First recitation SIO203B/MAE294B, 2025

For discussion in the recitation on Friday April 4th

Problem 1.2 Pythagoras

(i) A triangle in the plane can be specified uniquely by giving the length of the longest side – call it c – and the acute angles θ and ϕ that the two shorter sides make with the longest side. Use dimensional analysis to say what you can about the area of the triangle in terms of c, θ and ϕ. (Pretend you don't know trigonometry: leave an undetermined dimensionless function in the answer.) *(ii)* Consider the special case of a right-angled triangle with sides a, b and c. Divide the triangle into two sub-triangles by dropping a perpendicular onto the long side with length c. The total area is the sum of the areas of two right-angled subtriangles. Use this observation to prove Pythagoras's theorem. *(iii)* Spherical triangles don't satisfy Pythagoras's theorem. How far can you proceed with the spherical version this problem?

Problem 1.3 back to high school

You can use the high school formula to exactly solve the quadratic equation

$$x^2 - \pi x + 2 = 0 \,. \tag{1}$$

But if you replace π by the approximation 3 then you can solve the equation by inspection. Define ϵ by $\pi = 3 + \epsilon$ and use an RPS to solve (1) neglecting terms of order ϵ^3 and smaller. Assess the accuracy of this solution.

Problem 1.12 keeping time

A grandfather clock swings at a maximum angle $\theta_{\max} = 5^\circ$ to the vertical. How many seconds does the clock lose or gain each day if it is adjusted to keep perfect time when the swing is $\theta_{\max} = 2^\circ$? (Use results from section 1.5 of the notes.)

Problem 2.1 a cubic equation

Find two terms in the $\epsilon \to 0$ expansion of the roots of $\epsilon x^3 + x - 1 = 0$.

Problem 2.2 a problem from the 2017 mid-term

$x(t)$ is defined via the initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \exp\left(\frac{x}{10}\right) - x \,, \qquad \text{with IC} \qquad x(0) = 0 \,. \tag{2}$$

Estimate $\lim_{t\to\infty} x(t)$. Convince me that the error in your approximation is probably less than 0.01.

Problem 2.5 how to solve the most difficult quadratic equations

Consider a quadratic equation, $ax^2 + bx + c = 0$, and suppose that $b^2 \gg 4ac$ (all coefficients are real). Use dominant balance (not the exact solution) to obtain a simple approximation to both roots. Test drive your approximation on $x^2 + 3x + 1/2 = 0$.

Problem 2.6 a quintic

(i) Find a two-term approximation to all five roots of

$$x^5 - x + \epsilon = 0\,. \tag{3}$$

Take $\epsilon = 1/4$ and compare your approximation to a numerical solution (e.g. use the MATLAB command `roots`). *(ii)* Suppose that $\eta = \epsilon^{-1}$. Find a two-term approximation to the five roots in the limit $\eta \to 0$ (which is the same as the limit $\epsilon \to \infty$).

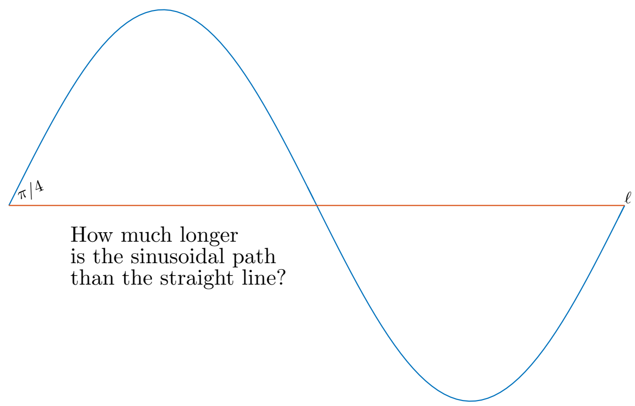

Figure 1: A tipsy walk.

Problem 1.9 a tipsy walk

The figure above shows the path followed by a tipsy sailor from a bar at the origin of the (x, y)-plane to home at $(x, y) = (\ell, 0)$. The path is a sinusoid leaving the bar at an angle α; in the figure 1 $\alpha = \pi/4$. How much longer is the sinusoidal path than the straight line? Answer this question by: *(i)* eyeballing the curve in figure 1 and guessing; *(ii)* constructing the integral that gives the arclength and evaluating it numerically with MATLAB; *(iii)* devising an approximation to the arc-length integral based on $\alpha \ll 1$, and then pressing your luck by using this approximation with $\alpha = \pi/4$.

Problem 1.10 the ocean surface

Suppose that the surface of the ocean is disturbed by a superposition of two surface gravity waves:

$$z = a_1 \cos(k_1 x - \sigma_1 t) + a_2 \cos(k_2 x - \sigma_2 t)\,. \tag{4}$$

Assuming small wave steepness (i.e. $a_n k_n \ll 1$), estimate the increase in ocean surface area produced by these waves. Generalize to more waves, $+a_3 \cos(k_3 x - \sigma_3 t)$ etc.

A problem from Bender & Orszag

Consider $(x + 1)^7 = \epsilon x$ with $\epsilon \ll 1$. How rapidly do the 7 roots vary from $x = -1$ as ϵ increases from zero? Give the first three terms in the expansion.