SIO203C/MAE294C Assignment 37

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An oscillator problem

Consider the nonlinearly damped oscillator

$$\ddot{x} + \beta \dot{x}^3 + x = 0\,, \qquad x(0) = 1\,, \qquad \dot{x}(0) = 0\,. \tag{1}$$

Assuming that $\beta \ll 1$, use the energy equation and the method of averaging to determine the slow evolution of the amplitude a in the approximate solution (8.24). Take $\beta = 1$ and use `ode45` to compare a numerical solution with the method of averaging.

Answer

The energy equation for the oscillator in (1) is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\beta \dot{x}^4\,, \qquad \text{with} \qquad E \stackrel{\text{def}}{=} \frac{1}{2}\left(\dot{x}^2 + x^2\right)\,. \tag{2}$$

Using the approximate solution $x(t) \approx a \cos t$, with a a function of the slow time βt, we have

$$E \approx \langle \dot{x}^2 \rangle \approx \langle x^2 \rangle = \frac{1}{2} a^2\,, \tag{3}$$

and[1]

$$\langle \dot{x}^4 \rangle \approx a^4 \langle \sin^4 t \rangle = \frac{3}{8} a^4 = \frac{3}{2} E^2\,. \tag{4}$$

Thus the averaged energy equation is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{3\beta}{2} E^2\,, \tag{5}$$

[1] You can look up this integral, or calculate it like this

$$\langle \sin^4 t \rangle = \frac{1}{(2\mathrm{i})^4} \left\langle \mathrm{e}^{4\mathrm{i}t} + 4\mathrm{e}^{2\mathrm{i}t} + 6 + 4\mathrm{e}^{-2\mathrm{i}t} + \mathrm{e}^{-4\mathrm{i}t} \right\rangle = \frac{3}{8}\,.$$

with solution

$$E = \frac{2E_0}{2 + 3\beta t E_0}\,. \tag{6}$$

Using the initial condition in (1), the initial energy is $E_0 = 1/2$ and the approximate solution is therefore

$$x = \underbrace{\sqrt{\frac{4}{4 + 3\beta t}}}_{a(t)} \cos t\,. \tag{7}$$

Notice that $a \sim t^{-1/2}$ as $t \to \infty$. This algebraic decay is much slower than exponential decay characteristic of linear damping. The frictional force varies with the cube of the amplitude and thus as the oscillations get smaller the damping becomes increasingly weak.

Note the use of roman d for differentials $\mathrm{d}E/\mathrm{d}t$ etc. Common mathematical functions should be typeset with in roman using special latex definitions such as

`\cos` `\tan` `\ln` `\log`

This is how it should look

$$\cos\theta\,, \qquad \tan\phi\,, \qquad \ln x\,, \qquad \log(2\pi r)\,, \qquad \text{etc.} \tag{8}$$

Not like this ugly mess

$$cos\theta\,, \qquad tan\phi\,, \qquad lnx\,, \qquad log(2\pi r)\,, \qquad \text{etc.} \tag{9}$$

Text, such as etc. above, can be put into equations with the“text” command. All equations should be punctuated with commas, periods and so on. Use “quads” and “qquads” to put spaces between different mathematical expressions on the same line, as above. Without the qquad’s it’s a real mess:

$$cos\theta\,, tan\phi\,, lnx\,, log(2\pi r)\,, \text{etc.} \tag{10}$$

Acknowledgments

I collaborated with Gloria Teasdale and Harpo ‘Pinky’ Marx on this homework.

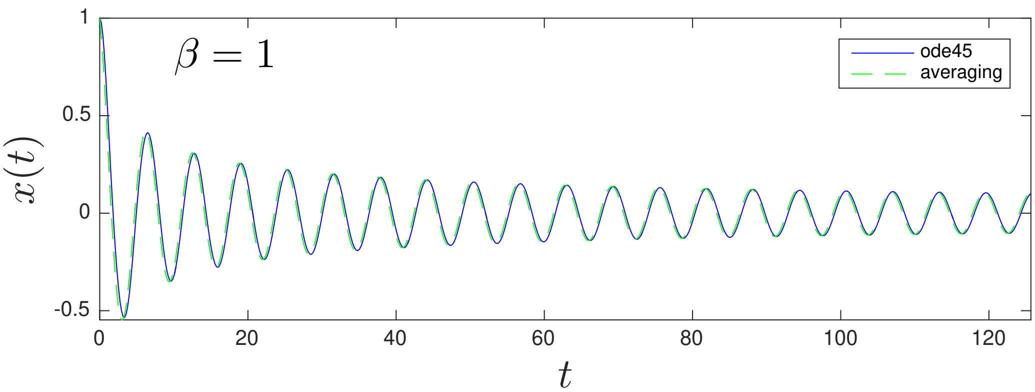

Figure 1: Comparison of the numerical solution of (1) with the approximate solution in (7). Even though $\beta = 1$ there is good agreement between the approximate solution (7) and the `matlab` numerical solution.

```
function nonLinDamp
%oscillator with cubic damping and the method of averaging
tspan = [0  40*pi];    t = linspace(0,max(tspan),600);
beta = 1;
options = odeset('AbsTol',1e-7, 'RelTol',1e-4);
aZero = [ 1 0 ];
sola = ode45(@ndamp,tspan,aZero,options);
xa = deval(sola,t);
subplot(2,1,1)
% The method of averaging approxiamtion:
a = sqrt(4./(4+3*beta*t)); xApprox = a.*cos(t);
plot(t,xa(1,:),'b-',t,xApprox,'g--')
xlabel('$t$','interpreter','latex','fontsize',20)
ylabel('$x(t)$','interpreter','latex','fontsize',20)
legend('ode45','averaging')
text(10,0.8,'$\beta=1$','interpreter','latex','fontsize',20)
axis tight

%------- nested function --------%
    function dxdt = ndamp(t,x)
    dxdt = [ x(2); - beta*x(2).^3 - x(1)];
    end

end
```