Assignment 3 SIO203B/MAE294B, 2025

Due by mid-night Friday April 25th
Submit by email to `wryoung@ucsd.edu`
with subject line `Third asymptotology assignment`

A corrugated boundary problem

Consider the partial differential equation

$$\kappa \left(C_{xx} + C_{zz} \right) - \mu C = 0 \tag{1}$$

in the region above $z = h(x)$, with $h(x) = a \cos kx$. The boundary conditions are $C(x, a\cos kx) = C_*$ and $C(x,z) \to 0$ as $z \to \infty$. *(i)* Describe a physical situation governed by this boundary value problem (ten or twenty words). *(ii)* Solve the problem with $a = 0$. *(iii)* Based on your exact solution, non-dimensionalize the problem with non-zero a and determine the non-dimensional control parameters. The small parameter, call it ϵ, should be linearly proportional to a. *(iv)* Use perturbation theory to find the first effects of small non-zero a on the "inventory"

$$A \stackrel{\text{def}}{=} \frac{k}{2\pi} \int_{h(x)}^{\infty} \int_0^{2\pi/k} C(x,z) \, \mathrm{d}x \, \mathrm{d}z \, . \tag{2}$$

(I think you'll have to go to second order in the relevant small parameter.)

Problem 2.6 a quintic. Use RPS or iteration – your choice.

(i) Find a two-term approximation to all five roots of

$$x^5 - x + \epsilon = 0 \, . \tag{3}$$

(ii) Suppose that $\eta = \epsilon^{-1}$. Find a two-term approximation to the five roots in the limit $\eta \to 0$ (which is the same as the limit $\epsilon \to \infty$).

Optional: Take $\epsilon = 1/4$ and compare your $\epsilon \ll 1$ approximation to a numerical solution e.g. use the handy MATLAB command `roots`.

A gift from John Hinch

Find the asymptotic behaviour of the Bessel function

$$K_\nu(z) \stackrel{\text{def}}{=} \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{e}^{\nu t - z \cosh t} \, \mathrm{d}t \, ,$$

with z fixed and $\nu \to \infty$. Use MATLAB `besselk` to make a comparison between the approximation and MATLABs evaluation of the Bessel function e.g. take $z = 1$ and $0 < \nu < 10$. Optional: play around with z and see if the approximations degrades as you make z larger and smaller.

Advice: Use the notation

$$\phi = -\nu t + z \sinh t \, . \tag{4}$$

Following the examples in the lecture let $t_*(\nu, z)$ denote the location of the peak and use the notation ϕ_* and ϕ_*'' for ϕ' and ϕ'' evaluated at t_*. Don't write out the details in terms of ν and z till the end of the calculation.