Third recitation SIO203B/MAE294B, 2025

For discussion in the recitation on Friday April 18th

An ODE

Consider

$$y' - 2xy = -1\,, \qquad \text{with the requirement that } \lim_{x\to\infty} y(x) = 0. \tag{1}$$

(i) Solve the ODE using the integrating factor method and write the solution in terms of $\operatorname{erfc}(x)$. *(ii)* Apply dominant balance and iteration to (9) to obtain three or four terms in the $x \to \infty$ expansion of $y(x)$. Compare to the asymptotic expansion of erfc in equation (4.28) of the notes.

An integral

Show that

$$A(x) \stackrel{\text{def}}{=} \int_0^\pi e^{x\cosh t}\,\mathrm{d}t \sim \frac{e^{x\cosh\pi}}{x\sinh\pi}\,, \qquad \text{as } x \to \infty. \tag{2}$$

(No need to justify asymptoticity – bash out the answer.)

More integrals

Find the leading-order $x \to \infty$ asymptotic approximation to

$$I(x) = \int_0^\infty e^{-xt}\ln(1+t^2)\,\mathrm{d}t\,, \qquad J(x) = \int_0^\infty \frac{e^{-xt}}{t^{1/3}(1+t)}\,, \qquad K(x) \stackrel{\text{def}}{=} \int_1^\infty e^{-xt^{1/2}}\,\mathrm{d}t\,. \tag{3}$$

A corrugated boundary problem

Consider the partial differential equation

$$\kappa\,(C_{xx} + C_{zz}) - \mu C = 0 \tag{4}$$

in the region above $z = h(x)$, with $h(x) = a\cos kx$. The boundary conditions are $C(x, a\cos kx) = C_*$ and $C(x,z) \to 0$ as $z \to \infty$. *(i)* Describe a physical situation governed by this boundary value problem. *(ii)* Solve the problem with $a = 0$. *(iii)* Based on your exact solution, non-dimensionalize the problem with non-zero a and determine the non-dimensional control parameters. *(iv)* Use perturbation theory to find the first effects of small non-zero a on the "inventory"

$$A \stackrel{\text{def}}{=} \frac{k}{2\pi}\int_{h(x)}^\infty \int_0^{2\pi/k} C(x,z)\,\mathrm{d}x\,\mathrm{d}z\,. \tag{5}$$

(I think you'll have to go to second order in the relevant small parameter.)

Time aloft (again)

Refer to section 3.1 of the notes for the notation here e.g. $\epsilon = u^2/Rg_0$. Let's use energy conservation to obtain a better solution to the projectile problem. *(i)* From the non-dimensional equation of motion (3.2) in the notes, show that

$$\tfrac{1}{2}\dot{z}^2 - \frac{1}{\epsilon}\frac{1}{1+\epsilon z} = \tfrac{1}{2} - \frac{1}{\epsilon}\,. \tag{6}$$

(ii) Find the maximum height reached by the projectile, $z_{\max}$, in terms of ϵ. *(iii)* Show that the time aloft is given exactly by

$$\tau = 2z_{\max} \int_0^1 \sqrt{\frac{1+a\xi}{1-\xi}}\, d\xi\,, \qquad \text{with} \qquad a(\epsilon) \stackrel{\text{def}}{=} \frac{\epsilon}{2-\epsilon}\,. \tag{7}$$

(iv) Evaluate the ξ-integral above exactly – the answer involves $\arctan(\sqrt{a})$. You can use MATHEMATICA or you might be able to look this integral up e.g. in Gradshteyn & Ryzhik. *(v)* In the lectures we found that the non-dimensional time aloft is

$$\tau = 2 + \frac{4\epsilon}{3} + O(\epsilon^3)\,. \tag{8}$$

Using $a \ll 1$, obtain an alternative approximation (just the first two terms) by simplifying the ξ-integral in (7). Make a comparison with the exact answer from *(iv)*. Which approximation is superior, a or ϵ?

An elementary boundary-layer example

In the lectures we saw that the exact solution of

$$\epsilon^2 v'' - v = 0\,, \qquad \text{and} \qquad v(\pm 1) = 1\,, \tag{9}$$

is

$$v = \frac{\cosh(x/\epsilon)}{\cosh(1/\epsilon)}\,. \tag{10}$$

(i) Draw a graph of the solution with $\epsilon = 1$, $1/4$, and $1/16$. In the limit $\epsilon \to 0$ simplify the solution in (10) in the neighbourhood of $x = -1$. Hint: hold a magnifying glass to this region by introducing the "stretched coordinate"

$$X \stackrel{\text{def}}{=} \frac{x+1}{\epsilon}\,, \qquad \Rightarrow \qquad x = -1 + \epsilon X\,. \tag{11}$$

(ii) Derive the simplified version of $v(x,\epsilon)$ by direct solution of (9) in vicinity of $x = -1$.

Another boundary-layer problem

Find an approximate solution of the boundary value problem:

$$10^{-12} w_{xx} = e^{x^4}\, w\,, \qquad \text{with BCs} \qquad v(\pm 1) = 1\,. \tag{12}$$

From the 2017 mid-term

$x(t)$ is defined via the initial value problem

$$\frac{dx}{dt} = \exp\left(\frac{x}{10}\right) - x\,, \qquad \text{with IC} \qquad x(0) = 0\,. \tag{13}$$

Estimate $\lim_{t\to\infty} x(t)$. Convince me that the error in your approximation is probably less than 0.01.

The last quadratic equation problem

By inspection find a simple approximation to the solutions of the quadratic equation $x^2 - 19x + 1 = 0$. How about $x^3 - 19x + 1 = 0$? How about $x^{n+1} - 19x + 1 = 0$ as $n \to \infty$?