}^{-a}a\big(X - \tfrac{1}{2}X^2\big) + \epsilon^2 A_2\big(1-\mathrm{e}^{-X}\big) + O(\epsilon^3)\,. \tag{6.57}$$

Above, we have satisfied the boundary condition at $x=0$.

The next step is to determine A_n by matching to the outer solution. In the matching region the interior solution (6.51) expands to

$$h = \underbrace{\frac{1-\mathrm{e}^{-a}}{a} - \underline{\mathrm{e}^{-a}x} - \underline{\tfrac{1}{2}\mathrm{e}^{-a}ax^2}}_{h_0} - \epsilon(1-\mathrm{e}^{-a}) + \underline{\epsilon a\mathrm{e}^{-a}x} + \epsilon^2 \underbrace{a(1-\mathrm{e}^{-a})}_{h_2} + O()\,. \tag{6.58}$$

The boundary-layer solution (6.57), written in terms of the outer variable x, is

$$h = A_0 - \underline{\mathrm{e}^{-a}x} + \epsilon A_1 + \underline{\epsilon\mathrm{e}^{-a}ax} - \underline{\tfrac{1}{2}\mathrm{e}^{-a}ax^2} + \epsilon^2 A_2 + O()\,. \tag{6.59}$$

The underlined terms already match, and matching the others gives

$$A_0 = \frac{1-\mathrm{e}^{-a}}{a}\,, \qquad A_1 = \,, \qquad A_2 = \tag{6.60}$$

Unfortunately it is not clear that this example is simpler than the general case.....

6.4 Variable speed

Suppose the conveyor belt is a stretchy membrane which moves with non-uniform speed $-c(x)$. With non-constant c, the dirt conservation equation in (6.1) generalizes to

$$h_t - (ch)_x = s + \kappa h_{xx}\,, \tag{6.61}$$

with boundary conditions unchanged: $h(0,t)=0$ and $h(\ell,t)=0$.

Exercise: Make sure you understand why it is $(ch)_x$, rather than ch_x, in (6.61). Nondimensionalize (6.61) so that the steady state problem is

$$\epsilon h_{xx} - (ch)_x = -s\,, \qquad h(0) = h(1) = 0\,, \tag{6.62}$$

with $\max c(x) = 1$ and $\max s(x)$=1.

Example: Slow-down and pile-up

Suppose that the belt slows near $x=0$. Specifically, let's assume that the belt speed is

$$c = \sqrt{x}\,. \tag{6.63}$$

The speed is zero at $x=0$, so we expect that dirt will start to pile up. If the source is uniform then the steady-state problem is

$$\epsilon h_{xx} + \big(\sqrt{x}h\big)_x = -1\,, \qquad \text{with BCs } h(0) = h(1) = 0. \tag{6.64}$$

Exercise: Show that a particle starting at $x=1$ and moving with $\dot{x} = -x^\beta$, with $\beta < 1$, reaches $x=0$ in a finite time. What happens if $\beta \geq 1$?

The first two terms in the interior solution are

$$h(x,\epsilon) = \left(x^{-1/2} - x^{1/2}\right) + \epsilon\left(\frac{1}{2}x^{-2} + \frac{1}{2}x^{-1} - x^{-1/2}\right) + O(\epsilon^2)\,. \tag{6.65}$$

We've satisfied the BC at $x=1$ and the pile-up at $x=0$ is evident via the divergence of the outer solution as $x \to 0$. The divergence is stronger at higher orders, and the RPS above is disordered as $x \to 0$.

Turning to the boundary layer at $x = 0$, we introduce

$$X \stackrel{\text{def}}{=} \frac{x}{\delta} \tag{6.66}$$

so that

$$\frac{\epsilon}{\delta^2} h_{XX} + \frac{1}{\delta^{1/2}} \left(\sqrt{X} h\right)_X = -1 \,. \tag{6.67}$$

A dominant balance between the first two terms is achieved with $\epsilon = \delta^{3/2}$, or

$$\delta = \epsilon^{2/3} \,. \tag{6.68}$$

With this definition of δ, and

$$h = H(X, \epsilon) \,, \tag{6.69}$$

the boundary layer equation is

$$H_{XX} + \left(\sqrt{X} H\right)_X = -\epsilon^{1/3} \,. \tag{6.70}$$

We attack with an RPS: $h = H_0(X) + \epsilon^{1/3} H_1(x) + \cdots$

At leading order

$$H_{0XX} + \left(\sqrt{X} H_0\right)_X = 0 \,, \tag{6.71}$$

with first integral

$$H_{0X} + \sqrt{X} H_0 = A_0 \,. \tag{6.72}$$

Solving this first-order equation with an integrating factor we obtain

$$H_0(X) = A_0 \mathrm{e}^{-2X^{3/2}/3} \int_0^X \mathrm{e}^{2t^{3/2}/3} \, \mathrm{d}t \,. \tag{6.73}$$

We've satisfied the boundary condition at $x = 0$, and we must determine the remaining constant of integration A_0 by matching to the interior solution.

To match the interior, we need the asymptotic expansion of (6.73) as $X \to \infty$: this can be obtained by following our earlier discussion of Dawson's integral:

$$H_0(X) \sim \frac{A_0}{\sqrt{X}} \,, \qquad \text{as } X \to \infty, \tag{6.74}$$

$$= \frac{\epsilon^{1/3} A_0}{x^{1/2}} \,. \tag{6.75}$$

An alternative, and more efficient, derivation of (6.75) is to write (6.72) as

$$H_0 = \frac{A_0}{X^{1/2}} - \frac{H_{0X}}{X^{1/2}} \tag{6.76}$$

and proceed iteratively starting with $H_0 \sim A_0 X^{-1/2}$ as $X \to \infty$.

On the other hand the inner expansion of the outer solution in (6.65) is

$$h = \frac{1}{x^{1/2}} + O\left(x^{1/2}, \epsilon x^{-2}\right) \,. \tag{6.77}$$

We *almost* have a match — it seems we should take $A_0 = \epsilon^{-1/3}$ in (6.75) so that both functions are equal to $x^{-1/2}$ in the matching region. But remember that we assumed that $H_0(x)$ is independent of ϵ, so A_0 *cannot* depend on ϵ. Our expansion has failed.

Exercise: How would you gear so that the term ϵx^{-2} in (6.75) is asymptotically negligible relative to $x^{-1/2}$ in the matching region?

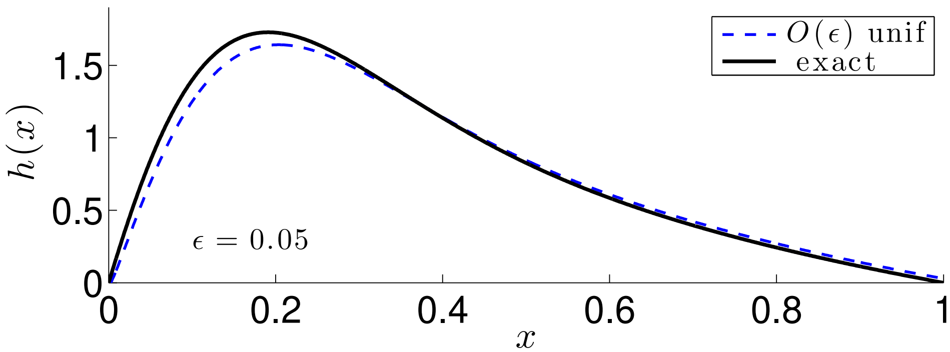

Figure 6.4: Comparison of (6.83) with the exact solution of (6.64).

Fortunately there is a simple cure: the correct definition of the boundary layer solution — which replaces (6.69) — is

$$h = \epsilon^{-1/3}\mathcal{H}(X, \epsilon)\,. \tag{6.78}$$

In retrospect perhaps the rescaling in (6.78) is obvious — the interior RPS in (6.65) is becoming disordered as $x \to 0$. The problem is acute once the second term in the expansion is comparable to the first term, which happens once

$$x^{-1/2} \sim \epsilon x^{-2} \qquad \text{or} \qquad x \sim \epsilon^{2/3} = \delta\,. \tag{6.79}$$

This the boundary layer scale, and as we enter this region the interior solution is of order $x^{-1/2} \sim \epsilon^{-1/3}$ — this is why the rescaling in (6.78) is required. If we'd been smart we would have made this argument immediately after (6.65) and avoided the mis-steps in (6.69) and (6.70).

Using the rescaled variable in (6.75), the boundary layer equation that replaces (6.70) is

$$\mathcal{H}_{XX} + \left(\sqrt{X}\mathcal{H}\right)_X = -\epsilon^{2/3}\,. \tag{6.80}$$

Now we can try the RPS

$$\mathcal{H}(X, \epsilon) = \mathcal{H}_0(X) + \epsilon^{2/3}\mathcal{H}_1(X) + \cdots \tag{6.81}$$

We quickly find the leading order solution

$$\mathcal{H}_0 = \mathrm{e}^{-2X^{3/2}/3}\int_0^X \mathrm{e}^{2t^{3/2}/3}\,\mathrm{d}t\,. \tag{6.82}$$

This satisfies the $x = 0$ boundary condition and also matches the $x^{-1/2}$ from the interior.

We can now construct a leading-order uniformly valid solution as

$$h_{\mathrm{uni}}(x, \epsilon) = \epsilon^{-1/3}\mathrm{e}^{-2X^{3/2}/3}\int_0^X \mathrm{e}^{2t^{3/2}/3}\,\mathrm{d}t - x^{1/2}\,. \tag{6.83}$$

Figure 6.4 compares the uniformly valid approximation (6.83) with an exact solution of (6.64).

Exercise: evaluate the integral $\int_0^1 h(x, \epsilon)\,\mathrm{d}x$ to leading order as $\epsilon \to 0$.

Example: higher-order corrections

To illustrate how to bash out higher order corrections let's calculate the first two terms in the BL solution of the BVP

$$\epsilon h_{xx} + [\mathrm{e}^x h]_x = -2\mathrm{e}^{2x}\,, \tag{6.84}$$

with BCs

$$h(0) = h(1) = 0\,. \tag{6.85}$$

We suspect there is a BL at $x = 0$. So we first develop the interior solution

$$h(x, \epsilon) = h_0(x) + \epsilon h_1(x) + \epsilon h_2(x) + \cdots \tag{6.86}$$

by satisfying the boundary condition at $x = 1$ at every order.

The leading-order term is

$$[\mathrm{e}^x h_0]_x = -2\mathrm{e}^{2x}, \qquad \Rightarrow \qquad h_0 = \mathrm{e}^{2-x} - \mathrm{e}^x. \tag{6.87}$$

The next two orders are

$$[\mathrm{e}^x h_1]_x = -h_{0xx}, \qquad \Rightarrow \qquad h_1 = 1 - 2\mathrm{e}^{1-x} + \mathrm{e}^{2-2x}, \tag{6.88}$$

$$[\mathrm{e}^x h_2]_x = -h_{1xx}, \qquad \Rightarrow \qquad h_2 = 2\left(\mathrm{e}^{2-3x} - \mathrm{e}^{1-2x}\right). \tag{6.89}$$

Later, to perform the match, we will need the inner limit of this outer solution. So in preparation for that, as $x \to 0$,

$$\begin{aligned} h_0 + \epsilon h_1 + \epsilon^2 h_2 = (\mathrm{e}^2 - 1) - (\mathrm{e}^2 + 1)x + \tfrac{1}{2}(\mathrm{e}^2 - 1)x^2 \\ + \epsilon(1 - \mathrm{e})^2 - \epsilon x 2(\mathrm{e}^2 - \mathrm{e}) \\ + \epsilon^2 2(\mathrm{e}^2 - \mathrm{e}) + O(x^3, \epsilon x^2, \epsilon^2 x). \end{aligned} \tag{6.90}$$

Turning to the boundary layer, we use the inner variable $X = x/\epsilon$ so that the rescaled differential equation is

$$h_{XX} + \left[\mathrm{e}^{\epsilon X} h\right]_X = -2\epsilon \mathrm{e}^{2\epsilon X}. \tag{6.91}$$

We substitute the inner expansion

$$h = H_0(X) + \epsilon H_1(X) + \epsilon^2 H_2(X) + \cdots \tag{6.92}$$

into the differential equation and collect powers of ϵ. The first three orders of the boundary-layer problem are

$$H_{0XX} + H_{0X} = 0, \tag{6.93}$$

$$H_{1XX} + [H_1 + XH_0]_X = -2, \tag{6.94}$$

$$H_{2XX} + \left[H_2 + XH_1 + \tfrac{1}{2}X^2 H_0\right]_X = -4X. \tag{6.95}$$

Note that it is necessary to expand the exponentials within the boundary layer, otherwise we cannot ensure that the H_n's do not depend on ϵ.

The solution for H_0 that satisfies the boundary condition at $x = 0$, and also matches the first term on the right of (6.90), is

$$H_0 = (\mathrm{e}^2 - 1)\left(1 - \mathrm{e}^{-X}\right). \tag{6.96}$$

The solution for H_1 that satisfies the boundary condition at $x = 0$ is

$$H_1 = A_1(1 - \mathrm{e}^{-X}) + (\mathrm{e}^2 + 1)\left(1 - X - \mathrm{e}^{-X}\right) + \tfrac{1}{2}(\mathrm{e}^2 - 1)X^2\mathrm{e}^{-X}. \tag{6.97}$$

The constant A_1 is determined by matching to the interior solution. We can do this by taking the limit as $X \to \infty$ in the boundary layer solution $H_0 + \epsilon H_1$. Effectively this means that all terms involving e^{-X} are exponentially small and therefore negligible in the matching. To help with pattern recognition we rewrite the outer limit of the boundary-layer solution in terms of the outer variable x. Thus, in the matching region where $X \gg 1$ and $x \ll 1$, the boundary-layer solution in (6.96) and (6.97) is:

$$H_0 + \epsilon H_1 \to (\mathrm{e}^2 - 1) + \epsilon A_1 + \epsilon(1 + \mathrm{e}^2) - (1 + \mathrm{e}^2)x. \tag{6.98}$$

To match the first term on the second line of (6.90) with (6.98) we require

$$\epsilon A_1 + \epsilon(1 + \mathrm{e}^2) = \epsilon(1 - \mathrm{e})^2, \qquad \Rightarrow \qquad A_1 = -2\mathrm{e}. \tag{6.99}$$

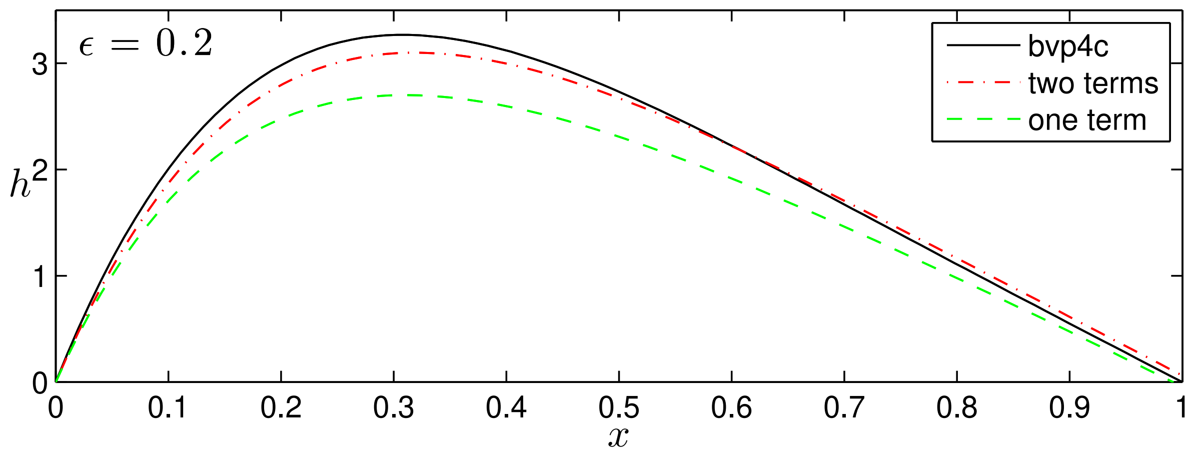

Figure 6.5: Comparison of the one and two term uniform approximations in (6.101) with the numerical solution of (6.84).

The final term in (6.98), namely $-(1+\mathrm{e}^2)x$, matches against a term on the first line of (6.90). That's interesting, because $-(1+\mathrm{e}^2)x$ comes from H_1 and matches against h_0.

There are many remaining ummatched terms in (6.90) e..g, $\frac{1}{2}(\mathrm{e}^2-1)x^2$ on the first line. This term will match against terms from H_2 i.e., it will require an infinite number of terms in the boundary layer expansion just to match terms arising from the expansion of the leading-order interior solution.

Now we construct a uniformly valid approximation using the recipe

$$\text{uniform} = \text{outer} + \text{inner} - \text{match}\,. \tag{6.100}$$

This gives

$$\begin{aligned} h^{\text{uni}} &= \mathrm{e}^{2-x} - \mathrm{e}^{x} - (\mathrm{e}^2-1)\mathrm{e}^{-X} \\ &\quad + \epsilon\left[1 - 2\mathrm{e}^{1-x} + \mathrm{e}^{2-2x} + \mathrm{e}^{-X}\left(\tfrac{1}{2}X^2(\mathrm{e}^2-1) - (\mathrm{e}-1)^2\right)\right]. \end{aligned} \tag{6.101}$$

This construction satisfies the $x=0$ boundary condition exactly. But there is an exponentially small embarrassment at $x=1$. Figure 6.5 compares the numerical solution of (6.84) with the approximation in (6.101). At $\epsilon = 0.2$ the two-term approximation is significantly better than just the leading-order term. We don't get line-width agreement — the ϵ^2 term would help.

```
function StommelBL
% Solution of epsilon h_{xx} +(exp(x) h)_x = - 2 \exp(2 x)
epsilon = 0.2;
solinit = bvpinit(linspace(0 , 1 , 10) , @guess);
sol     = bvp4c(@odez,@bcs,solinit);
% My fine mesh
xx = linspace(0,1,100);    hh = deval(sol,xx);
figure;                    subplot(2,1,1)
plot(xx , hh(1,:),'k')
hold on
xlabel('$x$','interpreter','latex','fontsize',16)
ylabel('$h$','interpreter','latex','fontsize',16,'rotation',0)
axis([0 1 0 3.5])
% The BL solution
XX = xx/epsilon;         EE =exp(-XX);
hZero = exp(2-xx) - exp(xx) - (exp(2) - 1).*EE;
hOne = 1 - 2*exp(1-xx) + exp(2-2*xx)...
               + EE.*( 0.5*XX.^2*(exp(2) - 1) - (exp(1) - 1)^2);
plot(xx, hZero+epsilon*hOne,'-.r' , xx,hZero,'--g')
legend('bvp4c','two terms' , 'one term')
text(0.02,3.2,'$\epsilon =0.2$','interpreter','latex','fontsize',16)

%% Inline functions
    %The differential equations
    function dhdx = odez(x,h)
        dhdx = [h(2)/epsilon ; ...
                    - exp(x).*h(2)/epsilon - exp(x).*h(1) - 2*exp(2*x)];
    end

    %residual in the boundary condition
    function res = bcs(um,up)
        res = [um(1) ; up(1) ];
    end

    % Inital guess at the solution
    function hinit = guess(x)
        hinit = [(1-x^2) ; 2*x];
    end
end
```

6.5 A nonlinear Stommel problem

Consider the Stommel model with nonlinear diffusivity:

$$\epsilon\left(\tfrac{1}{2}h^2\right)_{xx} + h_x = -1\,, \qquad \text{with BCs:} \qquad h(0) = h(1) = 0\,. \tag{6.102}$$

Diffusion is bigger where the pile is deeper because there is more height for diffusion to move dirt around.

If we assume that the boundary layer is at $x = 0$ then easy calculations show that the leading-order interior solution is

$$h_0 = 1 - x\,, \tag{6.103}$$

and that the series continues as

$$h = \left(1 + \epsilon + 2\epsilon^2 + \cdots\right)(1 - x)\,. \tag{6.104}$$

This perturbation series indicates that there is a simple exact solution that satisfies the $x = 1$ boundary condition:

$$h = A(\epsilon)\,(1 - x)\,, \qquad \text{where} \qquad \epsilon A^2 - A + 1 = 0\,. \tag{6.105}$$

This is pleasant, but it does not help with the boundary condition at $x = 0$.

Introducing the boundary layer variable

$$X \stackrel{\text{def}}{=} x/\epsilon\,, \tag{6.106}$$

we have the re-scaled equation

$$\left(\tfrac{1}{2}h^2\right)_{XX} + h_X = -\epsilon\,. \tag{6.107}$$

We try for a solution with $h = H_0(X) + \epsilon H_1(X) + \cdots$ The leading-order equation is

$$\left(\tfrac{1}{2}H_0^2\right)_{XX} + H_{0X} = 0\,, \tag{6.108}$$

which integrates to

$$H_0 H_{0X} + H_0 = C\,. \tag{6.109}$$

This leading-order solution must satisfy both the $X = 0$ boundary condition and the matching condition

$$H_0(0) = 0\,, \qquad \text{and} \qquad \lim_{X\to\infty} H_0(X) = 1\,. \tag{6.110}$$

If we apply the $x = 0$ boundary condition to (6.109), and assume that

$$\lim_{X\to 0} H_0 H_{0X} \stackrel{?}{=} 0\,, \tag{6.111}$$

then we conclude that $C = 0$. But $C = 0$ in (6.109) quickly leads to $H_0 = -X$. This satisfies the boundary condition at $x = 0$, but not the matching condition. We are forced to consider that the limit above is non-zero. In that case we can determine the constant C in (6.109) by matching to the interior. Thus $C = 1$ and

$$H_{0X} = \frac{1}{H_0} - 1\,. \tag{6.112}$$

We solve (6.112) via separation of variables

$$\frac{H_0\,\mathrm{d}H_0}{1 - H_0} = \mathrm{d}X\,, \tag{6.113}$$

integrating to

$$-H_0 + \ln\frac{1}{1 - H_0} = X + A\,. \tag{6.114}$$

Applying the boundary condition at $X = 0$ shows that $A = 0$, and thus $H_0(X)$ is determined implicitly by

$$H_0 = 1 - \mathrm{e}^{-X - H_0}\,. \tag{6.115}$$

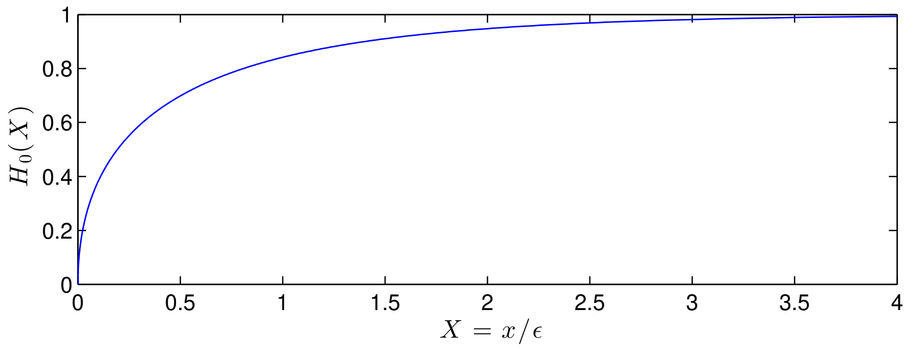

Figure 6.6: The boundary-layer solution in (6.115) of the nonlinear Stommel problem in (6.102).

This implicit solution is shown in figure 6.6. As $X \to \infty$ we use iteration to obtain the large-X behaviour of the boundary layer solution

$$H_0(X) \sim 1 - \mathrm{e}^{-X-1} + \mathrm{e}^{-2X-2} + \cdots \quad \text{as } X \to \infty . \tag{6.116}$$

This demonstrates matching to the leading-order interior solution.

Might we find another solution of (6.102) with a boundary layer at $x = 1$? The answer is **yes**: (6.102) has both the reflection symmetry

$$h \to -h , \qquad \text{and} \qquad x \to -x , \tag{6.117}$$

and the translation symmetry

$$x \to x + a . \tag{6.118}$$

Thus we can define $x_1 = x - 1/2$ so that the boundary conditions are applied at $x_1 = \pm 1/2$. The reflection symmetry then implies that if $h(x_1)$ is a solution then so is $-h(-x_1)$. With this trickery the solution we've just described is transformed into a perfectly acceptable solution but with a boundary layer at the other end of the domain.

Exercise: assume that the boundary layer is at $x = 1$, so that the leading-order outer solution is now $h_0 = -x$. Construct the boundary-layer solution using the inner variable $X = (x-1)/\epsilon$ – you'll be able to satisfy both the $x = 1$ boundary condition and match onto the inner limit of the outer solution. This solution has $h(x) \leq 0$.

Reformulation of the nonlinear diffusion model

As a solution of the dirt-pile model the second solution above makes no sense: dirt piles can't have negative height. And the physical intuition that put the boundary layer at $x = 0$ can't be wrong simply because we use a more complicated model of diffusion. The problem is that the nonlinear diffusion equation in (6.102) should be

$$\epsilon \left(\tfrac{1}{2}|h|h\right)_{xx} + h_x = -1 , \qquad h(0) = h(1) = 0 . \tag{6.119}$$

In other words, the diffusivity should vary with $|h|$, not h. Back in (6.102), our translation of the physical problem into mathematics was faulty. Changing h to $|h|$ in destroys the symmetry in (6.117).

Now let's use the correct model in (6.119) and show that the boundary layer cannot be at $x = 1$. If we try to put the boundary layer at $x = 1$ then the leading-order interior solution is

$$h_0 = -x . \tag{6.120}$$

Using the boundary layer coordinate

$$X \stackrel{\text{def}}{=} \frac{x-1}{\epsilon} , \tag{6.121}$$

the leading-order boundary layer equation is

$$-\left(\tfrac{1}{2}H_0^2\right)_{XX} + H_{0X} = 0\,, \tag{6.122}$$

Above we have assumed that $H_0(X) < 0$ so that $|H_0| = -H_0$. The differential equation in (6.125) must be solved with boundary and matching conditions

$$H_0(0) = 0\,, \qquad \text{and} \qquad \lim_{X\to-\infty} H_0 = -1\,. \tag{6.123}$$

The second condition above is matching onto the inner limit of the outer solution. We can integrate (6.125) and apply the matching condition to obtain

$$\frac{\mathrm{d}H_0}{\mathrm{d}X} = \frac{H_0+1}{H_0}\,. \tag{6.124}$$

Now if $-1 < H_0 < 0$ then the equation above implies that

$$\frac{\mathrm{d}H_0}{\mathrm{d}X} < 0\,. \tag{6.125}$$

The sign in (6.125) is not consistent with a solution that increases monotonically from $H_0(-\infty) = -1$ to $H_0(0) = 1$. Moreover if we integrate (6.124) with separation of variables we obtain an implicit solution

$$X = H_0 - \ln(1+H_0)\,, \qquad \text{or equivalently} \qquad H_0 = -1 + \mathrm{e}^{-X+H_0}\,. \tag{6.126}$$

But as $X \to -\infty$ we do not get a match — the boundary layer cannot be at $x = 1$. Thus we cannot construct a solution of the $|h|$-model in (6.119) with a boundary layer at $x = 1$

6.6 Problems

Problem 6.1. *(i)* Find a leading order uniformly valid solution of

$$-h_x = \epsilon h_{xx} + x\,, \qquad h(0) = h(1) = 0\,. \tag{6.127}$$

(ii) Solve the BVP above exactly and compare the exact solution to the boundary layer approximation with $\epsilon = 0.1$.

Problem 6.2. *(i)* Solve the boundary value problem

$$h_x = \epsilon h_{xx} + \sin x\,, \qquad h(0) = h(\pi) = 0\,, \tag{6.128}$$

exactly. To assist communication, please use the notation

$$X \stackrel{\text{def}}{=} \frac{x-\pi}{\epsilon}\,, \qquad \text{and} \qquad E \stackrel{\text{def}}{=} \mathrm{e}^{-\pi/\epsilon}\,. \tag{6.129}$$

This should enable you to write the exact solution in a compact form. *(ii)* Now solve the problem with boundary-layer theory. Begin with the interior. Applying the $x = 0$ boundary condition, find the first three terms in the regular perturbation expansion:

$$h(x) = h_0(x) + \epsilon h_1(x) + \epsilon^2 h_2(x) + O\left(\epsilon^3\right)\,. \tag{6.130}$$

(iii) There is a boundary layer at $x = \pi$. "Rescale" the equation using X above as the independent variable and denote the solution in the boundary layer by $H(X)$. Find the first three terms in the regular perturbation expansion of the boundary-layer equation:

$$H = H_0(X) + \epsilon H_1(X) + \epsilon^2 H_2(X) + O\left(\epsilon^3\right)\,. \tag{6.131}$$

(iv) The H_n's above will each contain an unknown constant. Determine the three constants by matching to the interior solution. *(v)* Construct a uniformly valid solution, up to an including terms of order ϵ^2. You can check your algebra by comparing your boundary layer solution with the expansion of the exact solution from part *(i)*. *(vi)* With $\epsilon = 0.2$ and 0.5, use MATLAB to compare the exact solution from part *(i)* with the approximation in part *(v)*.

Problem 6.3. Suppose ϵ is a real number with $|\epsilon| \ll 1$. (For the moment we relax our usual rule that $\epsilon > 0$.) Consider the boundary value problem

$$\epsilon y'' + y' = -e^{-x}, \tag{6.132}$$

posed on the interval $0 < x < 1$ with boundary conditions $y(0) = y(1) = 0$. *(i)* Solve the problem exactly. (You might need to consider the case $\epsilon = 1$ separately.) *(ii)* Plot the exact solution for $\epsilon = 1$, $1/4$, $1/10$. *(iii)* Plot the exact solution for $\epsilon = +1/100$ and $\epsilon = -1/100$. The limiting function depends on whether ϵ approaches zero through positive or negative values. *(iv)* Now assume that $\epsilon \to 0$ through positive values. Use boundary layer analysis to obtain a leading order, uniform approximation, $y_0(x)$. Make a graphical comparison of $y_0(x)$ with the exact solution at $\epsilon = 1/4$. *(v)* Now obtain the first-order in ϵ uniform approximation. Add this new and improved approximation to your figure.

Problem 6.4. Find the leading-order uniformly valid boundary-layer solution to the Stommel problem

$$-\left(e^x g\right)_x = \epsilon g_{xx} + 1, \qquad \text{with BCs} \quad g(0) = g(1) = 0 \ . \tag{6.133}$$

Do the same for

$$\left(e^x f\right)_x = \epsilon f_{xx} + 1, \qquad \text{with BCs} \quad f(0) = f(1) = 0 \, . \tag{6.134}$$

Problem 6.5. Analyze the variable-speed Stommel problem

$$\epsilon h'' + \left(x^a h\right)_x = -1, \qquad \text{with BCs} \quad h(0) = h(1) = 0 \, , \tag{6.135}$$

using boundary layer theory. (The case $a = 1/2$ was discussed in the lecture.) How thick is the boundary layer at $x = 0$, and how large is the solution in the boundary layer? Check your reasoning by constructing the leading-order uniformly valid solution when $a = -1$, $a = 1$ and $a = 2$.

Problem 6.6. Find the leading-order, uniformly valid solution of

$$\epsilon h'' + \left(\sin x \, h\right)_x = -1, \qquad \text{with BCs} \quad h(0) = h\left(\frac{\pi}{2}\right) = 0 \, . \tag{6.136}$$

Problem 6.7. Find a leading-order boundary layer solution to

$$\epsilon h'' + (\sin x \, h)_x = -1, \qquad \text{with BCs} \quad h(0) = h(\pi) = 0 \, . \tag{6.137}$$

(I think there are boundary layers at both $x = 0$ and $x = 1$.)

Problem 6.8. Considering the pile-up example (6.64), find the next term in the boundary-layer solution of this problem. Make sure you explain how the term ϵx^{-2} in the outer expansion is matched as $x \to 0$.

Problem 6.9. Find a leading-order boundary layer solution to the forced Burgers equation

$$\epsilon h_{xx} + \left(\tfrac{1}{2} h^2\right)_x = -1, \qquad h(0) = h(1) = 0 \, . \tag{6.138}$$

Use `bvp4c` to solve this problem numerically, and compare your leading order solution to the numerical solution: see figure 6.7.

Problem 6.10. The result of problem 6.9 is disappointing: even though $\epsilon = 0.05$ seems rather small, the approximation in Figure 6.7 is only so-so. Calculate the next correction and compare the new improved solution with the `bvp4c` solution. (The numerical solution seems to have finite slope at $x = 1$, while the leading-order outer solution has infinite slope as $x \to 1$: perhaps there a higher-order boundary layer at $x = 1$ is required to heal this singularity?)

Problem 6.11. Use boundary layer theory to find leading order solution of

$$h_x = \epsilon \left(\tfrac{1}{3} h^3\right)_{xx} + 1, \tag{6.139}$$

on the domain $0 < x < 1$ with boundary conditions $h(0) = h(1) = 0$. You can check your answer by showing that $h = 1/2$ at $x \approx 1 - (\ln 2 - 5/8)\epsilon$.

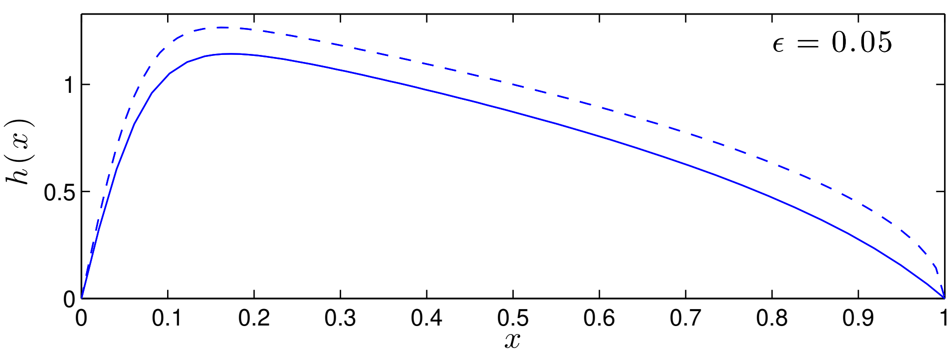

Figure 6.7: Figure for problem 6.9.

Lecture 7

More boundary layer theory

7.1 A second-order BVP with a boundary layer

At the risk of repetition, let's discuss another elementary example of boundary layer theory, focussing on intuitive concepts and on finding the leading-order uniformly valid solution. We use the BVP

$$\epsilon y'' + a(x)y' + b(x)y = 0\,, \tag{7.1}$$

with BCs

$$y(0) = p\,, \qquad y(1) = q\,, \tag{7.2}$$

as our model.

The case $a(x) > 0$

We start with a the special case in which $a(x)$ is strictly positive throughout the interval $0 < x < 1$. We begin with some heuristic considerations. Suppose we drop $\epsilon y''$ and start by solving

$$a(x)y' + b(x)y \approx 0\,. \tag{7.3}$$

There are at least two possible interior solutions

$$L(x) \stackrel{\text{def}}{=} p\exp\left(-\int_0^x \frac{a(t)}{b(t)}\,\mathrm{d}t\right)\,, \qquad \text{or} \qquad R(x) \stackrel{\text{def}}{=} q\exp\left(+\int_x^1 \frac{a(t)}{b(t)}\,\mathrm{d}t\right)\,. \tag{7.4}$$

$L(x)$ satisfies the left hand BC at $x = 0$ and $R(x)$ satisfies the right hand BC at $x = 1$.

Do we use $R(x)$ or $L(x)$ as an interior solution? If we use $L(x)$ then we will need a BL at $x = 1$. We anticipate that within this BL the ODE simplifies to

$$\epsilon y'' + a(1)y' \approx 0. \tag{7.5}$$

The general solution is

$$y = C + D\exp\left(-a(1)\frac{x-1}{\epsilon}\right)\,. \tag{7.6}$$

But this is not a BL solution – the exponential term above explodes $(x-1)/\epsilon \to -\infty$ i.e. as we move into the interior of the domain where $x - 1$ is negative. OK – it seems that if $a(x) > 0$ we must use $R(x)$ as the interior solution.

The argument above is correct: if $a(x) > 0$ we use $R(x)$ as the interior solution. And if $a(x) < 0$ then the boundary layer is at $x = 1$ and we should use $L(x)$ as the interior solution.

Let's now proceed more formally. In the interior we can look for a solution with the expansion

$$y(x, \epsilon) = y_0(x) + \epsilon y_1(x) + \cdots \tag{7.7}$$

The leading-order term satisfies

$$a y_0' + b y_0 = 0 \,. \tag{7.8}$$

With the construction

$$y_0(x) = q \exp\left(\int_x^1 \frac{b(v)}{a(v)} \mathrm{d}v \right) \tag{7.9}$$

we have satisfied the boundary condition at $x = 1$. We return later to discuss why this is the correct choice if $a(x) > 0$.

Unless we're very lucky, (7.9) will not satisfy the boundary condition at $x = 0$. We fix this problem by building a boundary layer at $x = 0$. Introduce a boundary layer coordinate

$$X \stackrel{\text{def}}{=} \frac{x}{\epsilon} \,, \tag{7.10}$$

and write

$$y(x, \epsilon) = Y(X, \epsilon) \,. \tag{7.11}$$

Then "re-scale" the differential equation (7.1) using the boundary-layer variables:

$$Y_{XX} + a(\epsilon X) Y_X + \epsilon b(\epsilon X) Y = 0 \,. \tag{7.12}$$

Within the boundary layer, where $X = O(1)$,

$$a(\epsilon X) = a(0) + \epsilon X a'(0) + \tfrac{1}{2} \epsilon^2 X^2 a''(0) + O\left(\epsilon^3\right) \,. \tag{7.13}$$

There is an analogous expansion for $b(\epsilon X)$.

In the boundary layer we use the "inner expansion":

$$Y(X, \epsilon) = Y_0(X) + \epsilon Y_1(X) + \cdots \tag{7.14}$$

The leading-order term is

$$\epsilon^0 : \qquad Y_{0XX} + a(0) Y_{0X} = 0 \,, \tag{7.15}$$

and, for good measure, the next term is

$$\epsilon^1 : \qquad Y_{1XX} + a(0) Y_{1X} + a'(0) X Y_{0X} + b(0) Y_0 = 0 \,. \tag{7.16}$$

Terms in the Taylor series (7.13) will impact the higher orders.

The solution of (7.15) that satisfies the boundary condition at $X = 0$ is

$$Y_0 = p + A_0 \left(1 - \mathrm{e}^{-a(0)X}\right) \,, \tag{7.17}$$

where A_0 is a constant of integration. We are assuming that $a(0) > 0$ so that the exponential in (7.17) decays to zero as $X \to \infty$. This is why the boundary layer *must* be at $x = 0$. The constant A_0 can then be determined by demanding that in the outer solution (7.9) agrees with the inner solution (7.17) in the *matching region* where $X \gg 1$ and $x \ll 1$. This requirement determines A_0:

$$p + A_0 = q \exp\left(\int_0^1 \frac{b(v)}{a(v)} \mathrm{d}v \right) \,. \tag{7.18}$$

Hence the leading order boundary-layer solution is

$$Y_0 = p + \left[q \exp\left(\int_0^1 \frac{b(v)}{a(v)} \mathrm{d}v \right) - p \right] \left(1 - \mathrm{e}^{-a(0)X}\right) \,, \tag{7.19}$$

$$= p \mathrm{e}^{-a(0)X} + q \exp\left(\int_0^1 \frac{b(v)}{a(v)} \mathrm{d}v \right) \left(1 - \mathrm{e}^{-a(0)X}\right) \,. \tag{7.20}$$

We construct a uniformly valid solutions using the earlier recipe

$$\text{uniformly valid} = \text{outer} + \text{inner} - \text{match} \,. \tag{7.21}$$

In this case we obtain

$$y_{\text{uni}} = q \exp\left(\int_x^1 \frac{b(v)}{a(v)} \mathrm{d}v \right) + \left[p - q \exp\left(\int_0^1 \frac{b(v)}{a(v)} \mathrm{d}v \right) \right] \mathrm{e}^{-a(0)x/\epsilon} \,. \tag{7.22}$$

Exercise: Find the analog of (7.22) if $a(x) < 0$ throughout the interval $0 < x < 1$.

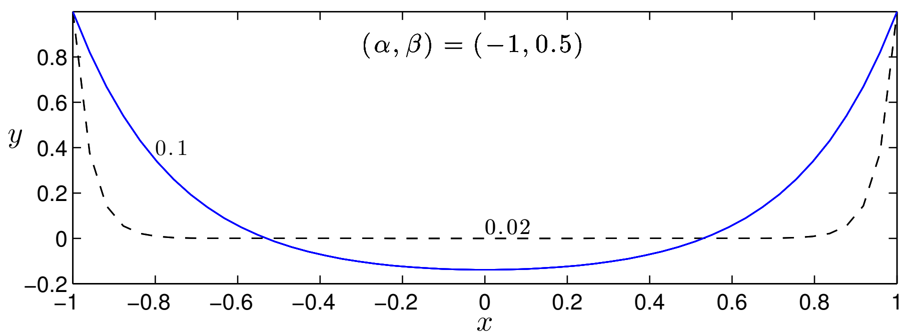

Figure 7.1: Solutions of (7.25) with $\alpha = -1 < 0$ and $\epsilon = 0.1$ (solid blue) and 0.02 (dashed black). There are boundary layers at $x = \pm 1$. The interior solution is zero to all orders in ϵ. There is no internal boundary layer at $x = 0$.

7.2 Internal boundary layers

BO section 9.6 has an extensive discussion of the boundary layer problem

$$y'' + ay' + by = 0\,, \tag{7.23}$$

in which $a(x)$ has an internal zero. The problem is posed on $-1 < x < 1$ with boundary conditions

$$y(-1) = p\,, \qquad y(+1) = q\,. \tag{7.24}$$

The zero of $a(x)$ is at $x = 0$ where $a(x) = \alpha x + O(x^2)$. The differential equation

$$\epsilon y'' + \frac{\alpha x}{1 + x^2} y + \beta y = 0 \tag{7.25}$$

is a typical example. I'll give a simplified treatment of (7.25) and defer to **BO** for more details of the general case.

Case 1: $\alpha < 0$ (easy)

There are boundary layers at *both* $x = 1$ and $x = -1$: see figure 7.1 for numerical solutions of (7.25) using the MATLAB routine `bvp4c`.

Exercise: Use BL theory to construct $\epsilon \ll 1$ approximations to the numerical solution in figure 7.1 .

Case 2: $\alpha > 0$ (hard)

Next consider (7.25) with $\alpha > 0$ and boundary conditions

$$y(-1) = 1\,, \qquad y(+1) = 0\,. \tag{7.26}$$

This example will reveal all the main features of the general case. Earlier arguments indicate that boundary layers not possible at either end of the domain. Instead we have two interior solutions – one for $0 < x < 1$ and the other for $-1 < x < 0$.

The left interior solution, $u(x)$, satisfies the boundary condition at $x = -1$:

$$y = u_0 + \epsilon u_1 + \cdots \tag{7.27}$$

with leading order

$$\frac{\alpha x}{1 + x^2} u_{0x} + \beta u_0 = 0\,, \qquad \Rightarrow \qquad u_0 = |x|^{-\beta/\alpha} \exp\left[-\frac{\beta}{2\alpha}\left(x^2 - 1\right)\right]. \tag{7.28}$$

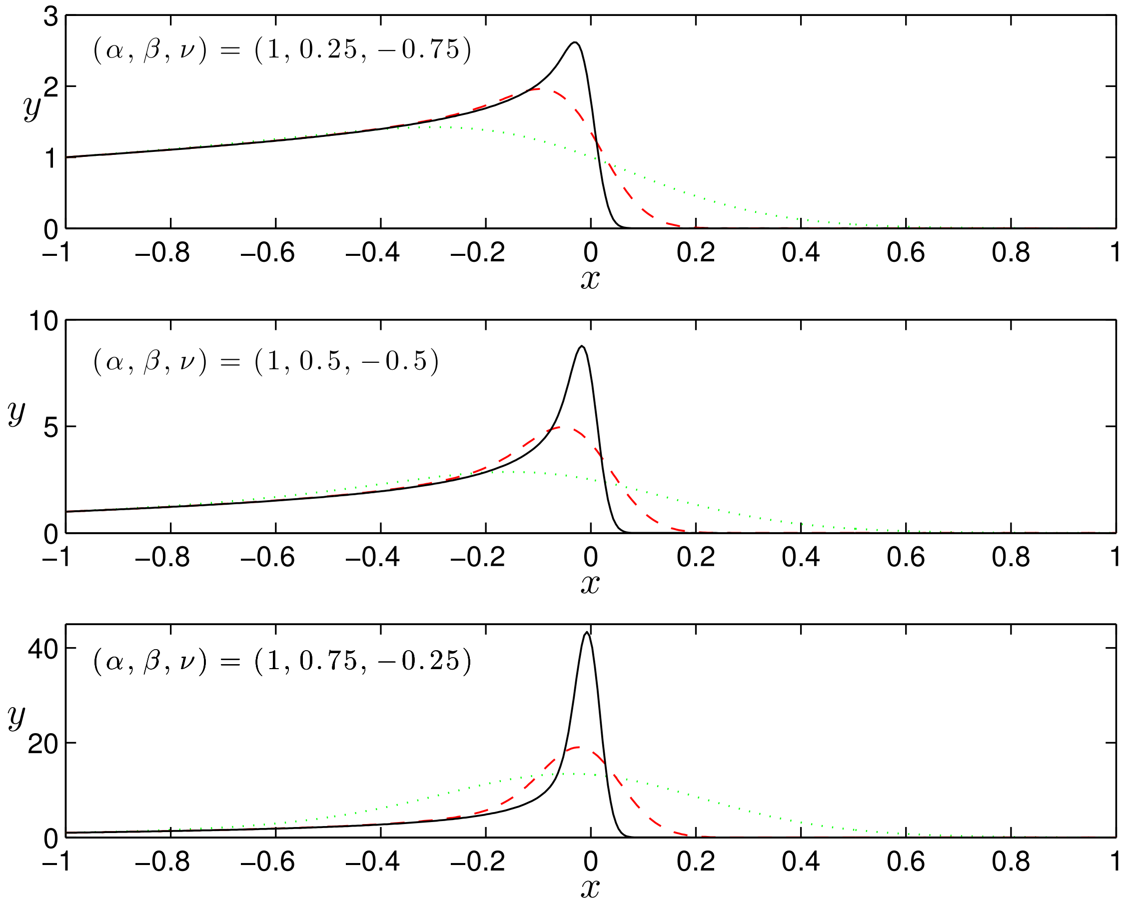

Figure 7.2: Internal boundary layer solution of (7.25) with $p = 1$ and $q = 0$, and $\epsilon = 0.05$ (green dotted) and 0.005 (red dashed) and 0.0005 (solid black).

There is also a right interior solution $v(x)$, satisfying the boundary condition at $x = +1$:

$$y = v_0 + \epsilon v_1 + \cdots \tag{7.29}$$

In this case, with $y(1) = 0$, the right interior solution is zero at all orders

$$v_n = 0\,. \tag{7.30}$$

We need a boundary layer at $x = 0$ to heal the $x \to 0$ singularity in (7.28) and to connect the right interior solution to the left interior solution. A distinguished-limit shows that the correct boundary-layer coordinate is

$$X \stackrel{\text{def}}{=} \frac{x}{\sqrt{\epsilon}}\,. \tag{7.31}$$

We must also re-scale the solution:

$$y = \epsilon^{-\beta/2\alpha} Y(X)\,. \tag{7.32}$$

The scaling above is indicated because the interior solution in (7.28) is order $\epsilon^{-\beta/2\alpha}$ once $x \sim \sqrt{\epsilon}$. Without much work we have now determined the boundary layer thickness and the amplitude of the solution within the boundary layer. This is valuable information in interpreting the numerical solution in figure 7.2 — we now understand how the vertical axis must be rescaled if we reduce ϵ further.

Using the boundary-layer variables, the BL equation is

$$Y_{XX} + \frac{\alpha X}{1 + \epsilon X^2} Y_X + \beta Y = 0\,. \tag{7.33}$$

We solve (7.33) with the RPS

$$Y = Y_0(X) + \epsilon Y_1(X) + \cdots \tag{7.34}$$

Leading order is the three-term balance

$$Y_{0XX} + \alpha X Y_{0X} + \beta Y_0 = 0\,, \tag{7.35}$$

with matching conditions

$$Y_0 \to |X|^{-\beta/\alpha} \mathrm{e}^{\beta/2\alpha}\,, \qquad \text{as } X \to -\infty, \tag{7.36}$$

$$Y_0 \to 0\,, \qquad \text{as } X \to +\infty. \tag{7.37}$$

We have to solve (7.35) exactly. When confronted with a second-order differential equation it is always a good idea to remove the first derivative term with the standard multiplicative substitution. In this case the substitution

$$Y_0 = W \mathrm{e}^{-\alpha X^2/4} \tag{7.38}$$

into (7.35) results in

$$W_{XX} + \left(\beta - \tfrac{1}{2}\alpha - \tfrac{1}{4}\alpha^2 X^2\right) W = 0\,. \tag{7.39}$$

Then, with $Z \stackrel{\text{def}}{=} \sqrt{\alpha} X$, we obtain the parabolic cylinder equation

$$W_{ZZ} + \Bigg(\underbrace{\frac{\beta}{\alpha} - 1}_{\nu + \frac{1}{2}} - \tfrac{1}{4} Z^2\Bigg) W = 0\,, \tag{7.40}$$

of order

$$\nu \stackrel{\text{def}}{=} \frac{\beta}{\alpha} - 1\,. \tag{7.41}$$

Provided that

$$\frac{\beta}{\alpha} \neq 1,\ 2,\ 3, \cdots \tag{7.42}$$

the general solution of (7.35) is

$$Y_0 = \mathrm{e}^{-\alpha X^2/4} \left[A D_\nu\left(\sqrt{\alpha} X\right) + B D_\nu\left(-\sqrt{\alpha} X\right)\right]\,. \tag{7.43}$$

We return to the exceptional case, in which $\nu = 0,\ 1,\ 2 \cdots$, later.

Parabolic Cylinder Functions

The parabolic cylinder equation is

$$y'' + \left(\nu + \tfrac{1}{2} - \tfrac{1}{4}x^2\right) y = 0\,.$$

If ν is not an integer then the general solution can be constructed as $y = c_1 D_\nu(x) + c_2 D_\nu(-x)$. Values at the origin are

$$D_\nu(0) = \sqrt{\pi}2^{\nu/2}/\Gamma\left(\tfrac{1}{2} - \tfrac{1}{2}\nu\right)\,, \qquad \text{and} \qquad D'_\nu(0) = -\sqrt{\pi}2^{(\nu+1)/2}/\Gamma\left(-\tfrac{1}{2}\nu\right)\,.$$

Asymptotic expansions on the real axis are

$$D_\nu(x) \sim x^\nu \mathrm{e}^{-x^2/4}\,, \qquad \text{as } x \to \infty,$$

and

$$D_\nu(x) \sim |x|^\nu \mathrm{e}^{-x^2/4} - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} \mathrm{e}^{\mathrm{i}\pi\nu} |x|^{-\nu-1} \mathrm{e}^{x^2/4}\,, \qquad \text{as } x \to -\infty.$$

If ν is a nonnegative integer $n = 0,\ 1,\ 2, \cdots$ then

$$D_n = \mathrm{e}^{-x^2/4}\, 2^{-n/2} \mathrm{H}_n\left(x/\sqrt{2}\right)\,,$$

where $\mathrm{H}_n(x)$ is a Hermite polynomial.

To take the outer limits, $X \to \pm\infty$, of the internal boundary layer solution in (7.43) we look up the asymptotic expansion of the parabolic cylinder functions e.g., in the appendix of **BO**, or in the **DLMF**:

$$D_\nu(t) \sim t^\nu \mathrm{e}^{-t^2/4}\,, \qquad \text{as } t \to \infty, \tag{7.44}$$

$$D_\nu(-t) \sim \tfrac{\sqrt{2\pi}}{\Gamma(-\nu)}\, t^{-\nu-1}\mathrm{e}^{t^2/4}\,, \qquad \text{as } t \to \infty. \tag{7.45}$$

Matching in the right-hand outer limit, $X \to +\infty$, implies that $B = 0$. Matching in the left-hand outer limit $X \to -\infty$ requires that

$$A = (\alpha \mathrm{e})^{(\nu+1)/2}\,\frac{\Gamma(-\nu)}{\sqrt{2\pi}}\,. \tag{7.46}$$

7.3 Initial layers

The over-damped oscillator

With our knowledge of boundary layer theory, let's reconsider the over-damped harmonic oscillator from problem 3.9. With a change of notation, the problem in (3.115) is:

$$\epsilon x_{tt} + x_t + x = 0\,, \qquad \text{with the IC:} \quad x(0) = 0\,, \quad x_t(0) = 1\,. \tag{7.47}$$

This scaling is convenient for the long-time solution, but not for satisfying the two initial conditions.

We are going to use a boundary-layer-in-time, also known as an initial layer, to solve this problem. To address the initial layer we introduce

$$T \overset{\text{def}}{=} t/\epsilon\,, \qquad \text{and} \qquad X(T,\epsilon) = x(t,\epsilon)\,. \tag{7.48}$$

The rescaled problem is

$$X_{TT} + X_T + \epsilon X = 0\,, \qquad \text{with the IC:} \quad X(0) = 0\,, \quad X_t(0) = \epsilon\,. \tag{7.49}$$

Because X satisfies both the initial conditions it is convenient to attack this problem by first solving the initial-layer equation with

$$X(T,\epsilon) = \epsilon X_1(T) + \epsilon^2 X_2(T) + \cdots \tag{7.50}$$

One finds

$$X_{1TT} + X_{1T} = 0\,, \qquad \Rightarrow \qquad X_1 = 1 - \mathrm{e}^{-T}\,, \tag{7.51}$$

$$X_{2TT} + X_{2T} = -X_1\,, \qquad \Rightarrow \qquad X_2 = 2(1-\mathrm{e}^{-T}) - T - T\mathrm{e}^{-T}\,, \tag{7.52}$$

$$X_{3TT} + X_{3T} = -X_2\,, \qquad \Rightarrow \qquad X_3 = 6(1-\mathrm{e}^{-T}) - 3T(1+\mathrm{e}^{-T}) - \tfrac{1}{2}T^2\mathrm{e}^{-T} + \tfrac{1}{2}T^2\,. \tag{7.53}$$

All the constants of integration are determined because the initial-layer solution satisfies both initial conditions. Once $T \gg 1$, the initial-layer solution is

$$X \to \epsilon + \epsilon^2\,(2 - T) + \epsilon^3(6 - 3T + \tfrac{1}{2}T^2) + O(\epsilon^4)\,, \tag{7.54}$$

$$= \epsilon(1 - t + \tfrac{1}{2}t^2) + \epsilon^2(2 - 3t) + 6\epsilon^3 + O(\epsilon^4)\,. \tag{7.55}$$

To facilitate matching at higher order in (7.55) we've written the solution in terms of the outer time t. Terms switch order in passing from (7.54) to (7.55). We can anticipate that there are further switchbacks from the $O(\epsilon^4)$ terms.

We obtain the outer solution by solving (7.47) (without the initial conditions!) with the RPS

$$x(t,\epsilon) = \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots \tag{7.56}$$

The first two terms are

$$x_{1t} + x_1 = 0\,, \qquad \Rightarrow \qquad x_1 = A_1\mathrm{e}^{-t}\,, \tag{7.57}$$

$$x_{2t} + x_2 = -x_{1tt}\,, \qquad \Rightarrow \qquad x_2 = A_1 t\mathrm{e}^{-t} + A_2\mathrm{e}^{-t}\,, \tag{7.58}$$

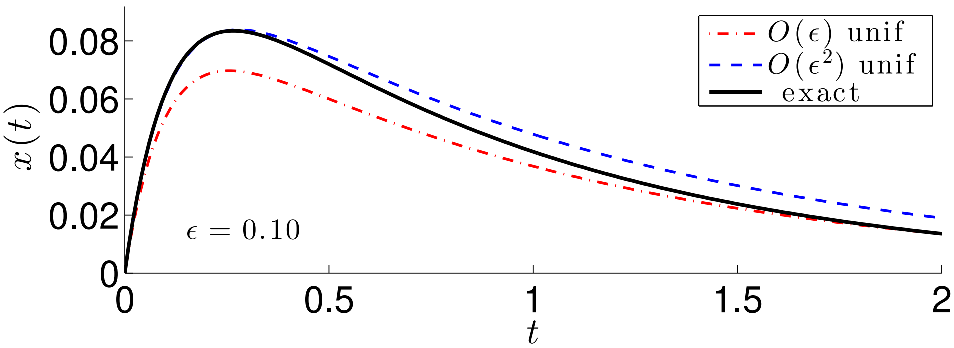

Figure 7.3: Comparison of (7.62) with the exact solution of (7.47).

and the reconstituted outer solution is

$$x = \epsilon A_1 \mathrm{e}^{-t} + \epsilon^2 \left(A_1 t \mathrm{e}^{-t} + A_2 \mathrm{e}^{-t} \right) . \tag{7.59}$$

In the matching region, $\epsilon \ll t \ll 1$, the outer solution in (7.59) is

$$x \to \epsilon A_1 \left(1 - t + \tfrac{1}{2}t^2\right) + \epsilon^2 \left(A_2 + (A_1 - A_2)t\right) + O(\epsilon^3, \epsilon t^2, \epsilon^2 t^2) \tag{7.60}$$

Comparing (7.60) with (7.55) we see that

$$A_1 = 1 , \qquad \text{and} \qquad A_2 = 2 . \tag{7.61}$$

Finally we can construct a uniformly valid solution as

$$x^{\mathrm{uni}} = \epsilon \left(\mathrm{e}^{-t} - \mathrm{e}^{-T} \right) + \epsilon^2 \left(t\mathrm{e}^{-t} - T\mathrm{e}^{-T} + 2\mathrm{e}^{-t} - 2\mathrm{e}^{-T} \right) + O(\epsilon^3) . \tag{7.62}$$

Figure 7.3 compares (7.62) with the exact solution

$$x = \frac{2\epsilon}{\sqrt{1-4\epsilon}} \mathrm{e}^{-t/2\epsilon} \sinh\left(\frac{\sqrt{1-4\epsilon}\, t}{2\epsilon} \right) . \tag{7.63}$$

Remark: Might there be a problem in figure 7.3 – seems like the second-order uniform solution is not accurate in the outer region?

Example: Consider

$$\dot{x} = -x - xy + \epsilon\kappa y , \qquad \epsilon\dot{y} = x - xy - \epsilon\kappa y . \tag{7.64}$$

7.4 Other BL examples

Not all boundary layers have thickness ϵ. Let's consider a medley of examples.

Example:

$$\epsilon y'' - y = -f(x) , \qquad y(-1) = y(1) = 0 , \tag{7.65}$$

If we solve the simple case with $f(x) = 1$ exactly we quickly see that $y \approx 1$, except that there are boundary layers with thickness $\sqrt{\epsilon}$ at *both* $x = 0$ and $x = 1$.

Thus we might hope to construct the outer solution of (7.65) via the RPS

$$y = f + \epsilon f'' + \epsilon^2 f'''' + O(\epsilon^3) . \tag{7.66}$$

The outer solution above doesn't satisfy either boundary condition: we need boundary layers at $x = -1$, and at $x = +1$.

Turning to the boundary layer at $x = -1$ we introduce

$$X \stackrel{\text{def}}{=} \frac{x+1}{\sqrt{\epsilon}}, \qquad \text{and} \qquad y(x,\epsilon) = Y(X, \sqrt{\epsilon}). \tag{7.67}$$

The re-scaled differential equation is

$$Y_{XX} - Y = f\left(-1 + \sqrt{\epsilon}X\right), \tag{7.68}$$

and we look for a solution with

$$Y = Y_0(X) + \sqrt{\epsilon}Y_1(X) + \epsilon Y_2(X) + \cdots \tag{7.69}$$

The leading-order problem is

$$Y_{0XX} - Y_0 = -f(-1), \tag{7.70}$$

with solution

$$Y_0 = f(-1) + A_0 e^{-X} + \underbrace{B_0}_{=0} e^X. \tag{7.71}$$

We quickly set the constant of integration B_0 to zero — the alternative would prevent matching with the interior solution. Then the other constant of integration A_0 is determined so that the boundary condition at $X = 0$ is satisfied:

$$Y_0 = f(-1)\left(1 - e^{-X}\right). \tag{7.72}$$

The boundary condition at $x = +1$ is satisfied with an analogous construction using the coordinate $X \stackrel{\text{def}}{=} (x-1)/\sqrt{\epsilon}$. One finds

$$Y_0 = f(1)\left(1 - e^{X}\right). \tag{7.73}$$

The outer limit of this boundary layer is obtained by taking $X \to -\infty$.

Finally we can construct a uniformly valid solutions via

$$y^{\text{uni}}(x) = f(x) - f(-1)e^{-(x+1)/\sqrt{\epsilon}} - f(+1)e^{(x-1)/\sqrt{\epsilon}}. \tag{7.74}$$

Example:

$$\epsilon y'' + y = f(x), \qquad y(0) = y(1) = 0, \tag{7.75}$$

If we solve the simple case with $f(x) = 1$ exactly we quickly see that this is not a boundary layer problem. This belongs in the WKB lecture.

Example: Find the leading order BL approximation to

$$\epsilon u'' - u = -\frac{1}{\sqrt{1-x^2}}, \qquad \text{with BCs} \qquad u(\pm 1) = 0. \tag{7.76}$$

The leading-order outer solution is

$$u_0 = \frac{1}{\sqrt{1-x^2}}. \tag{7.77}$$

Obviously this singular solution doesn't satisfy the boundary conditions. We suspect that there are boundary layers of thickness $\sqrt{\epsilon}$ at $x = \pm 1$. NThe interior solution (7.77) is $\sim \epsilon^{-1/4}$ as x moves into this BL. Moreover, considering the BL at $x = -1$, we use $X = (1+x)/\sqrt{\epsilon}$ as the boundary layer coordinate, so that

$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{\epsilon^{1/4}\sqrt{X(2-\sqrt{\epsilon}X)}}. \tag{7.78}$$

Hence we try a boundary-layer expansion with the form

$$u(x,\epsilon) = \epsilon^{-1/4}\left[U_0(X) + \sqrt{\epsilon}U_1(X) + O(\epsilon)\right]. \tag{7.79}$$

A main point of this example is that it is necessary to include the factor $\epsilon^{-1/4}$ above.

The leading-order term in the boundary layer expansion is then

$$U_0'' - U_0 = -\frac{1}{\sqrt{X}}, \tag{7.80}$$

which we solve using variation of parameters

$$U_0(X) = \tfrac{1}{2}\underbrace{\mathrm{e}^{-X}\int_0^X \frac{\mathrm{e}^{v}}{\sqrt{v}}\,\mathrm{d}v}_{\sim \mathrm{e}^X/\sqrt{X}} - \tfrac{1}{2}\underbrace{\mathrm{e}^{X}\int_0^X \frac{\mathrm{e}^{-v}}{\sqrt{v}}\,\mathrm{d}v}_{\sim\sqrt{\pi}-(\mathrm{e}^{-X}/\sqrt{X})} + P\mathrm{e}^{X} + Q\mathrm{e}^{-X}\,. \tag{7.81}$$

The boundary condition at $X = 0$ requires

$$P + Q = 0\,. \tag{7.82}$$

To match the outer solution as $X \to \infty$, we must use the $X \to \infty$ asymptotic expansion of the integrals in (7.81), indicated via the underbrace. We determine P so that the exponentially growing terms are eliminated, which requires that $P = \sqrt{\pi}/2$. Thus the boundary layer solution is

$$U_0(X) = \sqrt{\pi}\sinh X - \int_0^X \frac{\sinh(X-v)}{\sqrt{v}}\,\mathrm{d}v\,. \tag{7.83}$$

(Must check this, and then construct the uniformly valid solution!)

Example: Find the leading-order BL approximation to

$$\epsilon y'' + xy' + x^2 y = 0\,, \qquad \text{with BCs} \qquad y(0) = p\,, \qquad y(1) = q\,. \tag{7.84}$$

We divide and conquer by writing the solutions as

$$y = pf(x,\epsilon) + qg(x,\epsilon)\,, \tag{7.85}$$

where

$$\epsilon f'' + xf' + x^2 f = 0\,, \qquad \text{with BCs} \qquad f(0) = 1\,, \qquad f(1) = 0\,, \tag{7.86}$$

and

$$\epsilon g'' + xg' + x^2 g = 0\,, \qquad \text{with BCs} \qquad g(0) = 0\,, \qquad g(1) = 1\,, \tag{7.87}$$

The outer solution of the g-problem is

$$g = \mathrm{e}^{(1-x^2)/2} + \epsilon g_1 + \cdots \tag{7.88}$$

We need a BL at $x = 0$. A dominant balance argument shows that the correct BL variable is

$$X = \frac{x}{\sqrt{\epsilon}}\,. \tag{7.89}$$

If $g(x,\epsilon) = G(X,\sqrt{\epsilon})$ then the rescaled problem is

$$G_{XX} + XG_X + \epsilon X^2 G = 0\,. \tag{7.90}$$

The leading-order problem is

$$G_{0XX} + XG_{0X} = 0\,, \tag{7.91}$$

with general solution

$$G_0 = \underbrace{P}_{=0} + \underbrace{Q}_{\sqrt{2\mathrm{e}/\pi}} \int_0^X \mathrm{e}^{-v^2/2}\,\mathrm{d}v\,. \tag{7.92}$$

To satisfy the $X = 0$ boundary condition we take $P = 0$, and to match the outer solution we require

$$Q\int_0^\infty \mathrm{e}^{-v^2/2}\,\mathrm{d}v = \sqrt{\mathrm{e}}\,. \tag{7.93}$$

The uniformly valid solution is

$$g^{\mathrm{uni}}(x,\epsilon) = \mathrm{e}^{(1-x^2)/2} + \sqrt{\frac{2\mathrm{e}}{\pi}}\int_0^{x/\sqrt{\epsilon}} \mathrm{e}^{-v^2/2}\,\mathrm{d}v - \sqrt{\mathrm{e}}\,, \tag{7.94}$$

$$= \mathrm{e}^{(1-x^2)/2} - \sqrt{\frac{2\mathrm{e}}{\pi}}\int_{x/\sqrt{\epsilon}}^{\infty} \mathrm{e}^{-v^2/2}\,\mathrm{d}v\,. \tag{7.95}$$

Now turn to the f-problem. The outer solution is $f_n(x) = 0$ at all orders. The solution of the leading-order boundary-layer problem is

$$F_0(X) = \frac{1}{\sqrt{2\pi}}\int_X^\infty \mathrm{e}^{-v^2/2}\,\mathrm{d}v\,. \tag{7.96}$$

This is a stand-alone boundary layer.

Example: Let's analyze the higher-order terms in the BL solution of our earlier example

$$\epsilon y'' - y = -f(x)\,, \qquad y(-1) = y(1) = 0\,. \tag{7.97}$$

Our provisional outer solution is

$$y(x) = f(x) + \epsilon f''(x) + \epsilon^2 f''''(x) + O(\epsilon^3)\,. \tag{7.98}$$

Let's rewrite this outer solution in terms of the inner variable $X \stackrel{\text{def}}{=} (x-1)/\sqrt{\epsilon}$

$$y(x) = f(1+\sqrt{\epsilon}X) + \epsilon f''(1+\sqrt{\epsilon}X) + \epsilon^2 f''''(1+\sqrt{\epsilon}X) + O(\epsilon^3)\,. \tag{7.99}$$

Assuming that $\sqrt{\epsilon}X$ is small in the matching region, we expand the outer solution:

$$\begin{aligned} y(x) = f(1) + \sqrt{\epsilon}Xf'(1) + \epsilon\left(\tfrac{1}{2}X^2+1\right)f''(1) + \epsilon^{3/2}\left(X + \frac{1}{6}X^3\right)f'''(1) \\ + \epsilon^2\left(1 + \frac{1}{2}X^2 + \frac{1}{24}X^4\right)f''''(1) + O\left(\epsilon^{5/2}\right)\,. \end{aligned} \tag{7.100}$$

We hope that the outer expansion of the inner solution at $x = 1$ will match the series above. The rescaled inner problem is

$$Y_{XX} - Y = -f(1+\sqrt{\epsilon}X)\,, \tag{7.101}$$

$$= -f(1) - \sqrt{\epsilon}Xf'(1) - \epsilon\frac{1}{2}X^2 f''(1) + O(\epsilon^{3/2})\,. \tag{7.102}$$

The RPS is

$$Y = f(1)\left(1 - \mathrm{e}^X\right) + \sqrt{\epsilon}Y_1(X) + \epsilon Y_2(X) + \epsilon^{3/2}Y_3(X)O(\epsilon^2)\,, \tag{7.103}$$

with

$$Y_1'' - Y_1 = -Xf'(1)\,, \tag{7.104}$$

$$Y_2'' - Y_2 = -\tfrac{1}{2}X^2 f''(1)\,, \tag{7.105}$$

$$Y_3'' - Y_3 = -\frac{1}{6}X^3 f'''(1)\,. \tag{7.106}$$

We solve the equations above, applying the boundary condition $Y_n(0) = 0$, to obtain

$$Y_1(X) = Xf'(1)\,, \qquad Y_2(X) = \left(1 + \tfrac{1}{2}X^2 - \mathrm{e}^X\right)f''(1)\,, \tag{7.107}$$

$$\text{and} \qquad Y_3(X) = \left(X + \frac{1}{6}X^3\right)f'''(1)\,. \tag{7.108}$$

The inner limit of the leading-order outer solution, $y_0(x) = f(x)$, produces terms at all orders in the matching region. In order to match all of $y_0(x)$ one requires *all* the $Y_n(X)$'s.

7.5 Problems

Problem 7.1. Assuming that $a(x) < 0$, construct the uniformly valid leading-order approximation to the solution of

$$\epsilon y'' + ay' + by = 0\,, \qquad \text{with BCs} \quad y'(0) = p\,, \quad y'(1) = q\,. \tag{7.109}$$

(Consider using linear superposition by first taking $(p,q) = (1,0)$, and then $(p,q) = (0,1)$.)

Problem 7.2. Consider

$$\epsilon y'' + \sqrt{x}y' + y = 0\,, \qquad \text{with BCs} \qquad y(0) = p\,, \qquad y(1) = q\,. \tag{7.110}$$

(i) Find the rescaling for the boundary layer near $x = 0$, and obtain the leading order inner approximation. Then find the leading-order outer approximation and match to determine all constants of integration. *(ii)* Repeat for

$$\epsilon y'' - \sqrt{x}y' + y = 0\,, \qquad \text{with BCs} \qquad y(0) = p\,, \qquad y(1) = q\,. \tag{7.111}$$

Problem 7.3. (a) Consider the nonlinear boundary value problem

$$\epsilon u'' + u' + \frac{1}{2}u^2 = 0\,, \tag{7.112}$$

posed on $0 < x < 1$ with boundary conditions

$$u(0) = 1\,, \qquad \text{and} \qquad u(1) = 1\,. \tag{7.113}$$

(a) Use boundary layer theory to construct a leading order uniformly valid solution in the limit $\epsilon \to 0$.
(b) (b) Consider the boundary value problem

$$\epsilon v'' - v' + \frac{1}{2}v^2 = 0\,, \tag{7.114}$$

posed on $0 < x < 1$ with boundary conditions

$$v(0) = 1\,, \qquad \text{and} \qquad v(1) = 1\,. \tag{7.115}$$

Use boundary layer theory to construct a leading order uniformly valid solution in the limit $\epsilon \to 0$. (c) Find a leading order $\epsilon \to 0$ approximations to $\int_0^1 u(x)\,\mathrm{d}x$ and $\int_0^1 v(x)\,\mathrm{d}x$.

Problem 7.4. Find a leading order, uniformly valid solution of

$$\epsilon y'' + \sqrt{x}y' + y^2 = 0\,, \tag{7.116}$$

posed on $0 < x < 1$ with boundary conditions $y(0, \epsilon) = 2$ and $y(1, \epsilon) = 1/3$.

Problem 7.5. Find a leading order, uniformly valid solution of

$$\epsilon y'' - (1 + 3x^2)y = x\,, \qquad \text{with BCs} \qquad y\,,(0, \epsilon) = y(1, \epsilon) = 1\,. \tag{7.117}$$

Problem 7.6. Find a leading-order, uniformly valid solution of

$$\epsilon y'' - \frac{y'}{1 + 2x} - \frac{1}{y} = 0\,, \qquad \text{with BCs} \qquad y(0, \epsilon) = y(1, \epsilon) = 3\,. \tag{7.118}$$

Problem 7.7. In an earlier problem you were asked to construct a leading order, uniformly valid solution of

$$\epsilon y'' - (1 + 3x^2)y = x \qquad \text{with BCs} \qquad y(0, \epsilon) = y(1, \epsilon) = 1\,. \tag{7.119}$$

Now construct the uniformly valid two-term boundary layer approximation.

Problem 7.8. Consider

$$\epsilon y'' + (1 + \epsilon)y' + y = 0\,, \qquad y(0) = 0\,, \qquad y(1) = e^{-1}\,, \tag{7.120}$$

$$m' = y\,, \qquad m(1) = 0\,. \tag{7.121}$$

Find two terms in the outer expansion of $y(x)$ and $m(x)$, applying only boundary conditions at $x = 1$. Next find two terms in the inner approximation at $x = 0$, applying the boundary condition at $x = 0$. Determine the constants of integration by matching. Calculate $m(0)$ correct to order ϵ.

Problem 7.9. Use boundary-layer theory to construct a leading-order solution of the IVP

$$\epsilon x_{tt} + x_t + x = t\mathrm{e}^{-t}\,, \qquad \text{with} \qquad x(0) = \dot{x}(0) = 0\,, \qquad \text{as } \epsilon \to 0. \tag{7.122}$$

Problem 7.10. Find the leading order $\epsilon \to 0$ solution of

$$\frac{\mathrm{d}u}{\mathrm{d}t} = v\,, \qquad \epsilon\frac{\mathrm{d}v}{\mathrm{d}t} = -v - u^2\,, \tag{7.123}$$

for $t > 0$ with initial conditions $u(0) = 0$ and $v(0) = 1$.

Problem 7.11. Find the leading order $\epsilon \to 0$ solution of

$$\epsilon \ddot{u} + (1+t)\dot{u} + u = 1 \,, \tag{7.124}$$

for $t > 0$ with initial conditions $u(0) = 1$ and $\dot{u}(0) = -\epsilon^{-1}$.

Problem 7.12. A function $y(t, x)$ satisfies the integro-differential equation

$$\epsilon y_t = -y + f(t) + Y(t) \,, \tag{7.125}$$

where

$$Y(t) \stackrel{\text{def}}{=} \int_0^\infty y(t, x) \mathrm{e}^{-\beta x} \, \mathrm{d}x \,, \tag{7.126}$$

with $\beta > 1$. The initial condition is $y(0, x) = a(x)$. (This is the Grodsky model for insulin release.) Use boundary layer theory to find the composite solution on the interval $0 < t < \infty$. Compare this approximate solution with the exact solution of the model. To assist communication, use the notation

$$\alpha \stackrel{\text{def}}{=} 1 - \beta^{-1} \qquad \text{and} \qquad A \stackrel{\text{def}}{=} Y(0) \,, \qquad \text{and} \qquad \tau \stackrel{\text{def}}{=} \frac{t}{\epsilon} \,. \tag{7.127}$$

Problem 7.13. Solve the previous problem with $\beta = 1$.

Problem 7.14. The Michaelis-Menten model for an enzyme catalyzed reaction is

$$\dot{s} = -s + (s + k - 1)c \,, \qquad \epsilon \dot{c} = s - (s + k)c \,, \tag{7.128}$$

where $s(t)$ is the concentration of the substrate and $c(t)$ is the concentration of the catalyst. The initial conditions are

$$s(0) = 1 \,, \qquad c(0) = 0 \,. \tag{7.129}$$

Find the first term in the: *(i)* outer solution; *(ii)* the "initial layer" ($\tau \stackrel{\text{def}}{=} t/\epsilon$); *(iii)* the composite expansion.

Lecture 8

Multiple scale theory

8.1 Introduction to two-timing

Begin by considering the damped harmonic oscillator

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \beta \frac{\mathrm{d}x}{\mathrm{d}t} + x = 0\,, \tag{8.1}$$

with initial conditions

$$x(0) = 0\,, \qquad \text{and} \qquad \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 1\,. \tag{8.2}$$

You should recall that the exact solution is

$$x = \nu^{-1}\mathrm{e}^{-\beta t/2}\sin \nu t\,, \qquad \text{with} \qquad \nu \stackrel{\text{def}}{=} \sqrt{1 - \frac{\beta^2}{4}}\,. \tag{8.3}$$

A good or useful $\beta \ll 1$ approximation to this exact solution is

$$x \approx \mathrm{e}^{-\beta t/2}\sin t\,. \tag{8.4}$$

Let's use this example to motivate the multiple-scale method.

Failure of the regular perturbation expansion

If $\beta \ll 1$ we might be tempted to try an RPS on (8.1):

$$x(t,\beta) = x_0(t) + \beta x_1(t) + \beta^2 x_2(t) + \cdots \tag{8.5}$$

A reasonable goal is to produce the good approximation (8.4). The RPS will not be successful and this failure will drive us towards the method of multiple time a scales, also know as "two timing".

The leading-order problem is

$$\ddot{x}_0 + x_0 = 0\,, \qquad \text{with IC} \qquad x_0 = 0\,, \qquad \dot{x}_0(0) = 1\,. \tag{8.6}$$

The solution is

$$x_0 = \sin t\,. \tag{8.7}$$

The first-order problem is

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + x_1 = -\cos t\,, \qquad \text{with IC} \qquad \bar{x}_1(0) = 0\,, \qquad \frac{\mathrm{d}x_1}{\mathrm{d}t}(0) = 0\,. \tag{8.8}$$

This is a resonantly forced oscillator equation, with solution

$$x_1 = -\frac{t}{2}\sin t\,. \tag{8.9}$$

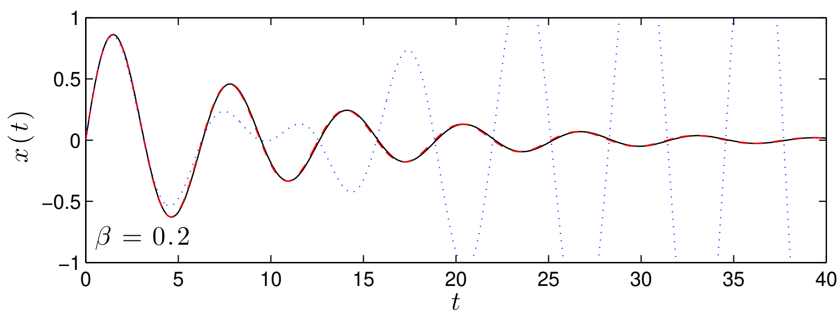

Figure 8.1: Comparison of the exact solution in (8.3) (the solid black curve), with the two-term RPS in (8.10) (the blue dotted curve) and the two-time approximation in (8.23) (the dashed red curve). It is difficult to distinguish the two-time approximation from the exact result.

Thus the developing RPS is

$$x(t,\beta) = \sin t - \frac{\beta t}{2}\sin t + \beta^2 x_2(t) + \cdots \tag{8.10}$$

At this point we recognize that the RPS is misleading: the exact solution damps to zero on a time scale $2/\beta$, while the RPS suggests that the solution is growing linearly with time. With hindsight we realize that the RPS is producing the Taylor series expansion of the exact solution in (8.3) about $\beta = 0$. Using MATHEMATICA, this series is

$$x(t,\beta) = \sin t - \frac{\beta t}{2}\sin t + \frac{\beta^2}{8}\left[t^2 \sin t + \sin t - t\cos t\right] + O(\beta^3)\,. \tag{8.11}$$

Calculating more terms in the RPS will not move us closer to the useful approximation in (8.4): instead we'll grind out the useless approximation in (8.11). In this example the small term in (8.1) is small relative to the other terms at all times. Yet the small error slowly accumulates over long times $\sim \beta^{-1}$. This is a *secular error*.

Two-timing

Looking at the good approximation in (8.4) we are inspired to introduce a *slow time*:

$$s \stackrel{\text{def}}{=} \beta t\,. \tag{8.12}$$

We assume that $x(t,\beta)$ has a perturbation expansion of the form

$$x(t,\beta) = x_0(t,s) + \beta x_1(t,s) + \beta^2 x_3(t,s) + \cdots \tag{8.13}$$

Notice how this differs from the RPS in (8.5).

At each order x_n is a function of both s and t a function of both t and s. To keep track of all the terms we use the rule

$$\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + \beta\partial_s\,, \tag{8.14}$$

and the equation of motion is

$$\left(\partial_t + \beta\partial_s\right)^2 x + \beta\left(\partial_t + \beta\partial_s\right)x + x = 0\,. \tag{8.15}$$

At leading order

$$\beta^0: \qquad \partial_t^2 x_0 + x_0 = 0\,, \qquad \text{with general solution} \qquad x_0 = A(s)\mathrm{e}^{\mathrm{i}t} + A^*(s)\mathrm{e}^{-\mathrm{i}t}\,. \tag{8.16}$$

The "constant of integration" is actually a function of the slow time s. We determine the evolution of this function $A(s)$ at next order[1].

At next order

$$\beta^1: \qquad \partial_t^2 x_1 + x_1 = -2x_{0ts} - x_{0t}\,, \tag{8.18}$$

$$= -2\mathrm{i}A_s \mathrm{e}^{\mathrm{i}t} - \mathrm{i}A\mathrm{e}^{\mathrm{i}t} + c.c. \tag{8.19}$$

Again we have a resonantly forced oscillator. but this time we can prevent the secular growth of x_1 on the fast time scale by requiring that

$$2A_s + A = 0\,. \tag{8.20}$$

Thus the leading-order solution is

$$x_0(s,t) = A_0 \mathrm{e}^{-s/2}\mathrm{e}^{\mathrm{i}t} + A_0^* \mathrm{e}^{-s/2}\mathrm{e}^{-\mathrm{i}t}\,. \tag{8.21}$$

The constant of integration A_0 is determined to satisfy the initial conditions. This requires

$$0 = A_0 + A_0^*\,, \qquad 1 = \mathrm{i}A_0 - \mathrm{i}A_0^*\,, \qquad \Rightarrow \qquad A_0 = \tfrac{1}{2}\mathrm{i}\,. \tag{8.22}$$

Thus we have obtained the good approximation

$$x_0 = \mathrm{e}^{-\beta t/2}\sin t\,. \tag{8.23}$$

Averaging

8.2 The Duffing oscillator

We consider an oscillator with a nonlinear spring

$$m\ddot{x} + k_1 x + k_3 x^3 = 0\,, \tag{8.24}$$

and an initial condition

$$x(0) = x_0 \qquad \dot{x}(0) = 0\,. \tag{8.25}$$

If $k_3 > 0$ then the restoring force is stronger than linear — this is a *stiff spring*. With $k_3 < 0$ we have a *soft spring*

We can non-dimensionalize this problem into the form

$$\ddot{x} + x + \epsilon x^3 = 0\,, \tag{8.26}$$

with the initial condition

$$x(0) = 1 \qquad \dot{x}(0) = 0\,. \tag{8.27}$$

We use this *Duffing oscillator* as an introductory example of the multiple time scale method.

Energy conservation,

$$\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^2 + \tfrac{1}{4}\epsilon x^4 = \underbrace{E}_{=\frac{1}{2}+\frac{1}{4}\epsilon}\,, \tag{8.28}$$

immediately provides a phase-plane visualization of the solution and shows that the oscillations are bounded.

Exercise: Show that in (8.26), $\epsilon = k_3 x_0^2/k_1$.

[1] We could alternatively write the general solution of the leading order problem as

$$x_0 = R\cos(t+\phi)\,, \tag{8.17}$$

where the amplitude R and the phase ϕ are as yet undetermined functions of s. I think the complex notation in (8.16) is a little simpler.

Exercise: Derive (8.28).

The naive RPS

$$x = x_0(t) + \epsilon x_1(t) + \cdots \tag{8.29}$$

leads to

$$\ddot{x}_0 + x_0 = 0\,, \qquad \Rightarrow \qquad x_0 = \cos t\,, \tag{8.30}$$

and at next order

$$\ddot{x}_1 + x_1 = -\cos^3 t\,, \tag{8.31}$$

$$= -\tfrac{1}{8}\left(\mathrm{e}^{3\mathrm{i}t} + 3\mathrm{e}^{\mathrm{i}t} + c.c.\right)\,, \tag{8.32}$$

$$= -\tfrac{1}{4}\cos 3t - \tfrac{3}{4}\cos t\,. \tag{8.33}$$

The x_1-oscillator problem is resonantly forced and the solution will grow secularly, with $x_1 \propto t\sin t$. Thus the RPS fails once $t \sim \epsilon^{-1}$.

Two-timing

Instead of an RPS we use the two-time expansion

$$x = x_0(s,t) + \epsilon x_1(s,t) + \cdots \tag{8.34}$$

where $s = \epsilon t$ is the slow time. Thus the expanded version of (8.26) is

$$\begin{aligned}(\partial_t + \epsilon\partial_s)^2\left(x_0(s,t) + \epsilon x_1(s,t) + \cdots\right) + \left(x_0(s,t) + \epsilon x_1(s,t) + \cdots\right) \\ + \epsilon\left(x_0(s,t) + \epsilon x_1(s,t) + \cdots\right)^3 = 0\end{aligned} \tag{8.35}$$

The leading order is

$$\partial_t^2 x_0 + x_0 = 0\,, \tag{8.36}$$

with general solution

$$x_0 = A(s)\mathrm{e}^{\mathrm{i}t} + A^*(s)\mathrm{e}^{-\mathrm{i}t}\,. \tag{8.37}$$

The amplitude A is a function of the slow time s. At next order, ϵ^1, we have

$$\partial_t^2 x_1 + x_1 = -2\partial_t\partial_s x_0 - x_0^3\,, \tag{8.38}$$

$$= -2\mathrm{i}A_s\mathrm{e}^{\mathrm{i}t} - A^3\mathrm{e}^{3\mathrm{i}t} - 3A^2A^*\mathrm{e}^{\mathrm{i}t} + c.c. \tag{8.39}$$

To prevent the secular growth of x_1 we must remove the resonant terms, $\mathrm{e}^{\pm\mathrm{i}t}$ on the right of (8.39) – this prescription determines the evolution of the slow time:

$$2\mathrm{i}A_s + 3|A|^2A = 0\,. \tag{8.40}$$

The remaining terms in (8.39) are

$$\partial_t^2 x_1 + x_1 = -A^3\mathrm{e}^{3\mathrm{i}t} + c.c. \tag{8.41}$$

The solution will be $x_1 \propto \mathrm{e}^{\pm 3\mathrm{i}t}$ and will remain bounded.

Polar coordinates

To solve (8.40) it is best to transform to polar coordinates

$$A = r(s)\mathrm{e}^{\mathrm{i}\theta(s)} \qquad \text{and} \qquad A_s = (r_s + \mathrm{i}r\theta_s)\,\mathrm{e}^{\mathrm{i}\theta}\,; \tag{8.42}$$

substituting into the amplitude equation (8.40)

$$r_s = 0\,, \qquad \text{and} \qquad \theta_s = \frac{3}{2}r^2\,. \tag{8.43}$$

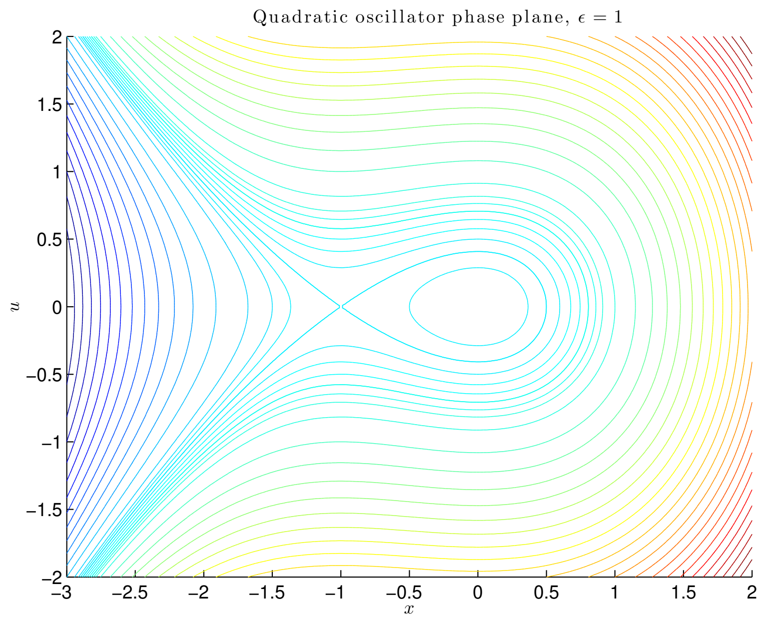

Figure 8.2: Quadratic oscillator phase plane.

The energy of this nonlinear oscillator is constant and thus r is constant, $r(s) = r_0$. The phase $\theta(s)$ therefore evolves as $\theta = \theta_0 + 3r_0^2 s/2$.

The reconstituted solution is

$$x = r_0 \exp\left[\mathrm{i}\left(1 + \tfrac{3}{2}\epsilon r_0^2\right) t + \mathrm{i}\theta_0\right] + c.c. + O(\epsilon)\,. \tag{8.44}$$

The velocity of the oscillator is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{i} r_0 \exp\left[\mathrm{i}\left(1 + \tfrac{3}{2}\epsilon r_0^2\right) t + \mathrm{i}\theta_0\right] + c.c. + O(\epsilon)\,. \tag{8.45}$$

To satisfy the initial condition in (8.27) at leading order, we take $\theta_0 = 0$ and $r_0 = 1/2$. Thus, with this particular initial condition,

$$x = \cos\left[\left(1 + \frac{3\epsilon}{8}\right) t\right] + O(\epsilon)\,. \tag{8.46}$$

The frequency of the oscillator in (8.44),

$$\nu = 1 + \tfrac{3}{2}\epsilon r_0^2\,, \tag{8.47}$$

depends on the amplitude r_0 and the sign of ϵ. If the spring is stiff (i.e., $k_3 > 0$) then ϵ is positive and bigger oscillations have higher frequency.

Exercise: Now investigate nonlinear damping

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \epsilon\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^3 + x = 0\,. \tag{8.48}$$

8.3 The quadratic oscillator

The quadratic oscillator is

$$\ddot{x} + x + \epsilon x^2 = 0\,. \tag{8.49}$$

The conserved energy is

$$E = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^2 + \tfrac{1}{3}\epsilon x^3\,, \tag{8.50}$$

and the curves of constant energy in the phase plane are shown in Figure 8.2.

Following our experience with the Duffing oscillator we try the two-time expansion

$$(\partial_t + \epsilon\partial_s)^2 \left(x_0(s,t) + \epsilon x_1(s,t) + \cdots\right) + \left(x_0(s,t) + \epsilon x_1(s,t) + \cdots\right) \\ + \epsilon\left(x_0(s,t) + \epsilon x_1(s,t) + \cdots\right)^2 = 0\,. \tag{8.51}$$

The leading-order solution is again

$$x_0 = A(s)\mathrm{e}^{\mathrm{i}t} + A^*(s)\mathrm{e}^{-\mathrm{i}t}\,, \tag{8.52}$$

and at next order

$$\partial_t^2 x_1 + x_1 = -2\left(\mathrm{i}A_s\mathrm{e}^{\mathrm{i}t} - \mathrm{i}A_s^*\mathrm{e}^{-\mathrm{i}t}\right) - \underbrace{\left(A^2\mathrm{e}^{2\mathrm{i}t} + 2|A|^2 + A^{*2}\mathrm{e}^{-2\mathrm{i}t}\right)}_{x_0^2}\,. \tag{8.53}$$

Elimination of the resonant terms $\mathrm{e}^{\pm \mathrm{i}t}$ requires simply $A_s = 0$, and then the solution of the remaining equation is

$$x_1 = \tfrac{1}{3}A^2\mathrm{e}^{2\mathrm{i}t} - 2|A|^2 + \tfrac{1}{3}A^{*2}\mathrm{e}^{-2\mathrm{i}t}\,. \tag{8.54}$$

This is why the quadratic oscillator is not used as an introductory example: there is no secular forcing at order ϵ.

To see the effects of nonlinearity in the quadratic oscillator, we must press on to higher orders, and use a slower slow time:

$$s = \epsilon^2 t\,. \tag{8.55}$$

Thus we revise (8.51) to

$$\big(\partial_t + \underbrace{\epsilon^2\partial_s}_{NB}\big)^2 \left(x_0(s,t) + \epsilon x_1(s,t) + \cdots\right) + \left(x_0(s,t) + \epsilon x_1(s,t) + \cdots\right) \\ + \epsilon\left(x_0(s,t) + \epsilon x_1(s,t) + \cdots\right)^2 = 0\,. \tag{8.56}$$

The solutions at the first two orders are the same as (8.52) and (8.54). At order ϵ^2 we have

$$\partial_t^2 x_2 + x_2 = -2\partial_t\partial_s x_0 - 2x_0x_1\,, \tag{8.57}$$

$$= -2\left(\mathrm{i}A_s\mathrm{e}^{\mathrm{i}t} - \mathrm{i}A_s^*\mathrm{e}^{-\mathrm{i}t}\right) - 2\underbrace{\left(A\mathrm{e}^{\mathrm{i}t} + A^*\mathrm{e}^{-\mathrm{i}t}\right)\left(\tfrac{1}{3}A^2\mathrm{e}^{2\mathrm{i}t} - 2|A|^2 + \tfrac{1}{3}A^{*2}\mathrm{e}^{-2\mathrm{i}t}\right)}_{x_0x_1}\,. \tag{8.58}$$

Eliminating the $\mathrm{e}^{\pm\mathrm{i}t}$ resonant terms produces the amplitude equation

$$\mathrm{i}A_s = \frac{5}{3}|A|^2A\,. \tag{8.59}$$

Although the nonlinearity in (8.49) is quadratic, the final amplitude equation in (8.59) is cubic. In fact, despite the difference in the original nonlinear term, the amplitude equation in (8.59) is essentially the same as that of the Duffing oscillator in (8.40).

Example: The Morse oscillator. Using dimensional variables, the Morse oscillator is

$$\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x} = 0 \qquad \text{with the potential} \qquad V = \frac{\nu}{2}\left(1 - \mathrm{e}^{-\alpha x}\right)^2\,. \tag{8.60}$$

The phase plane is shown in figure **??** — the orbits are curves of constant energy

$$E = \frac{1}{2}\dot{x}^2 + \frac{\nu}{2}\left(1 - \mathrm{e}^{-\alpha x}\right)^2\,. \tag{8.61}$$

There is a turning point at $x = \infty$ corresponding to the "energy of escape" $E_{\mathrm{escape}} = \nu/2$.

A "natural" choice of non-dimensional variables is

$$\bar{x} \stackrel{\mathrm{def}}{=} \alpha x\,, \qquad \text{and} \qquad \bar{t} = \alpha\sqrt{\nu}t\,. \tag{8.62}$$

In these variables, $\nu \to 1$ and $\alpha \to 1$ in the barred equations. Thus, the non-dimensional equation of motion is

$$\ddot{x} + \mathrm{e}^{-x}\left(1 - \mathrm{e}^{-x}\right) = 0\,. \tag{8.63}$$

If we're interested in small oscillations around the minimum of the potential at $x = 0$, then the small parameter is supplied by an initial condition such as

$$x(0) = \epsilon\,, \qquad \text{and} \qquad \dot{x}(0) = 0\,. \tag{8.64}$$

We rescale with

$$x = \epsilon X\,, \tag{8.65}$$

so that the equation is

$$\epsilon\ddot{X} + \mathrm{e}^{-\epsilon X}\left(1 - \mathrm{e}^{-\epsilon X}\right) = 0\,, \tag{8.66}$$

or

$$\ddot{X} + X - \epsilon\frac{3}{2}X^2 + \frac{7}{6}\epsilon^2 X^3 = O\left(\epsilon^3\right)\,. \tag{8.67}$$

The multiple time scale expansion is now

$$X = X_0(s,t) + \epsilon X_1(s,t) + \epsilon^2 X_2(s,t) + \cdots \tag{8.68}$$

with slow time $s = \epsilon^2 t$.

The main point of this example is that it is necessary to proceed to order ϵ^2, and therefore to retain the term $7\epsilon^2 X^3/6$, to obtain the amplitude equation. One finds

$$\mathrm{i}A_s = \tag{8.69}$$

8.4 Symmetry and universality of the Landau equation

So far the two-time expansion always leads to the Landau equation

$$A_s = pA + q|A|^2 A\,. \tag{8.70}$$

If you dutifully solve some of the early problems in this lecture you'll obtain (8.70) again and again. Why is that? If we simply list all the terms up to cubic order that *might* occur in an amplitude equation we have

$$\underline{A_s} = \underline{?A} + ?A^* + ?A^2 + ?|A|^2 + ?A^{*2} + ?A^3 + \underline{?|A|^2 A} + ?|A|^2 A^* + ?A^{*3} + \cdots \tag{8.71}$$

The coefficients are denoted by "?" and we're not interested in the precise value of these numbers, except in so far as most of them turn out to be zero. The answer in (8.70) is simple because we have two terms on the right, instead of the nine in (8.71). We've been down in the weeds calculating, but we have not asked the big question why do we have to calculate only the two coefficients p and q?

Remark: Why no terms such as $A^2|A|$ in (8.71)? They're "nonanalytic" because $|A| = \sqrt{AA^*}$. Can one devise an example in which such terms appear?

We have been considering only *autonomous* differential equations, such as

$$\frac{\mathrm{d}x}{\mathrm{d}t} + x + \epsilon x^5 = 0\,. \tag{8.72}$$

This means that if $x(t)$ is a solution of (8.72) then so is $x(t-\alpha)$, where α is *any* constant. In other words, the equations we've been considering are unchanged ("invariant") if

$$t \to t + \alpha\,. \tag{8.73}$$

Now if we try to solve (8.70) with a solution of the form

$$x(t) = A(s)\mathrm{e}^{\mathrm{i}t} + A^*(s)\mathrm{e}^{-\mathrm{i}t} + \epsilon x_1(t,s) + \cdots \tag{8.74}$$

then

$$x(t+\alpha) = A\mathrm{e}^{\mathrm{i}\alpha}\,\mathrm{e}^{\mathrm{i}t} + A^*\mathrm{e}^{-\mathrm{i}\alpha}\,\mathrm{e}^{-\mathrm{i}t} + \epsilon x_1 + \cdots \tag{8.75}$$

Thus the time-translation symmetry of the original differential equation implies that the amplitude equation should be invariant under the rotation

$$A \to A\mathrm{e}^{\mathrm{i}\alpha}\,, \tag{8.76}$$

where α is any constant. Only the underlined terms in (8.71) respect this symmetry and therefore only the underlined terms can appear in the amplitude equation.

Exercise: many of our examples have time reversal symmetry i.e. the equation is invariant under $t \to -t$. For example, the nonlinear oscillator (with no damping) is invariant under $t \to -t$. Show that this implies that p and q in (8.70) must be pure imaginary.

8.5 Parametric instability

Consider an oscillator whose frequency is changing slightly

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \left(\omega^2 + \nu^2 \cos\sigma t\right) x = 0\,. \tag{8.77}$$

How does the small fluctuation $\nu^2 \cos\sigma t$ affect the oscillations?

We live dangerously by rewriting (8.77) as

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \left(\omega^2 + \epsilon \cos\sigma t\right) x = 0\,. \tag{8.78}$$

where $\epsilon = \nu^2$ is the perturbation parameter. Note $\dim\epsilon = \dim\omega^2 = \dim\sigma^2 = T^{-2}$. Let's apply a multiple time scale approximation to (8.77). We'll write the unperturbed ($\epsilon = 0$) solution as

$$x = A\mathrm{e}^{\mathrm{i}\omega t} + A^*\mathrm{e}^{-\mathrm{i}\omega t}\,. \tag{8.79}$$

When we switch on the perturbation the amplitude A becomes a function of slow time $s = \epsilon t$.

Remark: Before doing the algebra, we show that the only amplitude equation consistent with all symmetries of (8.77) is

$$A_s = \mathrm{i}nA + \mathrm{i}mA^*\,, \tag{8.80}$$

where m and n are real.

Because (8.77) is invariant under $x \to \alpha x$ and so the amplitude equation must also be linear:

$$A_s = pA + qA^*\,. \tag{8.81}$$

But (8.77) is also invariant under time reversal $t \to -t$. Time reversal applied to the amplitude-equation ansatz in (8.79) is equivalent to $A \to A^*$. Hence time reversing (8.81)

$$-A_s^* = pA^* + qA\,, \qquad \Rightarrow \qquad A_s^* = -pA^* - qA\,. \tag{8.82}$$

But the complex conjugate of (8.81) is

$$A_s^* = p^*A^* + q^*A\,. \tag{8.83}$$

Comparing (8.82) with (8.83) we conclude that p and q in (8.81) must be pure imaginary.

We attack (8.78) with the multiple scale expansion

$$x = \underbrace{A(s)\mathrm{e}^{\mathrm{i}\omega t} + A^*(s)\mathrm{e}^{-\mathrm{i}\omega t}}_{x_0(s,t)} + \epsilon x_1(s,t) + \cdots \tag{8.84}$$

where $s = \epsilon t$. At order ϵ^1:

$$\left(\partial_t^2 + \omega^2\right) x_1 + 2\mathrm{i}\omega A_s \mathrm{e}^{\mathrm{i}\omega t} - 2\mathrm{i}\omega A_s^* \mathrm{e}^{-\mathrm{i}\omega t} + \tfrac{1}{2}\left(\mathrm{e}^{\mathrm{i}\sigma t} + \mathrm{e}^{-\mathrm{i}\sigma t}\right)\left(A\mathrm{e}^{\mathrm{i}\omega t} + A^*\mathrm{e}^{-\mathrm{i}\omega t}\right) = 0\,. \tag{8.85}$$

Resonant terms are $e^{\pm i\omega t}$. The final term contains frequencies

$$\omega+\sigma\,,\qquad -\omega+\sigma\,,\qquad \omega-\sigma\qquad -\omega-\sigma\,. \tag{8.86}$$

Can any of these four frequencies equal ω and resonantly force x_1? Yes:

$$\omega=-\omega\pm\sigma\quad\Rightarrow\quad \omega=\pm\tfrac{1}{2}\sigma\,. \tag{8.87}$$

Taking $\omega=\frac{1}{2}\sigma$ and eliminating the resonant terms we emerge triumphantly with the amplitude equation

$$2i\tfrac{1}{2}\sigma A_s+\tfrac{1}{2}A_*=0\,,\qquad \text{or}\qquad \sigma A_s=\tfrac{1}{2}iA^*\,, \tag{8.88}$$

implying that

$$\sigma^2 A_{ss}-\tfrac{1}{4}A=0\,. \tag{8.89}$$

Thus A will grow exponentially $\sim e^{s/2\sigma}=e^{\nu^2 t/2\sigma}$. This is *parametric subharmonic instability*: the natural frequency of the oscillator, ω, is one-half is a *subharmonic* of the forcing frequency σ. Equivalently the period of the forcing is half the period of the oscillator. (You pump a swing twice in one period.)

Example: What happens if ω is not exactly equal to one half?

We investigate this by introducing a slight *de-tuning*:

$$\frac{d^2x}{d\bar t^2}+\left(\tfrac{1}{4}+\epsilon\beta+\epsilon\cos\bar t\right)x=0\,. \tag{8.90}$$

Above we have non-dimensionalized time via $\bar t=\sigma t$ and the order parameter is $\epsilon\overset{\text{def}}{=}\nu^2/\sigma^2$. The natural frequency is written as $\omega=\sigma\sqrt{\frac{1}{4}+\epsilon\beta}$. Proceeding as before, we find the amplitude equation

$$A_s=i\beta A+\tfrac{1}{2}iA^*\,,\qquad\Rightarrow\qquad A_{ss}+\left(\beta^2-\tfrac{1}{4}\right)A=0\,. \tag{8.91}$$

If $|\beta|\geq\frac{1}{2}$ then the de-tuning quenches the parametric subharmonic instability.

8.6 The resonantly forced Duffing oscillator

The linear oscillator

First consider the forced linear oscillator

$$\ddot x+\mu\dot x+\omega^2x=f\cos\sigma t\,. \tag{8.92}$$

We can find the "permanent oscillation" with

$$x=Xe^{i\sigma t}+X^*e^{-i\sigma t}\,. \tag{8.93}$$

After some algebra

$$X=\frac{f}{2}\frac{1}{\omega^2-\sigma^2+i\mu\sigma}\,, \tag{8.94}$$

and the squared amplitude of the response is

$$|X|^2=\frac{f^2}{4}\frac{1}{(\omega^2-\sigma^2)^2+\mu^2\sigma^2}\,. \tag{8.95}$$

We view $|X|^2$ as a function of the forcing frequency σ and notice there is a maximum at $\sigma=\omega$ i.e., when the oscillator is resonantly forced. The maximum response, namely

$$\max_{\forall\sigma}|X|=\frac{f}{2\mu\sigma}\,, \tag{8.96}$$

is limited by the damping μ. In the neighbourhood of this peak, where $\omega\approx\sigma$, the amplitude in (8.95) can approximated by the Lorentzian

$$|X|^2\approx\frac{f^2}{4\sigma^2}\frac{1}{4(\omega-\sigma)^2+\mu^2}\,. \tag{8.97}$$

The difference between ω and σ is *de-tuning*.

Nondimensionalization of the nonlinear oscillator

Now consider the forced and damped Duffing oscillator:

$$\ddot{x} + \mu\dot{x} + \omega^2 x + \eta x^3 = f\cos\sigma t\,. \tag{8.98}$$

We're interested in the weakly damped and nearly resonant problem. That is μ/ω is small and σ is close to ω. Inspired by the linear solution we define non-dimensional variables

$$\bar{t} = \sigma t\,, \qquad \text{and therefore} \qquad \frac{\mathrm{d}}{\mathrm{d}t} = \sigma\frac{\mathrm{d}}{\mathrm{d}\bar{t}}\,, \tag{8.99}$$

and the amplitude scaling

$$x = \frac{f}{\mu\sigma}\bar{x}\,. \tag{8.100}$$

(Note particularly the definition of the non-dimensional displacement $\bar{x}$ in (8.100). Naively we might have balanced the restoring force $\omega^2 x$ against the forcing and introduced the non-dimensional displacement

$$\hat{x} \stackrel{\text{def}}{=} \frac{\omega^2 x}{f} = \frac{\mu}{\sigma}\bar{x}\,. \tag{8.101}$$

This is not the most convenient scaling for analysis of the near-resonant excitation of a weakly damped oscillator. As an exercise you can try to repeat the following calculations using this alternative scaling — you'll encounter a problem right at leading order.)

The non-dimensional equation is then

$$\bar{x}_{\bar{t}\bar{t}} + \frac{\mu}{\sigma}\,\bar{x}_{\bar{t}} + \left(\frac{\omega}{\sigma}\right)^2\bar{x} + \frac{\eta f^2}{\mu^2\sigma^4}\,\bar{x}^3 = \frac{\mu}{\sigma}\cos\bar{t}\,. \tag{8.102}$$

To proceed with the perturbation expansion we define the small parameter

$$\epsilon \stackrel{\text{def}}{=} \frac{\mu}{\sigma}\,. \tag{8.103}$$

We must also ensure that the nonlinearity and de-tuning appear at order ϵ^1 in the expansion. We do this by introducing the detuning parameter β and the nonlinearity parameter γ defined by

$$\left(\frac{\omega}{\sigma}\right)^2 = 1 + \epsilon\beta\,, \qquad \text{and} \qquad \frac{\eta f^2}{\mu^2\sigma^4} = \epsilon\gamma\,. \tag{8.104}$$

Dropping the decoration, the non-dimensional equation (8.102) is now

$$x_{tt} + \epsilon x_t + (1+\epsilon\beta)x + \epsilon\gamma x^3 = \epsilon\cos t\,. \tag{8.105}$$

We have used the exact solution of the linear problem to make a non-obvious definition of the non-dimensional amplitude in (8.100). Even in the linear case – $\eta = 0$ in (8.98) – one might not guess that the forcing should be scaled so that it appears at order ϵ in (8.105). In (8.105) we can now take the *distinguished limit* $\epsilon \to 0$ with β and γ fixed and use two-timing to understand the different effects of nonlinearity and de-tuning on a resonantly forced oscillator.

Exercise: Suppose one naively balances the restoring force $\omega^2 x$ against the forcing $f\cos\sigma t$ in (8.98) and therefore introduces the non-dimensional displacement

$$\hat{x} \stackrel{\text{def}}{=} \frac{\omega^2}{f}x = \frac{\omega^2}{\mu\sigma}\bar{x}\,. \tag{8.106}$$

Find the $\hat{x}$-version of (8.105) and explain why it is not suitable for the two time expansion.

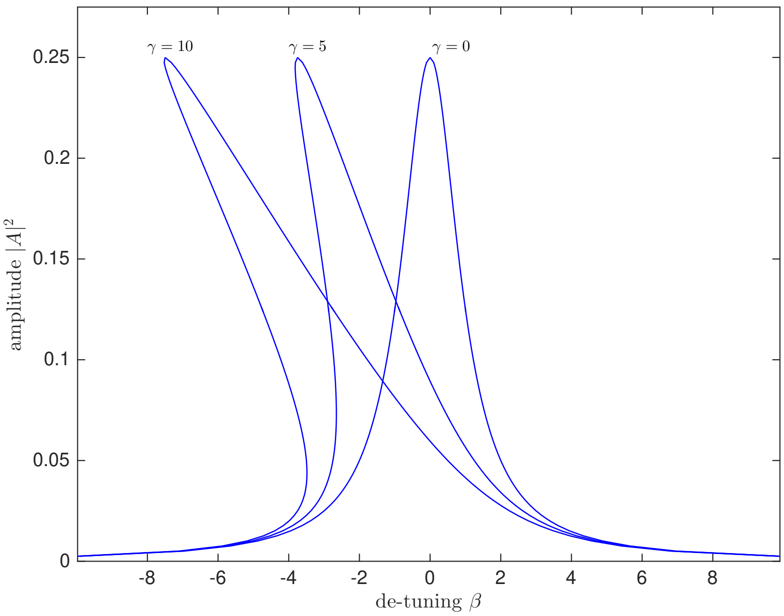

Figure 8.3: The amplitude as a function of detuning obtained from (8.112).

The amplitude equation and its solution

We attack (8.105) with our multiple-scale expansion

$$x = x_0(t,s) + \epsilon x_1(t,s) + \cdots \tag{8.107}$$

with slow time $s = \epsilon t$. The leading-order balance is

$$\left(\partial_t^2 + 1\right) x_0 = 0\,. \tag{8.108}$$

Because of our amplitude scaling in (8.100) the forcing does not appear at this order. The familiar leading order solution is therefore

$$x_0 = A\mathrm{e}^{\mathrm{i}t} + A^*\mathrm{e}^{-\mathrm{i}t}\,. \tag{8.109}$$

At order ϵ^1 we have

$$\left(\partial_t^2 + 1\right) x_1 + 2x_{0st} + x_{0t} + \beta x_0 + \gamma x_0^3 = \tfrac{1}{2}\mathrm{e}^{\mathrm{i}t} + \tfrac{1}{2}\mathrm{e}^{-\mathrm{i}t}\,. \tag{8.110}$$

Eliminating the resonant terms we obtain the amplitude equation

$$2A_s + (1 - \mathrm{i}\beta)A - 3\mathrm{i}\gamma|A|^2 A = -\tfrac{1}{2}\mathrm{i}\,. \tag{8.111}$$

The scaled problem (8.105) has three non-dimensional parameters, ϵ, β and γ. But in the amplitude equation (8.111) only β and γ appear. (Of course ϵ is hidden in the definition of the slow time s.) These perturbation expansions are called *reductive* because they reduce the number of non-dimensional parameters by taking a distinguished limit.

Steady solutions of the amplitude equation

Although (8.111) is simpler than the original forced Duffing equation (8.102), it is still difficult to solve. We begin by looking for special solutions, namely *steady* solution $A_s = 0$. In this case we find

$$|A|^2 = \frac{1}{4}\,\frac{1}{1 + (\beta + 3\gamma|A|^2)^2}\,. \tag{8.112}$$

If we set $\gamma = 0$ we recover a non-dimensional version of our earlier Lorentzian approximation (8.97) to the response curve of a linear oscillator. With non-zero β we can exhibit the response curve, while avoiding the solution of a cubic equation, by solving (8.112) for the de-tuning β as a function of $|A|^2$:

$$\beta = -3\gamma|A|^2 \pm \sqrt{\frac{1}{4|A|^2} - 1}\,. \tag{8.113}$$

Figure 8.4 is constructed by specifying $|A|^2$ and then calculating β from (8.113). There are "multiple solutions" i.e., for the same detuning β there as many as three solutions for $|B|^2$. The middle branch is unstable — the system ends up on either the lower or upper branch, depending on initial conditions. Figure 8.4 illustrates the two different attracting solutions.

Example: Show that the multiple equilibria found in the numerical solution in figure 8.4 agree with the predictons of (8.112).

We have to bite the bullet and solve the cubic equation

$$Q^3 + \cdots \tag{8.114}$$

where $Q \stackrel{\text{def}}{=} |B|^2$.

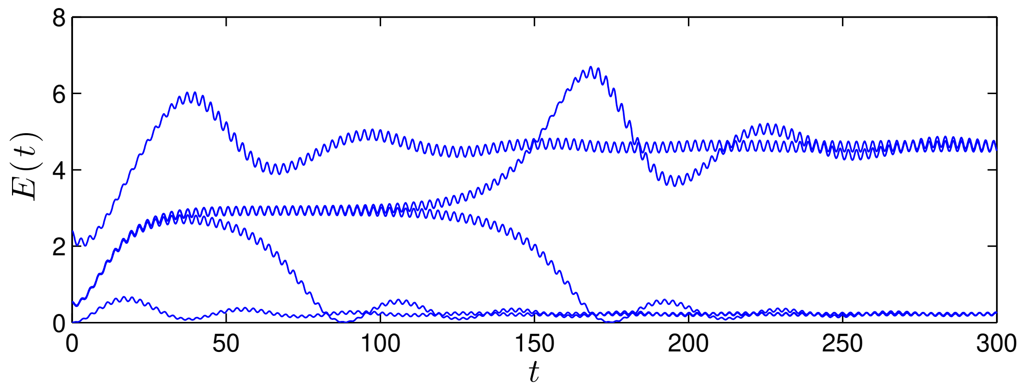

Figure 8.4: Energy $E = (\dot{x}^2 + \omega^2 x^2)/2 + \beta x^4/4$ as a function of time for five `ode45` solutions of the forced Duffing equation (8.98) differing only in initial conditions. There is a high energy attractor that collects two of the solutions, and a low energy attractor that gets the other three solutions. The MATLAB code is below. Note how the differential equation is defined in the nested function `oscill` so that the parameters `om`, `mu` defined in the main function `ForcedDuffing` are passed.

```
function ForcedDuffing
% Multiple solutions of the forced Duffing equation
% Slightly different initial conditions fall on different limit cycles
tspan = [0 300];   om =1;  mu =0.05;   beta = 0.1;   f = 0.25;
sig = 1.2*om;        yinit = [0 1 1.0188 1.0189  2];
for n=1:1:length(yinit)
    yZero=[yinit(n) 0];
    [t,y] = ode45(@oscill,tspan,yZero);
    %Use the energy E as an index of amplitude
    E = 0.5*( om*om* y(:,1).^2 + 0.5*beta*y(:,1).^4 + y(:,2).^2 );
    subplot(2,1,1)         plot(t,E(:))
    xlabel('$t$','interpreter','latex','fontsize',16)
    ylabel('$E(t)$','interpreter','latex','fontsize',16)
    hold on
end
%------------- nested function --------------------%
    function dydt = oscill(t,y)
        dydt = zeros(2,1);
        dydt(1) = y(2);
        dydt(2) = - mu*y(2) - om^2*y(1) - beta*y(1)^3 + f*cos( sig*t );
    end
end
```

8.7 Problems

Problem 8.1. In an early lecture we compared the exact solution of the initial value problem

$$\ddot{f} + (1+\epsilon)f = 0\,, \qquad \text{with ICs} \quad f(0) = 1\,, \quad \text{and} \quad \dot{f}(0) = 0\,, \tag{8.115}$$

with an approximation based on a regular perturbation expansion — see the discussion surrounding (3.69). Redo this problem with a two-time expansion. Compare your answer with the exact solution and explain the limitations of the two-time expansion.

Problem 8.2. Consider

$$\frac{\mathrm{d}^2 g}{\mathrm{d}t^2} + \left[1 + \epsilon\left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)^2\right] g = 0\,, \qquad \text{with ICs} \qquad g(0) = 1\,, \quad \text{and} \quad \frac{\mathrm{d}g}{\mathrm{d}t}(0) = 0\,. \tag{8.116}$$

(i) Show that a RPS fails once $t \sim \epsilon^{-1}$. *(ii)* Use the two-timing method to obtain the solution on the long time scale.

Problem 8.3. Consider the initial value problem:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + u = 2 + 2\epsilon u^2\,, \qquad \text{with ICs} \qquad u(0) = \frac{\mathrm{d}u}{\mathrm{d}t}(0) = 0\,. \tag{8.117}$$

(i) Supposing that $\epsilon \ll 1$, use the method of multiple time scales ($s = \epsilon t$) to obtain an approximate solution valid on times of order ϵ^{-1}. *(ii)* Consider

$$\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + v = u\,, \qquad \text{with ICs} \qquad v(0) = \frac{\mathrm{d}v}{\mathrm{d}t}(0) = 0\,, \tag{8.118}$$

where $u(t, \epsilon)$ on the right is the solution from part *(i)*. Find a leading-order approximation to $v(t, \epsilon)$, valid on the long time scale $t \sim \epsilon^{-1}$.

Problem 8.4. Consider the initial value problem:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} + w = 2\cos(\epsilon t) + 2\epsilon w^2\,, \qquad \text{with ICs} \qquad w(0) = \frac{\mathrm{d}w}{\mathrm{d}t}(0) = 0\,. \tag{8.119}$$

Supposing that $\epsilon \ll 1$, use the method of multiple time scales ($s = \epsilon t$) to obtain an approximate solution valid on times of order ϵ^{-1}.

Problem 8.5. Use multiple scale theory to find an approximate solution of

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x = \mathrm{e}^{\epsilon^2 t} + \epsilon \mathrm{e}^{-\epsilon t} x^2\,, \qquad \text{with ICs} \qquad x(0) = \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0\,, \tag{8.120}$$

valid on the time scale $t \sim \epsilon^{-1} \ll \epsilon^{-2}$.

Problem 8.6. Consider an oscillator parametrically forced at its natural frequency. The model is:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + (1 + \epsilon\cos t)\, x = 0\,; \tag{8.121}$$

show that $x(t)$ will grow exponentially, and calculate the growth-rate. (Following the discussion of parametric instability in the lecture, you won't find resonance at order ϵ^1. So go to higher order.) Study the effect of slightly detuning the frequency: $1 \to 1 + \epsilon^{?}\beta$. How large must β be to prevent exponential growth?

Problem 8.7. *(a)* Use multiple scales to derive a set of amplitude equations for the two coupled, linear oscillators:

$$\begin{aligned} \ddot{x} + 2\epsilon\alpha\dot{x} + (1 + k\epsilon)x &= 2\epsilon\mu(x - y)\,, \\ \ddot{y} + 2\epsilon\beta\dot{y} + (1 - k\epsilon)y &= 2\epsilon\mu(y - x)\,. \end{aligned} \tag{8.122}$$

(b) Consider the special case $\alpha = \beta = k = 0$. Solve both the amplitude equations and the exact equation with the initial condition $x(0) = 1$, $y(0) = \dot{y}(0) = \dot{x}(0) = 0$. Show that both methods give

$$x(t) \approx \cos\left[(1 - \epsilon\mu)t\right]\cos(\epsilon\mu t)\,. \tag{8.123}$$

Problem 8.8. Consider two nonlinearly coupled oscillators:

$$\ddot{x} + 4x = \epsilon y^2\,, \qquad \ddot{y} + y = -\epsilon\alpha x y\,, \tag{8.124}$$

where $\epsilon \ll 1$. *(a)* Show that the nonlinearly coupled oscillators in (1) have an energy conservation law. *(b)* The multiple scale method begins with

$$x(t) = A(s)e^{2it} + \text{c.c.}\,, \qquad y(t) = B(s)e^{it} + \text{c.c.}\,, \tag{8.125}$$

where $s \stackrel{\text{def}}{=} \epsilon t$ is the "slow time" and A and B are "amplitudes". Find the coupled evolution equations for A and B using the method of multiple scales. *(c)* Show that the amplitude equations have a conservation law

$$|B|^2 - 2\alpha|A|^2 = E\,, \tag{8.126}$$

and use this result to show that

$$4A_{ss} - \alpha E A - 2\alpha^2|A|^2 A = 0\,. \tag{8.127}$$

Obtain the analogous equation for $B(s)$. *(d)* Describe the solutions of (8.127) in qualitative terms. Does the sign of α have a qualitative impact on the solution?

Problem 8.9. The equation of motion of a pendulum with length ℓ in a gravitational field g is

$$\ddot{\theta} + \omega^2 \sin\theta = 0\,, \qquad \text{with} \qquad \omega^2 \stackrel{\text{def}}{=} \frac{g}{\ell}\,. \tag{8.128}$$

Suppose that the maximum displacement is $\theta_{\max} = \phi$. *(a)* Show that the period P of the oscillation is

$$\omega P = 2\sqrt{2}\int_0^{\phi} \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\phi}}\,.$$

(b) Suppose that $\phi \ll 1$. By approximating the integral above, obtain the coefficient of ϕ^2 in the expansion:

$$\omega P = 2\pi\left[1 + ?\phi^2 + O(\phi^3)\right]$$

(c) Check this result by re-derving it via a multiple scale expansion applied to (8.128). *(d)* A grandfather clock swings to a maximum angle $\phi = 5^\circ$ from the vertical. How many seconds does the clock lose or gain each day if the clock is adjusted to keep perfect time when the swing is $\phi = 2^\circ$?

Problem 8.10. (**H**) Find a leading order approximation to the general solution $x(t,\epsilon)$ and $y(t,\epsilon)$ of the system

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\epsilon y\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0\,, \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \tfrac{1}{2}\epsilon\ln x^2\,, \tag{8.129}$$

which is valid for $t = O(\epsilon^{-1})$. You can quote the result

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\cos^2\theta\,\mathrm{d}\theta = -\ln 4\,. \tag{8.130}$$

Problem 8.11. (**H**) Find the leading order approximation, valid for times of order ϵ^{-1}, to the solution $x(t,\epsilon)$ and $y(t,\epsilon)$ of the system

$$\ddot{x} + \epsilon y\dot{x} + x = y^2\,, \qquad \text{and} \qquad \dot{y} = \epsilon(1 + x - y - y^2)\,, \tag{8.131}$$

with initial conditions $x = 1$, $\dot{x} = 0$ and $y = 0$.

Lecture 9

Rapid fluctuations, Stokes drift and averaging

9.1 A Lotka-Volterra Example

Consider the Lotka-Volterra equation with a sinusoidally varying carrying capacity. Using non-dimensional variables the problem is

$$\frac{\mathrm{d}n}{\mathrm{d}t} = n\left(1 - \frac{n}{1+\kappa\cos\omega t}\right) . \tag{9.1}$$

Problem 3.5 asked you to analyze this equation with $\kappa \ll 1$ i.e., small fluctuations in the carrying capacity. Here we consider the case of rapid fluctuations: $\omega \to \infty$, with κ fixed and order unity. In this limit the small parameter is

$$\epsilon \overset{\text{def}}{=} \frac{1}{\omega} . \tag{9.2}$$

Figure 9.1 shows a numerical solution with the carrying capacity $1+\kappa\cos\omega t$ varying by a factor of seven over a cycle:. After an initial spin-up the population fluctuates about an average value which is close to $\sqrt{7/16} = 0.6614$. This average population is quite different from the average carrying capacity, namely 1.

Method 1: heuristic averaging

We introduce an average over the fast oscillation

$$\langle A(t)\rangle \overset{\text{def}}{=} \frac{\omega}{2\pi}\int_{t-\pi/\omega}^{t+\pi/\omega} A(t')\,\mathrm{d}t' . \tag{9.3}$$

This average is a low-pass filter: $\langle\rangle$ removes variability with frequencies greater than ω.

The numerical solution $n(t)$ exhibits two time scales: fast wiggles with small amplitude superposed on a slower evolution that looks like a Lotka-Volterra solution. Although the carrying capacity is varying rapidly, the population $n(t)$ hardly reacts to these fast, large-amplitude fluctuations. Clouds blowing overhead lead to modulations in sunlight on the scale of minutes. But plants don't die when the sun is momentarily obscured by a cloud.

It is important that the fast wiggles in $n(t)$ have small amplitude so that

$$\langle n\rangle \approx n , \tag{9.4}$$

and

$$\left\langle \frac{\mathrm{d}n}{\mathrm{d}t}\right\rangle \approx \frac{\mathrm{d}\langle n\rangle}{\mathrm{d}t} . \tag{9.5}$$

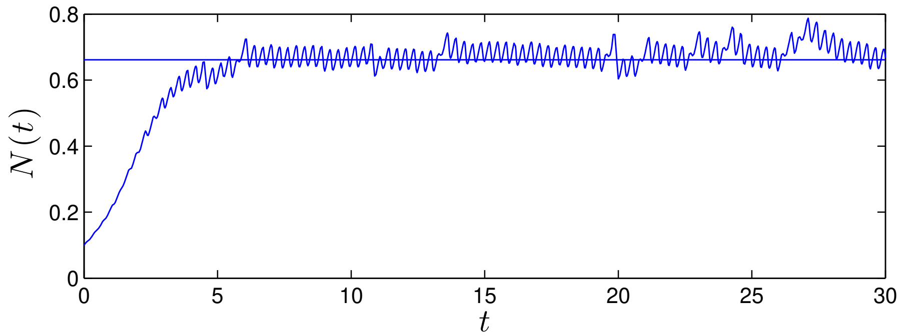

Figure 9.1: Solution of (9.1) with $\omega = 20$ and $\kappa = 3/4$. The blue line is at $\sqrt{7}/4 = 0.6614$.

Thus averaging (9.1)

$$\frac{\mathrm{d}\langle n\rangle}{\mathrm{d}t} = \left\langle n\left(1 - \frac{n}{1+\kappa\cos\omega t}\right)\right\rangle . \tag{9.6}$$

But the fast oscillations in $n(t)$ have small amplitude, so

$$\left\langle n\left(1 - \frac{n}{1+\kappa\cos\omega t}\right)\right\rangle \approx \langle n\rangle\left(1 - \left\langle\frac{1}{1+\kappa\cos\omega t}\right\rangle\langle n\rangle\right) . \tag{9.7}$$

The average of the reciprocal carrying capacity is calculated by invoking a favourite textbook example of the residue theorem

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\tau}{1+\kappa\cos\tau} = \frac{1}{\sqrt{1-\kappa^2}} . \tag{9.8}$$

Putting it all together, (9.6) becomes

$$\frac{\mathrm{d}\langle n\rangle}{\mathrm{d}t} \approx \langle n\rangle\left(1 - \frac{\langle n\rangle}{\sqrt{1-\kappa^2}}\right) . \tag{9.9}$$

Thus, the long time limit is

$$\lim_{t\to\infty}\langle n\rangle = \sqrt{1-\kappa^2} . \tag{9.10}$$

This prediction is in agreement with the MATLAB solution in Figure 9.1.

Method 2: two-timing

Define a fast time

$$\tau \stackrel{\text{def}}{=} \omega t = \frac{t}{\epsilon} , \tag{9.11}$$

and assume that the solution has the multiple time scale expansion

$$n = n_0(t,\tau) + \epsilon n_1(t,\tau) + \cdots \tag{9.12}$$

depending on both t and τ. Thus the expanded version of (9.1) is

$$(\partial_\tau + \epsilon\partial_t)(n_0 + \epsilon n_1) = \epsilon n_0\left(1 - \frac{n_0}{1+\kappa\cos\tau}\right) + O\left(\epsilon^2\right) . \tag{9.13}$$

At leading order

$$\epsilon^0 : \qquad \partial_\tau n_0 = 0 , \qquad \text{with solution} \qquad n_0 = g(t) . \tag{9.14}$$

We decide to define $g(t)$ so that all subsequent terms in the expansion have zero average i.e.,

$$g = \langle n\rangle . \tag{9.15}$$

This definition has the implication that $g(t)$ does not satisfy the initial condition on $n(t)$ i.e., $g(0) \neq n(0)$. (See the discussion of the "guiding center" in section 9.2.)

At next order

$$\epsilon^1: \qquad \partial_\tau n_1 + \frac{\mathrm{d}g}{\mathrm{d}t} = g\left(1 - \frac{g}{1+\kappa\cos\tau}\right). \tag{9.16}$$

Using (9.8), we average the equation above over the fast time scale to obtain

$$\frac{\mathrm{d}g}{\mathrm{d}t} = g\left(1 - \frac{g}{\sqrt{1-\kappa^2}}\right). \tag{9.17}$$

This confirms the earlier heuristic average.

To determine the fluctuations about this average $g(t)$ we can subtract (9.17) from (9.16) to obtain

$$\partial_\tau n_1 = \left(\frac{1}{\sqrt{1-\kappa^2}} - \frac{1}{1+\kappa\cos\tau}\right) g^2. \tag{9.18}$$

To perform the integration and obtain $n_1(\tau)$ one can invoke the Fourier series

$$\frac{1}{\sqrt{1-\kappa^2}} - \frac{1}{1+\kappa\cos\tau} = -\frac{2}{\sqrt{1-\kappa^2}}\sum_{n=1}^{\infty}\left(\frac{\sqrt{1-\kappa^2}-1}{\kappa}\right)^n \cos n\tau. \tag{9.19}$$

This may be more than we need to know. The main point is that the fast fluctuations about the mean solution $g(t)$ scale with ω^{-1} i.e. faster fluctuations in the carrying capacity induce smaller variations in population.

Example: Consider

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\alpha x\cos\omega t. \tag{9.20}$$

Example: Consider

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\alpha(x + \cos^2\omega t). \tag{9.21}$$

9.2 Stokes drift

Consider the motion along the x-axis of a fluid particle in a simple compressive wave e.g., a sound wave. The position of the particle is determined by solving the nonlinear differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u\cos(kx - \omega t), \tag{9.22}$$

with an initial condition $x(0) = a$. We non-dimensionalize this problem by defining

$$\bar{x} \overset{\text{def}}{=} kx \qquad \text{and} \qquad \bar{t} \overset{\text{def}}{=} \omega t. \tag{9.23}$$

The non-dimensional problem is

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}\bar{t}} = \epsilon\cos(\bar{x} - \bar{t}), \qquad \text{with IC} \quad \bar{x}(0) = \bar{a}. \tag{9.24}$$

The non-dimensional wave amplitude,

$$\epsilon \overset{\text{def}}{=} \frac{uk}{\omega}, \tag{9.25}$$

is the ratio of the maximum particle speed u to the phase speed ω/k. We proceed dropping all bars.

Figure 9.2 shows some numerical solutions of (9.24) with $\epsilon = 0.3$. Even though the time-average velocity at a fixed point is zero there is a slow motion of the particles along the x-axis with constant average velocity. If one waits long enough then a particle will move very far from its initial position and travel through many wavelengths.

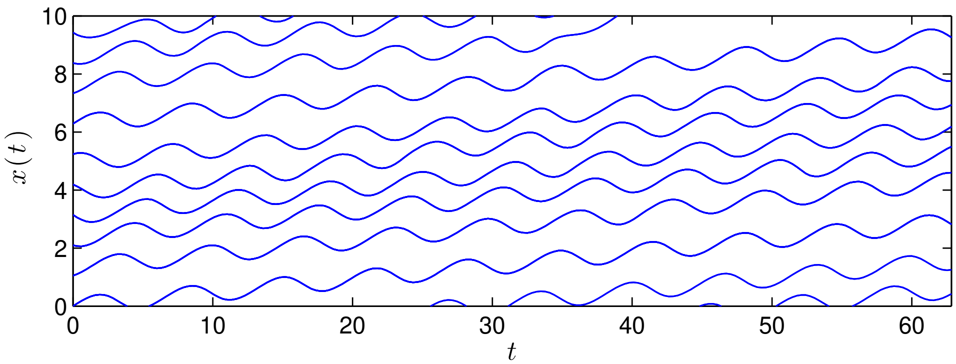

Figure 9.2: Solutions of $\dot{x} = 0.3\cos(x - t)$.

Method 1: RPS to obtain a one-period map

Fortunately the velocity is a periodic function of time. Given the initial condition $x(0)$, we use a straightforward RPS to determine the *one-period map* F:

$$x(2\pi) = F\left[x(0), \epsilon\right] . \tag{9.26}$$

The position of the particle over many periods follows by iteration of this map. That is

$$x(4\pi) = F\left[x(2\pi), \epsilon\right] , \qquad x(6\pi) = F\left[x(4\pi), \epsilon\right] , \qquad \text{etc.} \tag{9.27}$$

We don't have to worry about secular errors because F is determined by solving the differential period over a *finite* time $0 < t < 2\pi$. The second period is just the same as the first.

Method 2: two-timing

To analyze this problem with multiple scale theory we introduce

$$s \stackrel{\text{def}}{=} \epsilon^2 t \,. \tag{9.28}$$

Why ϵ^2 above? Because we tried ϵ^1 and found that there were no secular terms on this time scale.

Exercise: Assume that $s = \epsilon t$ and repeat the following calculation. Does it work?

With the slow time s, the dressed-up problem is

$$\epsilon^2 x_s + x_t = \epsilon\cos(x - t) \,. \tag{9.29}$$

We now go to town with the RPS:

$$x = x_0(s,t) + \epsilon x_1(s,t) + \epsilon^2 x_2(s,t) + \cdots \tag{9.30}$$

Notice that

$$\begin{aligned}\cos(x - t) = \cos(x_0 - t) &- \sin(x_0 - t)\left[\epsilon x_1(s,t) + \epsilon^2 x_2(s,t) + \cdots\right] \\ &- \cos(x_0 - t)\left[\epsilon x_1(s,t) + \epsilon^2 x_2(s,t) + \cdots\right]^2 + \cdots\end{aligned} \tag{9.31}$$

We cannot assume that x_0 is smaller than one, so must keep $\cos(x_0 - t)$ and $\sin(x_0 - t)$. We are assuming the higher order x_n's are bounded, and since $\epsilon \ll 1$ we can expand the sinusoids as above.

At leading order, ϵ^0:

$$x_{0t} = 0 \,, \qquad \Rightarrow \qquad x_0 = f(s) \,. \tag{9.32}$$

The function $f(s)$ is the slow drift. At next order, ϵ^1:

$$x_{1t} = \cos(f - t) \qquad \Rightarrow \qquad x_1 = \sin f - \sin(f - t)\,. \tag{9.33}$$

We determined the constant of integration above so that x_1 is zero initially i.e., we are saying that $f(0)$ is equal to the initial position of the particle.

At ϵ^2

$$f_s + x_{2t} = -\sin(f - t)\underbrace{\left[\sin f - \sin(f - t)\right]}_{=x_1}\,. \tag{9.34}$$

Averaging over the fast time t we obtain

$$f_s = \left\langle \sin^2(f - t)\right\rangle = \frac{1}{2}\,. \tag{9.35}$$

Thus the average position of the particle is

$$f = \frac{s}{2} + a = \frac{\epsilon^2}{2}t + a\,. \tag{9.36}$$

The prediction is that the averaged velocity in figure 9.2 is $(0.3)^2/2 = 0.045$. You can check this by noting that the final time is 20π.

Subtracting (9.35) from (9.34) we have the remaining oscillatory terms:

$$x_{2t} = -\sin(f - t)\sin f - \sin(2f - 2t)\,. \tag{9.37}$$

Integrating and applying the initial condition we have

$$x_2 = -\cos(f - t)\sin f + \tfrac{1}{4}\cos(2f - 2t) + \cos f \sin f - \tfrac{1}{4}\cos 2f\,. \tag{9.38}$$

This is bounded and all is well.

The solution we've constructed consists of a slow drift and a rapid oscillation about this slowly evolving mean position. Note however that the mean position of the particle is

$$\langle x\rangle = f + \epsilon\underbrace{\sin f}_{\langle x_1\rangle} + \epsilon^2\underbrace{\left[\tfrac{1}{2}\sin 2f - \tfrac{1}{4}\cos 2f\right]}_{\langle x_2\rangle} + O(\epsilon^2) \tag{9.39}$$

In other words, the mean position is not the same as the leading-order term.

Method 3: two-timing and the "guiding center"

In this variant we use the two-timing but insist that the leading-order term is the mean position of the particle. This means that the leading-order solution no longer satisfies the initial condition, and that constants of integration at higher orders are determined by insisting that

$$\forall n \geq 1: \qquad \langle x_n\rangle = 0\,. \tag{9.40}$$

OK, let's do it, starting with the scaled two-time equation in (9.29). The leading order is

$$x_{0t} = 0\,, \qquad \Rightarrow \qquad x_0 = g(s)\,. \tag{9.41}$$

The function $g(s)$ is the "guiding center" — it's different from $f(s)$ in the previous method.

At next order, ϵ^1:

$$x_{1t} = \cos(g - t) \qquad \Rightarrow \qquad x_1 = -\sin(g - t)\,. \tag{9.42}$$

This is not the same as the first-order term in (9.33): in (9.42) we have determined the constant of integration so that $\langle x_1\rangle = 0$.

At order ϵ^2 we have

$$g_s + x_{2t} = \sin^2(g - t) = \tfrac{1}{2} - \tfrac{1}{2}\cos(2g - 2t)\,. \tag{9.43}$$

The average of (9.43) is the motion of the guiding center:

$$g_s = \tfrac{1}{2} \qquad \Rightarrow \qquad g = \frac{\epsilon^2}{2} t + g(0)\,. \tag{9.44}$$

The oscillatory part of the solution, with zero time average, is

$$x_2 = \tfrac{1}{4}\sin(2g - 2t)\,. \tag{9.45}$$

Now we must satisfy the initial conditions by requiring that

$$a = g(0) - \epsilon \sin\left(g(0)\right) + \epsilon^2 \tfrac{1}{4} \sin\left(2g(0)\right) + \cdots \tag{9.46}$$

We can invert this series to obtain

$$g(0) = a + \epsilon \sin a + \cdots \tag{9.47}$$

I prefer this guiding-center method. But in either case the essential point is that the leading-order drift velocity is $\epsilon^2/2$.

9.3 The Kapitsa pendulum

If you google the "Kapitsa pendulum" you'll find videos which show the stabilization of an inverted pendulum by a rapidly vibrating point of support. The equation of motion of this system is

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} + \left(\frac{g}{\ell} + \frac{a\nu^2}{\ell}\cos\nu t\right)\sin\varphi = 0\,, \tag{9.48}$$

where a is the amplitude of vibration and ν is the frequency. There is a steady solution $\varphi = \pi$ and in the absence of vibration ($a = 0$) this solution is unstable.

Let's scale the equation and show that if ν is large enough then solution $\varphi = \pi$ becomes stable. We begin the analysis by introducing a nondimensional time

$$\bar{t} \stackrel{\text{def}}{=} \omega t\,, \tag{9.49}$$

where $\omega \stackrel{\text{def}}{=} \sqrt{g/\ell}$ is the linear frequency of the pendulum. The non-dimensional equation of motion can then be written as

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\bar{t}^2} + \left[1 + \frac{\alpha}{\epsilon}\cos\left(\frac{\bar{t}}{\epsilon}\right)\right]\sin\varphi = 0\,, \tag{9.50}$$

where

$$\epsilon \stackrel{\text{def}}{=} \frac{\omega}{\nu}\,, \qquad \text{and} \qquad \alpha \stackrel{\text{def}}{=} \frac{a\omega\nu}{g}\,. \tag{9.51}$$

To investigate the stability of $\varphi = \pi$ we introduce $\theta = \pi - \varphi$ so that

$$\sin\varphi = -\sin\theta = -\theta + O(\theta^3)\,. \tag{9.52}$$

Dropping the bar on the nondimensional time, the linearized problem is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} - \left(1 + \frac{\alpha}{\epsilon}\cos\left(\frac{t}{\epsilon}\right)\right)\theta = 0\,. \tag{9.53}$$

We take the distinguished limit $\epsilon \to 0$ with α fixed.

Introduce the fast time $\tau = t/\epsilon$ so that

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{\epsilon}\partial_\tau + \partial_t\,, \tag{9.54}$$

and use the two-time expansion

$$\theta = \theta_0(t,\tau) + \epsilon\theta_1(t,\tau) + \cdots \tag{9.55}$$

The expanded equation of motion is

$$\left(\partial_\tau + 2\epsilon\partial_t\partial_\tau + \epsilon^2\partial_t^2\right)\left(\theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2\right) - \epsilon^2\theta_0 - \epsilon\alpha\cos\tau\,\left(\theta_0 + \epsilon\theta_1\right) = O\left(\epsilon^3\right)\,. \tag{9.56}$$

At order ϵ^0 we have

$$\partial_\tau^2\theta_0 = 0\,, \qquad \Rightarrow \qquad \theta_0 = S(t)\,. \tag{9.57}$$

Above, at leading order, the leading order displacement is a function of only the slow time t. At next order ϵ^1:

$$\partial_\tau^2\theta_1 + 2\underbrace{\partial_t\partial_\tau S}_{=0} - \alpha S\,\cos\tau = 0\,, \qquad \Rightarrow \qquad \theta_1 = -\alpha S\cos\tau\,. \tag{9.58}$$

At order ϵ^2:

$$\partial_\tau^2\theta_2 + 2\underbrace{\partial_t\partial_\tau\theta_1}_{\alpha S_t\sin\tau} + \partial_t^2 S - S - \underbrace{\alpha\cos\tau\,\theta_1}_{-\alpha^2 S\cos^2\tau} = 0\,. \tag{9.59}$$

Averaging the equation above we find that

$$S_{tt} - \left(1 - \tfrac{1}{2}\alpha^2\right)S = 0\,. \tag{9.60}$$

The inverted pendulum is stable if $\alpha > \sqrt{2}$.

The averaging theorem

There is theorem[a] that can be used to justify the method of averaging. Consider the differential equation

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \epsilon \boldsymbol{f}(\boldsymbol{x}, t, \epsilon)\,, \tag{1}$$

where $\boldsymbol{f}$ is periodic in time with period p:

$$\boldsymbol{f}(\boldsymbol{x}, t + p, \epsilon) = \boldsymbol{f}(\boldsymbol{x}, t + p, \epsilon)\,. \tag{2}$$

With enough assumptions regarding the smoothness of $\boldsymbol{f}$, there exists an $\epsilon_0 > 0$ and a function $\boldsymbol{u}(\boldsymbol{x}, t, \epsilon)$ such that if $|\epsilon| < \epsilon_0$ then the change of variables

$$\boldsymbol{x} = \boldsymbol{y} + \epsilon \boldsymbol{u}(\boldsymbol{x}, t, \epsilon)\,, \tag{3}$$

transforms (1) into

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = \epsilon \bar{\boldsymbol{f}}(\boldsymbol{y}) + \epsilon^2 \boldsymbol{f}_1(\boldsymbol{y}, t, \epsilon)\,. \tag{4}$$

In (4)

$$\bar{\boldsymbol{f}}(\boldsymbol{y}) \stackrel{\text{def}}{=} \frac{1}{p}\int_0^p \boldsymbol{f}(\boldsymbol{y}, t, 0)\,\mathrm{d}t \tag{5}$$

is the average of $\boldsymbol{f}$. The change of variables from $\boldsymbol{x}$ to $\boldsymbol{y}$ in (3) is a *near-identity transformation* that takes (1) into (4) exactly. Neglecting the ϵ^2 term in (4) we obtain an approximate *autonomous* differential equation

$$\frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}t} = \epsilon \bar{\boldsymbol{f}}(\boldsymbol{z})\,. \tag{6}$$

We hope that the solution $\boldsymbol{z}(t)$ of (6) approximates the solution $\boldsymbol{y}(t)$ of (4) over some long time interval. This hope is justified if the solutions of (4) are stable when subject to the small, order ϵ^2, persistent $\boldsymbol{f}_1(\boldsymbol{y}, t, \epsilon)$ perturbations.

[a] See Sanders & Verhulst *Averaging Methods in Nonlinear Dynamical Systems*

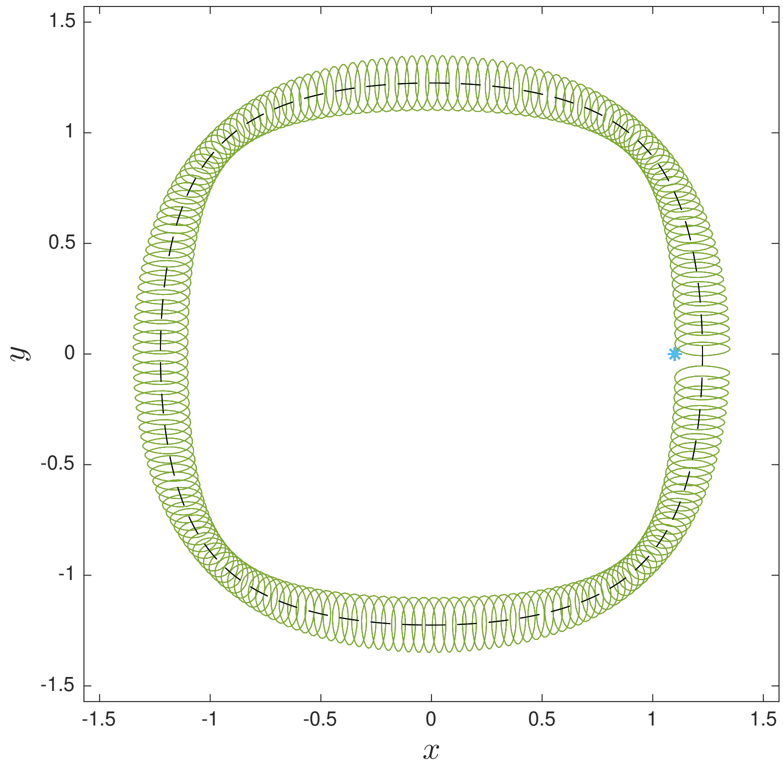

Figure 9.3: The solid curve is the numerical solution of (9.62), with $\omega = 8$; the initial condition is indicated by the blue $*$; the system has been integrated from $t = 0$ to $t = 203.7 \times 2\pi/8$ i.e., almost 204 periods of $\cos \omega t$. The black dashed curve is the guiding center.

9.4 Problems

Problem 9.1. Consider the *nonlinear* inverted pendulum

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} - \left[1 + \frac{\alpha}{\epsilon}\cos\left(\frac{t}{\epsilon}\right)\right]\sin\theta = 0\,. \tag{9.61}$$

Apply the two-time method to this nonlinear equation and find the effective potential resulting from averaging the rapid oscillations. Calculate the phase-plane orbits in the effective potential.

Problem 9.2. A solution of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sin\omega t\,\cos y\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \cos\omega t\,\cos x\,, \tag{9.62}$$

with $\omega = 8$ and initial condition $[x(0), y(0)] = [1.1, 0]$ is shown in figure 9.3. Find an expression for the trajectory of the guiding center (the black dashed curve).

Problem 9.3. As a generalization of problem 9.2, investigate Stokes drift in the two-dimensional incompressible velocity field with streamfunction

$$\psi = a(x,y)\cos(t/\epsilon) + b(x,y)\sin(t/\epsilon)\,, \tag{9.63}$$

and velocity

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\psi_y\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \psi_x\,. \tag{9.64}$$

Obtain an expression for the streamfunction of the Stokes flow in terms of a and b. Check your answer by showing that if $a = b$ there is no Stokes drift.

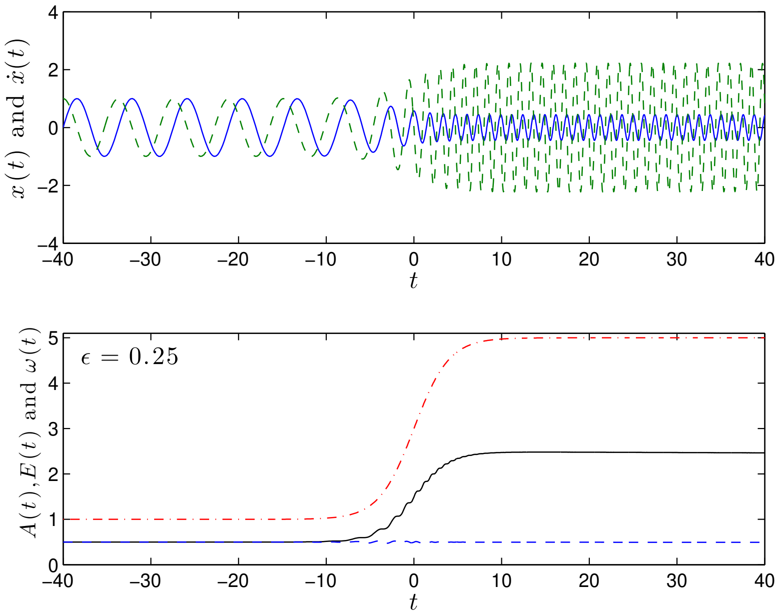

Figure 9.4: Solution of (9.65) and (9.66) with `ode45`. In the lower panel the red dashed curve is $\omega(t)$, the black curve is the energy $E(t)$ and the almost constant action is the blue dashed curve.

Problem 9.4. Consider an oscillator with a slowly changing frequency $\omega(\epsilon t)$:

$$\ddot{x} + \omega^2 x = 0\,. \tag{9.65}$$

Use the method of averaging to show that the action $A \stackrel{\text{def}}{=} E/\omega$ is approximately constant. Test this result with `ode45` using the frequency

$$\omega(t) = 3 + 2\tanh\left(\epsilon t\right)\,, \tag{9.66}$$

and the initial condition $x(-40) = 0$ and $\dot{x}(-40) = 1$ e.g. see Figure 9.4. Use several values of ϵ to test action conservation e.g. try to break the constant-action approximation with large ϵ.

Problem 9.5. A multiple scale ($0 < \epsilon \ll 1$) reduction of Hinch's crazy oscillator system

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\epsilon y \frac{\mathrm{d}x}{\mathrm{d}t} + x = 0\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \tfrac{1}{2}\epsilon x^2\,, \tag{9.67}$$

begins with

$$x = \left[A(s)\mathrm{e}^{\mathrm{i}t} + A^*(s)\mathrm{e}^{-\mathrm{i}t}\right] + \epsilon x_1(t,s) + \cdots\,, \qquad y = B(s) + \epsilon y_1(t,s) + \cdots \tag{9.68}$$

where $s = \epsilon t$ is the slow time. *(i)* Find coupled evolution equations for $A(s)$ and $B(s)$. *(ii)* Figure 9.5 shows a numercial solution of (9.67) with the initial conditions

$$x(0) = 2\,, \qquad \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0\,, \qquad y(0) = 0\,. \tag{9.69}$$

Explain why $\lim_{t\to\infty} y = 1$.

Problem 9.6. Consider particle motion in the incompressible velocity field $(u, v) = (-\psi_y, \psi_x)$ obtained from

$$\psi(x, y, t) = \alpha \sin y \cos(x - t) \tag{9.70}$$

with $\alpha \ll 1$. Some Lagrangian trajectories, computed with MATLAB, are shown in the figure 9.6. Find the mean Lagrangian velocity and discuss the agreement with the numerical solution.

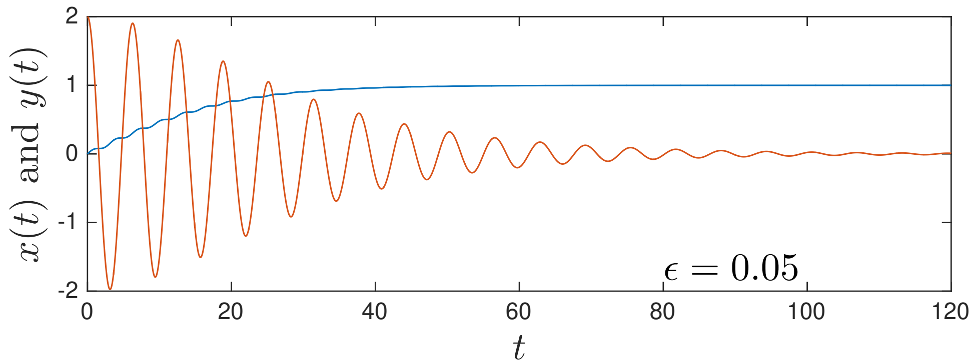

Figure 9.5: Numerical solution of (9.67).

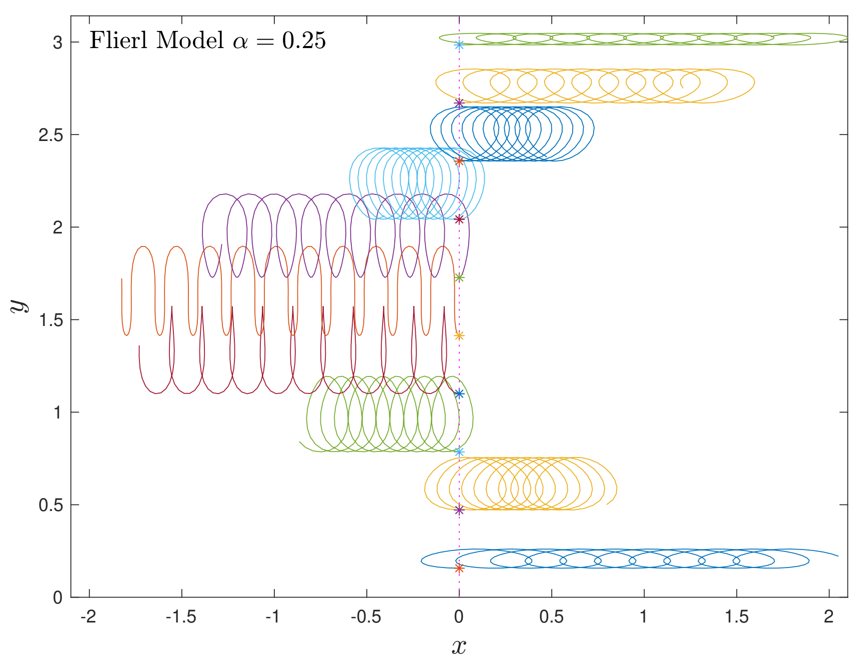

Figure 9.6: Nine particle trajectories computed over the interval $0 < t < 20\pi$. Initial positions are $*$.

Lecture 10

WKB

10.1 The WKB approximation

Suppose we need to solve

$$\epsilon^2 y'' + q(x)y = 0\,, \quad \text{as } \epsilon \to 0. \tag{10.1}$$

The approximate WKB solution to this singular perturbation problem is

$$y \approx \frac{A}{q^{1/4}} \exp\left[\frac{\mathrm{i}}{\epsilon}\int^x \sqrt{q(t)}\,\mathrm{d}t\right] + \frac{B}{q^{1/4}} \exp\left[-\frac{\mathrm{i}}{\epsilon}\int^x \sqrt{q(t)}\,\mathrm{d}t\right]\,, \tag{10.2}$$

or equivalently

$$y \approx \frac{E}{q^{1/4}} \cos\left[\frac{1}{\epsilon}\int^x \sqrt{q(t)}\,\mathrm{d}t\right] + \frac{F}{q^{1/4}} \sin\left[\frac{1}{\epsilon}\int^x \sqrt{q(t)}\,\mathrm{d}t\right]\,. \tag{10.3}$$

The constructions above are most convenient if $q(x) > 0$ so that the solution of (10.1) is oscillatory. But WKB also works if $q(x) < 0$ and in that case the approximation is

$$y \approx \frac{A}{|q|^{1/4}} \exp\left[\frac{1}{\epsilon}\int^x \sqrt{|q(t)|}\,\mathrm{d}t\right] + \frac{B}{|q|^{1/4}} \exp\left[-\frac{1}{\epsilon}\int^x \sqrt{|q(t)|}\,\mathrm{d}t\right]\,. \tag{10.4}$$

We have not specified the lower limits of the integrals in (10.1) through (10.4). Different choices amount to altering the constants A, B et cetera.

The approximations in (10.2) through (10.4) fail in the neighbourhood of x_* where $q(x_*) = 0$. The point x_* is called a *turning point*. We'll need a different approximation in the vicinity of a turning point. But everywhere else WKB provides a spectacular $\epsilon \to 0$ approximation.

Exercise: Check the special cases $q = 1$ and $q = -1$ and commit the WKB approximation to memory.

10.2 The WKB series

Following **BO**, an expeditious route to the approximations in (10.2) through (10.4) is to make the exponential substitution

$$y = \exp\left(\frac{\mathrm{i}s}{\epsilon}\right) \tag{10.5}$$

in (10.1). One finds that the phase function $s(x)$ satisfies the Ricatti equation

$$\mathrm{i}\epsilon s'' - s'^2 + q = 0\,. \tag{10.6}$$

We've "nonlinearized" the linear equation (10.1). The advantage is that (10.6) has an obvious $\epsilon \to 0$ two-term dominant balance. This motivates the RPS

$$s = s_0(x) + \epsilon s_1(x) + \epsilon^2 s_2(x) + \cdots \tag{10.7}$$

The first four terms in this "WKB hierarchy" are

$$s_0'^2 = q\,, \tag{10.8}$$

$$\mathrm{i}s_0'' - 2s_0's_1' = 0\,, \tag{10.9}$$

$$\mathrm{i}s_1'' - 2s_0's_2' - s_1'^2 = 0\,, \tag{10.10}$$

$$\mathrm{i}s_2'' - 2s_0's_3' - 2s_1's_2' = 0\,. \tag{10.11}$$

The solution of the first two equations is

$$s_0 = \pm \int^x \sqrt{q(t)}\,\mathrm{d}t\,, \tag{10.12}$$

$$s_1 = \tfrac{\mathrm{i}}{4}\ln q\,. \tag{10.13}$$

Using the two terms above, we have

$$y = \exp\left[\pm\frac{\mathrm{i}}{\epsilon}\int^x \sqrt{q(t)}\,\mathrm{d}t - \frac{1}{4}\ln q + O(\epsilon)\right]\,, \tag{10.14}$$

$$= q^{-1/4}\exp\left[\pm\frac{\mathrm{i}}{\epsilon}\int^x \sqrt{q(t)}\,\mathrm{d}t\right]\,[1 + O(\epsilon)]\,. \tag{10.15}$$

Linearly combining the two solutions above we obtain (10.2) and (10.3). **BO** refer to the two term approximation above as *physical optics* (PO).

A necessary condition for the validity of the WKB approximation

We launched our peturbation expansion by neglecting the term $\epsilon s''$ relative to s'^2. Thus a necessary condition for the validity of the PO approximation is that

$$\frac{\epsilon s_0''}{{s_0'}^2} \to 0 \qquad \text{as } \epsilon \to 0. \tag{10.16}$$

Suppose that $q(x)$ has a simple zero at x_* i.e. $q \propto x - x_*$. Then $s_0' \propto (x - x_*)^{1/2}$ and $s_0'' \propto (x - x_*)^{-1/2}$. The condition in (10.16) is therefore

$$\frac{\epsilon}{(x - x_*)^{3/2}} \to 0 \qquad \text{as } \epsilon \to 0. \tag{10.17}$$

So, in order to apply the PO approximation, we must ensure that x is at distance greater than $\epsilon^{2/3}$ from the turning point at x_*.

Exercise: Suppose that $q \propto (x - x_*)^m$. Show that validity of PO requires that $x - x_* \gg \epsilon^{2/(m+2)}$.

In physical problems involving wave propagation through a spatially inhomogeneous medium (e.g., sound in the ocean) x has the dimensions of length and

$$k \overset{\text{def}}{=} \frac{\sqrt{q}}{\epsilon} \tag{10.18}$$

is a spatially varying wavenumber with dimensions (length)$^{-1}$. The argument of the sinusoidal functions in (10.3), namely

$$\int^x k(t)\,\mathrm{d}t\,, \tag{10.19}$$

is dimensionless. Suppose the wavenumber $k(x)$ changes over a length ℓ and $k(x)$ has a typical order of magnitude K. This means that waves have a typical wavelength $2\pi/K$. The medium is *slowly changing* if

$$\epsilon \overset{\text{def}}{=} \frac{1}{K\ell} \ll 1 \tag{10.20}$$

i.e. if the length of the waves is much less than the scale ℓ over which medium varies. Another way to look at this is to rewrite the condition for the validity of PO in (10.16) as

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{k} \ll 1\,. \tag{10.21}$$

It is easy remember (10.21) because the left hand side of (10.21) is dimensionless and the inequality says that the rate of change of the local wavelength with distance x is much less than one. We return to this perspective on WKB in the problems.

We'll examine the validity of the WKB approximation in much more detail below. But first we develop some confidence in the approximation by comparing WKB solutions with numerical solutions.

10.3 Some examples

The oscillatory case

Let's apply the WKB approximation to

$$\epsilon^2 y'' + \underbrace{\left(1+\beta \mathrm{e}^{-x}\right)}_{q(x)} y = 0\,, \tag{10.22}$$

with initial conditions

$$y(0) = 0\,, \qquad \text{and} \qquad y'(0) = 1\,. \tag{10.23}$$

In (10.22), β is a parameter which is fixed as $\epsilon \to 0$; the wavenumber $\sqrt{q}/\epsilon$ varies from $\sqrt{1+\beta}/\epsilon$ at $x = 0$ to $1/\epsilon$ as $x \to \infty$.

The leading-order phase function is

$$s_0 = \int_0^x \sqrt{1+\beta \mathrm{e}^{-t}}\,\mathrm{d}t\,, \tag{10.24}$$

$$= x + 2\sqrt{1+\beta} - 2\sqrt{1+\beta\mathrm{e}^{-x}} + 2\ln\frac{1+\sqrt{1+\beta\mathrm{e}^{-x}}}{1+\sqrt{1+\beta}}\,. \tag{10.25}$$

Because the initial conditions are imposed at $x = 0$, it is convenient to use 0 as the lower limit in the phase integral on the right of (10.24): $s_0(0) = 0$. We construct the WKB approximation using the sinusoidal form in (10.3), and secure the initial condition $y(0) = 0$ by setting $E = 0$:

$$y_{\mathrm{WKB}} = \frac{F}{\sqrt{s_0'}}\sin\left(\frac{s_0}{\epsilon}\right)\,. \tag{10.26}$$

We must determine F so that $y_{\mathrm{WKB}}'(0) = 1$. This calculation is easy: to leading order

$$y_{\mathrm{WKB}}' = \frac{F\sqrt{s_0'}}{\epsilon}\cos\left(\frac{s_0}{\epsilon}\right)\,. \tag{10.27}$$

Mercifully, to obtain a consistent approximation to y_{WKB}' we don't differentiate the $1/\sqrt{s_0'}$ amplitude in (10.26): those terms are much less than the $1/\epsilon$ produced by differentiating the phase s_0/ϵ.

At $x = 0$ we have $s_0' = \sqrt{1+\beta}$ and therefore (10.26) implies $1 = \sqrt{1+\beta}F/\epsilon$. Thus the physical optics approximation is

$$y_{\mathrm{WKB}} = \frac{\epsilon\,\sin\left(s_0/\epsilon\right)}{\sqrt{(1+\beta)s_0'}}\,, \tag{10.28}$$

where the phase $s_0(x)$ is given in (10.25).

Example: Solve the differential (10.22) exactly and compare y_{WKB} to a numerical solution of the initial value problem.

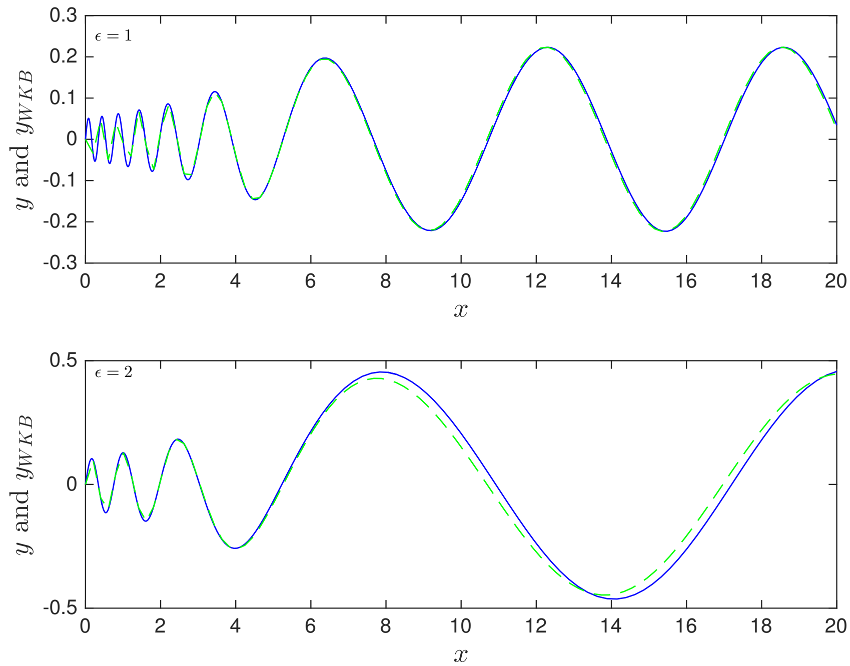

Figure 10.1: Comparison of the WKB approximation (green dashed) in (10.28) with a MATLAB integration (solid blue) of the initial value problem (10.22) and (10.23). The upper panel shows $\epsilon = 1$ and the lower panel $\epsilon = 2$. We use $\beta = 399$ so that the wavenumber varies by a factor of 20 between $x = 0$ and ∞.

We observe that the exact solution of

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(\lambda^2 \mathrm{e}^{2z} - \nu^2\right) w = 0\,, \qquad \text{is} \qquad w = J_{\pm\nu}(\lambda \mathrm{e}^z)\,. \tag{10.29}$$

If ν is not an integer then the $\pm\nu$ in (10.29) provides a linearly independent pair. Changing variables to $x = -2z$, the w equation becomes

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{1}{4}\left(\lambda^2 \mathrm{e}^{-x} - \nu^2\right) w = 0\,. \tag{10.30}$$

Comparing this with (10.23), we see that $\lambda^2/4 = 8/\epsilon^2$ and $-\nu^2/4 = 1/\epsilon^2$. Thus

$$J_{\pm 2\mathrm{i}/\epsilon}\left(\frac{4\sqrt{2}}{\epsilon\,\mathrm{e}^x}\right) \tag{10.31}$$

is a linearly independent pair of solutions to (10.22). For MATLAB enthusiasts this is a pyrrhic victory: MATLAB Bessel function routines do not include complex orders. Instead we make the comparison using `ode45`: see Figure 10.1.

The exponential case

Next, consider

$$\epsilon^2 y'' - \left(1 + \beta \mathrm{e}^{-x}\right) y = 0\,, \tag{10.32}$$

with conditions

$$y(0) = 1\,, \qquad \text{and} \qquad \lim_{x\to\infty} y(x) = 0\,. \tag{10.33}$$

The e-folding scale varies from $\sqrt{1+\beta}/\epsilon$ near $x = 0$ to $1/\epsilon$ as $x \to \infty$. The exact solution is

$$y = \frac{I_{2/\epsilon}[\zeta(x)]}{I_{2/\epsilon}[\zeta(0)]}\,, \qquad \text{where} \qquad \zeta(x) \stackrel{\text{def}}{=} \frac{2\sqrt{\beta}}{\epsilon \mathrm{e}^{x/2}}\,, \tag{10.34}$$

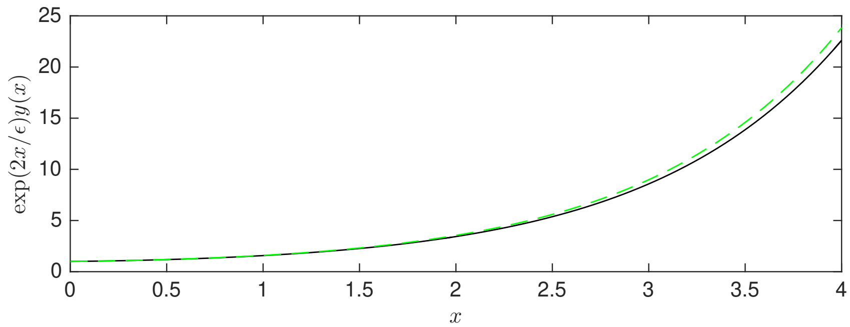

Figure 10.2: An $\epsilon = 1$ and $\beta = 3$ comparison of the WKB approximation (green dashed) in (10.35) with the analytic solution in (10.34). To reveal the solutions when $x > 1$ we "compensate" by dividing $y(x)$ by the initial decay $\exp(-2x/\epsilon)$.

and I_ν is the modified Bessel function.

The WKB approximation is

$$y_{\mathrm{WKB}} = \left(\frac{1+\beta}{1+\beta \mathrm{e}^{-x}}\right)^{1/4} \mathrm{e}^{-s_0/\epsilon}\,, \tag{10.35}$$

where $s_0(x)$ is given in (10.25). Figure 10.2 compares the exact and approximate solutions.

There is also a simple boundary-layer type approximation:

$$y_{BL} = \mathrm{e}^{-2x/\epsilon}\,. \tag{10.36}$$

Figure 10.2 shows that this approximation is not as accurate as WKB.

An example of WKB without an obvious ϵ

Next we use WKB to obtain an approximate solution of

$$y'' + \sqrt{x} y = 0\,, \qquad \text{with initial conditions } y(1) = 0 \text{ and } y'(1) = 1. \tag{10.37}$$

In this example $\epsilon = 1$ and $q = \sqrt{x}$. The phase function is therefore

$$\pm \int_1^x \sqrt{q(t)}\,\mathrm{d}t = \pm i \frac{4}{5}\left(x^{5/4} - 1\right) \tag{10.38}$$

We're going to apply the initial conditions at $x = 1$, so it is wise use $t = 1$ as the lower limit in the integral above. Thus the solution satisfying $y(1) = 0$ is

$$y = \frac{A}{x^{1/8}} \sin\left[\frac{1}{\epsilon}\frac{4}{5}\left(x^{5/4} - 1\right)\right] + O(\epsilon)\,. \tag{10.39}$$

The cosine is eliminated by the requirement that $y(1) = 0$. We've included the factor ϵ^{-1} because when we take the derivative of (10.39) we have

$$y' = \frac{A}{\epsilon} x^{1/8} \cos\left[\frac{1}{\epsilon}\frac{4}{5}\left(x^{5/4} - 1\right)\right] + O(1)\,. \tag{10.40}$$

When we take the derivative we only differentiate the cosine and not the amplitude $x^{-1/8}$ — the derivative of the amplitude factor is order ϵ smaller. Requiring that $y'(1) = 1$ we see that $A = \epsilon = 1$. Thus the WKB approximation is

$$y^{WKB}(x) = x^{-1/8} \sin\left[\frac{4}{5}\left(x^{5/4} - 1\right)\right]\,. \tag{10.41}$$

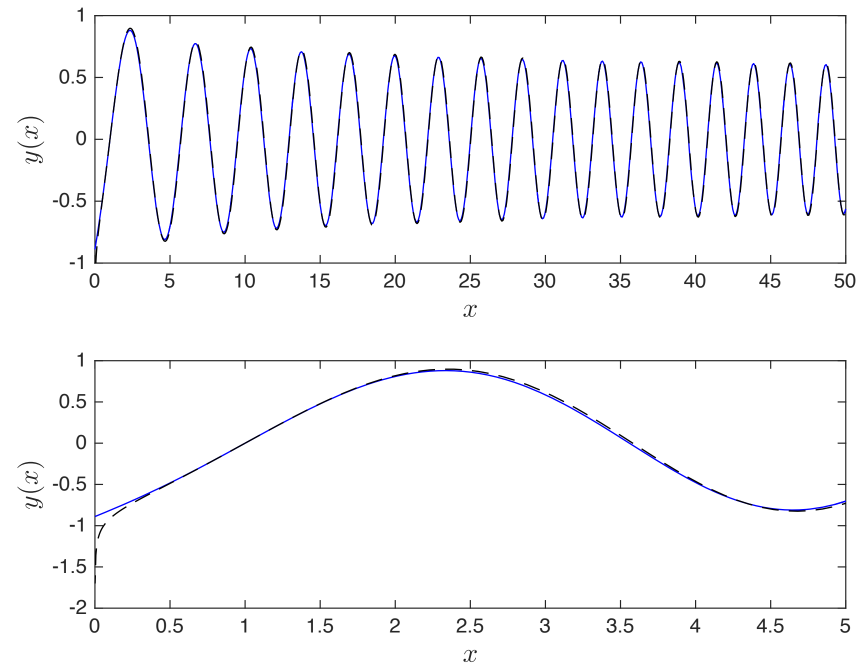

Figure 10.3: Comparison of the WKB approximation (black dashed) in (10.41) with a MATLAB integration (solid blue) of the initial value problem (10.37) (shooting both ways starting at $x = 1$). The upper panel shows close agreement over a large interval and the lower panel shows failure of the WKB approximation close to the turning point at $x = 0$.

Figure 10.3 shows that this is an excellent approximation to the solution to the initial value problem — unless we get too close to the turning point at $x = 0$.

Why does the WKB approximation work when there is no ϵ? Suppose we're interested in the solutions of (10.37) with x large. If we introduce

$$X \stackrel{\text{def}}{=} \delta\, x \tag{10.42}$$

then the rescaled equation is

$$\delta^{5/2} y_{XX} + \sqrt{X} y = 0\,. \tag{10.43}$$

As $\delta \to 0$ with X fixed we obtain the standard WKB problem with the small parameter

$$\epsilon = \delta^{5/4}\,. \tag{10.44}$$

Thus we expect that the PO approximation (10.41) is asymptotic as $x \to \infty$.

Now consider the more general equation

$$y'' + x^a y = 0\,. \tag{10.45}$$

Is the WKB approximation valid as $x \to \infty$? The problems invite you to explore this issue in detail e.g., by examining the higher order corrections to the physical optics approximation. But the re-scaling argument shows very quickly that WKB is valid provided that $a > -2$.

Exercise: Show that as $x \to \infty$ the WKB approximation applies to (10.45) provided that $a > -2$.

Failure of WKB

Consider

$$y'' + x^{-3} y = 0\,, \qquad \text{with ICs } y(1) = 1 \text{ and } y'(1) = 0\,. \tag{10.46}$$

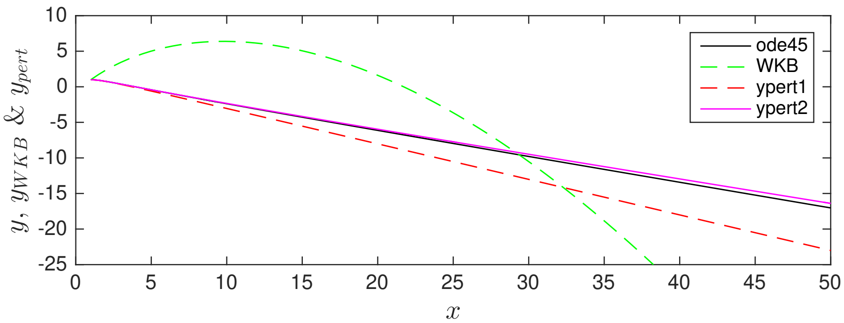

Figure 10.4: Failure of the WKB approximation. Should plot an expanded view near $x = 1$ to show that the perturbation solutions satisfy both initial conditions.

The condition for the validity of PO in (10.16) is strongly violated:

$$\left| \frac{s_0''}{s_0'^2} \right| = \frac{3x^{1/2}}{2} \to \infty \,, \qquad \text{as } x \to \infty. \tag{10.47}$$

Ignoring this red flag, we proceed to construct the PO approximation

$$y_{\text{WKB}} = x^{3/4} \cos\left[2\left(1 - x^{-1/2}\right)\right] . \tag{10.48}$$

Differentiating only the cos in (10.48).

$$y_{\text{WKB}}' = x^{-3/4} \sin\left[2\left(1 - x^{-1/2}\right)\right] . \tag{10.49}$$

The comparison with the numerical solution in Figure 10.4 is disastrous.

Exercise: In figure 10.4 why doesn't y_{WKB}' satisfy the initial condition $y'(1) = 0$?

Remark: Instead of WKB there is a simple iterative solution starting with $y_0 = 1$, and then proceeding with

$$y_{n+1}'' = x^{-3} y_n \,. \tag{10.50}$$

Iterating once and twice we have

$$y_1 = \frac{-1 + 4x - x^2}{2x} \,, \qquad \text{and} \qquad y_2 = \frac{1 - 12x + 27x^2 - 4x^3}{12x^2} - \frac{\ln x}{2} \,. \tag{10.51}$$

And once more for good measure

$$y_3 = \frac{1 - 24x + 108x^2 + 8x^3 + 51x^4}{144x^3} + \frac{(4x - 3)\ln x}{12x} \tag{10.52}$$

These iterative approximations compare well with the numerics in Figure 10.4.

The iterative solution is based on the idea that x^{-3} goes to zero rather rapidly as $x \to \infty$, and thus the term $x^{-3}y$ has a small effect on the initial condition. But eventually as $x \to \infty$ the approximation y_2 will deviate significantly from the exact solution. To see this failure of iteration we might extend the plot in Figure 10.4 to x larger than 50. Alternatively we solve (10.37) exactly in terms of Bessel functions.

10.4 An example: radiation of waves on a string

Consider shaking a string stretched along the half-line $x > 0$. The problem is

$$\rho \zeta_{tt} - T \zeta_{xx} = 0 \,, \tag{10.53}$$

The tension T (Newtons) is constant and the density ρ (kilograms per meter) is some positive function of x. The wave speed c (meters per second) is

$$c(x) \stackrel{\text{def}}{=} \sqrt{\frac{T}{\rho(x)}} \,. \tag{10.54}$$

The energy conservation equation follows from ζ_t (10.53):

$$E_t + J_x = 0 \,, \tag{10.55}$$

where the energy density E and the energy flux J are

$$E \stackrel{\text{def}}{=} \tfrac{1}{2}\rho\zeta_t^2 + \tfrac{1}{2}T\zeta_x^2 \,, \qquad \text{and} \qquad J \stackrel{\text{def}}{=} -T\zeta_t\zeta_x \tag{10.56}$$

Imagine forcing the string by shaking the end. A simple example of shaking is the boundary condition

$$x = 0: \qquad \zeta(0,t) = a\cos\omega t \,. \tag{10.57}$$

Suppose that this $x = 0$ shaking has gone on forever so that there is a permanent wave with frequency ω propagating towards $x = \infty$.

If we first assume that ρ is constant then we can easily solve this problem with

$$\zeta(x,t) = a\cos(\omega t - kx) \,, \tag{10.58}$$

where the wwavenumber k is

$$k \stackrel{\text{def}}{=} \frac{\omega}{c} \tag{10.59}$$

Remark: If we try to solve the problem above with separation of variables, $\zeta \stackrel{?}{=} \cos\omega t\, Z(x)$, then we quickly find $\zeta = a\cos\omega t\,\cos kx$. This separable solution satisfies wave equation (10.53) and the $x = 0$ boundary condition (10.57). Nonetheless the separable solution $a\cos\omega t\,\cos kx$ does not correspond to the physical situation. Why?

The solution $\zeta = a\cos\omega t\,\cos kx$ corresponds to a standing wave which might be established by reflection of a wave incident from $x = +\infty$ being reflected from a free boundary at $x = 0$. The free boundary condition is $\zeta_x(0,t) = 0$ – this boundary condition is satisfied by $\zeta = a\cos\omega t\,\cos kx$.

To more deeply understand the physical situation we should calculate E and J. The standing wave solution has $\langle J\rangle = 0$ i.e. there is no source of energy at $x = 0$. The radiating solution in (10.58) has non-zero constant flux, $\langle J\rangle$. In the radiating case the hand shaking the end of the string is doing work.

INCOMPLETE – now consider non-constant ρ as an example of WKB

10.5 Eigenproblems

Consider the problem of determining the eigenfrequencies of a vibrating string with non-uniform mass density $\rho(x)$ and uniform tension T:

$$\phi_{xx} + \omega^2 \underbrace{\frac{\rho}{T}}_{\stackrel{\text{def}}{=}\sigma^2} \phi = 0 \,. \tag{10.60}$$

The wave speed is σ^{-1} — let's follow our friend the seismologists and refer to the inverse wave speed σ as the "slowness". The ends of the string at $x = 0$ and ℓ are "clamped" i.e. we have the Dirichlet boundary conditions

$$\phi(0) = \phi(\ell) = 0 \,. \tag{10.61}$$

The eigenfrequencies are ordered

$$0 < \omega_1 < \omega_2 < \cdots \tag{10.62}$$

and we know that $\omega_n \to \infty$ as $n \to \infty$. To find an approximation to these large eigenvalues, we say that $\omega_n = \epsilon^{-1}$ and rewrite the differential equation as

$$\epsilon^2 \phi_{xx} + \sigma^2 \phi = 0\,. \tag{10.63}$$

The PO approximation is then:

$$\phi^{PO} = a\sigma^{-1/2} \sin\left[\omega \int_0^x \sigma(x')\,\mathrm{d}x'\right]\,, \tag{10.64}$$

where a is a normalization constant. The construction above secures the boundary condition at $x = 0$. The other boundary condition at $x = 1$ provides the eigenfrequency

$$\omega_n^{PO} = \frac{n\pi}{\int_0^\ell \sigma(x)\,\mathrm{d}x}\,. \tag{10.65}$$

A standard way of normalizing the eigenfunctions is to require

$$1 = \int_0^\ell \phi_n^2 \sigma^2\,\mathrm{d}x\,. \tag{10.66}$$

This normalization determines a in (10.64) as

$$1 = a^2 \int_0^\ell \sin^2\left[\omega^{PO} \int_0^x \sigma(x')\,\mathrm{d}x'\right] \sigma(x)\,\mathrm{d}x\,. \tag{10.67}$$

We can evaluate the integral above using the "WKB coordinate"

$$\xi(x) \stackrel{\text{def}}{=} \frac{\int_0^x \sigma(x')\,\mathrm{d}x'}{\int_0^\ell \sigma(x')\,\mathrm{d}x'} = \frac{\omega_n^{PO}}{n\pi} \int_0^x \sigma(x')\,\mathrm{d}x' \qquad \Rightarrow \qquad \sigma \mathrm{d}x = \left(\int_0^\ell \sigma\,\mathrm{d}x\right) \mathrm{d}\xi\,. \tag{10.68}$$

In terms of ξ, the normalization condition (10.66) is

$$1 = a^2 \left(\int_0^\ell \sigma\,\mathrm{d}x\right) \underbrace{\int_0^1 \sin^2(n\pi\xi)\,\mathrm{d}\xi}_{=1/2}\,. \tag{10.69}$$

Example: If $\sigma = 1 + x^2$ and $\ell = 1$ then (10.65) gives

$$\omega_n^{PO} = \frac{3\pi n}{4}\,. \tag{10.70}$$

The upper panel of Figure 10.5 shows the percentage error

$$e \stackrel{\text{def}}{=} 100 \frac{\omega^2 - (3\pi n/4)^2}{\omega^2}\,, \tag{10.71}$$

with ω determined numerically using `bvp4c`. The error is less than 3% even for the first mode.

Example: Let's use the WKB approximation to estimate the eigenvalues of the Sturm-Liouville eigenproblem

$$y'' + \lambda \underbrace{\left(x + x^{-1}\right)}_{w(x)} y = 0\,, \quad \text{with BCs} \qquad y'(1) = 0\,, \quad y(L) = 0\,. \tag{10.72}$$

The eigenvalues are function of the parameter L. The physical optics approximation is

$$y = w^{-1/4} \sin\left(\lambda^{1/2} \underbrace{\int_x^L \sqrt{w(x')}\,\mathrm{d}x'}_{\text{phase}}\right)\,, \tag{10.73}$$

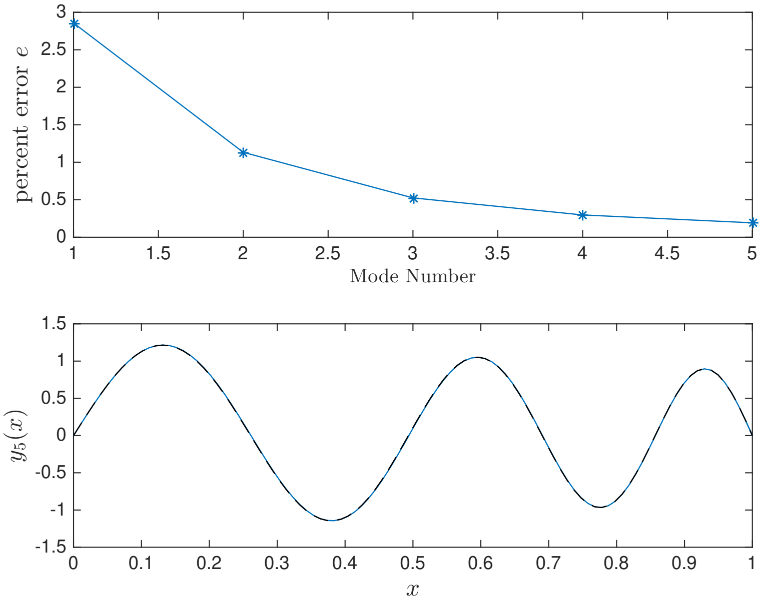

Figure 10.5: Upper panel shows the percentage error in (10.70). The lower panel compares the fifth eiegnfunction determined by `bvp4c` (the blue solid curve) with the WKB eigenfunction (the black dashed curve).

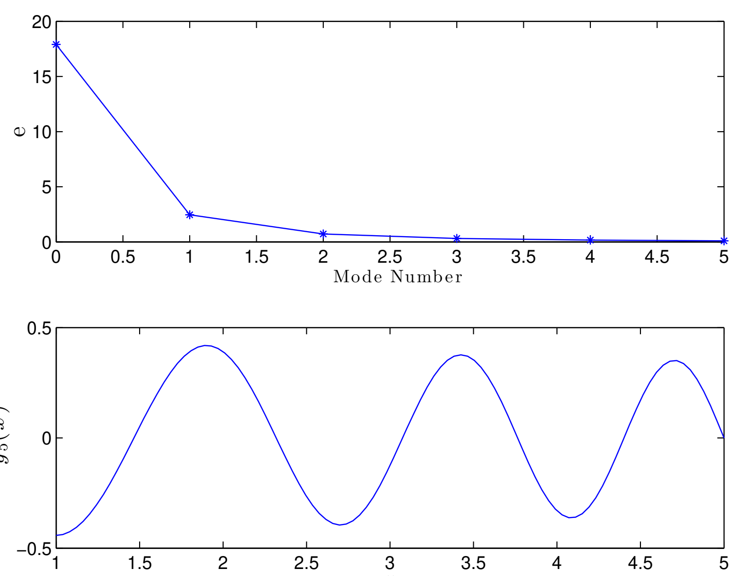

Figure 10.6: Solution with $L = 5$. There are five interior zeros.

and the leading-order derivative is

$$y' = -\lambda^{1/2} w^{+1/4} \cos\left(\lambda^{1/2} \int_x^L \sqrt{w(x')}\, \mathrm{d}x'\right) . \tag{10.74}$$

The phase in (10.73) has been constructed so that the boundary condition at $x = L$ is already satisfied. To apply the derivative boundary condition at $x = 1$ we have from (10.74)

$$\sqrt{\lambda_n^{\mathrm{WKB}}}\, J(L) = \pi\left(n + \frac{1}{2}\right), \qquad n = 0, 1, \cdots \tag{10.75}$$

where

$$J(L) \stackrel{\mathrm{def}}{=} \int_1^L \sqrt{x + x^{-1}}\, \mathrm{d}x . \tag{10.76}$$

In Figure 10.6 we take $L = 5$ and compare the WKB eigenvalue with those obtained from the MATLAB routine `bvp4c`. It is not easy to analytically evaluate $J(L)$, so instead we calculate $J(L)$ using `quad`. Figure 10.6 shows the relative percentage error,

$$e \equiv 100 \times \frac{\lambda_{\mathtt{bvp4c}} - \lambda_{\mathrm{WKB}}}{\lambda_{\mathtt{bvp4c}}} , \tag{10.77}$$

as a function of $n = 0, 2, \cdots 5$. The WKB approximation has about 18% error for λ_0, but the higher eigenvalues are accurate.

Example: Compute the next WKB correction to the $n = 0$ eigenvalue and compare both (10.75) and the improved eigenvalue to the numerical solution for $1 \le L \le 10$.

10.6 Airy's equation and turning points

Airy's equation,

$$y'' - xy = 0 , \tag{10.78}$$

is the simplest second-order differential equation with a turning point. There are two linearly independent solutions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$, shown on the real axis in figure 10.7.

Although there is no obvious ϵ we still attack (10.78) with the WKB approximation. We find that as $x \to +\infty$

$$y \sim A x^{-1/4} \exp\left(\frac{2}{3} x^{3/2}\right) + B x^{-1/4} \exp\left(-\frac{2}{3} x^{3/2}\right) . \tag{10.79}$$

And as $x \to -\infty$

$$y \sim E|x|^{-1/4} \cos\left(\frac{2}{3}|x|^{3/2}\right) + F|x|^{-1/4} \sin\left(\frac{2}{3}|x|^{3/2}\right) . \tag{10.80}$$

The approximations above don't work at the turning point $x = 0$. But they do tell us that if we numerically integrate (10.78) from $x = 0$ then we might hope to find special values of $y(0)$ and $y'(0)$ such that $\lim_{x\to\infty} y(x) = 0$. In other words, if we use the right initial conditions then we can arrange things so that when we arrive at $x = \infty$, $A = 0$ in (10.79). These "right initial conditions" produce the Airy function, $\mathrm{Ai}(x)$, shown in figure 10.7. The other solution, with A nonzero in (10.79), is the Bairy function $\mathrm{Bi}(x)$. The Airy function, $\mathrm{Ai}(x)$, is defined as the solution that decays as $x \to \infty$, with the normalization

$$\int_{-\infty}^{\infty} \mathrm{Ai}(x)\, \mathrm{d}x = 1 . \tag{10.81}$$

An integral representation

We obtain an integral representation of $\mathrm{Ai}(x)$ by attacking (10.78) with the Fourier transform. Denote the Fourier transform of $\mathrm{Ai}(x)$ by

$$\widetilde{\mathrm{Ai}}(k) = \int_{-\infty}^{\infty} \mathrm{Ai}(x) \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x . \tag{10.82}$$

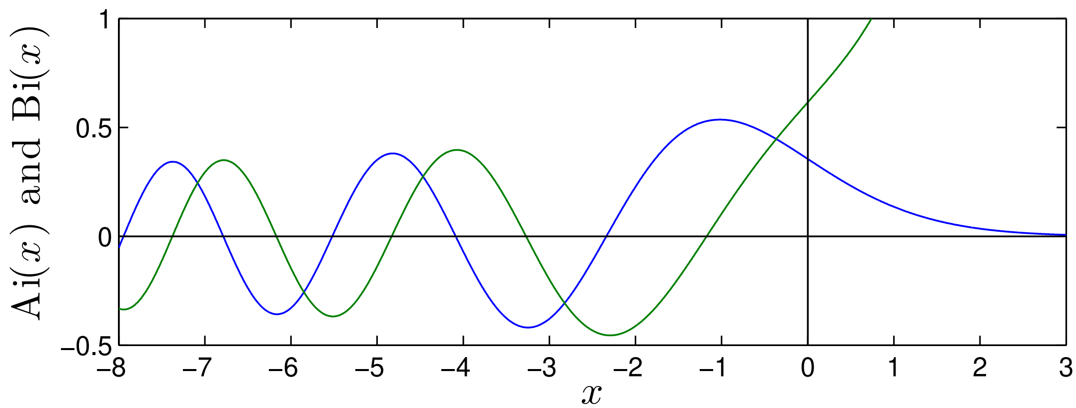

Figure 10.7: The functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$. The Airy function decays rapidly as $x \to \infty$ and rather slowly as $x \to -\infty$.

Fourier transforming (10.78), we eventually find

$$\widetilde{\mathrm{Ai}}(k) = \mathrm{e}^{\mathrm{i}k^3/3} . \tag{10.83}$$

Using the Fourier integral theorem

$$\mathrm{Ai}(x) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx+\mathrm{i}k^3/3} \frac{\mathrm{d}k}{2\pi} , \tag{10.84}$$

$$= \frac{1}{\pi} \int_0^{\infty} \cos\left(kx + \frac{k^3}{3}\right) \mathrm{d}k . \tag{10.85}$$

The integral converges at $k = \infty$ because of destructive interference or catastrophic cancellation.

From the integral representation (10.85) we find the magic initial conditions that produce rapid decay of $\mathrm{Ai}(x)$ as $x \to \infty$:

$$\mathrm{Ai}(0) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{k^3}{3}\right) \mathrm{d}k = \frac{1}{3^{2/3}\Gamma(2/3)} , \tag{10.86}$$

$$\mathrm{Ai}'(0) = -\frac{1}{\pi} \int_0^{\infty} k \sin\left(\frac{k^3}{3}\right) \mathrm{d}k = -\frac{1}{3^{1/3}\Gamma(1/3)} . \tag{10.87}$$

Exercise: Fill in the details between (10.82) and (10.83). Why does the Fourier transform provide only one solution of the second order equation (10.82)?

Applying asymptotic techniques to the integral representations in (10.85) one obtains as $x \to -\infty$

$$\mathrm{Ai}(x) \sim \frac{1}{\sqrt{\pi}|x|^{1/4}} \sin\left(\frac{2|x|^{3/2}}{3} + \frac{\pi}{4}\right) . \tag{10.88}$$

And as $x \to +\infty$:

$$\mathrm{Ai}(x) \sim \frac{\mathrm{e}^{-\frac{2x^{3/2}}{3}}}{2\sqrt{\pi}x^{1/4}} . \tag{10.89}$$

In figure 10.8 we compare the asymptotic approximations in (10.88) and (10.89) with the $\mathrm{Ai}(x)$. The approximations (10.88) and (10.89) are splendid, except close to the turning point.

The Bairy function $\mathrm{Bi}(x)$ is defined so that as $x \to -\infty$:

$$\mathrm{Bi}(x) \sim \frac{1}{\sqrt{\pi}|x|^{1/4}} \cos\left(\frac{2|x|^{3/2}}{3} + \frac{\pi}{4}\right) . \tag{10.90}$$

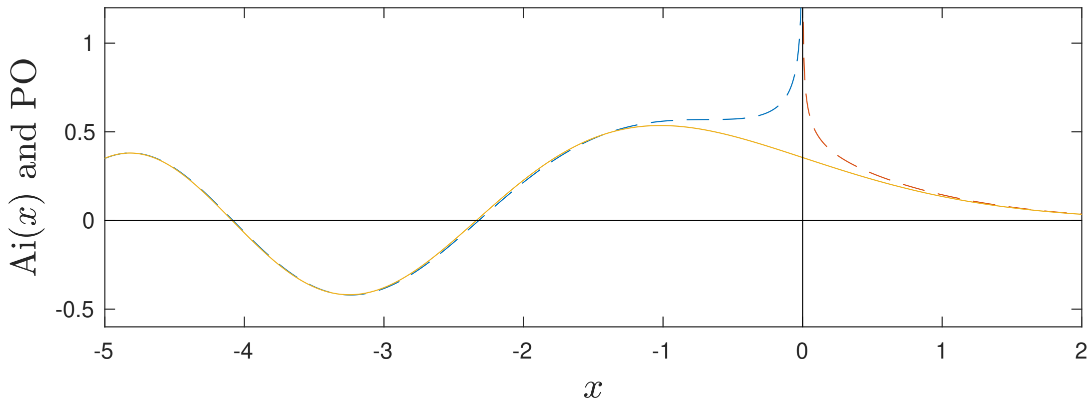

Figure 10.8: Comparison of the PO approximations in (10.88) and (10.89) with Ai(x).

And as $x \to +\infty$:

$$\mathrm{Bi}(x) \sim \frac{\mathrm{e}^{+\frac{2x^{3/2}}{3}}}{\sqrt{\pi}x^{1/4}}\,. \tag{10.91}$$

Further Bairy factoids are

$$\mathrm{Bi}(0) = \sqrt{3}\mathrm{Ai}(0)\,, \qquad \text{and} \qquad \mathrm{Bi}'(0) = -\sqrt{3}\mathrm{Ai}'(0)\,. \tag{10.92}$$

We can use Airy and Bairy to extend the utility of the WKB approximation.

Example: An eigenproblem with a turning point

Let's apply the WKB approximation to estimate the large eigenvalues of the Sturm-Liouville eigenproblem

$$\phi'' + \lambda \sin x\, \phi = 0\,, \qquad \phi(0) = \phi\left(\frac{\pi}{2}\right) = 0\,. \tag{10.93}$$

There is a turning point at $x = 0$ so the WKB approximation does not apply close to the boundary.

Hope is eternal and we begin by ignoring the turning point and constructing a physical optics approximation:

$$\phi^{\mathrm{hope}} = (\sin x)^{-1/4} \sin\left(\sqrt{\lambda}\int_0^x \sqrt{\sin v}\,\mathrm{d}v\right)\,. \tag{10.94}$$

The construction above satisfies the boundary condition at $x = 0$ and then the other boundary condition at $\pi/2$ determines our hopeful approximation to the eigenvalue. To ensure that $\phi^{\mathrm{hope}}(\pi/2) = 0$, the argument of the sin must be $n\pi$ and thus the approximate eigenvalue is

$$\lambda_n^{\mathrm{hope}} = \left(\frac{n\pi}{J}\right)^2\,, \quad n = 1\,,\ 2\,, \cdots \tag{10.95}$$

In the expression above the integral of the phase function is

$$J \stackrel{\mathrm{def}}{=} \int_0^{\pi/2} \sqrt{\sin v}\,\mathrm{d}v = \sqrt{\frac{2}{\pi}}\Gamma^2\left(\frac{3}{4}\right) = 1.19814\cdots \tag{10.96}$$

We'll see later that the approximation in (10.95) is not very accurate — we can't ignore the turning point and hope for the best. Instead we use a combination of WKB and asymptotic matching to account for the turning point and obtain a better approximation to the eigenvalues.

The outer solution — use WKB: we apply the WKB approximation where it is guaranteed to work. This is in the outer region defined by $\lambda^{1/3}x \gg 1$. The construction that satisfies the boundary condition at $x = \pi/2$ is

$$\phi^{WKB} = (\sin x)^{-1/4} \sin\left(\sqrt{\lambda}\int_x^{\pi/2} \sqrt{\sin t}\,\mathrm{d}t\right)\,. \tag{10.97}$$

To perform the match we will need the "inner limit" of the approximation above. In the region where

$$\lambda^{-1/3} \ll x \ll 1 \tag{10.98}$$

the WKB approximation is valid *and* we can simplify the phase function in (10.97):

$$\phi^{WKB} = (\sin x)^{-1/4} \sin\left(\sqrt{\lambda}J - \int_0^x \sqrt{\sin v}\,\mathrm{d}v\right), \tag{10.99}$$

$$\sim x^{-1/4} \sin\left(\sqrt{\lambda}J - \tfrac{2}{3}\sqrt{\lambda}x^{3/2} + O\left(x^{7/2}\right)\right). \tag{10.100}$$

The inner solution: close to $x = 0$ — specifically in the region where $x\lambda^{1/3}$ is order unity — we can approximate the differential equation by

$$\phi_{xx} + \lambda\left(x + O(x^3)\right)\phi = 0\,. \tag{10.101}$$

As an inner variable we use

$$X = \lambda^{1/3}x\,, \tag{10.102}$$

so that the leading-order inner approximation is a variant of Airy's equation

$$\Phi_{XX} + X\Phi = 0\,. \tag{10.103}$$

The solution that satisfies the boundary condition at $X = 0$ is

$$\Phi = Q\left[\frac{\mathrm{Ai}(-X)}{\mathrm{Ai}(0)} - \frac{\mathrm{Bi}(-X)}{\mathrm{Bi}(0)}\right]. \tag{10.104}$$

Matching: To take the outer limit of the inner solution in (10.104) we look up the relevant asymptotic expansions of the Airy functions. Then we write the outer limit of (10.104) as

$$\Phi \sim \frac{2Q}{\sqrt{3\pi}\mathrm{Ai}(0)}\frac{1}{X^{1/4}}\Big[\underbrace{\tfrac{\sqrt{3}}{2}}_{\cos\frac{\pi}{6}} \sin\left(\frac{2}{3}X^{3/2} + \frac{\pi}{4}\right) - \underbrace{\tfrac{1}{2}}_{\sin\frac{\pi}{6}} \cos\left(\frac{2}{3}X^{3/2} + \frac{\pi}{4}\right)\Big], \tag{10.105}$$

$$= \frac{2Q}{\sqrt{3\pi}\mathrm{Ai}(0)}\frac{1}{X^{1/4}} \sin\left(\frac{2}{3}X^{3/2} + \frac{\pi}{12}\right), \tag{10.106}$$

$$= -\frac{2Q}{\sqrt{3\pi}\mathrm{Ai}(0)}\frac{1}{X^{1/4}} \sin\left(-\frac{\pi}{12} - \frac{2}{3}X^{3/2}\right) \tag{10.107}$$

We now match the phase in (10.100) with that in (10.107). This requires

$$\sqrt{\lambda}J - n\pi = -\frac{\pi}{12}, \tag{10.108}$$

or

$$\lambda^{\mathrm{WKB}} = \left((n - \tfrac{1}{12})\frac{\pi}{J}\right)^2, \quad n = 1\,,\ 2\,,3\cdots \tag{10.109}$$

With $n = 1$ the hopeful approximation in (10.95) is about 18% larger than the correct WKB-Airy approximation in (10.109). The numerical comparison below shows that (10.109) is good even for $n = 1$:

$\lambda_{\mathtt{bvp4c}}$	5.7414	25.2094	58.4349	105.4114	166.1422	240.6232
λ_{WKB}	5.7771	25.2568	58.4341	105.4673	166.1456	240.6793

The `bvp4c` results fluctuate in the final decimal place as I play with the resolution and the initial guess.

Some ODEs that can be solved using Bessel functions

We are going to encounter some second-order differential equations that can solved exactly in terms of Bessel functions. Here are results extracted from Abramowitz & Stegun **9.1.49** to **9.1.54**.

Denote any solution of Bessel's equation

$$z^2\frac{\mathrm{d}^2y}{\mathrm{d}z^2}+x\frac{\mathrm{d}y}{\mathrm{d}z}+(z^2-\nu^2)y=0$$

by $\mathcal{C}_\nu(z)$. For example, $C_\nu = J_\nu$ or Y_ν, or a linear combination of J_ν and Y_ν. Then

$$\begin{aligned}
w''+\left(\lambda^2-\tfrac{\nu^2-\frac{1}{4}}{z^2}\right)w=0\,, &\qquad \Rightarrow \qquad w=z^{1/2}\mathcal{C}_\nu(\lambda z)\\
w''+\left(\tfrac{\lambda^2}{4z}-\tfrac{\nu^2-1}{4z^2}\right)=0\,, &\qquad \Rightarrow \qquad w=z^{1/2}\mathcal{C}_\nu(\lambda z^{1/2})\\
w''+\lambda^2z^{p-2}w=0\,, &\qquad \Rightarrow \qquad w=z^{1/2}\mathcal{C}_{1/p}(2\lambda z^{p/2}/p)\,,\\
w''-\tfrac{2\nu-1}{z}w'+\lambda^2w=0\,, &\qquad \Rightarrow \qquad w=z^\nu\mathcal{C}_\nu(2\lambda z)\,,\\
z^2w''+(1-2p)zw'+(\lambda^2q^2z^{2q}+p^2-\nu^2q^2)w=0\,, &\qquad \Rightarrow \qquad w=z^p\mathcal{C}_{1/p}(\lambda z^q)\,.\\
w''+(\lambda^2\mathrm{e}^{2z}-\nu^2)w=0\,, &\qquad \Rightarrow \qquad w=\mathcal{C}_\nu(\lambda\mathrm{e}^z)\,,
\end{aligned}$$

Denote any solution of the modified Bessel equation

$$z^2\frac{\mathrm{d}^2y}{\mathrm{d}z^2}+x\frac{\mathrm{d}y}{\mathrm{d}z}-(z^2+\nu^2)y=0$$

by $\mathcal{Z}_\nu(z)$. For example, $\mathcal{Z}_\nu = K_\nu$ or I_ν, or a linear combination of K_ν and I_ν. Then λ^2 in the differential equations above can be replaced by $-\lambda^2$ if $\mathcal{C}_\nu$ is replaced by $\mathcal{Z}_\nu$. For example

$$w''-\lambda^2z^{p-2}w=0\,, \qquad \Rightarrow \qquad w=z^{1/2}\mathcal{Z}_{1/p}(2\lambda z^{p/2}/p)\,.$$

10.7 Higher order terms in the WKB approximation

In (10.16) we obtained a necessary condition for the validity of the PO approximation. In this section we provide further elaboration of the conditions required for the PO approximation to work.

The solutions of the WKB hierarchy at the next two orders are

$$s_2 = \mp \int^x \frac{q''}{8q^{3/2}} - \frac{5}{32}\frac{q'^2}{q^{5/2}}\,\mathrm{d}t\,, \tag{10.110}$$

$$\mathrm{i}s_3 = \frac{1}{16}\frac{q''}{q^2} - \frac{5}{64}\frac{q'^2}{q^3}\,. \tag{10.111}$$

These formulas are equivalent to

$$s_2 = \pm\frac{1}{2}\int^x q^{-1/4}\left(q^{-1/4}\right)''\,\mathrm{d}t \tag{10.112}$$

and

$$\mathrm{i}s_3 = -\frac{1}{4}q^{-1/2}\left(q^{-1/4}\right)''\,. \tag{10.113}$$

Let's apply these formulas to Airy's equation

$$y'' = xy \tag{10.114}$$

with $x \to \infty$.

Example: Consider

$$\epsilon^2 y'' - x^{-1}y = 0\,. \tag{10.115}$$

How small must ϵ be in order for the physical optics approximation to within 5% when $x \geq 1$?

Example: Consider

$$y'' + kx^{-\alpha}y = 0\,, \qquad y(1) = 0\,, \qquad y'(1) = 1\,. \tag{10.116}$$

Is WKB valid as $x \to \infty$?

With $k = 1$, I found

$$s_0 = \pm\frac{2}{2-\alpha}\left(x^{1-\frac{\alpha}{2}} - 1\right)\,, \qquad \text{and} \qquad s_1 = \frac{\alpha}{4}\ln x\,, \qquad \text{and} \qquad s_2 = \frac{\alpha(\alpha-4)}{16(\alpha-2)}\left(x^{\frac{\alpha}{2}-1} - 1\right)\,. \tag{10.117}$$

The calculation of s_2 should be checked (and should do general k). But the tentative conclusion is that WKB works if $\alpha < 2$. (This includes $\alpha < 0$ e.g. $\alpha = -1/2$ is the example in figure 10.3.) Note $\alpha = 2$ is a special case with an elementary solution.

10.8 Using `bvp4c`

In this section I discuss the MATLAB solution of (10.72)

$$y'' + \lambda\underbrace{\left(x + x^{-1}\right)}_{w(x)}y = 0\,, \quad \text{with BCs} \qquad y'(1) = 0\,, \quad y(L) = 0\,.$$

To use `bvp4c` we let $y_1(x) = y(x)$ and write the eigenproblem as the first-order system

$$y_1' = y_2\,, \tag{10.118}$$

$$y_2' = -\lambda\left(x + x^{-1}\right)y_1\,, \tag{10.119}$$

$$y_3' = \left(x + x^{-1}\right)y_1^2\,. \tag{10.120}$$

This Sturm-Liouville boundary value problem always has a trivial solution viz., $y(x) = 0$ and λ arbitrary. We realize that this is trivial, but perhaps `bvp4c` isn't that smart. So with (10.120) we force `bvp4c` to look for a nontrivial solution by adding an extra equation with the boundary conditions

$$y_3(0) = 0\,, \qquad \text{and} \qquad y_3(L) = 1\,. \tag{10.121}$$

We also have $y_2(1) = 0$ and $y_1(L) = 0$, so there are four boundary conditions on a third-order problem. This is OK because we also have the unknown parameter λ. The addition of $y_3(x)$ also ensures that `bvp4c` returns a normalized solution:

$$\int_1^L y^2 \left(x + x^{-1}\right) \, \mathrm{d}x = 1 \,. \tag{10.122}$$

An alternative that avoids the introduction of $y_3(x)$ is to use $y_1(1) = 1$ as a normalization, and as an additional boundary condition. However the normalization in (10.122) is standard.

In summary, the system for $[y_1, y_2, y_3]$ now only has nontrivial solutions at special values of the eigenvalue λ.

The MATLAB function `billzWKBeig`, with neither input nor output arguments, solves the eigenproblem with $L = 5$. The code is written as an argumentless function so that three nested functions can be embedded. This is particularly convenient for passing the parameter L – avoid global variables. All functions are concluded with `end`. In this relatively simple application of `bvp4c` there are only three arguments:

1. a function `odez` that evaluates the right of (10.118) through (10.120);
2. a function `bcz` for evaluating the residual error in the boundary conditions;
3. a MATLAB structure `solinit` that provides a guess for the mesh and the solution on this mesh.

`solinit` is set-up with the utility function `bvpinit`, which calls the nested function `initz`. `bvp4c` returns a MATLAB structure that I've imaginatively called `sol`. In this structure, `sol.x` contains the mesh and `sol.y` contains the solution on that mesh. `bvp4c` uses the smallest number of mesh points it can. So, if you want to make a smooth plot of the solution, as in the lower panel of Figure 10.6, then you need the solution on a finer mesh, called `xx` in this example. Fortunately `sol` contains all the information needed to compute the smooth solution on the fine mesh, which is done with the auxiliary function `deval`.

```
function billzWKBeig
L = 5;          J = quad(@(x)sqrt(x+x.^(-1)),1,L);
%The first 6 eigenvalues; n = 0 is the ground state.
nEig = [0 1 2 3 4 5];                 lamWKB = (nEig+0.5).^2*(pi/J)^2;
lamNum = zeros(1,length(lamWKB));

    for N = 1:1:length(nEig)
        lamGuess = lamWKB(N);
        x = linspace(1,L,10);
        solinit = bvpinit(x,@initz,lamGuess);
        sol = bvp4c(@odez,@bcz,solinit);
        lambda = sol.parameters;
        lamNum(N) = lambda;
     end

err = 100*(lamNum - lamWKB)./lamNum;
figure
subplot(2,1,1)
plot(nEig,err,'*-')
xlabel('Mode Number','interpreter','latex')
ylabel('$e$','interpreter','latex','fontsize',16)
% Plot the last eigenfunction
xx = linspace(1,L);                ssol = deval(sol,xx);
subplot(2,1,2)
plot(xx,ssol(1,:))
xlabel('$x$','interpreter','latex','fontsize',16)
ylabel('$y_5(x)$','interpreter','latex','fontsize',16)

        %--------------- Nested Functions ----------------%
   function dydx = odez(x,y,lambda)
        %ODEZ evalates the derivatives
         dydx = [ y(2); -lambda*(x+x^(-1))*y(1);
                                (x+x^(-1))*y(1)*y(1)];
    end

    %% BCs applied
    function res = bcz(ya, yb, lambda)
         res = [ ya(2) ; yb(1); ya(3) ; yb(3) - 1];
         %Four BCs: solve three first-order
         %equations and also determine lambda.
    end

    %% Use a simple guess for the Nth eigenmode
    function yinit = initz(x)
        alpha = (N + 1/2)*pi/(L-1);
        yinit = [ sin(alpha*(L - x))
                  alpha*cos(alpha*(L - x))
                  (x - 1)/(L - 1)     ];
    end
end
```

10.9 Problems

Problem 10.1. Consider the IVP

$$\ddot{x} + 256\mathrm{e}^{4t}x = 0\,, \qquad x(0) = 0\,, \qquad \dot{x}(0) = 1\,. \tag{10.123}$$

Estimate the position and magnitude of the first positive maximum of $y(t)$. Compare the WKB approximation with a numerical solution on the interval $0 < t \leq 1$.

Problem 10.2. Consider the differential equation

$$y'' + \underbrace{\frac{400}{400 + x^2}}_{Q(x)}\, y = 0\,. \tag{10.124}$$

How can we apply the WKB approximation to this equation? Compare the physical optics approximation to a numerical solution with the initial conditions $y(0) = 1$ and $y'(0) = 0$.

Problem 10.3. Consider

$$y'' + \frac{a}{x^2}y = 0\,. \tag{10.125}$$

Take $a > 0$ and obtain the physical-optics approximation. Compare to the exact solution. Is the physical-optics approximation asymptotically valid as $x \to \infty$? As $x \to 0$? Is the physical-optics approximation ever valid?

Problem 10.4. Find an approximation to the large eigenvalues of the Sturm-Liouville problem

$$\phi'' + \lambda\mathrm{e}^{2x}\phi = 0\,, \qquad \text{posed on } 0 < x < 1\text{, with BCs:} \qquad \phi(0) = 0\,, \qquad \phi'(1) = 0\,. \tag{10.126}$$

(Bonus for comparison with a numerical solution.)

Problem 10.5. Substitute the WKB ansatz $y = \mathrm{e}^{S/\epsilon}$ into the fourth- order differential equation

$$\epsilon^4\frac{\mathrm{d}^4y}{\mathrm{d}x^4} + Qy = 0\,, \tag{10.127}$$

and obtain a nonlinear equation for S. Using the expansion $S = S_0 + \epsilon S_1 + \epsilon^2 S_2 + \cdots$ find S_0 and S_1 in terms of Q. (Consider both signs of Q.)

Problem 10.6. Put Bessel's differential equation

$$r^2\frac{\mathrm{d}^2y}{\mathrm{d}r^2} + r\frac{\mathrm{d}y}{\mathrm{d}r} + (r^2 - \nu^2)y = 0 \tag{10.128}$$

into Schrödinger form

$$\frac{\mathrm{d}^2Y}{\mathrm{d}r^2} + \left(1 - \frac{\nu^2 - \frac{1}{4}}{r^2}\right)Y = 0\,. \tag{10.129}$$

Consider $r = R/\epsilon$ with $\epsilon \to 0$ and R fixed. Obtain the physical optics approximation to (10.129) in this limit. Compare your answer to Bessel-function asymptotics in some convenient reference.

Problem 10.7. Consider the differential equation

$$y'' + x^2y = 0\,, \qquad \text{with ICs} \qquad y(1) = 0\,, \qquad y(1) = 1\,, \tag{10.130}$$

posed on the semi-infinite interval $x > 1$. Solve the differential equation using the PO approximation and assess the accuracy of the large-x PO approximation by considering the third term in the WKB expansion. Find the exact solution in terms of Bessel functions. Use MATLAB to compare the Bessel function solution with the WKB approximation.

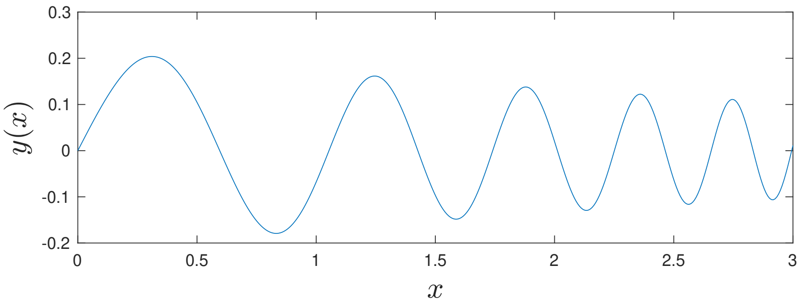

Figure 10.9: A figure for problem 10.8.

Problem 10.8. The top panel of figure 10.9 shows the solution to one of the four initial value problems:

$$\begin{aligned}
\epsilon^2 y_1'' - \mathrm{e}^{-x} y_1 = 0\,, \quad y_1(0) = 0\,, \quad y_1'(0) = 1\,,\\
\epsilon^2 y_2'' - \mathrm{e}^{x} y_2 = 0\,, \quad y_2(0) = 0\,, \quad y_2'(0) = 1\,,\\
\epsilon^2 y_3'' + \mathrm{e}^{-x} y_3 = 0\,, \quad y_3(0) = 0\,, \quad y_3'(0) = 1\,,\\
\epsilon^2 y_4'' + \mathrm{e}^{x} y_4 = 0\,, \quad y_4(0) = 0\,, \quad y_4'(0) = 1\,.
\end{aligned}$$

(a) Which $y_n(x)$ is shown in figure 10.9? (b) Use the WKB approximation to estimate the value of ϵ used in figure 10.9.

Problem 10.9. Estimate the large eigenvalues of

$$\psi'' + \lambda \sin^2 x\, \psi = 0\,, \qquad \text{with BCs } \psi(0) = \psi(\pi/2) = 0. \tag{10.131}$$

Compute the first five eigenvalues with `bvp5c` and compare the numerical estimate with your approximation.

Problem 10.10. Use the exponential substitution $y = \exp(S/\epsilon)$ to construct a WKB approximation to the differential equation

$$\epsilon^2 \left(py'\right)' + qy = 0\,. \tag{10.132}$$

Above $p(x)$ and $q(x)$ are coefficient functions, independent of the small parameter ϵ.

Problem 10.11. Consider the eigenproblem

$$\phi'' + \lambda w \phi = 0\,, \qquad \phi(0) = 0\,, \qquad \phi'(1) + \phi(1) = 0\,. \tag{10.133}$$

The weight function, $w(x)$ above, is positive for $0 \le x \le 1$. *(i)* Show that the eigenvalues λ_n are real and positive. *(ii)* Show that eigenfunctions with distinct eigenvalues are orthogonal

$$(\lambda_n - \lambda_m) \int_0^1 \phi_n \phi_m w \, \mathrm{d}x = 0\,. \tag{10.134}$$

(iii) With $w = 1$, find the first five eigenvalues and plot the first five eigenfunctions. You should obtain transcendental equation for λ, and then solve that equation with MATLAB. *(iv)* Next, with non-constant $w(x)$, use the WKB approximation to obtain a formula for λ_n. *(v)* Consider

$$w = (a + x)^2\,. \tag{10.135}$$

Take $a = 1$ and use `bvp4c` to calculate the first five eigenvalues and compare λ^{WKB} with $\lambda^{\texttt{bvp4c}}$. *(vi)* Is the WKB approximation better or worse if a increases?

Problem 10.12. Consider the Sturm-Liouville problem

$$(wy')' + \lambda y = 0\,, \tag{10.136}$$

with boundary conditions

$$\lim_{x\to 0} wy' = 0\,, \qquad y(1) = 0\,. \tag{10.137}$$

Assume that $w(x)$ increases monotonically with $w(0) = 0$ and $w(1) = 1$ e.g., $w(x) = \sin \pi x/2$. Further, suppose that if $x \ll 1$ then

$$w(x) = w_1 x + \frac{w_2}{2}x^2 + \frac{w_3}{6}x^3 + \cdots\,. \tag{10.138}$$

There is a regular singular point at $x = 0$, and thus we require only that $y(0)$ is not infinite.

Show that the transformation $y = w^{-1/2}Y$ puts the equation into the Schrödinger form

$$Y'' + \left[\frac{\lambda}{w} - \frac{w''}{2w} + \frac{w'^2}{4w^2}\right] Y = 0\,. \tag{10.139}$$

Use the WKB method and matching to find an approximation for the large eigenvalues ($\lambda = \epsilon^{-2} \gg 1$) in terms of the w_n's and the constant

$$q \equiv \int_0^1 \frac{\mathrm{d}x}{\sqrt{w(x)}}\,. \tag{10.140}$$

Some useful information from DLMF: The solution of

$$u'' + \left[\frac{a^2}{4z} + \frac{1-\nu^2}{4z^2}\right] u = 0 \tag{10.141}$$

is

$$u(z) = A\sqrt{z}J_\nu(a\sqrt{z}) + B\sqrt{z}Y_\nu(a\sqrt{z})\,, \tag{10.142}$$

where J_ν and Y_ν are Bessel functions. You will need to look up basic properties of Bessel functions.

Problem 10.13. Consider the epsilonless Schrödinger equation

$$y'' + p^2 y = 0\,, \tag{10.143}$$

where $p(x) > 0$. *(i)* Try to solve the equation by substituting

$$Y \equiv \exp\left(\pm \mathrm{i}\int_0^x p(t)\,\mathrm{d}t\right)\,. \tag{10.144}$$

Unfortunately this doesn't work: $Y(x)$ is not an exact solution of (10.143) unless p is constant. Instead, show that Y satisfies

$$Y'' + \left(p^2 \mp \mathrm{i}p'\right) Y = 0\,. \tag{10.145}$$

(ii) Compare (10.145) with (10.143), and explain why $Y(x)$ is an approximate solution of (10.143) if

$$\left|\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{p}\right| \ll 1\,. \tag{10.146}$$

(iii) Prove that if y_1 and y_2 are two linearly independent solutions of (10.143) then the Wronskian

$$W \equiv y_1 y_2' - y_1' y_2 \tag{10.147}$$

is constant. *(iv)* Show that the Wronskian of

$$Y_1 \equiv \exp\left(+\mathrm{i}\int_0^x p(t)\,\mathrm{d}t\right) \qquad \text{and} \qquad Y_2 \equiv \exp\left(-\mathrm{i}\int_0^x p(t)\,\mathrm{d}t\right) \tag{10.148}$$

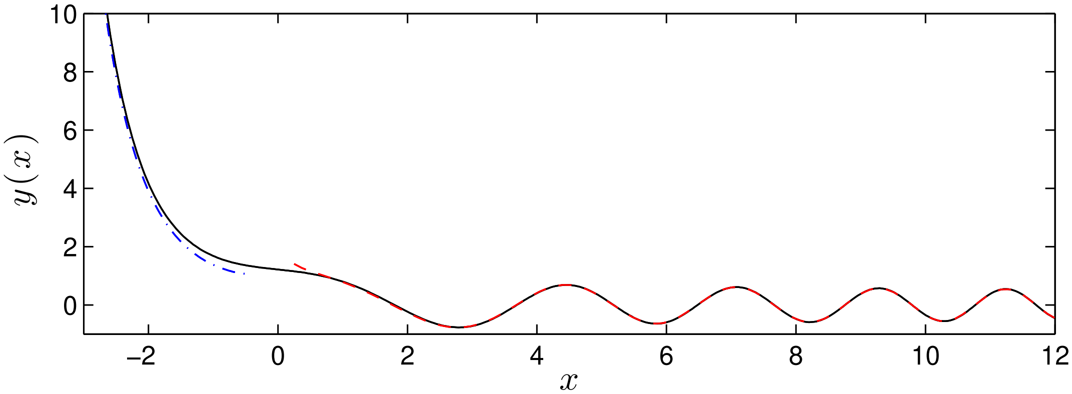

Figure 10.10: Figure for the problem 10.14 showing a comparison of the exact solution (the solid black curve) with the asymptotic expansions as $x \to -\infty$ (the dot-dash blue curve) and (10.151) as $x \to +\infty$ (the dashed red curve).

is equal to $2\mathrm{i}p$. This suggests that if we modify the amplitude of $Y(x)$ like this:

$$Y_3 \equiv \frac{1}{\sqrt{p}} \exp\left(+\mathrm{i}\int_0^x p(t)\,\mathrm{d}t\right) \qquad \text{and} \qquad Y_4 \equiv \frac{1}{\sqrt{p}} \exp\left(-\mathrm{i}\int_0^x p(t)\,\mathrm{d}t\right), \tag{10.149}$$

then we might have a better approximation. *(v)* Show that the Wronskian of Y_3 and Y_4 is a constant. *(vi)* Find a Schrödinger equation satisfied by Y_3 and Y_4 and discuss the circumstances in which this equation is close to (10.143).

Problem 10.14. Consider

$$y'' + xy = 0\,, \tag{10.150}$$

and suppose that

$$y(x) \sim x^{-1/4}\cos(2x^{3/2}/3) \qquad \text{as} \qquad x \to +\infty\,. \tag{10.151}$$

Solve this problem in terms of well known special functions. Find the asymptotic behaviour of $y(x)$ as $x \to -\infty$. Check your answer with MATLAB (see Figure 10.14)

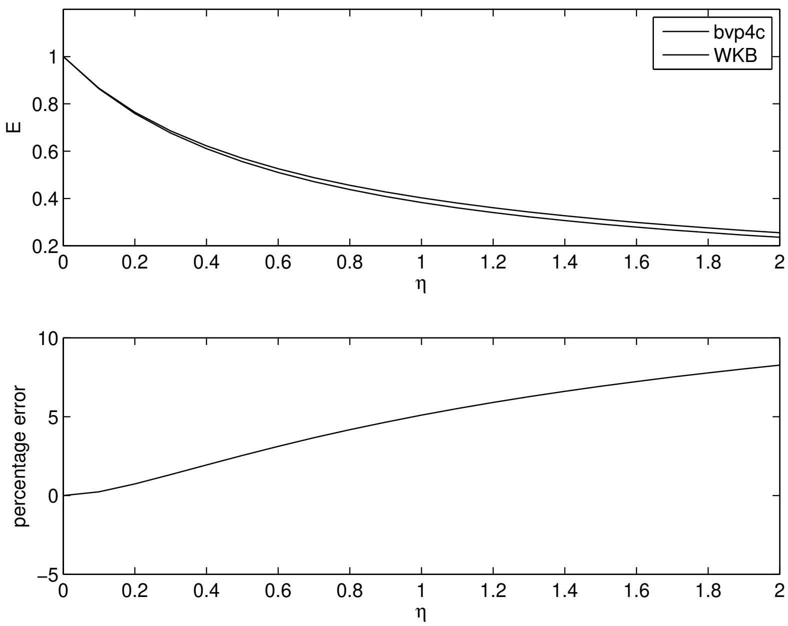

Figure 10.11: Figure for the problem with (10.152).

Problem 10.15. *(i)* Consider the eigenproblem

$$y'' + E\,(1 + \eta x)\,y = 0\,, \qquad y(0) = 0\,, \qquad y(\pi) = 0\,, \tag{10.152}$$

where η is a parameter and E is an eigenvalue. With $\eta = 0$ the gravest mode is

$$y = \sin x\,, \qquad E = 1\,. \tag{10.153}$$

(i) Suppose $|\eta| \ll 1$. Find the $O(\eta)$ shift in the eigenvalue using perturbation theory. If you're energetic, calculate the $O(\eta^2)$ term for good measure (optional). *(ii)* In equation (10.1.31) of BO, there is a WKB approximation to the eigenvalue $E(\eta)$. Take $n = 1$, and expand this formula for E up to and including terms of order η^2; compare this with your answer to part *(i)*. *(iii)* Use `bvp4c` in MATLAB to calculate $E(\eta)$, with $0 < \eta < 2$, numerically. Compare the WKB approximation in (10.1.31) with your numerical answer by plotting $E_{\texttt{bvp4c}}(\eta)$ and $E_{\mathrm{WKB}}(\eta)$ in the same figure.

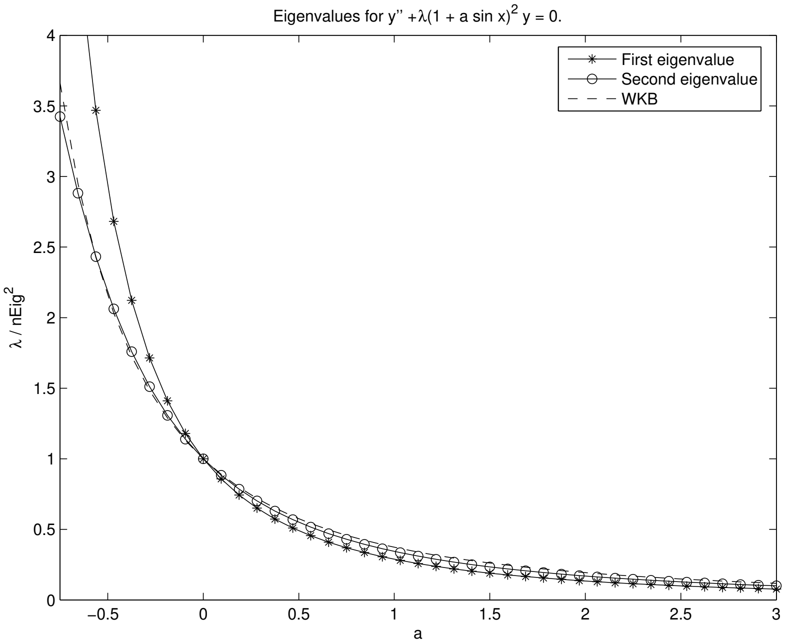

Figure 10.12: Figure for the problem containing (10.154).

Problem 10.16. Consider the Sturm-Liouville eigenproblem

$$y'' + \lambda \left(1 + a \sin x\right)^2 y = 0\,, \quad y(0) = y(\pi) = 0\,. \tag{10.154}$$

(a) Using `bvp4c`, compute the first two eigenvalues, λ_1 and λ_2, as a functions of a in the range $-3/4 < a < 3$. *(b)* Estimate $\lambda_1(a)$ and $\lambda_2(a)$ using the WKB approximation. *(c)* Assuming $|a| \ll 1$ use perturbation theory to compute the first two nonzero terms in the expansion of $\lambda_1(a)$ and $\lambda_2(a)$ about $a = 0$. Compare these approximations with the WKB solution — do they agree? *(d)* Compare the WKB approximation to those from `bvp4c` by plotting the various results for $\lambda_n(a)/n^2$ on the interval $-3/4 < a < 3$.

Remark: If $a = -1$ the differential equation has a turning point at $x = \pi/2$. This requires special analysis — so we're staying well away from this ticklish situation by taking $a > -3/4$.

Lecture 11

Boundary layers in fourth-order problems

11.1 A fourth-order differential equation

Let us consider a fourth-order boundary value problem which is similar to problems occurring in the theory of elasticity:

$$-\epsilon^2 u_{xxxx} + u_{xx} = 1\,, \tag{11.1}$$

with boundary conditions

$$u(-1) = u'(-1) = u(1) = u'(1) = 0\,. \tag{11.2}$$

The outer solution might be obtained with the RPS such as

$$u(x,\epsilon) = u_0(x) + \epsilon^{?} u_1(x) + \cdots \tag{11.3}$$

At leading order

$$u_{0xx} = 1\,, \qquad \Rightarrow \qquad u_0 = \frac{x^2-1}{2}\,. \tag{11.4}$$

We've applied only two of the four boundary conditions above.

Before worrying about higher order terms in (11.3), let's turn to the boundary layer at $x = -1$. We assume that the solution is an even function of x so the boundary layer at $x = +1$ can be constructed by symmetry.

If we look for a dominant balance with $X = (x+1)/\delta$ we find that $\delta = \epsilon$. Thus we consider a boundary layer rescaling

$$u(x,\epsilon) = U\left(X,\epsilon\right)\,, \qquad \text{where} \qquad X \stackrel{\text{def}}{=} \frac{x+1}{\epsilon}\,. \tag{11.5}$$

The boundary layer problem is then

$$-U_{XXXX} + U_{XX} = \epsilon^2\,. \tag{11.6}$$

Writing the leading-order outer solution in (11.4) in terms of X, we have

$$u_0(x,\epsilon) = -\epsilon X + \frac{1}{2}\epsilon^2 X^2\,. \tag{11.7}$$

Anticipating that we'll ultimately need to match the term $-\epsilon X$ in (11.7), we pose the boundary-layer expansion

$$U(X,\epsilon) = \epsilon U_1(X) + \epsilon^2 U_2(X) + \epsilon^3 U_3(X) + \cdots \tag{11.8}$$

There is no term $U_0(X)$ because the outer solution is $O(\epsilon)$ in the matching region.

Thus we have the hierarchy

$$-U_{1XXXX} + U_{1XX} = 0\,, \tag{11.9}$$
$$-U_{2XXXX} + U_{2XX} = 1\,, \tag{11.10}$$
$$-U_{3XXXX} + U_{3XX} = 0\,, \tag{11.11}$$

and so on.

The general solution of (11.9) is

$$U_1 = A_1 + B_1 X + C_1 \mathrm{e}^{-X} + \underbrace{D_1}_{=0} \mathrm{e}^{X}\,. \tag{11.12}$$

Above we've anticipated that $D_0 = 0$ to remove the exponentially growing solution. Then applying the boundary conditions at $X = 0$ we find

$$U_1 = A_1 \left(1 - X - \mathrm{e}^{-X}\right)\,. \tag{11.13}$$

To match (11.13) against the term $-\epsilon X$ in the interior solution in (11.7) we take

$$A_1 = 1\,. \tag{11.14}$$

Now we can construct a leading-order solution that is uniformly valid in the region near $x = -1$:

$$u_{\mathrm{uni}}(x) = \frac{x^2 - 1}{2} + \epsilon\left(1 - \mathrm{e}^{-(x+1)/\epsilon}\right)\,. \tag{11.15}$$

The derivative is

$$u_{\mathrm{uni}x}(x) = x + \mathrm{e}^{-(x+1)/\epsilon}\,, \tag{11.16}$$

which is indeed zero at $x = -1$.

Higher order terms

The equation for U_2, (11.10), has a solution

$$U_2(X) = \frac{X^2}{2} + A_2\left(1 - X - \mathrm{e}^{-X}\right)\,. \tag{11.17}$$

Above, we've satisfied both boundary conditions at $X = 0$. We've also matched the term $\epsilon^2 X^2/2$ in (11.7). To summarize, our boundary layer solution is

$$U(X) = \epsilon\Big(\underbrace{1}_{\text{orphan}} - X - \mathrm{e}^{-X}\Big) + \epsilon^2 \frac{X^2}{2} + \epsilon^2 A_2\left(1 - X - \mathrm{e}^{-X}\right) + O(\epsilon^3)\,. \tag{11.18}$$

But we have unfinished business: we have not matched the orphan above with any term in the leading-order outer solution $u_0(x)$.

To take care of the orphan we must go to next order in the interior expansion:

$$u(x,\epsilon) = \frac{x^2 - 1}{2} + \epsilon u_1(x) + O\left(\epsilon^2\right)\,. \tag{11.19}$$

Thus

$$u_{1xx} = 0\,, \qquad \Rightarrow \qquad u_1(x) = \underbrace{P_1}_{=1} + \underbrace{Q_1}_{=0} x \tag{11.20}$$

We take $Q_1 = 0$ because the solution is even, and $P_1 = 1$ to take care of the orphan. The solution $u_1(x)$ does not satisfy any of the four boundary conditions. To summarize, the outer solution is

$$u(x,\epsilon) = \frac{x^2 - 1}{2} + \epsilon + O(\epsilon^2)\,. \tag{11.21}$$

The $O(\epsilon)$ term above was accidently included in the uniform solution (11.16): in the outer region the expansion of (11.15) already agrees with all terms in (11.21).

Because $u_{0xxxx} = 0$, there are now no more non-zero terms in the outer region i.e., $u_2 = 0$, and therefore $A_2 = 0$ in (11.18). Moreover, all terms U_3, U_4 etcetera are also zero. Thus we have constructed an infinite-order asymptotic expansion. Using symmetry we can construct a uniformly valid solution throughout the whole domain

$$u_{\text{uni}}(x) = \frac{x^2 - 1}{2} + \epsilon \left(1 - \mathrm{e}^{-(x+1)/\epsilon} - \mathrm{e}^{(x-1)/\epsilon}\right) . \tag{11.22}$$

11.2 Problems

Problem 11.1. Solve (11.1) exactly and use MATLAB to compare the exact solution with the asymptotic solution in (11.22).

Problem 11.2. Find two terms in ϵ in the outer region and match to the inner solution at both boundaries for

$$\epsilon^2 u'''' - u'' = \mathrm{e}^{ax} . \tag{11.23}$$

The domain is $-1 \le x \le 1$ with BCs

$$u(-1) = u'(-1) = 0 , \qquad \text{and} \qquad u(1) = u'(1) = 0 . \tag{11.24}$$

Problem 11.3. Find two terms in ϵ in the outer region and match to the inner solution at both boundaries for

$$\epsilon^2 u'''' - u'' = 0 . \tag{11.25}$$

The domain is $0 \le x \le 1$ with BCs

$$u(0) = 0 , \quad u'(0) = 1 , \qquad \text{and} \qquad u(1) = u'(1) = 0 . \tag{11.26}$$

Problem 11.4. Considering the eigenproblem

$$-\epsilon^2 u'''' + u'' = \lambda u , \tag{11.27}$$

on the domain is $0 \le x \le \pi$ with BCs

$$u(0) = u'(0) = 0 , \qquad \text{and} \qquad u(\pi) = u'(\pi) = 0 . \tag{11.28}$$

(i) Prove that all eigenvalues are real and positive. *(ii)* Show that with a suitable definition of inner product, that eigenfunctions with different eigenvalues are orthogonal. *(iii)* Use boundary layer theory to find the shift in the unperturbed spectrum, $\lambda = 1, 2, 3 \cdots$, induced by ϵ.